Series on Grey System

Sifeng Liu

# Grey Systems Analysis

## Methods, Models and Applications

*Second Edition*

OPEN ACCESS

Springer

# Series on Grey System

**Series Editors**

This series aims to publish books on grey system and various applications in the fields of natural sciences, social sciences and engineering.

This series is devoted to the international advancement of the theory and application of grey system. It seeks to foster professional exchanges between scientists and practitioners who are interested in the models, methods and applications of grey system. Through the pioneering work completed over 40 years, grey data analysis methods have become powerful tools in addressing system with poor information.

Books published with this series will explore the models and applications of grey system, in order to tackle poor information more effectively and efficiently. The series aims to provide state-of-the-art information and case studies on new developments and trends in grey system research and its potential application to solve practical problems.

Coverage includes, but is not limited to:

- Foundations of grey systems theory
- Grey sequence operators
- Grey relational analysis models
- Grey clustering evaluations models
- Techniques for grey system forecasting
- Grey models for decision-making
- Combined grey models
- Grey input-output models
- Techniques for grey control
- Various applications of grey system models in the fields of natural sciences, social sciences and engineering.

Sifeng Liu

# Grey Systems Analysis

## Methods, Models and Applications

Second Edition

Springer

Sifeng Liu
Center for Grey Systems Studies
Northwestern Polytechnical University
Xi'an, China

ISSN 2731-4936 ISSN 2731-4944 (electronic)
Series on Grey System
ISBN 978-981-97-8726-5 ISBN 978-981-97-8727-2 (eBook)
https://doi.org/10.1007/978-981-97-8727-2

This work was made possible due to projects supported by the national major talent programme of China, the Marie Curie International Incoming Fellowship of the European Union, the National Natural Science Foundation of China, the Leverhulme Trust International Network, the joint projects supported by the NSFC and the RS in the UK, the Fundamental Research Funds for the Central Universities and the Publishing Fund of Excellence Academic Works of NPU.



This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

# Series Preface

This series will publish the books on grey system theory and various applications in the fields of natural sciences, social sciences and engineering.

It is devoted to the international advancement of the theory and application of grey system theory, and seeks to foster professional exchanges between scientists and practitioners who are interested in the models, methods and applications of grey system theory. Through the pioneering work completed over 40 years, grey system analysis methods have become powerful tools in addressing system with poor information.

Books published with this series will explore the models and applications of grey system theory, in order to tackle poor information more effectively and efficiently. The series aims to provide state-of-the-art information and case studies on new developments and trends in grey system research and its potential application to solve practical problems.

In the era of big data, the grey system theory based on poor information data mining has sprung up. It has become an effective tool for people to extract valuable information from massive data. In the past 40 years, grey system method and model have been widely used in many fields, such as social science, natural science and engineering technology, which has led to innovation and progress in various fields. More and more people interested in grey system theory and a lot of new results have been obtained in recent years. In particular, successful applications in many fields have won the attention of the international world of learning.

Scholars from more than 100 countries and regions in the world have published more than 300,000 documents of grey system research and applications.

On the 7th of September, 2019, Angela Dorothea Merkel, then German Chancellor, praised grey system theory in her speech at Huazhong University of Science and Technology. She said that the work of Prof. Deng Julong, the founder of grey system theory and Prof. Liu Sifeng, the editor of this series, “have made a profound impact on the world.”

The Coverage of this series includes, but is not limited to:

- Foundations of grey systems theory
- Grey sequence operators
- Grey relational analysis models
- Grey clustering evaluations models
- Techniques for grey system forecasting
- Grey models for decision-making
- Combined grey models
- Grey input-output models
- Techniques for grey control
- Various applications of grey system models in the fields of natural sciences, social sciences and engineering.

If you are interested in the series on grey systems, please contact with Ms. Emily Zhang at emily.zhang@springernature.com or Prof. Sifeng Liu at sfliu@nwpu.edu.cn.

Xi'an, China

Prof. Sifeng Liu, Ph.D.
Editor of the Book Series on Grey System, Director of Center for Grey Systems Studies, NPU, President of International Association of Grey System and Uncertain Analysis

# Preface

In this book we answer the calls of the readers of our previous publications, and systematically present the main advances in grey system theory and applications. By following our readers' feedback and suggestions, this volume introduces the most recent research results and updates on what is presented in our earlier books. In particular, the following content, which represents the author's recent research, is highlighted in the book: general grey numbers and their operations, negative grey relational analysis models and grey relational analysis models based on similarity and closeness, three dimensional grey relational analysis models, grey clustering evaluation models based on mixed possibility functions, original difference grey model (ODGM), even difference grey model (EDGM), discrete grey model (DGM), fractional grey models, self-memory grey models, multi-attribute weighted intelligent grey target decision models, weight vector group with kernel and weighted comprehensive clustering coefficient vector. We also attach a software designed for grey system modelling, which was developed by Bo Zeng using Visual C#, the widely employed C/S software tool. This user-friendly software allows users to conveniently input and/or upload data and clearly distinguish module functions. Also, the software has the ability to present users with operational details, as well as periodic and partial results. Additionally, users can adjust the levels of computational accuracy based on their practical needs.

During the writing of this book, we prioritized theoretical simplicity and clarity to make it easy for the reader to follow the main arguments made. With a good number of practical applications, we intended to illustrate the methodology of grey system theory and modelling techniques so that we could emphasize the practical applicability of grey system thinking. We drew on the most recent research developments from various research groups around the world and tried to present the most complete picture of this new area of scientific endeavor in a concise manner.

The overall planning and organization of topics contained in this book were carried out by Sifeng Liu, who also authored Chaps. 1, 2, 4, 6, 10 and 12. Yingjie Yang produced Chaps. 3, and 11, Jeffrey Forrest composed Chaps. 7 and 8, Naiming Xie wrote Chap. 9, and the Appendix and the attached computer software were developed by Zeng Bo. Zhigeng Fang, Yaoguo Dang, Lirong Jian and Chunhua Su and colleagues also worked with the authors to refine some of the book's content. Sifeng Liu was responsible for unifying the terms used throughout the book and for finalizing the manuscript.

Finally, we would like to encourage you to communicate with us and send us any comments you might have about this book. After decades or more of continuous polishing and improvement, an academic work can become increasingly perfect and recognized as a masterpiece. Sifeng Liu can be reached at sfliu@nwpu.edu.cn.

Xi'an, China
August 2024

Sifeng Liu

# Foreword by Prof. Alain Bernard

In the era of big data, the paradigm of scientific research is undergoing fundamental changes. The Fourth Paradigm: Data-Intensive Scientific Discovery which proposed by Jim Gray, a Turing Award winner, is increasingly becoming the mainstream paradigm in scientific research. The significant feature of big data is its low information density. The characteristics and operation rules of various systems are like gold buried in a sea of sand, deeply concealed by the big chaotic and complex uncertain data.

In 1982, Prof. Julong Deng founded the grey system theory, which is a distinctive method for modelling and analysing uncertain data. Grey system theory takes the "poor data" uncertain system with "some information known and some information unknown" as the research object. It mainly extracts valuable information through the mining of "some" known information, and realizes the correct description of the system operation behavior and evolution law, so that people can use mathematical models to analyse and assess the "poor data" uncertain system, then realize high-precision prediction, scientific decision-making and optimal control of the "poor data" uncertain system.

Prof. Liu has been dedicated to grey system research for 40 years. The series of concepts and models he proposed have become classics in this field. Such as kernel, degree of greyness of grey number, simplified form of grey number, general grey numbers and their algebraic systems; sequence operator, weakening and strengthening buffer operators; A series of grey relational analysis models based on a global perspective; The grey evaluation model based on mixed possibility function of endpoints and center points, a multi-objective weighted intelligent grey target decision-making model, and a two-stage grey decision-making model based on a kernel weight vector group; And various original poverty information data prediction models such as original difference models, mean difference models, discrete grey models, fractional order grey models, and self memory models proposed in collaboration with his students. These original achievements have greatly enriched the knowledge system of grey system theory. Various editions of his seminal book on Grey system theory have been published in different languages such as Chinese, English, Romanian and Korean. Hundreds of universities from around the world adopted them as textbooks. There are more than one million audiences of his books, videos and software of grey modelling. In 2024, he was selected as one of the top 0.05% Lifetime Highly Ranked Scholar in Systems Theory by Scholar GPS. His publications have been cited 51270 times with an H-Index of 95 in Huezhi Scholar.

As a new edition of his book of *Grey Systems Analysis* is about to be published, I am great honored to write the foreword for this classic work. This book will undoubtedly benefit more grey system theory learners and researchers as it has been funded by Publishing Fund of Excellence Academic Works of NPU and will be published as OA book. It is expected that it will be widely spread around the world, promote the in-depth application of grey system methods and models and benefit all mankind.

Prof. Alain Bernard
Honorary Professor Centrale Nantes
Digital Sciences Laboratory of Nantes (LS2N UMR CNRS 6004)
Fellow member of the National Academy of Technologies of France
CIRP Fellow Emeritus
France

# Foreword by Prof. Edmundas Kazimieras Zavadskas

As a new edition of *Grey Systems Analysis* by Prof. Sifeng Liu is about to be published, I am great honored to write the preface for this classic work in the field of grey systems research.

In the mid to late 20th century, human society began to move towards the information age. People are beginning to deeply realize that data analysis methods have become an indispensable skill for everyone. The characteristics and operating rules of various systems are like gold buried in a sea of sand, deeply concealed by chaotic and complex data information, and there is an urgent need for effective scientific methods to explore and reveal. In response to the needs of the times, as a poverty information data analysis method, grey system theory has emerged. Grey system theory takes the "poor data" uncertain system with "some information known and some information unknown" as the research object. It mainly extracts valuable information through the mining of "some" known information and realizes the correct description of the system operation behavior and evolution law, so that people can use mathematical models to analyse and assess the "poor data" uncertain system, then realize high-precision prediction, scientific decision-making and optimal control of the "poor data" uncertain system.

Prof. Liu has been dedicated to grey system research for 40 years, and his series of original concepts and models have become classics in the field. Such as general grey numbers, simplified forms of grey numbers, and their algebraic systems; Construction and properties of sequence operators and practical buffer operators; A series of grey relational analysis models based on a global perspective; The grey evaluation model based on a mixed possibility function of endpoints and center points, a multi-objective weighted intelligent grey target decision-making model, and a two-stage grey decision-making model based on a kernel weight vector group; And various original poverty information data prediction models such as original difference models, mean difference models, discrete grey models, fractional order grey models and self memory models proposed in collaboration with his students.

Especially his seminal books greatly promoted the dissemination and development of grey system theory. *The Grey System Theory and Its Applications*, first published in 1991, were deeply loved by readers. In 2024, Science Press released its 10th edition, which was rated as the first highly cited book in pandect of Natural Science by China National Knowledge Infrastructure. Multiple English versions, such as *An Introduction to Grey System Theory* (1998, IIGSS Academic Publisher, USA), *Grey Information* (2006, Springer London Ltd., UK), *Grey Systems* (2011, Springer-Verlag, DE), *Grey Data Analysis* (2016, Springer, SG), *Grey Systems Analysis* (2022, Springer, SG), are the first choice for scholars from all over the world to understand grey system theory and its research progress.

Currently, scholars from over 130 countries or regions around the world have published papers on grey systems. My team has been conducting grey system theory research for over 20 years. And starting to publish papers related to grey systems in the early 21st century. We have successfully applied grey system methods and models to solve problems such as construction project evaluation and supplier selection, and proposed multiple combined grey models, such as COPRAS-G, ARAS-G and EDAS-G.

This book will undoubtedly benefit more grey system theory learners and researchers as it is published in OA format with the support of the Excellent Academic Works Publishing Fund of Northwestern Polytechnical University. Grey system theory is a powerful tool for analysing uncertain data in the era of big data. I look forward to its widespread dissemination worldwide, promoting the in-depth application of grey system theory in the fields of natural sciences, social sciences and engineering technology.

Prof. Edmundas Kazimieras Zavadskas, Ph.D., D.Sc., Dr h.c.mult
Founder of Journal *Technological and Economic Development of Economy*
Founder of *Journal of Civil Engineering and Management*
Member of Lithuanian Academy of Science
Honorary GSUA Fellow
Chief Researcher, Institute of Sustainable Construction
Faculty of Civil Engineering
Vilnius Gediminas Technical University
Vilnius, Lithuania

# Foreword by Dr. James M. Tien

It gives me great pleasure to be introducing this 8th edition of *Grey System Theory and Its Applications* by Prof. Sifeng Liu. The theory of grey systems was first introduced in 1982 by J. L. Deng (1933–2013) at Huazhong University of Science and Technology; it established a relatively new approach for addressing poorly defined problems with a high level of greyness or uncertainty. The theory enables one to model, analyse, monitor and control such partially defined systems by generating, excavating and extracting useful information from what is available. It built on the work of Dr. Lotfi A. Zadeh, who introduced the concept of fuzzy sets in the 1960s that in turn led to breakthroughs in neural networks and soft computing.

*Grey System Theory* actually combines two critical and overarching areas. The first concerns systems which attempt to synthesize the various components or subsystems into an overall functioning system or system of systems. Systems theory attempts to make transparent the deep connections and interactions among objects and events, all leading to the enrichment and progress of science and technology. Many of the historically difficult, hard-to-solve problems in the different scientific fields have been successfully resolved through the application of systems theory and its allied methodologies, including information theory, cybernetics, combinatorics, genetics, etc. The second concerns the greyness or uncertainty level that is implicit in all natural or man-made systems. Indeed, most modelling techniques assume the existence of uncertainty or stochasticity, as defined by either empirical evidence or assumed distributions, including fuzzy sets.

*Grey System Theory*, then, provides a realistic approach to modelling, analysing, monitoring and controlling systems. Professor Sifeng Liu has greatly extended, if not expanded, Prof. Deng's earlier efforts. In the 1980s, he put forward a series of new models and concepts, including sequence operator, absolute degree of grey incidence, grey cluster evaluation model with fixed weight, and positioned coefficient of grey matrix. In the 1990s, he proposed a buffer operator and its axiom, generalized degree of grey incidence, grey number and measurement of its information content, drifting and positioning solution, the grey-econometrics model GM(1,1), the grey Cobb-Douglass model, etc. More recently, he proposed the concept of general grey numbers, the grey algebraic system based on a kernel and degree of greyness, and different variations of the model GM(1,1).

The widespread recognition and application of grey system theory reflect its growing acceptance. A number of universities from around the world has adopted Prof. Sifeng Liu's monographs, both in Chinese and English, as their textbooks. In 2002, he won the World Organization of Systems and Cybernetics (WOSC) Prize. In 2008, as a preeminent Chinese scholar, he was elected an Honorary WOSC Fellow. In 2013, after a strict review by the European Commission, he was selected to be a Marie Curie International Fellow, thus honoring him as the first such Fellow with grey systems expertise.

As a systems scientist and engineer, I am honored to write this foreword for the 8th edition of *Grey System Theory and Its Applications*. I look forward to its widespread dissemination and its promulgation of grey system applications in science and engineering.

James M. Tien, Ph.D., DEng (h.c.), NAE
Distinguished Professor and Dean Emeritus
College of Engineering
University of Miami
Coral Gables, FL, USA

**Note** Professor James M. Tien prepared this note for 8th edition of *Grey System Theory and Its Applications* (in Chinese) by the same authors, published in 2016. With his permission, it is printed here as a foreword for this current book.

# Foreword by Dr. Keith William Hipel

**Grey Systems: Theory and Applications**
Written by Sifeng Liu and Yi Lin
Springer-Verlag: Berlin, Heidelberg
2010, 379 pages, ISBN 978-3-642-16158-2 (cloth)
DOI: 10.1007/978-3-642-16158-2

Professors Sifeng Liu and Yi Lin have written another pioneering book on the important topic of grey systems. In 2006, the same authors wrote the well-received book entitled *Grey Information: Theory and Practical Applications* which was also published by Springer-Verlag. I am pleased to say that their second book on Grey Systems constitutes a significant expansion and improvement of their previous fine book. Accordingly, if you already possess a copy of the 2006 book, you can make a worthwhile academic investment by obtaining a copy of their recent book in order to be cognizant of the latest ideas and advancements in the crucial field of grey systems.

The question that naturally arises is why grey systems are of such great import at this point in history. The answer is quite straightforward: many challenging problems facing society consist of interconnected complex systems of systems exhibiting high uncertainty and having few measurements. For example, in order to effectively combat climate change, one must understand as much as possible the complex interactions among natural systems such as atmospheric, oceanic, geological and hydrological systems, with societal systems including energy production, industrial, agricultural and city systems. The deep uncertainty involved with these interconnected systems of systems and their potential emergent behavior, coupled with a dearth of observations, mean that formal tools for handling this uncertainty are in high demand. Fortunately, an arsenal of mathematically based methodologies and techniques have been developed over the years: a rich variety of probabilistic-based tools, fuzzy sets founded by Lotfi Zadeh, rough sets started by Z. Pawlak, information-gap modelling perfected by Yakov Ben-Haim, uncertainty theory developed by Baoding Liu, and grey systems established by Julong Deng in 1982. The foregoing and other approaches to describing uncertainty are based upon different axioms and are thereby highly complementary for tackling a wide variety of uncertain situations.

Grey systems are purposefully designed for modelling uncertain systems, or systems of systems, problems having small samples and low-quality information. Grey systems are capable of dealing with partially known information through generating, excavating and extracting useful information from what is available. How this is accomplished is explained in depth in the timely grey systems book of Profs. Liu and Lin.

In their contemporary textbook, Liu and Lin systematically present the theory and practice of grey systems. In fact, the excellent ideas and applications contained in their book are based upon the authors' many years of developing theoretical concepts, applying their methods to real world applications, testing and refining their new techniques with actual data, carrying out stimulating research with their students and colleagues, teaching their students about their exciting work and delivering research papers at international conferences around the globe. Their comprehensive book contains the latest theoretical and applied advances created by the authors and other scholars around the world in order to place the readers at the forefront of international research in grey systems.

The main body of their book contains ten well-explained and interconnected chapters: Introduction to Grey Systems Theory, Basic Building Blocks, Grey Incidence and Evaluation, Grey Systems Modeling, Discrete Grey Prediction Models, Combined Grey Models, Grey Models for Decision Making, Grey Game Models, Grey Control Systems, and Introduction to Grey Systems Modeling Software. Moreover, the book includes a computer software package developed for grey systems modelling to permit both researchers and practitioners to use the new methodologies. Their book concludes with three appendices. The first appendix compares grey systems theory and interval analysis while revealing the fact that interval analysis is a part of grey mathematics. The second presents an array of different approaches to studying uncertainties. Finally, the last appendix shows how uncertainties occur using a general systems approach.

The book contains a wealth of mathematical results, techniques and algorithms which are presented by the authors for the first time. These contributions include an axiomatic system of buffer operators and a series of weakening and strengthening operators; axioms for measuring the greyness of grey numbers; general grey incidences (grey absolute incidence, grey relative incidence, grey comprehensive incidence, grey analogy incidence and grey nearness incidence); discrete grey models; fixed weight grey cluster evaluation; and grey evaluation methods based on triangular whitenization weight functions, multi-attribute intelligent grey target decision models, applicable range of the G(1,1), grey econometrics (G-E), grey Cobb-Douglass (G-C-D), grey input-output (G-I-O) and grey game models (G-G).

In their well-written book, Drs. Liu and Lin do a thorough job in their presentation of many difficult technical concepts. The authors are able to convince the readers of their book regarding the power and usefulness of their new theory by presenting many interesting examples of practical applications to real-life problems. The challenging practical problems addressed in their book include urban economic planning, downtown traffic design, natural disaster prediction, relative strength evaluation of a state, investment projection of a company and employee performance evaluation.

The depth and scope of the advancements in grey systems covered in this book, in conjunction with clarity of explanation, make this seminal book attractive to researchers, students, teachers and practitioners working in many different fields. These areas of endeavor include image processing, video processing, multimedia security, computer vision, machinery, control, agriculture, water resources, medicine, astronomy, earth science, economics and management. I personally found grey systems useful for accurately forecasting wastewater time series for which there is a scarcity of data. I intend to keep a copy of this valuable book easily accessible in my university office and purchase more copies of the book for use by my students.

Keith William Hipel
Ph.D., P.Eng., FIEEE, FINCOSE, FCAE, FEIC, FRSC, FAWRA
*University Professor* of Systems Design Engineering
University of Waterloo
Waterloo, ON, Canada
e-mail: kwhipel@uwaterloo.ca
Website: http://www.systems.uwaterloo/Faculty/Hipel

*Senior Fellow*
Centre for International Governance Innovation
Waterloo, ON, Canada

**Note** Professor Keith William Hipel prepared this note for one of the earlier book by the same authors, published in 2010. It is published in *Grey Systems: Theory and Application*, 2011, Vol. 1, No. 3. With his permission, it is printed here as a foreword for this current book.

# Foreword by Dr. Hermann Haken

With human knowledge maturing and scientific exploration deepening and largely expanding in the course of time, mankind finally realizes the fundamental fact that due to both internal and external disturbances and limitations of human and technical sensing organs, all information received or collected contains some kind of uncertainty. Accompanying the progress of science and technology and the aforementioned realization, our understanding about various kinds of uncertainties has gradually been deepened. Attesting to this end, in the second half of the 20th century, the continual appearance of several influential and different types of theories and methods on unascertained systems and information has become a major aspect of the modern world of learning. Each of these new theories was initiated and followed-up by some of the best minds of our modern time.

In their recent book, entitled *Grey Information: Theory and Practical Applications*, published in its traditionally excellent way by Springer, Profs. Sifeng Liu and Yi Lin presented in a systematic fashion the theory of grey system, which was first proposed by J. L. Deng in early 1980s and enthusiastically supported by hundreds of scientists and practitioners in the following years. Based on the hard work of these scholars in the past (nearly) thirty years, scholars from many countries currently are studying and working on the theory and various applications of this fruitful scientific endeavor. With this book published by such a prestigious leading publisher of the world, it can be expected that more scientific workers from different parts of the world will soon join hands and together make grey system and information a powerful theory capable of bringing forward practically beneficial impacts to the advancement of the human society.

This book focuses on the study of such unascertained systems that are known with small samples or "poor information." Different of all other relevant theories on uncertainties, this work introduces a system of many methods on how to deal with grey information. Starting off with a brief historical introduction, this book carries the reader through all the basics of the theory. And, each important method studied is accompanied with a real-life project the authors were involved in during their professional careers.

Many of the methods and techniques the reader will learn in this book were originally introduced by the authors. They show how from our knowledge based on partially and poorly known information can be obtained to accurate descriptions and effective controls of the systems of interest. Because this book shows how the theory of grey system and information was established and how each method could be practically applied, this book can easily be used as a reference by scholars who are interested in either theoretical exploration or practical applications or both. I recommend this book highly to anyone who has either a desire or a need to learn.

Stuttgart, Germany
July 2007

Professor Dr. Dr. h.c. mult. Hermann Haken
Founder of Synergetics

**Note** Professor Hermann Haken prepared this note for one of the earlier book by the same authors, published in 2006. It is published in *Grey Systems: Theory and Application*, 2011, Vol. 1, No. 1.

# Foreword by Dr. Robert Vallée

I am much interested and impressed by Dr. Sifeng Liu and Dr. Yi Lin's recently published monograph on grey information, dealing with the theory and practical applications.

This book encompasses many aspects of mathematics under the aegis of uncertain information. I am greatly in favour of this attitude, concerning the uncertainty of information, which has been mine since a long time ago. Also, this book focuses on practice and aims at explorations of new knowledge. It is a comprehensive, all-in-one exposition, detailing not only with the theoretical foundation but also real-life applications. Because of this characteristic of quality and usefulness, Liu and Lin's book possesses the value of the widest possible range of reference by the workers and practitioners from all corners of natural and social sciences and technology.

In this book, Liu and Lin present the theory of grey information and systems starting on such background information as the relevant history, an attempt to establish an unified information theory, the basics of grey elements, and reaching all the most advanced topics of the theory. Complemented by many first-hand and practical project-successes, the authors developed an organic theory and methodology of grey information and grey system, dealing with errors. In fact, there is much more to tell about error than about truth. Error (inexactitude) can be met everywhere and truth (exactitude) nowhere. But inexactitude contains a part of the truth. Greyness is the field we live in. Extremes, as whiteness and blackness, are inaccessible, but very useful, ideal concepts.

With the publication of such a book that contains not only a theory, aspects of magnificent real-life implications and explorations of new research, but also the history, the theorization of various difficult concepts, and directions for future works, there is no doubt that Drs. Liu and Lin have made a remarkable contribution to the development and applications of systems science.

June 2007

Prof. Robert Vallée
President of the World Organisation
of Systems and Cybernetics, Université
Paris-Nord
Paris, France

**Note** This note is a book review written by Prof. Robert Vallée for one of the earlier book by the same authors, published in 2006. It is published in *Kybernetes: The International Journal of Cybernetics, Systems and Management Science*, 2008, Vol. 37, No. 1.

# Acknowledgements

The work presented in this book is supported by a project of the national major talent programme of China (YQR20024), a project of Marie Curie International Incoming Fellowship of the European Union, (629051). It is also supported by a project of the Leverhulme Trust International Network (IN-2014-020), the joint projects of both the NSFC and the RS in the UK (71811530338, 71111130211) and projects linked to the National Natural Science Foundation of China (72071111, 71671091, 71671090, 91324003, 90924022, 70473037, 70971064). The projects of Intelligence Introduction base of the Ministry of Science and Technology of China (G2021181014L, G20190010178). A project of National Society Open Cooperation Demonstration Program of China Association for Science and Technology (2021KFHZ001). At the same time, the authors would like to acknowledge the partial support of the Fundamental Research Funds for the Central Universities of China (ILA220721A22, NC2019003, NP2015208). The Foundation for National Outstanding Teaching Group of China (No. 10td128), the Foundation for Doctoral Programs (grant No. 20020287001, 200802870020, 20093218120032) and the Key Project of Natural Science Foundation of Jiangsu Province (grant No. BK2003211). The Science Foundation for the Excellent and Creative group in Science and Technology (on Grey System) of Nanjing University of Aeronautics and Astronautics and Jiangsu Province (No.Y0553-091), the Foundation for the Excellent Young Scientists in Henan Province (No. 964040600) have also provided their support, a project for Innovation Team of Shaanxi Sanqin Talents Special Support Program, and a project of Publishing Fund of Excellence Academic Works of Northwestern Polytechnical University.

Over the years, our research has been highly commended by many first class scholars, such as Julong Deng, the founder of grey system theory, L. A. Zadeh, the founder of fuzzy mathematics, Hermann Haken, the founder of synergetics, Robert Vallee, former president of the World Organization of Cybernetics and Systems, James Tien, former vice-president of IEEE and member of the American Academy of Engineering, K. W. Hipel, former president of The Academy of Science of the Royal Society of Canada and, Jifa Gu, former president of the International Federation for Systems Research. And praised or cited positively by more than 100 academicians from academies of science and engineering all over the world, such as E. K. Zavadskas, A. Bernard, Guozhi Xu, Qun Lin, Da Chen, Chunsheng Zhao, Haiyan Hu, Suzi Yang, Youlun Xiong, Zhongtuo Wang, Shanlin Yang, Xiaohong Chen and others. Xuesen Qian, the first winner of the National Highest Science and Technology Award of China, Jinpeng Huai, minister of the Ministry of Education of China, Angela Dorothea Merkel, former German Chancellor, have also praised us for the achievements in this area of research. Indeed, such positive feedback on our work has given us the impetus to develop this book.

Finally, many colleagues and administrators have supported us in the process of writing this book, as the author have consulted many scholars and a wide range of relevant published literatures throughout the development of this project. In particular, they are deeply in debt to Dr. Caroline Moraes, who proofread all chapters of the book, and Ms. Emily Zhang and colleagues of Springer Nature for their great help. The authors would like to express their appreciation to them all.

# Contents

**1 Introduction** . . . . . 1
1.1 The Background of Grey System Theory . . . . . 1
1.2 The Founder of Grey System Theory . . . . . 2
1.3 Development of Grey Systems Theory . . . . . 3
1.3.1 Building a Basic Team . . . . . 3
1.3.2 Establishment of Academic Organizations . . . . . 4
1.3.3 Hold a Series of Grey System Academic Conferences . . . . . 5
1.3.4 Journals and Book Series on Grey System Theory . . . . . 6
1.3.5 Grey System Theory Curriculums . . . . . 7
1.3.6 Researchers of Grey System Theory Are All Over the World . . . . . 8
1.3.7 Papers of Grey Systems Theory Are Growing Rapidly . . . . . 9
1.4 Elementary Concepts of Grey System Theory . . . . . 12
1.5 Fundamental Principles of Grey System Theory . . . . . 13
1.5.1 Main Contents of Grey System Theory . . . . . 15
References . . . . . 17

**2 Characteristics of Grey System Theory** . . . . . 21
2.1 A Kind of Poor Data Analysis Method with Strong Penetration . . . . . 21
2.2 Characteristics of Uncertain Systems and the Simplicity Principle in Sciences . . . . . 22
2.3 Incomplete Information . . . . . 22
2.4 Inaccuracies in Data . . . . . 23
2.5 The Scientific Principle of Simplicity . . . . . 24
2.6 Precise Models Suffer from Inaccuracies . . . . . 25
2.7 Comparison of Several Uncertainty Methods . . . . . 26

2.8 Deep Applications of Grey System Theory in the Fields of Social Science, Natural Science and Engineering Technology ... 28
2.8.1 Successful Application of Grey System Theory in the Field of Social Sciences ... 28
2.9 Deep Application of Grey System Theory in the Field of Natural Science ... 29
2.10 A Large Number of Applications of Grey System Theory in the Field of Engineering Technology ... 33
References ... 37

**3 Grey Numbers and Their Operations** ... 45
3.1 Grey Numbers ... 45
3.2 The Whitenization of a Grey Number and Degree of Greyness ... 47
3.3 Degree of Greyness Defined by Axioms ... 50
3.4 The Operations of Interval Grey Numbers ... 52
3.5 General Grey Numbers and Their Operations ... 53
3.5.1 Simplified Form of Interval Grey Numbers ... 53
3.5.2 General Grey Number and Its Simplified Form ... 55
3.5.3 Synthesis of Degree of Greyness and Operations of General Grey Numbers ... 57
References ... 60

**4 Sequence Operators and Grey Data Mining** ... 63
4.1 Introduction ... 63
4.2 Systems Under Shocking Disturbances and Buffer Operators ... 65
4.2.1 The Trap for Shocking Disturbed System Forecasting ... 65
4.2.2 Axioms of Buffer Operators ... 66
4.2.3 Properties of Buffer Operators ... 68
4.3 Construction of Practically Useful Buffer Operators ... 69
4.3.1 Weakening Buffer Operators ... 69
4.3.2 Strengthening Buffer Operators ... 72
4.3.3 The General Form of Buffer Operator ... 73
4.4 Average Operator and Moving Average Denoise Operator ... 75
4.5 The Quasi-Smooth Sequence and Stepwise Ratio Operator ... 77
4.6 Accumulation and Inverse Accumulation Operators ... 78
4.7 Exponentiality of Accumulation Sequence ... 80
References ... 84

**5 Grey Relational Analysis Models** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 87
5.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 87
5.2 Grey Relational Factors and Set of Grey Relational Operators . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 92
5.3 Grey Relational Axioms and Deng's Grey Relational Analysis Model . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 94
5.4 Grey Absolute Relational Degree . . . . . . . . . . . . . . . . . . . . . . . . . . 98
5.5 Grey Relative and Synthetic Relational Degree . . . . . . . . . . . . . . . 103
5.5.1 Relative Grey Relational Degree . . . . . . . . . . . . . . . . . . . . 103
5.5.2 Grey Synthetic Relational Degree . . . . . . . . . . . . . . . . . . . 105
5.6 Grey Similarity, Closeness and Three-Dimensional Relational Degree . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 106
5.6.1 Grey Relational Analysis Models Based on Similarity and Closeness . . . . . . . . . . . . . . . . . . . . . . . . 106
5.6.2 Three Dimensional Grey Relational Analysis Models . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 110
5.7 Negative Grey Relational Analysis Models . . . . . . . . . . . . . . . . . . . 113
5.8 Grey Relational Analysis Model for Cross-Sequences . . . . . . . . . 120
5.8.1 Measure of Difference in Cross Sequences . . . . . . . . . . . 120
5.8.2 The Modified Grey Relational Models . . . . . . . . . . . . . . . 124
5.9 Superiority Analysis . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 127
5.10 Practical Application . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 133
References . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 149

**6 Grey Clustering Evaluation Models** . . . . . . . . . . . . . . . . . . . . . . . . . . . . 155
6.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 155
6.2 Grey Relational Clustering Model . . . . . . . . . . . . . . . . . . . . . . . . . 157
6.3 Common Possibility Functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . 160
6.4 Variable Weight Grey Clustering Model . . . . . . . . . . . . . . . . . . . . 163
6.5 The Fixed Weight Grey Clustering Model . . . . . . . . . . . . . . . . . . . 167
6.6 Grey Clustering Evaluation Models Based on Mixed Possibility Functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 172
6.6.1 Grey Clustering Evaluation Model Based on End-Point Mixed Possibility Functions . . . . . . . . . . . . 172
6.6.2 Grey Clustering Evaluation Model Based on Center-Point Mixed Possibility Functions . . . . . . . . . . 174
6.7 Practical Applications . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 176
References . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 182

**7 Series of GM Models** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 185
7.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 185
7.2 The Four Basic Forms of GM(1,1) . . . . . . . . . . . . . . . . . . . . . . . . . 188
7.2.1 The Basic Forms of Model GM(1,1) . . . . . . . . . . . . . . . . . 188
7.2.2 Properties and Characteristics of the Basic Model . . . . . 191

7.3 Suitable Ranges of Different GM(1,1) . . . . . 195
7.3.1 Suitable Sequences of Different GM(1,1) . . . . . 195
7.3.2 Applicable Ranges of EGM . . . . . 203
7.4 Remnant GM(1,1) Model . . . . . 207
7.5 Group of GM(1,1) Models . . . . . 211
7.6 The Fractional Grey Model . . . . . 215
7.7 The Models of GM($r$,$h$) . . . . . 218
7.7.1 The Model of GM(0,$N$) . . . . . 218
7.7.2 The Model of GM(1, $N$) . . . . . 220
7.7.3 The Grey Verhulst Model . . . . . 222
7.7.4 The Self-memory Grey Model . . . . . 225
7.7.5 The Models of GM(r,h) . . . . . 227
7.8 Practical Applications . . . . . 228
References . . . . . 235

**8 Combined Grey Models** . . . . . 241
8.1 Introduction . . . . . 241
8.2 Grey Econometrics Models . . . . . 244
8.2.1 Select Explanatory Variables Using Grey Relational Analysis Method . . . . . 244
8.2.2 Grey Econometrics Model . . . . . 245
8.3 Combined Grey Linear Regression Models . . . . . 248
8.4 Grey Cobb–Douglas Model . . . . . 250
8.5 Grey Artificial Neural Network Models . . . . . 252
8.5.1 BP Artificial Neural Model and Computational Schemes . . . . . 252
8.5.2 Steps in Grey BP Neural Network Modeling . . . . . 253
8.6 Grey Markov Model . . . . . 254
8.6.1 Grey Moving Probability Markov Model . . . . . 254
8.6.2 Grey State Markov Model . . . . . 256
8.7 Combined Grey-Rough Model . . . . . 257
8.7.1 Rough Membership, Grey Membership and Grey Numbers . . . . . 258
8.7.2 Grey Rough Approximation . . . . . 260
8.7.3 The Combined Grey Clustering and Rough Set Model . . . . . 262
8.8 Practical Applications . . . . . 264
References . . . . . 268

**9 Techniques for Grey Systems Forecasting** . . . . . 271
9.1 Introduction . . . . . 271
9.2 Criteria for Model Verification . . . . . 272
9.3 Interval Forecasting . . . . . 274
9.4 Grey Distortion Forecasting . . . . . 278
9.5 Wave Form Forecasting . . . . . 281
9.6 System Forecasting . . . . . 283

9.6.1 The Five-Step Modeling Process . . . . . 283
9.6.2 System Models for Prediction . . . . . 285
9.7 Practical Applications . . . . . 286
References . . . . . 289

**10 Grey Models for Decision-Making** . . . . . 291
10.1 Introduction . . . . . 291
10.2 Event and Decision Scheme . . . . . 293
10.3 Grey Target Decisions . . . . . 295
10.4 Other Approaches for Grey Decision . . . . . 299
10.4.1 Grey Relational Decision . . . . . 299
10.4.2 Grey Development Decision . . . . . 304
10.4.3 Grey Clustering Decision . . . . . 306
10.5 Multi-attribute Weighted Intelligent Grey Target Decision Model . . . . . 306
10.5.1 The Uniform Effect Measure . . . . . 307
10.5.2 The Weighted Synthetic Effect Measure . . . . . 309
10.6 On Paradox of Rule of Maximum Value and Its Solution . . . . . 312
10.6.1 The Weight Vector Group with Kernel . . . . . 313
10.6.2 The Weighted Coefficient Vector of Kernel Clustering for Decision-Making . . . . . 314
10.6.3 Several Functional Weight Vector Groups with Kernel . . . . . 315
10.7 Practical Applications . . . . . 316
References . . . . . 318

**11 Grey Control Systems** . . . . . 321
11.1 Introduction . . . . . 321
11.2 Controllability and Observability of Grey System . . . . . 324
11.3 Transfer Functions of Grey System . . . . . 326
11.3.1 Grey Transfer Function . . . . . 326
11.3.2 Transfer Functions of Typical Links . . . . . 327
11.3.3 Matrices of Grey Transfer Functions . . . . . 331
11.4 Robust Stability of Grey System . . . . . 332
11.4.1 Robust Stability of Grey Linear Systems . . . . . 332
11.4.2 Robust Stability of Grey Linear Time-Delay Systems . . . . . 335
11.4.3 Robust Stability of Grey Stochastic Linear Time-Delay System . . . . . 337
11.5 Several Typical Grey Control Models . . . . . 342
11.5.1 Control of Redundancy Removal . . . . . 342
11.5.2 Grey Relational Control . . . . . 343
11.5.3 Control of Grey Prediction . . . . . 344
References . . . . . 349

**12 Spectrum Analysis of Sequence Operators** .......... 353
12.1 Introduction .......... 353
12.2 Spectrum Analysis of Time Series Data .......... 355
12.3 Filtering Effect of Mean Operator and Accumulation Operator .......... 356
12.3.1 Filtering Effect of Mean Operator .......... 356
12.3.2 Filtering Effect of Accumulation Operator .......... 358
12.3.3 Filtering Effect of Series Operator .......... 359
12.4 Spectrum Analysis of Buffer Operator .......... 360
References .......... 362

**Appendix: Introduction to Grey Systems Modeling Software** .......... 363

**Memorabilia of the Establishment and Development of Grey System Theory (1982–2024)** .......... 379

**References** .......... 387

**Index** .......... 389

# About the Author

**Professor Sifeng Liu Ph.D.** distinguished professor at Northwestern Polytechnical University. He was awarded Marie Curie International Incoming Fellowship of European Union in 2013 and national leading talents of China in 2020, and he was elected as an Honorary Fellow of WOSC in 2007.

Prof. Liu is serving as the founding director of Center for Grey Systems Studies at NPU, the founding president of International Association of Grey Systems and Uncertainty Analysis, the founding president of Grey System Society of China. Prof. Liu is also serving as the founding Editor of *Grey Systems-Theory and Application* (Emerald) and the Editor-in-Chief of *The Journal of Grey Systems* (Research Information), at the same time, he is currently serving as the editor of *Series on Grey System* in Chinese (by Science Press) and in English (by Springer-Nature) respectively. He was the founding director of Institute for Grey Systems Studies at NUAA from 2000 to 2023.

Dr. Liu has been devoted to the research of grey system theory for 40 years, and has put forward a series of new concepts, models and methods, which enrich and develop the grey system theory. For example, he proposed the original grey relational analysis models based on the global perspective, which including absolute, relative, similar, closeness, negative and three-dimensional, etc.; deeply excavate the evolution law of grey data series, and cooperate with students to put forward a series of original concepts and models, such as original difference, mean difference, discrete, fractional order model, etc.; proposed the grey clustering evaluation model based on mixed possibility function; multi-objective weighted intelligent grey target decision-making model;and the original concept of kernel clustering weight vector group and a two-stage decision-making model; proposed the original concepts of the "kernel", greyness and the simplified form of grey number, general grey number, built a new operation system of grey numbers; proposed the original concepts of sequence operator, weakening and strengthening buffer operator, etc., which open up a new field of buffer operator research.

Based on the work of Prof. Sifeng Liu's team, a new grey system knowledge system has been constructed.

Prof. Liu is the chief expert of a national teaching team on quantitative method of China. He presided over the construction of 16 national first-class or national excellent courses, national excellent or planned textbooks. Various editions of his seminal book on grey system theory have been published in different languages such as Chinese, English, Romanian and Korean. Hundreds of universities from around the world adopted them as textbooks. There are more than one million audiences of his books, videos and software of grey modelling.

Prof. Liu has won several accolades such as the “National Excellent Scientist”, “National Excellent Teacher”, “National Advanced Individual for Returnee” and the Famous Professor of China, etc. In 2017, Prof. Sifeng Liu has been selected to be one of the top 10 shortlisted promising scientists in the MSCA 2017 Prizes award. In 2023, He was awarded the “Global Excellence Award” by the Pakistan Grey System Society. He has listed in Top 2% scientists of the world by the research team at Stanford University. And ranking first in the list of highly cited authors in the field of system science by Baidu Scholar. In 2024, he was selected as one of the top 0.05% Highly Ranked Scholar-Lifetime in Systems Theory by Scholar GPS. His publications have been cited 54688 times with an H-Index of 100 in Huezhi Scholar.

# Chapter 1
# Introduction

## 1.1 The Background of Grey System Theory

In the late twentieth century, with the development of computer and information technology, mankind began to enter the information age. Data and information have become very important resources and the main driving force of social development. The ability to acquire, transmit, analyze and process data, information and knowledge determines the distribution of social wealth and power to a large extent, and becomes the decisive factor of political, economic and cultural competitive advantage.

With the development of information technology and the progress of human society, people have gradually deepened their understanding of various system uncertainties, and the research on uncertain systems is also deepening day by day. Since the 1960s, a variety of uncertain system theories and methods have been proposed one after another. Among them, Fuzzy mathematics founded by Professor L. A. Zadeh in the 1960s (Zadeh, 1965), grey system theory advanced by Professor Julong Deng in the 1980s (Deng, 1982), rough sets theory devoloped by Professor Z. Pawlak in the 1980s (Pawlak, 1991), etc., are all important achievements in the study of uncertain systems with extensive international influence. These uncertain theories discussed the theories and methods of describing, processing and mining various kinds of uncertain data and information from different perspectives and aspects.

Grey system theory takes the "poor information" uncertain system with "some information known and some information unknown" as the research object. It mainly extracts valuable information through the mining of "some" known information, and realizes the correct description of the system operation behavior and evolution law, so that people can use mathematical models to analyze and assess the "poor information" uncertain system, then realize high-precision prediction, scientific decision-making and optimal control of the "poor information" uncertain system. The uncertainty system of "poor information" in the real world provides rich research resources and broad development space for grey system theory.



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_1

In the era of big data, the paradigm of scientific research is undergoing fundamental changes. The Fourth Paradigm: Data-Intensive Scientific Discovery which proposed by Jim Gray, a Turing Award winner, is increasingly becoming the mainstream paradigm in scientific research. The significant feature of big data is its low information density. The first core idea of big data is to fully utilize all available data, not limited to random sampling; The second is to eliminate confounding factors and gain insight into the general direction; The third is to attach importance to relevant relationships rather than pursuing causality (Mayer-Schönberger & Cukier, 2013). The basic principles of grey system theory is completely consistent with the core idea of big data.

## 1.2 The Founder of Grey System Theory

The birth of grey system theory is an outcome of Professor Julong Deng who has been working with perseverance for decades.

Prof. Deng was born in Lianyuan County, Hunan Province of China in 1933. He got his degree in electrical machinery from Huazhong Institute of Technology and then joined the same institute in 1955 as a teaching assistant. Prof. Deng used to keep an eye on new ideas related to his field which led to his later investigation into multi-variable system control problems. In the 1960s, he put forward a new method – "control by removing redundant". His paper entitled "multivariable linear system shunt calibration device of a comprehensive approach" was published in 1965 (Deng, 1965). By the early 1970s, the method of "control by remove redundant" has been widely recognized as a representative methodology in cybernetics.

In 1965, Prof. L. A. Zadeh proposed Fuzzy Sets (Zadeh, 1965). Prof. Deng was involved in research of fuzzy mathematics. He published some papers in fuzzy mathematics. And served as a member of editorial board for several journals on fuzzy mathematics. In the late 1970s, Prof. Deng devoted himself to the study of "prediction and control problems of economic system". In dealing with systems where "some information is known, and some information is unknown", the main challenge is to develop an effective method to represent such systems. Despite the difficulties, Professor Deng and his colleagues have made significant progress in their explorations. In 1982, his pioneering paper titled "The Control Problems of Grey Systems" published by Systems and Control Letters (Deng, 1982). The publication of this seminal article indicated that grey system theory, a new branch of research, came into being.

Since the birth of Grey System Theory, it has received significant attention from academic communities and industries both in China and overseas, especially in real world applications.

So far, Prof. Deng's works has been cited over 50 thousand times. Prof. Deng won the award of founder of Grey System Theory at the 2007 IEEE International Conference on Grey Systems and Intelligent Services which held in Nanjing. In 2011, he was elected as the honor fellow of the World Organisation of Systems and

Cybernetics at the joint conference of the 15th WOSC International Congress on Cybernetics and Systems and 2011 IEEE International Conference on Grey Systems and Intelligent Services.

## 1.3 Development of Grey Systems Theory

### *1.3.1 Building a Basic Team*

In the early 1990s, Professor Julong Deng began to recruit and train doctoral students in the field of grey system theory in the discipline of system engineering of Huazhong University of Science and Technology. He has recruited and trained 10 doctoral students, most of them are young scholars who have been engaged in grey system theory research for many years before entered Prof. Deng's group. These scholars naturally become the first generation of grey system theory. They actively participate in the research of grey system theory, consciously assume the responsibility of developing and disseminating grey system theory, and unswervingly take the research and inheritance of grey system theory as their lifelong career.

In 2000, as the first distinguished professor introduced by Nanjing University of Aeronautics and Astronautics (NUAA), one of Prof. Deng's PhD students, Professor Sifeng Liu joined this university with aerospace characteristics. In the same year, with Professor Sifeng Liu as the chief discipline leader, NUAA submitted an application to the Academic Degrees Committee of the State Council of China for the establishment of a doctoral degree authorization point in management science and engineering, which was successfully approved. Therefore, grey system theory has naturally become the characteristic and leading direction of the doctoral program of management science and engineering of NUAA. At the same time, as the founding director, Professor Liu established the Institute for Grey System Studies at NUAA. IGSS-NUAA has also become the center of grey system scholars. A group of outstanding young scholars gathered in IGSS-NUAA through talent introduction, entering the station to carry out post-doctoral research and pursuing doctoral degree, forming a highland of grey system research. IGSS-NUAA has 12 doctoral tutors (including 6 full-time doctoral tutors). Over the past 20 years, it has recruited and trained more than 200 doctoral students, post-doctors and visiting scholars at home and abroad in the field of grey system theory.

Professor Julong Deng's other doctoral students, such as Qishan Zhang with Fuzhou University, Xinping Xiao with Wuhan University of technology, Wenping Wang with Southeast University and Xuerui Tan with Shantou University, began to cultivate high-level talents engaged in grey system theory and application research after becoming doctoral supervisors.

Many other universities are recruiting and funding doctoral and postdoctoral researchers in grey system theory and its application. Examples include Southeast

University, Wuhan University of Technology, Fuzhou University, Shantou University, De Montfort University, Bucharest Economics University, Poznań University of Technology, Bogazici University, Cape Town University, Central Florida University, Nebraska-Lincoln University, University of Waterloo, Pablo de Olavide University, Kanagawa University, National Cheng Kung University, etc.

In 2023, Professor Sifeng Liu joined Northwestern Polytechnic University, where he established the Center for Grey System Studies (CGSS-NPU) and began to recruit and train doctoral students and postdoctoral researchers in grey system theory.

Hundreds of doctoral graduates constitute the basic team of grey system theory research. Each PhD graduates in grey system theory become a seed which take root in the new institution, then enlarge and spread one's power and influence gradually.

### *1.3.2 Establishment of Academic Organizations*

In 1987, Wuhan Grey System Society, with members from provinces, cities and autonomous regions all over the country of China, was approved by Wuhan Association for Science and Technology.

In 1997, The Grey System Society in Taiwan region was established.

In 2005, the Grey System Society of China, CSOOPEM, was approved by China Association for Science and Technology, and the Ministry of Civil Affairs of China. At the beginning of 2008, the Technical Committee of IEEE SMC on Grey Systems was established. In 2012, the first Workshop of European grey system research collaboration network was held by De Montfort University, and delegates from twelve member states of the European Union attended the event. In 2013, Professor Sifeng Liu was selected for a Marie Curie International Incoming Fellowship (FP7-PEOPLE- IIF-GA-2013-629051) of the 7th Research Framework Program of the European Union. Furthermore, in 2014 an international network project entitled "Grey Systems and Its Applications" (IN-2014-020) was funded by The Leverhulme Trust. Supported by this project, a series of grey system theory cooperative research and academic exchange activities have been held in Europe, North America and China. In 2015, Jointly sponsored by well-known scholars from China, the United Kingdom, the United States, Canada, Spain, Romania and other countries, the International Association of Grey System and Uncertainty Analysis (GSUA) was established.

In recent years, Poland, Pakistan, Turkey and other countries have established grey system academic organizations, and Iran, Sri Lanka and other countries have established the Preparatory Committee of the grey system society.

The construction and development of specialized academic organizations have played an important role in promoting the development of the new theory of grey system.

### 1.3.3 Hold a Series of Grey System Academic Conferences

From December 20 to 24, 1984, with the support of Shanxi Academy of Agricultural Sciences, the first national grey system academic conference “grey system and agriculture” was held in Taiyuan, Shanxi Province. Nearly 100 experts and scholars from colleges and universities in 16 provinces, autonomous regions and cities, as well as the Chinese Academy of Sciences, the Chinese Academy of Agricultural Sciences and other units attended the conference. Professor Deng Julong, the founder of grey system theory, attended the conference and delivered a keynote speech.

Since 1985, Wuhan (National) Grey System Research Association has held six national grey system academic conferences in Wuhan. Zhejiang Agricultural University and Henan Agricultural University have also held grey system academic conferences respectively.

In 1996, more than 20 scholars from universities in Taiwan attended the Ninth National grey system academic conference held at Huazhong University of science and technology. The Grey System Society in Taiwan region held an academic conference every year since 1997.

Since 2002, the Institute for Grey System Studies at Nanjing University of Aeronautics and Astronautics has taken on the responsibility of organizing grey system academic conferences. IGSS-NUAA has held 28 (11th–38th) domestic and 18 international conferences on grey system theory and its applications so far. Since 2006, the grey system academic conference has been funded by the China Center for Advanced Science and Technology (Professor Tsung Dao Lee, Nobel Prize winner, as the director of the center, and former president of the Chinese Academy of Sciences academicians Zhou Guangzhao and LuYongxiang as deputy directors) for 15 years. The grey system academic activities have also been supported by the National Natural Science Foundation of China, the China Association for Science and Technology, the Leverhulme trust foundation and the Jiangsu Provincial Department of Education for many times. Nanjing University of Aeronautics and Astronautics regards the grey system theory as the characteristic field of the University and provides continuous support. Shanghai Pudong Institute of Education and Wuhan University of Technology, Henan Agricultural University and Northwestern Polytechnic University have taken the initiative to undertake grey system academic activities. Such conferences have been supported by IEEE, WOSC, GSUA, University of Macao, De Montfort University, Stockholm University, and Huawei Technology of Thailand. A large number of young scholars has attracted to such events.

A group of young scholars who are committed to the study of grey system theory and have made important achievements have also spontaneously organized and regularly held young scholars’ forums to exchange ideas and enlighten each other.

Many special sessions and tracks on grey system theory have been organized at significant international conferences such as International Conference on Uncertain System Modeling, International Conference on System Forecast and Control, International Conference on General System Studies, International Congress of

World Organization of Systems and Cybernetics, IEEE International Conference on Systems, Man and Cybernetics, etc.. The topicality of grey systems theory and its popularity in such high-profile international conferences have certainly played an active role in furthering understanding of, and promoting this theory among peers in the world of systems science.

### *1.3.4 Journals and Book Series on Grey System Theory*

In 1989, The Journal of Grey System was launched by Research Information Ltd in the UK. In 2007, The Journal of Grey System is indexed in SCIE (Science Citation Index Expanded) and belongs to the categories of "Mathematics" and "Mathematics, Interdisciplinary Applications" in SCIE. Currently, Journal of Grey System with an impact factor of 1.6. This publication is indexed by Mathematical Review of the United States and other important indexing agencies from around the world. In 2011, Emerald launched a new journal named Grey Systems: Theory and Application, edited by the faculty of the Institute for Grey System Studies at Nanjing University of Aeronautics and Astronautics. In 2019, Grey Systems Theory and Application is indexed in SCIE (Science Citation Index Expanded) and belongs to the categories of "Mathematics"and "Mathematics, Interdisciplinary Applications" in SCIE. At present, Emerald/ Grey Systems Theory and Application (GS) belong to JCR Q1 with an impact factor of 3.2. This journal is indexed by EBSCO, Scopus, Ei Compendex, Summon, ReadCube Discover and other important indexing agencies from around the world. There are currently over one thousand different professional journals in the world that have published papers in grey systems theory, many of which are top journals in a variety of fields. As of this writing, many journals and publishers such as the journal of the Association for Computing Machinery (USA), Communications in Fuzzy Mathematics (Taiwan, China), Kybernetes: The International Journal of Systems & Cybernetics, Transaction of Nanjing University of Aeronautics and Astronautics, China Ocean Press, Chinese Agricultural Science Press, Henan University Press, Huazhong University of Science and Technology Press Co. Ltd, IEEE Press, Springer-Verlag have respectively published special issues or proceedings on grey system theory.

Numerous publishing agencies such as Science Press, Defense Industries Press, Huazhong University of Science and Technology Press Co. Ltd, Jiangsu Science and Technology Press, Shandong People's Press, Science and Technology Literature Press of China, Henan University Press, China Science and Technology Book Press of Taiwan, Gaoli Books Limited Company of Taiwan, ASE Press of Romania, Japan Polytechnic Press, IIGSS Academic Press, CRC of Taylor & Francis Group, Springer-Verlag, Springer-Verlag London Ltd, and John Wiley & Sons, Inc. have published hundreds of academic works on grey systems, in many different languages including Chinese, Traditional Chinese, English, Japanese, Korean, Romanian, Turkish and Persian.

In 1991, In 1990, Henan University Press published the first grey system theory work of Sifeng Liu and Tianbang Guo. The title of the book is Grey Systems Theory and Its Applications (Liu & Guo, 1991). The book is very popular with readers. Since then, it has been revised and republished many times and printed dozens of times. Since the second edition, the book has been published by Science Press (Liu et al., 1999). The third edition was supported by the science publishing fund of the Chinese Academy of Sciences (Liu et al., 2003). The fourth edition, as a popular edition of textbooks, was selected into the national planning textbooks for the "Eleventh Five Year Plan" and the supporting textbooks for national top quality course of China (Liu & Xie, 2006). In 2024, The 10th edition came out. It is a national planning textbook and a supporting textbook for national first-class course of China (Liu, 2024).

In 1998, the first English book on grey system theory was published in the United States (Liu & Lin, 1998), enabling interested readers around the world to systematically understand grey system theory. In 2006, Grey Information-Theory and Practical Applications published by Springer-Verlag (Liu & Lin, 2006), so that the grey system theory can be widely spread around the world. Since then, Springer- Verlag has launched several English versions, such as Grey Systems Theory and Applications (Liu & Lin, 2010), Advance in grey systems research (Liu & Lin edited, 2010), Grey Data Analysis (Liu et al., 2017), and Grey Systems Analysis (Liu et al., 2022a, 2022b). This book has been funded by Publishing Fund of Excellence Academic Works of Northwestern Polytechnical University and will be published as OA book. It is expected that it will be widely spread around the world, promote the in-depth application of grey system methods and models, and benefit all mankind.

Series on grey systems both in Chinese and English are published by Science Press and Springer-Nature Group respectively. Series on grey systems in Chinese was launched by Science Press in 2014. So far, 34 books have been published. Series on grey systems in English was launched by Springer-Nature Group in 2021. The six books have been published and other two books have passed the review, will come out soon.

### *1.3.5 Grey System Theory Curriculums*

Numerous universities around the world have set up grey system theory curriculums. For example, in Nanjing University of Aeronautics and Astronautics (NUAA), the curriculums of the grey system theory are found not only in Ph.D. and Master's programs, but also in undergraduate programs of many disciplines across the university, as an elective module. Prof. Sifeng Liu and his team at IGSS- NUAA did a lot of work to popularize and inherit the Grey System Theory. As a result, this course has been selected as the National Excellence Course beginning in 2008, the National Excellence Resource Sharing Course since 2013, the National Excellence Online Open Course starting in 2018, and the National first class courses of online and offline since 2020. Furthermore, Professor Sifeng Liu's team worked with a number of professors from universities in Europe, the United States and Canada, including Keith William Hipel, former president of the Royal Canadian Academy of Sciences, Professor Yingjie Yang, the executive president of the GSUA, to complete the online course in English, Grey Data Analysis, which became a free open learning resource for all grey system hobbyists since 2021.

### *1.3.6 Researchers of Grey System Theory Are All Over the World*

Many scholars from USA, UK, Germany, France, Italy, Korea, Canada, Romania, Poland, Turkey, South Africa, Iran, India, Pakistan, Egypt and Sri Lanka etc. have joined IGSS-NUAA or CGSS-NPU as visiting professor, research fellow or for joint project research. In recent years, some young scholars from different countries joining IGSS-NUAA or CGSS-NPU as PhD or Master students supported by Chinese government scholarship. It is helpful to promotion the popularization and international communication of grey system theory (Liu, et al., 2016; Liu, et al., 2016a).

According to the retrieval results by the database of web of science, scholars from more than 120 countries and regions in the world have carried out research on grey system theory and applications and published relevant academic papers.

Hundreds of thousands of master's and doctoral students around the world applying grey system thinking and methods to carry out scientific research and complete their dissertations.

Many prominent scholars have commended grey system research. Such scholars include Professor Lotfi A. Zadeh (USA), the founder of fuzzy mathematics, Professor Herman Haken (Germany), the founder of synergetics, Professor James M. Tien (USA), former vice-president of IEEE and member of the National Academy of Engineering, Professor Robert Vallee (France), former president of World Organization of Systems and Cybernetics, Professor Alex Andrew (UK), former secretary General of the World Organization of Systems and Cybernetics, Professor Keith William Hipel (CA), former president of the Canadian Royal Academy of Sciences, Professor Edmundas K. Zavadskas, former president of Vilnius Gediminas Technical University and Member of Lithuanian Academy of Science, and Professor Alain Bernard, Président d'Honneur France Additive and Membre de l'Académie des Technologies, as well as many Academicians of the Chinese Academy of Sciences and the Chinese Academy of Engineering, including Professor Qian Xuesen, famous scientist and winner of the national highest science award, China, Professor Huai Jinpeng, Minister of the Ministry of Education, China, Professor Yang Shuzi, Professor Xiong Youlun, Professor Lin Qun, Professor Chen Da, Professor Zhao Chunsheng, Professor Hu Haiyan, Professor Xu Guozhi, Professor Huang Wei, Professor Wang Zhongtuo, Professor Yang Shanlin, Professor Chen Xiaohong, Professor Shan Zhongde, Professor Guo Baozhu, and Professor Song Zhengyu, et al.

It attracts not only the affirmation and support from international leading scholars, but also many early career researcher from different disciplines of social sciences, natural sciences and engineering technology as well. Successful applications have been found in more than 120 countries and regions. It has been established as a new scientific branch in data analytics and uncertainty modelling.

On 7th September, 2019, during the visit to China, Angela Dorothea Merkel-then, German Chancellor praised Chinese original grey system theory. She said that the work of professor Julong Deng, the founder of grey system theory, and professor Sifeng Liu and three other Alumni of HUST "profoundly affecting the world."

### *1.3.7 Papers of Grey Systems Theory Are Growing Rapidly*

The rapid development of grey system theory benefits from the strong promotion of practical application needs.

In the information age, people in various fields begin to deeply realize that data analysis method has become an indispensable skill for everyone. Just like the gold buried in the sand sea, the laws and characteristics that people want to understand and control are deeply covered up by the chaotic and complicated data information with extremely low information density and value. There is an urgent need for effective scientific methods. To meet the needs of the times, grey system theory came into being.

Just like any new thing, the growth process of a new theory is naturally full of hardships and twists and turns. When the grey system theory came out, it was inevitably criticized and questioned by some people. The desire for poor information data analysis methods in human social practice has formed a strong driving force, so that the grey system theory can still attract the positive attention of a large number of people of insight in various fields.

According to the retrieval results of the web of science database, scholars from more than 120 countries and regions in the world have carried out research on grey system theory and its application, and published a large number of academic papers on grey system theory (Fig. 1.1) Many universities and research institutions recruit and train doctoral students and researchers in grey system theory and applied research. Hundreds of thousands of master and doctoral students around the world use the thinking and method of grey system theory to carry out scientific research and complete dissertations (Liu, et al, 2022).

According to the retrieval results of CNKI database, over the past 40 years, the research papers on grey system model and its application have increased rapidly In recent years, more than 15,000 papers have been published every year (Fig. 1.2). From 1982 to 2023, more than 260,000 papers were published.

As can be seen from Fig. 1.2, the papers of grey system theory included in CNKI database show a rapid growth trend after entering the new century. In 2001, 1856 papers were included in CNKI database. By 2004, the number of papers included in CNKI database had reached 4151, double that of 2001. In 2007, it doubled on

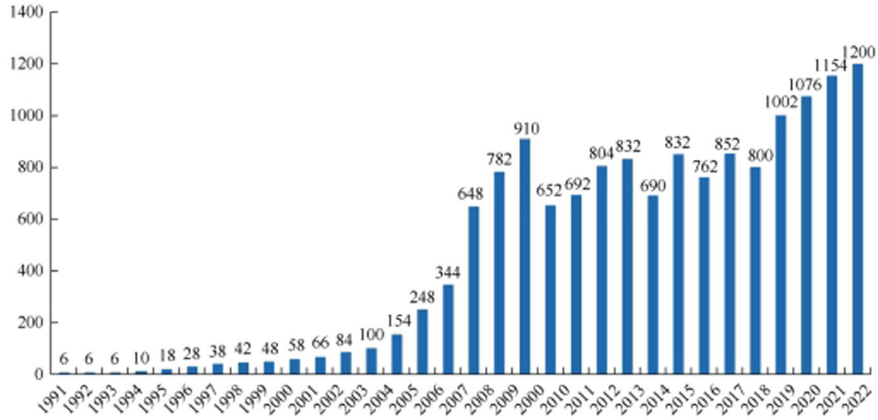



**Fig. 1.1** Numbers of grey system papers included in web of science database

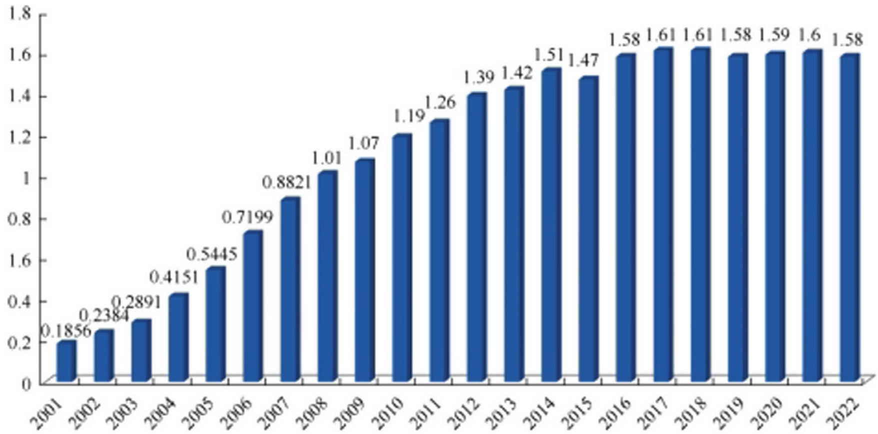



**Fig. 1.2** Numbers of grey system papers included in CNKI database (2001–2022)

the basis of 2004, reaching 8821. Since 2008, more than 10,000 papers have been included in CNKI database every year, and more than 15,000 papers have been included in CNKI database since 2014.

It can be found from the literatures included in CNKI database that a large number of the papers of grey system theory have been included in CNKI database in all double first-class universities and double first-class discipline construction universities in China. The top 20 universities of number of journal papers and dissertations of grey system theory included in CNKI database can be seen in Table 1.1. The data in Table 1.1 fully shows that the grey system theory has played an important role in the training of high-level talents in China.

**Table 1.1** Top 20 universities of number of grey system papers included in CNKI database

| Name of universities | NCEPU | CAU | SJU | WUT | CSU |
|---|---|---|---|---|---|
| No. of papers | 4018 | 2995 | 2970 | 2704 | 2684 |
| Name of universities | BJU | NUAA | JLU | CQU | TJU |
| No. of papers | 2644 | 2531 | 2526 | 2505 | 2427 |
| Name of universities | HUST | HNU | HHU | ZJU | DUT |
| No. of papers | 2016 | 1998 | 1987 | 1910 | 1857 |
| Name of universities | HUT | CUMT | DMU | XUAT | HIT |
| No. of papers | 1782 | 1762 | 1759 | 1740 | 1697 |

*Notes NCEPU* North China Electric Power University, *CNU* Chang' An University, *SJU* Southwest Jiaotong University; *WUT* Wuhan University of Technology, CSU: Central South University; *BJU* Beijing Jiaotong University, *NUAA* Nanjing University of Aeronautics and Astronautics, *JLU* Jilin University, *CQU* Chongqing University, *TJU* Tianjin University, *HUST* Huazhong University of Science and Technology, *HNU* Hunan University, *HHU* Hohai University, *ZJU* Zhejiang University, *DUT* Dalian University of Technology, *HUT* Hefei University of Technology, *CUMT* China University of Mining and Technology, *DMU* Dialian Maritime University, *XUAT* Xi'an University of Architecture and Technology, *HIT* Harbin Institute of Technology)

In China, the research group undertaking the national key research and development plan, the key and major projects of the National Natural Science Foundation, the national high tech research and development plan (863 plan), the national key basic research and development plan (973 Plan), the national major science and technology projects and the national science and technology support plan and other important national science and technology projects has published a large number of papers on the application of grey system models and methods to solve key scientific problems (Fig. 1.3).

It can be seen from Fig. 1.3 that the grey system theory has played an important role in promoting China's scientific and technological progress and innovation development. This was fully affirmed by academician Zhao Chunsheng of the Chinese Academy of Sciences (Zhao, 2015).

In the era of big data, the grey system theory based on poor data mining has sprung up and become an effective tool for people to extract valuable information from

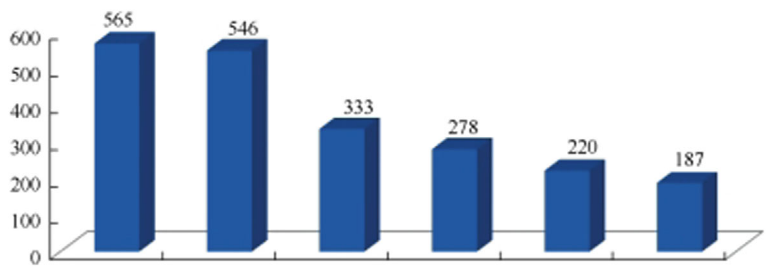



**Fig. 1.3** Application of key national science and technology programs in China

massive data. In the past 40 years, the wide application of grey system methods and models in many fields of social science, natural science and engineering technology has led to innovation and progress in various fields.

## 1.4 Elementary Concepts of Grey System Theory

Many social, economic, agricultural, industrial, ecological and biological systems are named by considering the features of classes of the research objects, while grey systems are labeled using the color of the systems of concern.

In the theory of control, scholars often make use of colors to describe the degree of clearness of available information. For instance, Ashby refers to objects with unknown internal information as black boxes. This terminology has been widely accepted in the scientific community. As another example, as a society moves toward democracy, citizens gradually demand more information regarding policies and the meanings of such policies. That is, citizens want to have an increased degree of information transparency (i.e. white information). Thus, we use “black” to indicate unknown information, “white” to indicate completely known information, and “grey” to convey partially known and partially unknown information. Accordingly, systems with completely known information are regarded as white, while systems with completely unknown information are considered black, and systems with partially known information and partially unknown information are seen as grey.

In this context, incompleteness in information is the fundamental meaning of “grey.” However, the meaning of “grey” can be expanded or stretched from different angles and in varied situations (see Table 1.2).

At this point, the difference between “system” and “box” Must be highlighted. On the one hand, the term “box” is used when one does not pay much attention, or does not attempt, to utilize information regarding the interior characteristics of an object, while focusing mainly on the external characteristics of such an object. In this case, the researcher generally investigates the properties and characteristics of the object through analyzing the input–output relation. On the other hand, the term

**Table 1.2** Extensions of the concept of “grey”

| Situation/concept | Black | Grey | White |
|---|---|---|---|
| Information | Unknown | Incomplete | Completely known |
| Appearance | Dark | Blurred | Clear |
| Processes | New | Changing | Old |
| Properties | Chaotic | Multivariate | Order |
| Methods | Negation | Change for the better | Confirmation |
| Attitude | Letting go | Tolerant | Rigorous |
| Outcomes | No solution | Multi-solutions | Unique solution |

"system" is employed to indicate the study of the object's structure and functions through the analysis of existing organic connections between the object, relevant factors, its environment, and related laws of change.

The research objects of grey systems theory consist of uncertain systems that are known only partially through small samples and poor information. The theory focuses on the generation and excavation of partially known information through grey sequence operators of possibility functions to enable an accurate description and understanding of the material world.

## 1.5 Fundamental Principles of Grey System Theory

In the process of developing grey systems theory, Julong Deng established six fundamental principles containing intrinsic philosophical intensions, as discussed below.

**Axiom 5.1 The Principle of Informational Differences.**

"Difference" implies the existence of information. Each piece of information must carry some kind of "difference".

When we say that object A is different from object B, we mean that there is some special information about object A that is not true for object B. All "differences" between natural objects and events have provided us with elementary information in order for us to understand their nature.

If information "I" has changed our understanding or impression of a complicated matter, then the piece of information "I" is definitely different from what we initially understood the complicated matter to be. Great breakthroughs in science and technology have provided us with necessary information, which we generally call knowledge and tools, to understand and change the world around us. Such advanced information is surely different from pre-scientific information. The more content a piece of information "I" contains, the more the differences from an earlier version of such information will become apparent.

**Axiom 5.2 The Principle of Non-Uniqueness.**

The solution to any problem with incomplete and indeterminate information is not unique.

Because of the principle of non-uniqueness, which is a basic law of the application of grey systems theory, one is set free to look at problems with flexibility. With flexibility, one becomes more effective in reaching their goals.

Strategically, the principle of non-uniqueness is realized through the concept of grey target. This concept is a unification of the concept of non-unique target and that of non-restrainable target. For example, on the one hand, if a high school graduate does not plan to enroll in any university except for one specific institution, then his chance of being accepted by a university is greatly limited. On the other hand, if a high school graduate with similar qualifications as the one in the previous example is willing to apply for several universities other than his preferred one, he will be more likely to succeed in being accepted by a university because he has multiple targets, which in turn leads to an improved chance of hitting one of the targets.

The principle of non-uniqueness can be seen as a comprehensive realization that each target can be approached, that any available information can be supplemented, that each plan made earlier can be further modified and improved, that each relationship can be harmonized, that each thinking logic can be multi-directional, that each understanding can be deepened, and that each path can be optimized. When faced with the possibility of multiple solutions, one can locate one or several satisfactory solutions through deterministic analysis and supplementation of information. Therefore, the method of finding solutions on the basis of "non-uniqueness" is one that combines both quantitative and qualitative analysis.

**Axiom 5.3 Principle of Minimal Information.**

. One characteristic of grey system theory is that it makes the most and best use of the "minimal amount of available information."

The "principle of minimal information" can be seen as a dialectic unification of "a little" and "a lot." One advantage of grey system theory is its ability to handle such uncertain problems with "small data" and/or "poor information." Its foundation of study is the concept of "spaces of limited information." "Minimal amount of information" is the basic territory for grey system theory to show its power. The amount of acquirable information is the dividing line between "grey" and "not grey". Making sufficient discovery and application of any available "minimal amount of information" is the basic thinking logic of problem-solving used in grey system theory.

**Axiom 5.4 Principle of Recognition Base.**

Information is the foundation on which people recognize and understand (nature).

This principle argues that all recognition must be based on information. Without information, there is no way for people to know anything. With complete and deterministic information, we can possibly gain firm understanding of nature. With incomplete and non-deterministic information, it is only possible to obtain incomplete and non-deterministic grey understanding of particular phenomena.

**Axiom 5.5 Principle of New Information Priority.**

The function of new pieces of information is greater than that of old pieces of information.

The "principle of new information priority" is the key idea behind information application in grey system theory. That is, by applying additional weights to new information, one can achieve a better result from grey modeling, grey prediction, grey analysis, grey evaluation, and grey decision making. The belief that "the new replaces the old" reflects our "principle of new information priority." With the availability of new information, the motivation for whitening grey elements is strengthened. The "principle of new information priority" reflects the fact that information in general is time sensitive.

**Axiom 5.6 Principle of Absolute Greyness.**

"Incompleteness" of information is absolute. Incompleteness and non-determinism of information have generality.

Completeness of information is relative and temporary. It is the moment when the original non-determinism has just disappeared, and new non-determinism is about to emerge. Human recognition and understanding of the objective world have been improved over time through continued supplementation of information. With endless supply of information, man's recognition and understanding of the world also become endless. That is, greyness of information is absolute and will never disappear.

### *1.5.1 Main Contents of Grey System Theory*

After 40 years' development, the grey system theory has basically established the knowledge system of a new discipline. Its main contents include: the basic theory of grey system such as grey number operation and grey algebraic system, grey equation, grey matrix, etc.; Sequence operator and grey information mining method; A series of grey relational analysis models for system diagnosis and analysis; A variety of grey clustering evaluation models are used to solve the classification problems of system elements and objects; Grey prediction series model and grey system prediction method and technology; Grey decision models such as grey target decision model and multi-objective weighted intelligent grey target decision model, which are mainly used for scheme evaluation and selection, as well as grey combination models characterized by multi-method fusion and innovation, such as grey programming model, Grey Input–output Model, grey game model, and grey control model, etc.

Grey number and its operation are the basis of grey system theory. From the perspective of self-improvement of discipline system, there are many problems worthy of further study, especially in the aspects of grey algebraic system, grey equation, grey matrix and so on ( HYPERLINK "sps:refid::bib40|bib44" ).

Sequence operator and grey information mining mainly include buffer operator (weakening buffer operator, strengthening buffer operator) ), mean operator, order ratio operator, accumulation operator, subtraction operator and spectrum analysis of sequence operators (Dang et al., 2004a, 2004b; Lin et al., 2021, 2022; Liu et al., 2020; Liu, 1991).

The grey correlation analysis model includes grey correlation axiom, Deng's grey relational analysis model, grey absolute relational analysis model, grey relative relational analysis model, grey comprehensive relational analysis model, grey relational analysis model based on similarity perspective, grey relational analysis model based on proximity perspective, grey relational analysis model for inverse sequences, grey relational analysis model for cross-sequences, and three-dimensional grey relational analysis model and so on (Liu, 1992, 2023; Liu et al., 2011, 2024; Lu et al., 2023; Zhang & Liu, 2009).

The grey clustering evaluation model includes grey relational clustering evaluation model, grey variable weight clustering model, grey fixed weight clustering model, grey clustering evaluation model based on mixed possibility function (center point mixed possibility function, endpoint mixed possibility function) and two-stage grey comprehensive measure decision model.(Deng, 1985; Liu, 1993; Liu & Zhu, 1993; Liu et al., 1998, 2015a; Liu & Xie, 2011).

The series of grey prediction models include even grey model GM (1,1), original difference grey model GM (1,1), even difference grey model GM (1,1), discrete grey prediction model, fractional grey prediction model, self-memory grey prediction model, Verhulst model and grey model GM (R, H), etc. (Deng, 1984, 1985; Guo et al., 2015; Liu et al., 2015b; Xie & Liu, 2005, 2009; Wu et al., 2013).

The grey combination model includes grey Econometrics (G-E) model, grey Cobb Douglas model, grey linear regression combination model, grey periodic extension combination model, grey Markov (G-M) model, grey artificial neural network model, grey clustering and dominance rough set combination model, etc. (Liu & Zhu, 1996; Liu et al., 2004; Zhu et al., 2011).

Grey prediction techniques are a series of quantitative prediction technique based on grey prediction thought, method and model. According to its function and characteristics, it can be divided into sequence prediction, interval prediction, distortion prediction, waveform prediction and system prediction (Dang & Liu, 2009; Deng, 1985; Liu et al., 2022a, 2022b).

The grey decision-making model includes grey target decision model and multi-objective weighted intelligent grey target decision model based on uniform effect measure functions ().

The grey programming model includes grey parameter linear programming, grey predictive linear programming, grey drift linear programming, grey 0–1 programming, grey multi-objective programming and grey nonlinear programming (Deng, 1985; Liu & Dang, 1997; Nasseri & Darvishi, 2018).

The grey input–output model includes the basic concept of grey input–output, grey input–output optimization model, grey dynamic input–output model, etc. (Li, 2009; Li & Liu, 2008; Li et al., 2012).

The grey game model mainly studies the game model based on limited rationality and limited knowledge and its solution (Fang & Liu, 2003; Fang et al, 2010).

Grey control model includes controllability and observability of grey system, transfer function of grey system, robust stability of grey system and several typical grey control models (Deng, 1982, 1985; Su & Liu, 2008, 2009; Zhou & Deng, 1986, 1989).

This book will focus on the most commonly used grey system methods and modeling technology.

Considering all the feedbacks from the readers of our earlier monographs, Grey Information (Liu & Lin, 2006), Grey Data Analysis (Liu et al., 2017) and Grey Systems Analysis (Liu et al., 2022a, 2022b), we have paid special attention to organize some of the most recent new results obtained by colleagues from around the world in this volume. Also, for the convenience of practical applications, this book is accompanied with a computer software on grey systems modeling, which is designed by Zeng Bo of our research group.

## References

Dang, Y. G., Liu, S. F., Liu, B., et al. (2004a). An improvement on the slope degree of grey incidence. *Engineering Science of China, 6*(3), 41–44. In Chinese.

Dang, Y. G., Liu, S. F., & Liu, B., et al. (2004b). About weakening buffer operators. *Management Science of China, 12*(2), 108–111 (In Chinese).

Dang, Y. G., & Liu, S. F. (2009). *On grey prediction and decision-making models.* Science. In Chinese.

Deng, J. L. (1965). Multivariable linear system shunt calibration device of a comprehensive approach. *Acta Automatica Sinica*, *3*(1), 13–26 (In Chinese).

Deng, J. L. (1982). Control problems of grey systems. *Systems & Control Letters, 1*(5), 288–294.

Deng, J. L. (1984). The theory and methodology of socio-economic gray systems. *Social Sciences in China*, (6):47–60.

Deng, J. L. (1985). *Grey control system.* Press of Huazhong University of Science and Technology (In Chinese).

Deng, J. L. (1986). *Grey prediction and decision-making.* Press of Huazhong University of Science and Technology.

Fang, Z. G., & Liu, S. F. (2003). Grey matrix game models based on pure strategies (1): Construction of standard grey matrix game models. *Journal of Southeast University, 33*(6), 796–800. In Chinese.

Fang, Z. G., Liu, S. F., & Shi, H. X. (2010). *Grey game theory and its applications in economic decision-making.* Taylor &Francis Group.

Guo, X. J., Liu, S. F., Wu, L. F., et al. (2015). A multi-variable grey model with a self-memory component and its application on engineering prediction. *Engineering Applications of Artificial Intelligence., 42*, 82–93.

Li, Q. X., & Liu, S. F. (2008). The foundation of the grey matrix and the grey input–output analysis. *Applied Mathematical Modelling, 32*(3), 267–291.

Li, Q. X. (2009). Grey dynamic input–output analysis. *Journal of Mathematical Analysis and Applications, 359*(2), 514–526.

Li, Q. X., Liu, S. F., & Lin, Y. (2012). Grey enterprise input-output analysis. *Journal of Computational and Applied Mathematics, 236*(7), 1862–1875.

Lin, C. H., Song, Z. Y., Liu, S. F., et al. (2021). Study on mechanism and filter efficacy of AGO/IAGO. *Grey Systems: Theory and Application, 11*(1), 1–21.

Lin, C. H., Liu, S. F., Fang, Z. G., & Yang, Y. J. (2022). Spectrum analysis of moving average operator and construction of time-frequency hybrid sequence operator. *Grey Systems: Theory and Application, 12*(1), 101–116.

Liu, S. F. (1991). The three axioms of buffer operator and their application. *Journal of Grey System, 3*(1), 39–48.

Liu, S. F., & Guo, T. B. (1991). *Grey systems theory and its applications.* Henan University Press.

Liu, S. F. (1992). Generalized degree of grey incidence. *Zhang Shengkai, Information and Systems* (pp. 113–116). DMU Publishing House.

Liu, S. F. (1993). Fixed weight grey clustering evaluation analysis. In: *New methods of Grey systems* (pp. 178–184). Agriculture Press.

Liu, S. F., & Zhu, Y. D. (1993). Models based on criteria's triangular membership functions for the evaluation of regional economics. *Journal of Agricultural Engineering, 9*(2), 8–13.

Liu, S. F., & Zhu, Y. D. (1996). Grey-econometrics combined model. *The Journal of Grey System, 8*(1), 103–110.

Liu, S. F., & Dang, Y. G. (1997). The drift of LPGP and the satisfaction of positioning solution. *Journal of Huazhong University of Technology, 25*(1), 28–31. in Chinese.

Liu, S. F., Li, B. J. & Dang, Y. G. (2004). The G-C-D model and technical advance. *Kybernetes: The International Journal of Systems & Cybernetics, 33*(2), 303–309.

Liu, S. F., Guo, T. B., & Dang, Y. G. (1999). *Grey systems theory and its applications* (2nd ed.). Science Press.

Liu, S. F., Dang, Y. G., & Fang, Z. G. (2003). *Grey systems theory and its applications* (3rd ed.). Science Press.
Liu, S. F., & Lin, Y. (2006). *Grey information: theory and practical applications.* Springer.
Liu, S. F., & Xie, N. M. (2006). *Grey systems theory and its applications* (4th ed.). Science Press.
Liu, S. F. (2024). *Grey systems theory and its applications* (10th ed.). Science Press.
Liu, S. F. (2004). Appearance and development of grey systems theory. *Journal of Nanjing University of Aeronautics and Astronautics*, *36*(2), 267–272.
Liu, S. F., & Lin, Y. (1998). An Introduction to Grey Systems: Foundation, Methodology and Applications. IIGSS Academic Publisher.
Liu, S. F., & Lin, Y. (Eds.), (2010). *Advance in grey systems research.* Springer.
Liu, S., Xie, N., & Forrest, J. (2011), Novel models of grey relational analysis based on visual angle of similarity and nearness. In *Grey systems: Theory and application* (Vol. 1 No. 1, pp. 8–18).
Liu, S. F., & Xie, N. M. (2011). A new grey evaluation method based on reformative triangular whitenization weight function. *Journal of Systems Engineering, 26*(2), 244–250 (In Chinese).
Liu, S. F., Fang, Z. G., & Yang, Y. J. et al. (2012). General grey numbers and its operations. In *Grey systems: Theory and application* (Vol. 2, No. 3, pp. 341–349).
Liu, S. F., Xu, B., Forrest, J., Chen, Y., & Yang, Y. J. (2013). On uniform effect measure functions and a weighted multi-attribute grey target decision model. *The Journal of Grey System, 25*(1), 1–11.
Liu, S. F., Yang, Y. J., & Fang, Z. G., et al. (2015a). Grey cluster evaluation models based on mixed triangular whitenization weight functions. In *Grey systems: Theory and application* (Vol.5, No. 3, pp. 410–418).
Liu, S. F., Zeng, B., Liu, J. F., et al. (2015b). Four basic models of GM(1, 1) and their suitable sequences. In *Grey systems: Theory and application* (Vol. 5, No. 2, pp. 141–156).
Liu, S., Tao, L., Xie, N., & Yang, Y. (2016), On the new model system and framework of grey system theory. *The Journal of Grey System, 28*(1), 1–15.
Liu, S. F., Yang, Y. J., Xie, N. M., & Forrest, J. (2016a). New progress of grey system theory in the new millennium. In *Grey systems theory and application* (Vol. 6, No. 1, pp. 2–31).
Liu, S. F., Lin, C. H., Tao, L. Y., et al. (2020). On spectral analysis and new research directions in grey system theory. *The Journal of Grey System, 32*(1), 108–117.
Liu, S. F. (2023). Negative grey relational model and measurement of the reverse incentive effect of Fields Medal. In *Grey systems: Theory and application* (Vol. 13, No. 1, pp. 1–13).
Liu, S. F., Yang, Y. J., & Lin, Y. (2022a). *Grey systems analysis.* Springer.
Liu, S. F., Li, B. J., & et al. (1998). Evaluation criteria of regional leading industries and mathematical models. *Management Science of China*, *6*(2), 8–13.
Liu, S. F., Yang, Y. J., & Forrest, J. (2017). *Grey data analysis.* Springer.
Liu, S. F., Tao, Y., Xie, N. M., Tao, L. Y., & Hu, M. L. (2022). Advance in grey system theory and applications in science and engineering. In *Grey systems: Theory and application* (Vol. 12, No. 4, pp. 1–18).
Liu, S. F.; Lu, N. N., Shang, Z. J., & Kapila Tharanga Rathnayaka, R. M. (2024). A new grey relational analysis model of cross-sequences. *Grey systems: Theory and application.* Published online https://doi.org/10.1108/GS-10-2023-0098
Lu, N. N.; Liu, S. F., & Du, J. L. (2023). Grey relational analysis model with cross-sequences and its application in evaluating air quality index. *Expert Systems with Applications, 233*, 120910.
Mayer-Schönberger, V., Cukier, K. (2013). *Big data: A revolution that will transform how we live, work, and think*. Houghton Mifflin Harcourt.
Nasseri, S. H., & Darvishi, D. (2018). Duality results on grey linear programming problems. *The Journal of Grey Science, 30*(3), 127–142.
Pawlak, Z. (1991). *Rough sets: Theoretical aspects of reasoning about data.* Kluwer Academic Publishers (References 15).
Su, C. H., & Liu, S. F. (2008). On the asymptotic stability of grey stochastic linear delay systems. *Control and Decision, 23*(5). 571–574 (In Chinese).

Su, C. H., & Liu, S. F. (2009). On robust stability of p-moment index of stochastic system with distributed delay and the interval parameters. *Applied Mathematics and Mechanics, 30*(7), 856–864.

Wu, L. F, Liu, S. F., Yao, L. G. et al. (2013). Grey system model with the fractional order accumulation. *Communications in Nonlinear Science and Numerical Simulation, 18*(7):1775–1785.

Xie, N. M., & Liu, S. F. (2005). Discrete GM(1,1) model and modeling mechanism of grey prediction models. *Systems Engineering: Theory and Practice, 28*(4), 93–99. In Chinese.

Xie, N. M., & Liu, S. F. (2009). Discrete grey forecasting model and its optimization. *Applied Mathematical Modelling, 33*(1), 1173–1186.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*, 338–353.

Zhang, K., & Liu, S. F. (2009). A novel algorithm of image edge detection based on matrix degree of grey incidences. *The Journal of Grey System, 9*(3), 265–276.

Zhao, C. S. (2015). Book reviews: Preface to the 7th edition of grey system theory and its applications. *The Journal of Grey System*, *27*(1), 127–129.

Zhou, C. S. & Deng, J. L. (1986). The stability of grey linear system. *International Journal of Control*, *43*(1), 313–320.

Zhou, C. S. & Deng, J. L. (1989). Stability analysis of grey discrete-time systems. *IEEE Trans. AC*, *34*(2), 173–175.

Zhu, D. H., Feng, J. C., Song, H. Q., et al. (2011). Application of grey neural network in life prediction of air to ship missile. *Tactical Missile Technology, 5*, 54–58.

# Chapter 2
# Characteristics of Grey System Theory

## 2.1 A Kind of Poor Data Analysis Method with Strong Penetration

Grey system theory takes the uncertain system with poor information as the research object. It is an interdisciplinary method with strong penetration.

At Nanjing University of Aeronautics and Astronautics, the teaching team of management quantitative method course group led by Professor Sifeng Liu has been committed to the construction of Chinese original grey system theory courses for a long time. With the strong support of peer experts, the grey system theory courses has been selected as the National Excellence Course beginning in 2008, and the National first class courses of online and offline since 2020. The teaching resources including textbooks, videos and modeling software are widely distributed. At the same time, the original elements such as grey sequence operator, grey relational analysis, grey clustering evaluation, grey prediction, grey decision-making and grey linear programming, etc. in the grey system theory and the latest achievements made by the course team and partnerships both at home and abroad are rewritten into teaching cases and injected into the courses of "Operations Research" "Applied Statistics" "Prediction Methods and Technologies" "Theory and Methods on Decision-making" "economic cybernetics" "system modeling and simulation", "input–output analysis" and "econometrics". It enriches the connotation of these courses, and greatly improved the overall construction level of the curriculum group. In 2010, the course team was selected into the national excellence teaching team. In 2018, "The construction of management quantitative method course group and teaching reform led by local original theory" won the prize of national teaching achievement.



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_2

## 2.2 Characteristics of Uncertain Systems and the Simplicity Principle in Sciences

The fundamental characteristic of uncertain systems is the incompleteness and inadequacy of their information. Due to the dynamics of system evolution, the biological limitations of the human sensory system, as well as the constraints of relevant economic conditions and technological availabilities, uncertain systems exist commonly.

## 2.3 Incomplete Information

Incompleteness in information is one of the fundamental characteristics of uncertain systems. The most common situations involving incomplete system information include cases where:

(1) Information about system elements (parameters) is incomplete;
(2) Information on the structure of the system is incomplete;
(3) Information about the boundary of the system is incomplete; and
(4) Information on the system's behaviors is incomplete.

Incomplete information is a common phenomenon in our social, economic, and scientific research activities. For instance, in agricultural production, even if we have exact information regarding plantation, seeds, fertilizers, and irrigation, uncertainties in areas such as labor quality, natural environment characteristics, weather conditions, and the commodity markets make it extremely difficult to precisely predict the production output and consequent economic value of agricultural fields. For biological prevention systems, even if we know the relationship between insects and their natural enemies, it is still really difficult to achieve the expected prevention effects due to uncertainty regarding the relationships between insects and their baits, insects' natural enemies and their baits, and a specific kind of natural enemy with another kind of natural enemy. As for the adjustment and reform of pricing systems, it is often difficult for policy makers to take actions because of the lack of information regarding price elasticity and consumer demand and how price changes on a certain commodity would affect the prices of other commodities. In security markets, even the brightest market analysts cannot be assured of winning constantly due to their inability to correctly predict economic policy and interest rate changes, management changes at various companies, the direction of political changes, investors' behavioral changes in international markets, and the effects of price changes in one block of commodities on another. As for the general economic system, because there are no clear relationships between the "inside" and the "outside" of the system, and between the system itself and its environment, and because the boundaries between the inside and the outside of the system are difficult to define, it is also difficult to analyze the effects of economic input on economic output.

Incompleteness in available information is absolute, while completeness in information is relative. Humans employ their limited cognitive ability to observe the infinite universe in order to try and obtain complete information. However, it is impossible for us to do so. In fact, the concept of large samples in statistics represents the degree of tolerance man has to incompleteness. In theory, when a sample contains at least 30 objects, it is considered "large." However, in some situations, even when a sample contains thousands or several tens of thousands of objects, the true statistical laws of a given system still cannot be successfully uncovered.

## 2.4 Inaccuracies in Data

Another fundamental characteristic of uncertain systems is naturally occurring inaccuracy in available data. In grey systems theory, the meanings of uncertain and inaccurate are roughly the same. Both terms stand for errors or deviations from actual data values. Based on the essence of how uncertainties are caused, inaccuracies can be categorized into three types: the conceptual, level, and prediction type inaccuracies.

(1) The Conceptual Type

Inaccuracies of the conceptual type emanate from the expression of a certain event, object, concept, or wish. For instance, all such frequently used concepts as "large," "small," "many," "few," "high," "low," "fat," "thin," "good," "bad," "young," and "beautiful" are inaccurate due to lack of clear definition. It is very difficult to use exact quantities to express these concepts. As a second example, suppose that a job seeker with an MBA degree wishes to get an annual salary offer of no less than ¥450,000, or that a manufacturing firm plans to control its rate of defective products to be less than 0.01%. These are all cases of conceptual type inaccuracies.

(2) The Level Type

This kind of data inaccuracy is caused by a change at the level of research or observation. This means that the available data might be accurate when seen at the level of the system of concern, that is, the macroscopic level, or at the level of the whole, that is, the cognitive conceptual level. However, when data are seen at a lower level, that is, a microscopic level, or at a partial localized level of the system, they generally become inaccurate. For example, the height of a person can be measured accurately to the unit of centimeters or millimeters. However, if the measurement has to be accurate to the level of one ten-thousandth micrometers, the former accurate reading will become extremely inaccurate.

(3) The Prediction (or Estimation) Type

Because it is difficult to have complete understanding of the laws of evolution, any prediction of the future tends to be inaccurate. For instance, it is estimated that two years from now, the GDP of a certain country will surpass $10 billion dollars; it is estimated that a certain bank will attract savings from individual residents of between

$7 billion and $9 billion for the year 2024; it is predicted that in the coming years the temperature in Leicester, UK, during the month of June will not go beyond 30º C, and so on. All these examples provide uncertain numbers of the prediction type. In statistics, it is often the case that samples are collected to estimate the whole. Therefore, much statistical data are inaccurate. As a matter of fact, no matter what method is used, it is very difficult for anyone to obtain any absolutely accurate (estimated) value. When we draw out plans for the future and make decisions about what course of action to take, we in general have to rely on inaccurate predictions and estimates.

## 2.5 The Scientific Principle of Simplicity

In the history of science, the achievement of simplicity has been a common goal among most scientists. As early as the sixth century BC, natural philosophers had a common wish to understand the material laws of nature: to build knowledge of the material world on the basis of a few common, simple elements. The ancient Pythagoras of Greece introduced the theory of four elements (earth, water, fire, and gas) at around 500 BC. The Greeks believed that all material matters in the universe were composed of these four simple elements. Around the same time, ancient Chinese philosophers also developed a theory of five elements including water, fire, wood, gold, and earth. These are the most primitive and elementary thoughts about simplicity.

The scientific principle of simplicity originates from the simplicity of thinking employed in the process of understanding nature. As the natural sciences matured over time, simplicity became the foundation and guiding principle of scientific research. For example, Newtonian laws of motion unify the macroscopic phenomena of objective movements in their form of extreme simplicity. In his *Mathematical Principles of Natural Philosophy*, Newton pointed out that nature does not do useless work; because nature is fond of simplicity, it does not like to employ extra reasons to flaunt itself. During the Era of relativity, Albert Einstein introduced two criteria for testing a theory: external confirmation and internal completeness, that is, logical simplicity. Einstein believed that a true scientific theory must comply with the principle of simplicity in order to reflect the harmony and orderliness of nature. In the 1870s, Ampere, Weber, Maxwell, and others established theories to explain the phenomenon of electromagnetism based on their different assumptions. Because Maxwell's theory is the one that best complies with the principle of simplicity, it became well accepted. Another example is the well-known Kepler's third law of planetary motion: $T^2 = D^3$. This formula is very concise in form.

According to the dominant principle of synergetics (Haken, 1978), one can transform an original high-dimensional equation into a low-dimensional evolution equation of order-parameters by eliminating the fast-relaxing variables in the high-dimensional nonlinear equation that describes the evolution process of a system. Because the order-parameters dominate the dynamic characteristics of the system near its boundary points, through dominant the evolution equation of order-parameters one can obtain the system's time structure, space structure or time–space structure, so that one can materialize efficient control over the system's behavior.

The simplicity of scientific models is actualized by employing simple expressions and by ignoring unimportant factors of the system of concern. In economics, the methods of using Gini coefficient to describe differences among consumers' incomes (Gini, 1972) and of employing Cobb-Dauglas production function to measure the contribution of advancing technology in economic growth are all introduced on the basis of simplifying realistic systems (Cobb & Douglas, 1928). Modigliani and Brumbergh (1954) use the following model to describe the average propensity to consume:

$$\frac{C_i}{y_i} = a + b\frac{y_0}{y_t}, a > 0, b > 0$$

The curve Phillips (1958) employs to describe the relationship between the rate of inflation $\frac{\Delta p}{p}$ and the unemployment rate $x$ is:

$$\frac{\Delta p}{p} = a + b\frac{1}{x}$$

Additionally, the well-known capital asset pricing model (CAPM, Sharpe, 1964) can be seen below:

$$E[E_i] = r_f + \beta_i(E[r_m] - r_f)$$

Essentially, all of these equations can be reduced to their simplest linear regression model with a few straightforward transformations.

## 2.6 Precise Models Suffer from Inaccuracies

When available information is incomplete and the collected data inaccurate, any pursuit of precise models in general becomes meaningless. This fact was well described by Lao Tzu more than two thousand years ago (Tau, 2007). The principle of incompatibility proposed by L. A. Zadeh, the founder of fuzzy mathematics, also addresses this matter: when the complexity of a system increases, our ability to precisely and meaningfully describe the characteristics of the system decreases accordingly until such a threshold that, as soon as it is surpassed, the preciseness and meaningfulness become two mutually excluding characteristics (Zadeh, 1994). This mutually antagonistic principle reveals that the pursuit of preciseness can reduce the operationality and meaningfulness of a cognitive outcome. Therefore, precise models are not necessarily an effective means to address complex matters.

**Table 2.1** Comparison between the prediction errors of a statistical model and a grey model

| Order No. | Type | Average error | |
|---|---|---|---|
| | | Statistical model | Grey model |
| 1 | Horizontal displacement | 0.862 | 0.809 |
| 2 | Horizontal displacement | 0.446 | 0.232 |
| 3 | Vertical displacement | 1.024 | 1.029 |
| 4 | Vertical displacement | 0.465 | 0.449 |
| 5 | Water level of pressure measurement hole | 6.297 | 3.842 |
| 6 | Water level of pressure measurement hole | 0.204 | 0.023 |

In 1994, Jiangping Yue and Xisheng Hua established both theoretically delicate statistical regression model and relatively coarse grey model based on the deformation data and leakage data of a certain large scale hydraulic dam. Their work shows that the grey model provided a better fit than the statistical regression model. When comparing the errors between the predictions of the two models with actual observations, it is found that the prediction accuracy of the grey model is generally better than that of the regression model; see Table 2.1 for details (Yue & Hua, 1994).

In 2001, Dr. Haiqing Guo as well as Zhongru Wu and colleagues respectively established a statistical regression model and a grey time series combined model using the observational data of displacement in the vertical direction of a certain large clay-rock filled dam of inclined walls. They compared the data fitting and predictions of the two models against actual observations and found that the data fitting eof the grey combined model was significantly superior to that of the statistical regression model (Guo et al., 2001).

On the other hand, Xiaobing Li, Haiyan Sun and colleagues employed fuzzy prediction functions (a type of uncertainty prediction) to dynamically trace and precisely control the fuel oil feeding temperature for anode baking. The control effect was clearly better than that obtained by utilizing the traditional PID control method (Li et al., 2009).

Finally, Caixing Sun and his research group made use of grey relational analysis, grey clustering, and various new types of grey prediction models to diagnose and predict insulation-related accidents related to electric transformers. Their substantial results indicate that these relatively coarse methods and models are operational and provide efficient results (Li et al., 2002; Sun et al., 2002, 2003).

## 2.7 Comparison of Several Uncertainty Methods

Probability and statistics, fuzzy mathematics, grey system theory and rough set theory are four of the most widely used research methods in the investigation of uncertain systems. Their research objects contain specific kinds of uncertainty, which represent their commonality. It is precisely the differences among the uncertainties in the research objects that make these four theories of uncertainty distinct from each other.

Probability and statistics study the phenomena of stochastic uncertainty with emphasis placed on revealing historical statistical laws. They investigate the chance of each possible outcome of the stochastic uncertain phenomenon to occur. Their starting point is the availability of large samples, which are required to satisfy a typical form of distribution.

Fuzzy mathematics emphasizes the investigation of problems with cognitive uncertainty, where research objects possess the characteristic of clear intension and unclear extension. For instance, "young man" is a fuzzy concept, because each person knows the intension of "young man." However, if we determine the exact age range within which everybody is young and outside which each person is not young, then we will have great difficulty. That is because the concept of young man does not have a clear extension. In fuzzy mathematics, this kind of cognitive uncertainty problem with clear intension and unclear extension is addressed by making use of experience and the so-called membership function.

Additionally, rough set theory tries to study uncertain systems by using the accuracy mathematical method. The main thought of rough set theory is to describe and address the inaccuracy or uncertain knowledge using a known knowledge library. Professor Z. Pawlak included all the units which cannot be acknowledged to have boundaries. He defined boundary as the difference set between upper approximate set and lower approximate set. The boundary is then described through the upper approximate set approaching the lower approximate set (Pawlak, 1991).

The focus of grey system theory, on the other hand, is on the uncertainty problems of small data sets and poor information, which are different to the problems addressed by probability, fuzzy mathematics or rough set theory. It explores and uncovers the realistic laws of evolution, motion of events and materials through information coverage by possibility function, and through the works of sequence operators. One of its characteristics is construct models with small amounts of data. What is clearly different about grey systems theory compared to fuzzy mathematics is that grey system theory emphasizes the investigation of objects that process clear extension and unclear intension. For example, by the year of 2035, The Per Capita GDP of Shenzhen, China will be within the range of USD 55000 to USD 65000 billion. This range from 55,000–65,000 is a grey concept. Its extension is definite and clear. However, if one inquires further regarding exactly which specific number within the said range it will be, then he will not be able to obtain any meaningful and definite answer before 2035. It's a grey number before 2035.

We summarize the differences among these four main uncertainty research methods in Table 2.2.

**Table 2.2** Comparison among the four methods of uncertainty research

| Uncertainty research | Grey system | Prob. statistics | Fuzzy math | Rough set |
|---|---|---|---|---|
| Research objects | Poor information | Stochastics | Cognitive | Boundary |
| Basic set | Grey number set | Cantor set | Fuzzy set | Approximate set |
| Describe method | Possibility func | Density func | Membership func | Upper, lower Appr |
| Procedure | Sequence operator | Frequency | Cut set | Dividing |
| Data requirement | Any distribution | Known distribution | Known membership | Equivalent Rel |
| Emphasis | Intension | Intension | Extension | Intension |
| Objective | Law of reality | Historical law | Cognitive expression | Approx. approaching |
| Characteristics | Small data | Large sample | Depend on experience | Information form |

## 2.8 Deep Applications of Grey System Theory in the Fields of Social Science, Natural Science and Engineering Technology

### *2.8.1 Successful Application of Grey System Theory in the Field of Social Sciences*

The rapid development of grey system theory in the early stage of its establishment largely benefited from its successful application in the field of economic management, that is, the strong impetus of the urgent need to carry out agricultural zoning and formulate economic development strategic planning all over the country of China in the 1980s. The reform of the economic system and the adjustment of the statistical system directly affected the integrity and continuity of economic data. The disconnected data posed a big problem for the planners at that time. How to complete the tasks of system analysis and modeling based on small samples and poor information data, so as to obtain the prediction results with high reliability and support the scientific decision-making of governments at all levels? The grey system theory characterized by small sample, poor information data modeling and analysis is just right. At that time, many government departments from the central to local governments tried to use grey system methods and models to analyze economic data and prepare development plans. Professor Deng Julong presided over and completed the research and preparation of the development plan of Yixian County, Hebei Province and Laohekou City, Hubei Province. The author has also presided over and participated in the completion of a number of key bidding projects of the National Development and Reform Commission of China, the Ministry of Science and Technology of China and the China Association for Science and Technology, as well as the development planning research of Henan Province, Jiangsu Province, Nanjing and Zhongyuan District of Zhengzhou, Hubin District of Sanmenxia, Changge City and Wuzhi county, etc. The data analysis mainly adopts the grey system method and model (Liu & Yang, 1994)

Academician Yang Shanlin and academician Chen Xiaohong of Chinese Academy of engineering, academician Zavadskas of Lithuanian Academy of Sciences and their team have successfully solved many major problems in management practice by using grey system model and method, and achieved a series of research results (Chen, 2018; Jahan & Zavadskas, 2019; Xu & Yang, 2013). Applied the grey relational analysis models, Kose et al., studied the problem of most livable city selection in Turkey (Kose et al., 2020). Peng, et al. analyzed the circular economy of E-commerce market based on grey model under the background of big data (Peng et al., 2022).

Emil Scarlat and Camelia Delcea with Bucharest University of Economics of Romania used the methods and models of grey system theory to study the control of economic system, achieved a series of achievements (Delcea et al., 2013; Scarlat & Delcea, 2011), and published a monograph in Romanian.

## 2.9 Deep Application of Grey System Theory in the Field of Natural Science

Enter physics, chemistry, biology, geology, hydrology, crops, and medicine etc. as subject words into CNKI database to search the literature with physics, chemistry and other subjects and accurately containing the phrase "grey system". The results are shown in Table 2.3.

Grey system theory has been applied to the fields of physics, chemistry, biology, geology, hydrology, crops, medicine and so on, a large number of valuable research results have been obtained (Liu et al., 2022).

**Table 2.3** Number of articles containing the phrase "grey system" accurately in various disciplines of Natural Science

| Discipline | Physics | Chemistry | Biology | Geology | Hydrology | Crops | Medicine |
|---|---|---|---|---|---|---|---|
| No. of papers | 2379 | 3127 | 3833 | 8565 | 3190 | 3010 | 565 |

**Retrieval time**: 26/7/2024

**For Example, in the Field of Physics**

Chen Lei et al. used the grey relational analysis model to study two sky light measurement methods based on ASD ground object spectrometer—standard gray plate inversion measurement method and direct measurement method, and defined the applicable scenarios of different methods(Chen et al., 2011). Wang Yue and Chen Zonghai studiedμparticles imaging of cosmic rays by using the method of grey correlation cluster analysis, the efficiency of material differentiation is improved (Wang et al., 2011a, 2011b).

Han Li et al. studied the geophysical characteristics of dynamic compaction fill foundation by using the grey correlation analysis model, and evaluated the quality and effect of dynamic compaction by analyzing the correlation between surface wave velocity, resistivity and geophysical characteristic parameters such as soil dry density and water content (Han et al., 2020).

Evans applied the grey system model to study the strength of British steel, and proposed a new method for parameter estimation of Generalized Grey Verhulst model (Evans, 2014).

Shi et al. conducted reliability analysis on passive residual heat removal of AP1000 nuclear power reactor based on grey model (Shi et al., 2017), and Wang Qin et al. used the grey correlation analysis method to study the optimal parameters of arc signal welding process (Wang et al., 2010a, 2010b), both have achieved important results.

**In the Field of Chemistry**

Liu Yaoxin et al. studied the formation reaction of calcium sulphoaluminate in high temperature sulfur fixation phase by using grey correlation analysis and prediction model (Liu et al., 2007). Pornnapa Kasemsiri et al. used Taguchi method and grey relational analysis model to optimize biodegradable foam composites made from cassava starch, oil palm fiber, chitosan and palm oil (Kasemsiri et al., 2017).

Gupta et al. applied the grey correlation analysis method to optimize the mechanical properties of hybrid filler pultruded glass fiber composites (Gupta et al., 2019).

Jena et al. applied Taguchi grey correlation analysis to optimize parameters for maximizing photocatalytic behaviour of Zn1-xFexO nanoparticles for methyl orange degradation using Taguchi and Grey relational analysis Approach (Jena et al., 2019).

**In the Field of Biology**

Zhang Fuli et al. studied the effect of BT insect resistant cotton straw returning on soil nutrient characteristics by using grey correlation analysis model. It is considered that straw returning is an ideal way for harmless treatment of Bt transgenic plant straw (Zhang et al., 2020). Yang et al. used the grey correlation analysis model to study the pigment content and standard deviation vegetation index in rice vegetative stage (Yang et al., 2012).

Luo Qin and others used the grey correlation analysis model to study the relationship between trace element content and lead content in the seed body of new irradiated

Pleurotus ostreatus, which provided a scientific basis for breeding Pleurotus ostreatus varieties with lower lead content (Luo et al., 2015).

Guo Ruilin has conducted in-depth research on crop grey breeding and cultivated some new crop varieties (Guo, 1995).

Based on hyperspectral data, Jin et al. used grey correlation analysis and partial least square method to estimate the leaf water content of winter wheat (Jin et al., 2013). Wei et al. used the grey correlation analysis method to evaluate the quality of Tibetan highland barley (Wei & Zhang 2019), has achieved important results.

**In the Field of Geology and Earth Sciences**

Academician Zhao Pengda constructed the theory and method system of quantitative prediction of mineral resources, and put forward "geological anomaly", "mathematical characteristics of geological body", "triple" quantitative metallogenic prediction, research on non-traditional mineral resources, new concepts, new contents and research methods. Two prospective metallogenic belts of copper nickel sulfide were found in Beishan area of Xinjiang and one gold belt was found in East Junggar (Zhao & Xia, 2009).

Research on safety analysis, evaluation, excavation and control measures design optimization and real-time monitoring of Geotechnical Engineering (including landslide) by Gao Wei and academician Feng XiaTing (Gao & Feng et al., 2004), study on limit displacement discrimination of stability and reliability analysis of surrounding rock of tunnel and underground engineering by Academician Li Xiaohong et al. (Li et al., 2005), have achieved results of great value.

Peng Fang and Wu Guoping established a new quantitative evaluation method of caprock based on grey programming cluster analysis. They used this method to evaluate 12 kinds of caprock objects in 4 sets of mudstone in 3 main exploration areas of southeast basin of Hainan. The conclusion is consistent with the exploration results (Peng et al., 2005). Liang Bing et al. optimized and ranked the exploration and development potential of complex geological parameter characteristic areas with evaluation index value of interval grey number by establishing a multi index grey correlation degree optimization model (Liang et al., 2014). Chen Ronghuan and others used the grey system theory to study logging, drilling coring, oil testing and relevant geological data. Through matching, fitting and extracting parameters, they studied and divided formation lithology, physical properties and oil bearing properties by statistical analysis of eigenvalues and their accuracy and resolution, which provided a geological basis for oilfield exploration and development (Chen et al., 2005). Wang yunyun et al. used the grey correlation analysis method to scientifically predict the Yaojialing zinc gold polymetallic deposit (Wang et al., 2013).

Fang Xiaotong and others used the multi-dimensional grey evaluation model to predict the risk of coal and gas outburst, which provided a basis for mine safety production (Fang et al., 2012). Zeng et al. predicted China's shale gas production based on weakening buffer operator and unbiased grey model (Zeng et al., 2018). Kose and Tasci predict geodetic deformation based on multivariable grey prediction model and regression model (Kose & Tasci, 2019).

**In the Field of Hydrology and Water Resources**

Lin Yuezhong and others established the grey prediction model of slope rock mass deformation based on the field slope test data of the Three Gorges, and drew the fitting and prediction curve of slope deformation, which provided a reliable guarantee and theoretical basis for the prediction of slope rock mass deformation (Lin et al., 2005). Academician Xia Jun's research on grey system hydrology (Xia & Zhao, 1996), academician Wu Zhongru's research on hydraulic structure and dam safety monitoring (Wu et al., 2012), and research on utilization of water resources by Wang and academician Hipel (Wang & Hipel, 2011b), have achieved a series of important results.

Hao et al. used the grey system model to analyze and predict the hydrological process of karst basin, and obtained high accuracy. They also used the segmented grey model to study the impact of human activities on the hydrological process of karst basin (Hao et al., 2013).

Peng Yong et al. studied the optimization algorithm of cascade reservoir operation based on the combination of grey prediction model and DDDP (Peng et al., 2018). With limited hydrogeological data, Mahmod et al. used the modified grey model to analyze the groundwater flow in Nubia sandstone area of Halga Oasis, Egypt (Mahmod et al., 2014).

**In the Field of Medicine**

Grey system method and model technology are widely used in modern medical fields such as disease prediction and control, health management evaluation, intelligent diagnosis system construction, drug efficacy evaluation and medical image processing, and have made gratifying achievements, forming a branch field of grey medical research in grey system theory (Zhang et al., 2015).

Professor Tan xuerui, Dean and doctoral supervisor of Medical College of Shantou University, and his research team have systematically studied the grey correlation methodology of clinical trials with the support of a number of National Natural Science Foundation of China and Guangdong Natural Science Foundation. The new clinical trial methods proposed, such as ergodic grey correlation space theory, polarity analysis theory and method of grey medical correlation factors, axiom system of multi-level grey medical correlation, grey correlation method comparison model, have been applied to many clinical medical disciplines, such as cardiovascular medicine, digestive medicine, neurology, infectious diseases and so on (Tan et al., 2011).

Wei Hang et al. Established the pattern recognition model of chromatographic fingerprint of traditional Chinese medicine by using the grey system theory. The results of high performance liquid chromatography analysis of 56 batches of different varieties of tangerine showed that the recognition rate exceeded 92.85% for different cultivated varieties of tangerine with very similar chemical composition and content (Wei et al., 2013).

Semra Icer et al. quantitatively graded the ultrasonic images of fatty liver based on grey correlation analysis, and obtained the scientific diagnosis results (Icer et al., 2012). Lai Hsin Yi et al. applied the unsupervised single chain clustering method based on grey correlation analysis to the automatic sorting of spike waves in extracellular electrophysiological records (Lai et al., 2011). Bhupendra Gupta and Mayank Tiwari have achieved good results in breast image brightness preserving contrast enhancement and quality segmentation based on histogram modified grey correlation analysis(Gupta & Tiwari, 2017).

## 2.10 A Large Number of Applications of Grey System Theory in the Field of Engineering Technology

Enter the subject words such as transportation, power and machinery, etc. respectively in the CNKI database, and to search the literatures with transportation, electric power and machinery, etc. as the subject and accurately including the phrase "grey system". The results are shown in Table 2.4.

Grey system theory has been applied to the fields of engineering technology such as transportation, electric power and machinery, etc., has achieved thousands or even tens of thousands of research results. Among them, there are more than 5000 papers containing "grey system" in the fields of power, computer and material science. There are more than 10,000 in the field of transportation, nearly 30,000 in the field of information science and more than 30,000 in the field of environmental science (Liu et al., 2022).

**For Example, in the Field of Transportation**

Liu Qiuyan and Zhong Zhangdui comprehensively used grey clustering and rough set model to optimize the planning scheme of railway digital mobile communication system with limited frequency, and improved the accuracy of electrical level and interference matrix estimation (Liu et al., 2010a, 2010b); Gao Fan and Zhang Youpeng designed the grey number of fitness according to the train operation target, and constructed the high-speed train speed controller model based on grey genetic algorithm (Gao et al, 2012); Lu Xiaohong and Wang Changlin studied the modeling and simulation of automatic train speed controller based on predictive grey control (Lu & Wang, 2013); Based on the data of Britain and the United States, Chirwa et al. used GM (1,1) model to estimate the accident risk (Chirwa & Mingzhi, 2006); Based on the diagnosis results of three diagnosis methods: fuzzy fault diagnosis method, genetic algorithm and grey system theory, Mi Gensuo et al. constructed the optimal combination model to diagnose the fault of 25 Hz phase sensitive track circuit (Mi et al., 2014).

After comparing the simulation results obtained by artificial neural network, classification and regression tree, k-nearest neighbor method, linear discriminant analysis method, naive Bayesian classifier, quasi optimal algorithm and support vector machine method with the grey correlation classifier algorithm, Twala found that the grey correlation classifier algorithm is most suitable for the modeling and analysis of road traffic accident data in Gauteng Province, South Africa (Twala, 2014).

**Table 2.4** Number of articles containing the phrase "grey system" accurately in various disciplines of Engineering Technology

| Discipline | Transportation | Power | Machinery | Motive power | Aviation | Architecture | Computer |
|---|---|---|---|---|---|---|---|
| No. of papers | 12,835 | 6832 | 3424 | 2714 | 1968 | 6244 | 6813 |
| Discipline | Electronics | Information | Petroleum | Chemical industry | Material | Irrigation works | Environment |
| No. of papers | 2727 | 29,159 | 3404 | 1280 | 9364 | 2073 | 34,853 |

**Retrieval time**: 26/7/2024

**In the Field of Power Engineering**

Research of academician Sun Caixin's team on the field of high voltage insulation and Fault Diagnosis Technology (Sun, 2005; Sun et al., 2003, 2002). Academician Li Licheng's research group on Power Grid Engineering, DC transmission and AC/DC parallel power grid operation technology (Huang et al., 2011).

Analysis of oil soluble gas content in power transformers by Liao et al. (Liao et al., 2012). According to the measured data of lubricating oil temperature and iron content of wind turbine gearbox, Yang et al. introduced multi-source information, improved the traditional grey system model, predicted the wear trend of wind turbine gearbox, and provided a scientific basis for gearbox maintenance and replacement decision-making (Yang et al, 2019).

Ossowski and Korzybski use grey system model to carry out analog circuit fault diagnosis (Ossowski & Korzybski, 2013); Jiang Wei diagnosed the fault of wind turbine drive chain based on grey rough set theory (Jiang, 2012);

Study on Modeling and prediction of non-stationary voltage fluctuations by Dejamkhooy et al. (Dejamkhooy et al, 2017).

**In the Field of Mechanical Engineering**

Academician Jia Zhenyuan's research on shape control machining theory, technology and equipment of high-end equipment and high-performance parts (Jia et al., 2009). Research on mechanical design and theory, computer aided design and graphics, digital design and manufacturing, etc. by academician Tan Jianrong's group (Fang et al., 2009). Research on submarine noise reduction technology by academician He Lin's group (Liao et al., 2017). Czeslaw Cempel used the grey prediction model to monitor the mechanical vibration state (Cempel, 2008). Wang Xuliang and Nie Hong used the grey system model to predict the fatigue life of mechanical parts, which greatly reduces the prediction error (Wang & Nie, 2008); Zhang Xueyuan et al. used GM (1,1) model to study the change law of robot emotional state, and realized the emotional robot interaction system (Zhang et al., 2006); Li Tong et al. used the grey prediction model to calculate the fatigue crack growth rate (Li et al., 2010).

Academician Zhang Jie et al. used the grey correlation analysis model to analyze the fault of two tooth difference swing movable teeth transmission, which provided a scientific basis for improving the reliability of two tooth difference swing movable teeth transmission system (Zhang et al., 2012). Xia Xintao and Wang Zhongyu used the grey correlation analysis model to study the relationship between rolling bearing processing quality and vibration, and found that the structural dimension error parameter is the factor that has a great impact on bearing vibration (Xia et al, 2005). Xie Yanmin et al. obtained the best parameters of each factor affecting the robustness of square box by analyzing the variance of the grey correlation degree between each factor and the target sequence (Xie et al., 2007).

Prakash et al. study on multi-objective optimization of turning stone powder reinforced aluminum matrix composites based on Taguchi method and grey correlation analysis model (Prakash et al., 2020). Loganathan et al. used the grey correlation analysis model to optimize the input parameters of progressive forming of AA6061 alloy (Loganathan et al., 2020). Pagar and Gawande (Pagar & Gawande, 2020) used the grey correlation analysis method to carry out parametric design analysis on the radial deflection stress of metal expansion bellows. Sharma used Taguchi and grey correlation analysis to study the accuracy and surface roughness of GFRP gears (Sharma et al., 2020). Khan et al. used the grey correlation analysis method to carry out multi-objective optimization of dry, wet and low temperature undercut titanium base alloys (Khan et al., 2020).

**In the Field of Aerospace**

Wang Yanyang and Cao Yihua established a nonlinear online prediction model of China's civil aviation operation risk by using the method of grey neural network (Wang et al., 2010a, 2010b); Yang et al. (2008) and Li Peihua (Yang & Li, 2011) used the grey system model to predict spacecraft faults and achieved high accuracy.

Xie Jianxi et al. solved the optimization decision-making problem of aircraft top-level design scheme by using the grey correlation analysis model (Xie et al., 2004); Zhang Cheng and Ding Songbin et al. studied the aircraft customization scheme based on the grey correlation analysis model (Zhang et al., 2014); Xiao Jun and Zhang Weiwei comprehensively used the grey correlation analysis and fault tree method to study the target crash fault, which provided a theoretical basis for diagnosing the cause of the target crash fault, controlling the occurrence of the fault and improving the system reliability (Xiao & Zhang, 2009).

Yu Fengjie and Ke Yinglin applied the grey clustering decision-making method to the automatic docking and assembly system of aircraft large parts, which improved the system stability, reduced the risk of equipment failure, and controlled the maintenance cost (Yu et al., 2009). Zhang Feng and Wang Pengwei used the grey clustering evaluation model to evaluate the safety of Shipborne aircraft system, which played a positive role in discovering system safety hazards in advance and preventing and reducing accidents (Zhang et al., 2010).

**In the Field of Intelligent Control**

Research on intelligent control theory and robot system, image recognition theory and machine vision application, intelligent control technology of advanced manufacturing equipment, and integrated automatic control system of major projects in power and electrical industry by Academician Wang Yaonan's team (Jia-qiang et al., 2005). Academician Liu Yexiang of the State Key Laboratory of powder metallurgy of Central South University have used the grey system method and model to study the control of aluminum electrolysis process, and achieved many results (Liu & Lin, 2004).

Tian Jianyan et al. established the grey prediction model of billet temperature in heating furnace and put forward the billet temperature control method (Tian & Lu, 2007); Wang Wei et al. proposed an improved fuzzy expert control method based on combined grey prediction model for the temperature control of coke oven flue with the characteristics of strong nonlinearity, large time delay and multi disturbance (Wang et al., 2010a, 2010b). Combining the traditional feedback control method and grey predictive control, Zhang Guangli et al. designed a self-adjusting grey predictive controller. The simulation results show that the new controller has better dynamic performance and robustness (Zhang et al., 2004).

In view of the randomness, nonlinearity and time variability of the deep-sea walking mechanism in the seabed complex operating environment, and it is difficult to establish an accurate mathematical model, Qiao Guiling et al. proposed a grey prediction fuzzy PID control method to realize the effective control of the deep-sea walking mechanism (Qiao et al., 2009). The research on pneumatic position servo control system based on grey correlation compensation control proposed by Zhu Jianmin et al. effectively improves the tracking accuracy of traditional control methods for pneumatic position servo control system (Zhu et al., 2012).

**In the Field of Weapon Equipment Development and Application**
Cui Jianpeng et al. studied the selection of surface to air missile weapon system by using multi-objective grey decision model (Cui et al., 2012); Li Xinqi et al. constructed a grey programming model for the optimal configuration of missile nuclear weapons, which provides a theoretical basis for the ordering, storage, position configuration and operational application of missile nuclear weapons (Li et al., 2007).

Han Xiaoming et al. used the grey clustering model to comprehensively evaluate the development scheme of air defense and anti-missile warhead (Han et al., 2014); Yao Junbo and Hu Weiwen applied the grey evaluation model to evaluate the operational effectiveness of over the horizon ground wave radar according to its characteristics and operational tasks (Yao & Hu, 2008).

Lin Jiajian used the grey relational analysis method to solve the main factors affecting the velocity of explosively formed projectile (EFP), and obtained the results that have important reference value for the design of EFP liner and charge structure (Lin et al., 2009). Zhao Guogang et al. established the threat assessment model of incoming missile in ship anti-missile operation by using the grey correlation analysis method, which provides a decision-making basis for the ship command and control system to judge the target threat in time (Zhao et al., 2007). And research on radar target tracking by Liu et al. (Liu et al., 2006).

## References

Cempel, C. (2008). Decomposition of the symptom observation matrix and grey forecasting in vibration condition monitoring of machines. *International Journal of Applied Mathematics and Computer Science, 18*(4), 569–579.

Chen, L., Qin, Y., & Deng, R. (2011). Comparative analysis of the two common methods for skylight measurement based on the ASD spectroradiometer. *Tropical Geography, 31*(2), 182–186.

Chen, R. H., Song, Z. Q., & Kang, L. M. (2005). Application of grey system in reservoir evaluation of the seventh middle area of Karamay Oilfield, Xinjiang. *Inner Mongolia Petrochemical Industry, 7*, 110–113.

Chen, X. H. (2018). Analysis on the trend of technology integration and application innovation in the era of digital economy. *Social Scientist, 8*, 8–23.

Chirwa, E. C., & Mingzhi, M. (2006). Application of grey model GM(1,1) to vehicle fatality risk estimation. *Technological Forecasting and Social Change, 73*(5), 588–605.

Cobb, C. W., & Douglas, P. H. (1928). A theory of production. *American Economic Review, 18*(1), 139–65.

Cui, J. P., Xin, Y. P., & Liu, X. J. (2012). Research on type selection of surface to air missile based on multi-objective grey decision. *Tactical Missile Technology, 1*, 7–10.

Dejamkhooy, A., Dastfan, A., & Ahmadyfard, A. (2017). Modeling and forecasting nonstationary voltage fluctuation based on grey system theory. *IEEE Transactions on Power Delivery, 32*(3), 1212–1219.

Delcea, C., Scarlat, E., & Liviu-Adrian, C. (2013). Grey relational analysis of the financial sector in Europe. *The Journal of Grey System, 25*(4), 19–30.

Jia-qiang, E., Wang, Y. N., Mei, Z., et al. (2005). Fuzzy adaptive variable weight combination prediction model for energy consumption in copper refining process and its application. *Mining and Metallurgy, 3*, 46–48.

Evans, M. (2014). An alternative approach to estimating the parameters of a generalised grey Verhulst model: An application to steel intensity of use in the UK. *Expert Systems with Applications, 41*(4), 1236–1244.

Fang, H., Tan, J. R, & Yin, G. F. et al. (2009). Research on user demand analysis method of house of quality based on Grey Theory. *Computer Integrated Manufacturing System, 15*(3), 576–584+591

Fang, X., Chen, Y., & Li, S. (2012). Application of multidimensional grey evaluation methods in coal and gas outburst prediction. *Industrial Safety and Environmental Protection, 38*(12), 81–83.

Gao, F., Zhang, Y., & Gao, P. (2012). Research on speed controller model for high-speed train based on grey genetic algorithm. *Computer Measurement & Control, 20*(5), 1272–1275.

Gao, W., & Feng, X. T. (2004). Study on landslide deformation prediction based on grey evolutionary neural network. *Geotechnical Mechanics, 25*(4), pp. 514–517.

Gini, C. (1972). Measurement of inequality and incomes. *General Information, 75*(1), 163–169.

Guo, H. Q., Wu, Z. R., Yang, J. (2001). Grey nonlinear time series combination model for deformation monitoring of Rockfill Dams. *Journal of Hohai University, 29*(6), 51–55.

Guo, R. L. (1995). *Crop grey breeding*. China Agricultural Science and Technology Press.

Gupta, A., Vaishya, R., Khan, K., et al. (2019). Multi-response optimization of the mechanical properties of pultruded glass fiber composite using optimized hybrid filler composition by the gray relation grade analysis. *Materials Research Express, 6*(12), 125322.

Gupta, B., & Tiwari, M. (2017). A tool supported approach for brightness preserving contrast enhancement and mass segmentation of mammogram images using histogram modified grey relational analysis. *Multidimensional Systems & Signal Processing, 28*(4), 1549–1567.

Haken, H. (1978). Introduction to *Synergetics*. Springer.

Han, L., Shuai, H. L., & Ge, Z. H., et al. (2020). Study on geophysical characteristics of dynamic compaction fill foundation. *Low Temperature Building Technology, 42*(01), 124–127.

Han, X., Nan, H., Chen, J., & Jiang, K. (2014). Grey cluster model used to evaluate the development scheme of warhead in antimissile missile of air defense. *Journal of Air Force, Engineering University, 1*(1), 29–33.

Hao, Y., Cao, B., Chen, X. et al. (2013). A piecewise grey system model for study the effects of anthropogenic activities on karst hydrological processes. *Water Resource Management, 27*(5), 1207–1220.

Huang, X. B., Ouyang, L., Wang, Y. N. et al. (2011). Analysis on key influencing factors of transmission line icing. *High Voltage Technology, 37*(7), 1677–1682.

Icer, S., Coskun, A., & Ikizceli, T. (2012). Quantitative grading using grey relational analysis on ultrasonographic images of a fatty liver. *Journal of Medical Systems, 36*(4), 2521–2528.

Jahan, A., & Zavadskas, E. K. (2019). ELECTRE-IDAT for design decision-making problems with interval data and target-based criteria. *Soft Computing, 23*, 129–143.

Jena, M., Manjunatha, C., & Shivaraj, B. W. (2019). Optimization of parameters for maximizing photocatalytic behaviour of Zn1-xFexO nanoparticles for methyl orange degradation using Taguchi and Grey relational analysis Approach. *Materials Today Chemistry, 12*, 187–199.

Jia, Z. Y., Ma, J. W., Wang, F. J., et al. (2009). Prediction method of assembly product characteristics under the influence of multi part geometric elements. *Journal of Mechanical Engineering, 45*(7), 168–173.

Jiang, W. (2012). An intelligent diagnosis method based on grey rough set theory for wind turbine driving chain. *Power System and Clean Energy, 28*(12), 79–83.

Jin, X., Xu, X., & Song, X. (2013). Estimation of leaf water content in winter wheat using grey relational analysis-partial least squares modeling with hyperspectral data. *Agronomy Journal, 105*(5), 1385–1392.

Khan, M. A., Jaffery, S., Khan, M., et al. (2020). Multi-objective optimization of turning titanium-based alloy Ti-6Al-4V under dry, wet, and cryogenic conditions using gray relational analysis (GRA). *International Journal of Advanced Manufacturing Technology, 106*(9–10), 3897–3911.

Kasemsiri, P., Dulsang, N., Pongsa, U., et al. (2017). Optimization of biodegradable foam composites from cassava starch, oil palm fiber, chitosan and palm oil using taguchi method and grey relational analysis. *Journal of Polymers and Environment, 25*(2), 378–390.

Kose, E., & Tasci, L. (2019). Geodetic deformation forecasting based on multi-variable grey prediction model and regression model. *Grey Systems: Theory and Application, 9*(4), 464–471.

Kose, E., Vural, D., & Canbulut, G. (2020). The most livable city selection in Turkey with the grey relational analysis. *Grey Systems: Theory and Application, 10*(4), 529–544.

Lai, H.-Y., Chen, Y.-Yin., Lin, S.-H., et al. (2011). Automatic spike sorting for extracellular electro-physiological recording using unsupervised single linkage clustering based on grey relational analysis. *Journal of Neural Engineering, 8*(3), 036003.

Lao, Tzu. (2007). *Tao Te Ching*. Signet Classics.

Li, J., Sun, C. X., & Chen, W. G.. (2002). Research on insulation fault diagnosis of oil filled power transformer based on grey clustering analysis. *Journal of Electrotechnics, 17*(4), 80–83

Li, T., Ren, M., & Chen, H. (2010). *Fatigue crack growth rate calculation method baseing on gray theory*. *Journal of Mechanical Strength, 32*(3), 472–475

Li, X. B., Sun, H. Y., & Wu, Y. X. (2009). Fuzzy predictive functional control of fuel supply temperature for anode baking. *Computer engineering and Application, 45*(9), 200–203.

Li, X., Tan, S., & Tang, B. (2007). The optimal collocation model of missile nuke based on the gray decision theory. *Fire Control and Command Control, 32*(2), 42–47.

Li, X., Yuan, Z., Zhang, G., & Cheng, S. H. (2014). Some properties of grey differential equation GM(1,1, β)". In *Systems engineering—Theory and practice* (Vol. 34, No. 5, pp. 1249–1255).

Li, X. H., Wang, H. T., Jia, J. Q., et al. (2005). Limit displacement discrimination of stability and reliability analysis of surrounding rock in tunnel and underground engineering. *Geotechnical Mechanics, 25*(6), 850–854.

Liang, B., Dai, Y., Chen, T., & Sun, W. J. (2014). Grey correlation optimization for shale gas exploration and development areas of complicated geological parameter features. *Journal of China Coal Society, 39*(3), 524–530.

Liao, J., He, L., Lv, Z. Q., et al. (2017). Research on comprehensive evaluation method of marine steering device system architecture scheme. *Machine Tools and Hydraulics, 45*(7), 59–63.

Liao, R. J., Yang, J. P., Grzybowski, S., et al. (2012). Forecasting dissolved gases content in power transformer oil based on weakening buffer operator and least square support vector machine-Markov. *IET Generation, Transmission & Distribution, 6*(2), 142–151.

Lin, J., Ren, H., & Shen, Z. (2009). Study on primary influence factors for application of Grey System theong to velocity of explosive forming projectile. *Journal of Projectiles, Rockets, Missiles and Guidance, 29*(3), 112–116.

Lin, C., Zhengyu, S., & Liu, S., et al. *Study on mechanism and filter efficacy of AGO/IAGO, grey systems: Theory and application* (Vol. 11, No. 1, pp. 1–21).

Lin, Y., Wang, T., Wang, L., & Chen, H. C. (2005). Stability analysis of high excavated slope in Three Gorges Project. *Journal of Tianjing University, 38*(10), 936–940.

Liu, Q., Zhong, Z., Ai, B. (2010a). Research on frequency planning of GSM-R system based on grey cluster theory with rough set. *Journal of the China Railway Society, 32*(5), 53–-58.

Liu, S. F., Fang, Z. G., Yang, Y. J., & Forrest, J. (2012). General grey numbers and its operations. *Grey Systems: Theory and Application, 2*(3), 341–349.

Liu, S. F., & Yang, L. (1994). *Regional economic evaluation, early warning and regulation.* Henan People's Publishing House.

Liu, S., & Lin, Y. (2004). An axiomatic definition of grey degree of grey numbers. *Engineering Science, 6*(8), 91–94.

Liu, S. F., Fang, Z. G., & Xie, N. M. (2010b). Algorithm rules of interval grey numbers based on the 'kernel' and the degree of greyness of grey numbers. *Systems Engineering and Electronics, 32*(2), 313–316.

Liu, S., Tao, Y., Xie, N., Tao, L., & Hu, M. (2022). Advance in grey system theory and applications in science and engineering. *Grey Systems: Theory and Application, 12*(4), 804–823.

Liu, Y., Yang, T., Li, R. and Wei, L.H. (2007), "Grey correlation analysis and prediction model for formation reaction of high-temperature sulphur-fixing phase sulphocalciumaluminate" Thermal Power Generation, Vol. 6, pp. 37–40

Liu, Y., Chen, S., Zhang, M., & Ma, X. (2006). The application in target tracking of buffer operator and data fusion technology. *Journal of Applied Sciences, 24*(2), 154–158.

Loganathan, D., Kumar, S. S., & Ramadoss, R. (2020). Grey relational analysis-based optimization of input parameters of incremental forming process applied to the AA6061 alloy. *Transactions of Famena, 44*(1), 93–104.

Lu, X., & Wang, C. (2013). Simulation on ATO speed controller based on grey prediction control. *City Track Traffic Development Research, 16*(2), 62–65.

Luo, Q., Jiang, Z. H., Xiao, S. X., et al. (2015). Grey correlation analysis of trace element content and lead content in new irradiated seeds of *Pleurotus ostreatus*. *Journal of Mountain Agricultural Biology, 34*(04), 18–21.

Mahmod, W. E., & Kunio, K. (2014). Modified Grey Model and its application to groundwater flow analysis with limited hydrogeological data: A case study of the Nubian Sandstone, Kharga Oasis, Egypt. *Environmental Monitoring and Assessment, 186*(2), 1063–1081.

Mi, G., Yang, R., Li, L. (2014), Research on the method of diagnosis of complex fault in track circuit based on combined model. *Journal of the China Railway Society, 36*(10), 65–69.

Modigliani, F., & Brumberg, R. H. (1954). Utility analysis and the consumption function: an interpretation of cross-section data. In K. K. Kurihara (Ed.), *Post-Keynesian economics* (pp. 388–436). Rutgers University Press.

Ossowski, M., & Korzybski, M. (2013). Data mining based algorithm for analog circuits fault diagnosis. *Przeglad Elektrotechniczny, 89*(2a), 285–287.

Pagar, N. D., & Gawande, S. H. (2020). Parametric design analysis of meridional deflection stresses in metal expansion bellows using grayrelational grade. *Journal of the Brazilian Society of Mechanical Sciences and Engineering, 42*(5), 256.

Pawlak, Z. (1991). *Rough sets: theoretical aspects of reasoning about data.* Kluwer Academic.

Peng, F., Wu, G., & Min, F. (2005). Grey programming cluster and application in evaluation of oil and gas cap layer. *Journal of Hunan University of Science and Technology, 20*(2), 5–10.

Peng, X., Li, X., & Yang, X. (2022). Analysis of circular economy of E-commerce market based on grey model under the background of big data. *Journal of Enterprise Information Management, 35*(4/5), 1148–1167.

Peng, Y., Zhang, X., Xu, W., et al. (2018). An optimal algorithm for cascaded reservoir operation by combining the grey forecasting model with DDDP. *Water Science and Technology-Water Supply, 18*(1), 142–150.

Phillips, A. W. (1958). The relation between unemployment and the rate of change of money wages in the United Kingdom, 1861–1951. *Economics, 3*, 283–299.

Prakash, K. S., Gopal, P. M., Karthik, S. (2020). Multi-objective optimization using Taguchi based grey relational analysis in turning of Rock dust reinforced Aluminum MMC. *Measurement, 157*(107664).

Qiao, G., Zhang, W., & Xue, S. (2009). Speed control based on fuzzy PID control with grey prediction in the deep sea stepping system. *Journal of China Coal Society, 34*(11), 1550–1553.

Scarlat, E., & Delcea, C. (2011). Complete analysis of bankruptcy syndrome using grey systems theory. *Grey Systems: Theory and Application, 1*(1), 19–32.
Sharma, A., Aggarwal, M. L., & Singh, L. (2020). Investigation of GFRP gear accuracy and surface roughness using Taguchi and grey relational analysis. *Journal of Advanced Manufacturing Systems, 19*(1), 147–165.
Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance, 19*(3), 425–442.
Shi, Q., & Tao, Z., & Shahzad, M. A. (2017). Reliability analysis on passive residual heat removal of AP1000 based on grey model. *ATW-International Journal for Nuclear Power, 62*(6), 408–417
Sun, C. X. (2005). Present situation and prospect of online monitoring and diagnosis technology of the state of power transmission and transformation equipment. *China Power, 38*(2), 1–7.
Sun, C. X., Bi, W. M., & Zhou, Q. (2003). A new model of grey prediction parameter model and its application in electrical insulation fault prediction. *Control Theory and Application, 20*(5), 797–802.
Sun, C. X., Li, J., & Zheng, H. P. (2002). Power transformer insulation fault diagnosis method based on grey area correlation analysis. *Power Grid Technology, 26*(7), 24–29.
Tan, X. R., Deng, J. L., & Ren, S. Y. et al. (2011). Study on grey correlation methodology of clinical trials, National scientific and technological achievements.
Tian, J., & Lu, Y. (2007). Research on grey prediction model of the slab temperature in heating furnace. *Journal of Northeast University, 28*(S1), 6–10.
Twala, B. (2014). Extracting grey relational systems from incomplete road traffic accidents data: The case of Gauteng Province in South Africa. *Expert Systems, 31*(3), 220–231.
Wang, W., Wu, M., Cao, W., & Lei, Q. (2010). Fuzzy-expert control based on combination grey prediction model for flue temperature in coke oven. *Control and Decision, 25*(2), 185–190
Wang, Q., Kuang, L. Z., & Zeng, S. B. (2010). Grey relational analysis of optimal parameters of welding process based on arc signal. *Electric Welding Machine, 40*(3), 75–78.
Wang, X., & Nie, H. (2008). On a method of fatigue life prediction based on Grey System model GM(1,1). *Transaction of Nanjing University of Aeronautics and Astronautics, 40*(6), 845–848.
Wang, Y., & Cao, Y. (2010). Gray neural network model of aviation safety risk. *Journal of Aerospace Power, 25*(5), 1036–1042.
Wang, Y., Chen, Z., Wang, H., & Yang, H. W. (2011a). Application of gray relational cluster method in muon tomography. *Nuclear Electronics & Detection Technology, 31*(8), 871–873.
Wang, Y., Zhou, T., Zhang, M., Li, X. H., & Ding, Y. (2013). Application of grey relational analysis in Yaojialing Zn–Au polymetallic deposit prediction. *Journal of Hefei University of Technology, 36*(10), 1236–1241.
Wang, Z. X., Hipel, K. W., Wang, Q., et al. (2011b). An optimized NGBM(1,1) model for forecasting the qualified discharge rate of industrial wastewater in China. *Applied Mathematical Modeling, 35*, 5524–5532.
Wei, H., Lin, X., Zhang, Y., Wang, L., & Chen, Q. (2013). Research on the application of Grey System theory in the pattern recognition for chromatographic fingerprints of traditional Chinese medicine. *Chinese Journal of Chromatography, 31*(2), 127–132.
Wei, N., & Zhang, T. L. (2019). Evaluate the quality of barley on the Qinghai Tibet Plateau through grey relational analysis. *Bengali Botanical Journal, 48*(3), 817–826.
Wu, Z., Xu, B., Gu, C., & Li, Z. C. (2012). On comprehensive evaluation methods of the service condition of the dam. *China Science: Technological Sciences, 42*(11), 1243–1254.
Xia, J., Zhao, H. Y. (1996). Grey artificial neural network model and its application in short-term runoff prediction. In *Systems engineering—Theory and practice* (Vol. 16, No. 11, pp. 82–90).
Xia, X., Wang, Z., & Chang, H. (2005). Degree of grey incidence for the quality of processing and vibration of rolling bearing. *Journal of Aerospace Power, 20*(2), 250–254.
Xiao, J., & Zhang, W. (2009). Grey incidence analysis applied to fault diagnosis of drone crash. *Journal of Sichuan Ordnance Engineering, 30*(9), 112–115.
Xie, J., Song, B., & Liu, D. (2004). Gray correlation analysis method for scheme selection decision of aircraft top-layer design. *Journal of Systems Engineering, 19*(4), 350–354.

Xie, N. M., & Liu, S. F. (2005). Discrete GM (1,1) and the modeling mechanism of grey prediction model. In *Systems engineering—Theory and practice* (Vol. 25, No. 1, pp. 93–99).
Xie, Y., Yu, H., & Chen, J. (2007). Application of grey theory in deep drawing robust design Chinese. *Journal of Mechanical Engineering, 43*(3), 54–59.
Xu, G. F., & Yang, S. L. (2013). Application of Grey Theory in fault diagnosis of sucker rod pumping wells. *Journal of Hefei University of Technology, 36*(10), 1265–1268.
Yang, T., Yang, P., Dong, X., & Huang, Y. X. (2008). Method for predicting fault status of satellite based on gray system theory. *Computer Measurement & Control, 16*(9), 1284–1285, 1307.
Yang, X., Fang, Z., & Yang, Y. (2019). A novel multi-information fusion grey model and its application in wear trend prediction of wind turbines. *Applied Mathematical Modeling, 71*, 543–557.
Yang, Z.-W., Hsu, M.-H., & Chen, J.-C. (2012). Grey Relational analysis of pigment levels and the normalized difference vegetation index during the vegetative phase of paddy rice. *The Journal of Grey System, 24*(3), 275–284.
Yao, J., & Hu, W. (2008). Gray evaluation of operational efficiency of OTH ground-wave radar. *Armament Automation, 27*(4), 12–14.
Yuan, Z., Sun, C., Yuan, Z., Li, J., & Liao, R. J. (2005). Method of grey clustering decision making to state assessmentof power transformer. *Journal of Zhongqing University, 28*(3), 22–25.
Yu, F., Ke, Y., & Ying, Z. (2009). Decision of fault repair in aircraft assembly system of automated docking. *Computer Integrated Manufacturing System, 15*(9), 1823–1830.
Yue, J. P., & Hua, X. S. (1994). An grey regression model and accuracy analysis. In *Dam & Safety* (Vol. 2, pp. 23–8).
Zadeh, L. A. (1994). Soft computing and fuzzy logic. *IEEE Software, 11*(6), 48–56.
Zhang, C., Ding, S. B., & Wang, B. (2014). Research on aircraft customization model based on grey correlation analysis. *Traffic Information and Safety, 32*(4), 131–136.
Zhang, F., Wang, P., Xiao, Z., & Sun, N. (2010). Application of grey theory in safety evaluation of carier aircraft system. *Aircraft Design, 30*(3), 56–61.
Zhang, F. L., Yin, Q., Wang, D., et al. (2020). Effects of Bt insect resistant cotton straw returning on soil nutrient characteristics. *Journal of Biosafety, 29*(01), 69–77.
Zhang, G., Fu, Y., & Yang, R. (2004). Novel self-adjustable grey prediction controller. *Control and Decision, 19*(2), 212–215.
Zhang, J., Liang, S., Zhou, R., Yao, L., & Luo, P. (2012). Fault tree analysis of two teeth difference swing movable teeth transmission based on grey correlation. *Machinery Design & Manufacture, 6*, 183–185.
Zhang, L. P., Zheng, Y. L., Zhang, X. L., et al. (2015). Application of grey system theory in modern medicine. *Journal of Mathematical Medicine, 28*(3), 430–434.
Zhang, X., Wang, Z., & Nagai, M. (2006). Research on affective interaction models of robot. *Computer Engineering, 32*(24), 6–12.
Zhao, C. S. (2015). Book reviews: preface to the 7th edition of grey system theory and its applications. *The Journal of Grey System, 27*(1), 127–129
Zhao, G., Sun, Y., Xu, Y., & Wang, L. Q. (2007). "Gray decision ana lysis of threat estimation in anti missile combat of surfacewarshipTactical Missile Technology. *No., 3*, 32–35.
Zhao, P. D., & Xia, Q. L. (2009). Achievements and contributions of Chinese scholars in the development of mathematical geology. *Earth Science—Journal of China University of Geosciences, 34*(2), 225–231.
Zhu, J. M., Lei, J. T., & Huang, Z. W. (2012). Pneumatic position servo control system based on grey correlation compensation control. *Journal of Mechanical Engineering, 48*(20), 159–167.

# Chapter 3
# Grey Numbers and Their Operations

A grey system is described with grey numbers, grey sequences, grey equations, or grey matrices. Here, grey numbers are the elementary "atoms" or "cells", and their exact values are unknown. In applications, a grey number stands for an indeterminate number that takes its possible value within an interval or a general set of numbers. Professor Deng proposed the concept of grey number and gave the algorithm of interval grey number referring to the operation of interval number (Deng, 1985). As a basis of grey system theory, the studies on grey number, operations of grey numbers and grey algebraic systems have drawn the attention of scholars for a long time. The author's research on grey numbers and their operations began in the 1980s. The definitions and properties of degree of greyness of grey number have been studied and proposed the simplified form of grey numbers and the algorithm in 1980s, 1990s and 2000s (Liu, 1989, 1996; Liu & Lin, 2004; Liu et al., 2010). Then, proposed the definition of general grey number and the synthesis axiom of degree of greyness (Liu et al., 2012). In 2017, Jiang Shiquan et al studied the distance measuring and sorting method of general grey number (Jiang et al., 2017). They proposed a decision-making model based on general grey number in 2021 (Jiang et al., 2021). In 2024, Li proposed the concept of the generalized whiteness of interval grey number based on the generalized greyness of interval grey number.

## 3.1 Grey Numbers

A grey number is generally represented using the symbol "$\otimes$." There are several types of grey numbers, as discussed below.

(1) Grey numbers with only a lower bound: This kind of grey number $\otimes$ is represented as $\otimes \in [\underline{a}, \infty)$ or $\otimes(\underline{a})$, where $\underline{a}$ stands for the definite, known lower bound of the grey number $\otimes$. The interval $[\underline{a}, \infty)$ is referred to as the field of $\otimes$.



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_3

For example, the weight of a celestial body which is far away from the Earth is a grey number containing only a lower bound, because the weight of the celestial body must be greater than zero. However, the exact value of the weight cannot be obtained through normal means. If we use the symbol $\otimes$ to represent the weight of the celestial body, we then have that $\otimes \in [0, \infty)$.

(2) Grey numbers with only an upper bound: This kind of grey number $\otimes$ is written as $\otimes \in (-\infty, \overline{a}]$ or $\otimes(\overline{a})$, where $\overline{a}$ stands for the definite, known upper bound of $\otimes$.

A grey number containing only an upper bound is a grey number with a negative value, but its absolute value is infinitely great. For example, the opposite number of the weight of the celestial body mentioned above is a grey number with only an upper bound.

(3) Interval grey numbers: This kind of grey number $\otimes$ has both a lower $\underline{a}$ and an upper bound $\overline{a}$, written $\otimes \in \left[\underline{a}, \overline{a}\right]$.

For example, for an investment opportunity, there always exists an upper limit representing the maximum amount of money that can be mobilized. For an electrical equipment, there must be a maximum critical value for the equipment to function normally. The critical value could be for a maximum voltage or for a maximum amount of current allowed to be applied to the equipment. At the same time, the values of investment, voltage, and current are all greater than zero. Therefore, the amount of dollars that can be used for a specific investment opportunity, and the voltage and the current requirements for the electrical equipment are all examples of interval grey numbers.

(4) Continuous and discrete grey numbers: This kind of grey number takes only a finite number or a countable number of potential values and is known as discrete. If a grey number can potentially take any value within an interval, then it is known as continuous.

For example, if a person's age is between 30 and 35, his or her age could be one of the values 30, 31, 32, 33, 34, 35. Thus, age is a discrete grey number. As for a person's height and weight, they are continuous grey numbers.

(5) Black and white numbers: Black numbers are represented as $\otimes \in (-\infty, +\infty)$; that is, when $\otimes$ has neither an upper nor a lower bound, then $\otimes$ is known as a black number. When $\otimes \in \left[\underline{a}, \overline{a}\right]$ and $\underline{a} = \overline{a}$, $\otimes$ is known as a white number.

For the sake of parsimony, in our discussion we treat black and white numbers as special grey numbers.

(6) Essential and non-essential grey numbers: The former stands for a grey number that temporarily cannot be represented by a white number; the latter entails a grey number that can be represented by a white number obtained either through experience or through a certain method. The definite white number is referred to as the whitenization (value) of the grey number, denoted $\tilde{\otimes}$. Also, we use $\otimes(a)$ to represent grey number(s) with $a$ as its whitenization.

A grey number is an uncertain number with its value in a specific range. The range can be regarded as a cover of the grey number. Therefore, an interval grey number $\otimes \in \left[\underline{a}, \overline{a}\right], \underline{a} < \overline{a}$ is very different from an interval number $\left[\underline{a}, \overline{a}\right], \underline{a} < \overline{a}$. An interval grey number $\otimes \in \left[\underline{a}, \overline{a}\right], \underline{a} < \overline{a}$ is only one value in interval $\left[\underline{a}, \overline{a}\right], \underline{a} < \overline{a}$. However, an interval number $\left[\underline{a}, \overline{a}\right], \underline{a} < \overline{a}$ is the whole interval $\left[\underline{a}, \overline{a}\right], \underline{a} < \overline{a}$.

## 3.2 The Whitenization of a Grey Number and Degree of Greyness

When a type of grey number vibrates around a certain fixed value, the whitenization of this kind of grey number is relatively easy. One can simply use that fixed value as its whitenization. A grey number that vibrates around $a$ can be written as $\otimes(a) = a + \delta_a$ or $\otimes(a) \in (-, a, +)$, where $\delta_a$ stands for the vibration. In this case, the whitenized value is $\tilde{\otimes}(a) = a$.

For the general interval grey number $\otimes \in [a, b]$, we can take its whitenization value $\tilde{\otimes}$ as indicated in (3.1), based on the possible value information:

$$\tilde{\otimes} = \alpha a + (1 - \alpha)b, \alpha \in [0, 1] \tag{3.1}$$

Here, $\alpha$ is called the positioned coefficient of the interval grey number $\otimes \in [a, b]$ (Liu, 1989).

**Definition 3.2.1** The whitenization of the form $\tilde{\otimes} = \alpha a + (1 - \alpha)b, \alpha \in [0, 1]$ is called a whitenization with positioned coefficient $\alpha$.

**Definition 3.2.2** Mean whitenization occurs when $\alpha = \frac{1}{2}$. When the distribution of an interval grey number is unknown, mean whitenization is often employed.

**Definition 3.2.3** Take the interval grey numbers $\otimes_1 \in [a, b]$, $\otimes_2 \in [a, b]$; let $\tilde{\otimes}_1 = \alpha a + (1 - \alpha)b, \alpha \in [0, 1]$; and $\tilde{\otimes}_2 = \beta a + (1 - \beta)b, \beta \in [0, 1]$.

If $\alpha = \beta$, we say that both $\otimes_1$ and $\otimes_2$ are synchronous. If $\alpha \neq \beta$, we say that the grey numbers $\otimes_1$ and $\otimes_2$ are non-synchronous. When two grey numbers $\otimes_1$ and $\otimes_2$ have the same value range in interval [a, b], it is only when they are synchronous that it is possible to have $\otimes_1 = \otimes_2$.

When the distribution of a grey number is known, mean whitenization is not used. For instance, a certain person's age is within the range of 35 to 45 years old. Thus, $\otimes \in [35, 45]$ is a grey number. It is also known that the person in question finished him 12 years of pre-college education and entered college at the end of 1990s. Hence, the chance of the person to be around 42 years old in 2024 is quite good. For this grey number, it is not reasonable for us to employ mean whitenization.

When the value information of a grey number is known to a certain extent, we can use a possibility function to describe the possibility of the grey number has taking its potential values.

The possibility function is different from the membership function in fuzzy mathematics. The membership function describes the degree to which an object belongs to a certain set. However, the possibility function describes the possibility that a grey number can take a certain value, or the possibility that a certain value is the true value of a grey number. The possibility function is similar to the density function of probability distribution, but there are essential differences between the two concepts. A grey number described by the possibility function is a number with incomplete value information. Once a number with complete value information can be treated as a random variable with a certain probability distribution, it is no longer a grey number with poor value information:

For any conceptual type of grey number that represents wishes, its possibility function generally increases monotonically. In Fig. 3.1, the possibility function $f(x)$ stands for, say, the grey number of the amount of funds for a research project (in ten thousand dollars) and its degree of preference. A straight line stands for the "normal desire," that is, the degree of preference is directly proportional to the amount of funds, with different slopes representing different intensities of desire. In particular, $f_1(x)$ represents a relatively mild intensity of desire, where a funds in amount of \$100,000 is not enough, a funds in the amount of \$200,000 will be more satisfying, and a funds of \$300,000 will be quite adequate. $f_2(x)$ stands for a desire with more intensity, where a funds in the amount of \$350,000 is only about 40% satisfactory. The curve of $f_3(x)$ means that even for a funds in the amount of \$400,000, the degree of satisfaction is only about 20%. To be satisfied, the amount of funds has to be somewhere around \$800,000.

Generally speaking, the possibility function of a grey number is designed according to what is known to the researcher. Therefore, it does not have a fixed form. The start and end of the curve should have its significance. For instance, in a trade negotiation, there is a process of changing from a grey state to a white state. The eventual agreed upon deal will be somewhere between the ask and the bid. Thus, the relevant possibility function should start at the level of the ask (or the bid) and end at the level of bid (or the ask).

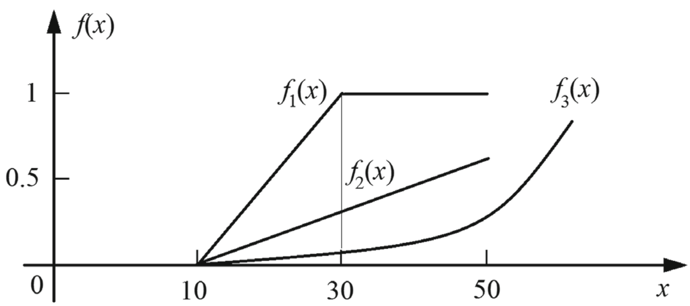



**Fig. 3.1** Different types of possibility functions

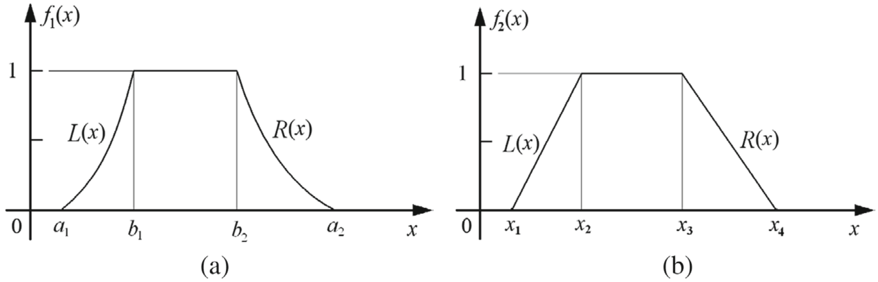



**Fig. 3.2** Typical possibility function

The typical possibility function is a continuous function with fixed starting and ending points so that the left-hand side increases and the right-hand side decreases, as seen in Fig. 3.2a, where:

$$f_1(x) = \begin{cases} L(x), & x \in [a_1, b_1) \\ 1, & x \in [b_1, b_2] \\ R(x), & x \in (b_2, a_2] \end{cases}.$$

For the convenience of computer programming and computation, in practical applications the left- and right-hand functions $L(x)$ and $R(x)$ are generally simplified into straight lines, as seen in Fig. 3.2b, where:

$$f_2(x) = \begin{cases} L(x) = \frac{x-x_1}{x_2-x_1}, & x \in [x_1, x_2) \\ 1, & x \in [x_2, x_3] \\ R(x) = \frac{x_4-x}{x_4-x_3}, & x \in (x_3, x_4] \end{cases}.$$

**Definition 3.2.4** For the possibility function shown in Fig. 3.2a, the following representation is referred to as the degree of greyness of ⊗ (Deng, 1985):

$$g^\circ = \frac{2|b_1 - b_2|}{b_1 + b_2} + \max\left\{\frac{|a_1 - b_1|}{b_1}, \frac{|a_2 - b_2|}{b_2}\right\} \tag{3.2}$$

The expression $g^\circ$ is a sum of two parts. The first part represents the greyness of the grey number as affected by the size of the peak area under the curve of the possibility function, while the second part shows the effect of the size of the area under the curves of $L(x)$ and $R(x)$. Generally, the greater the peak area and the area under $L(x)$ and $R(x)$, the greater the value of $g^\circ$. When $\max\left\{\frac{|a_1-b_1|}{b_1}, \frac{|a_2-b_2|}{b_2}\right\} = 0$, $g^\circ = \frac{2|b_1-b_2|}{b_1+b_2}$. In this case, the possibility function is a horizontal line. When $\frac{2|b_1-b_2|}{b_1+b_2} = 0$, grey number ⊗ is a grey number with its basic value $b = b_1 = b_2$. When $g^\circ = 0$, ⊗ is a white number.

## 3.3 Degree of Greyness Defined by Axioms

Professor Deng (1985) provided a definition of degree of greyness of a grey number with a typical possibility function, as shown in Fig. 3.2a. In 1996, The author of this book established an axiomatic definition of degree of greyness by using the length $l(\otimes)$ of the grey number interval and its kernel $\hat{\otimes}$ (Liu, 1996):

$$g^{\circ}(\otimes) = \frac{l(\otimes)}{\hat{\otimes}} \tag{3.3}$$

Such a definition is valid on the basis of the postulates of non-negativity, zero greyness, infinite greyness, and scalar multiplication. However, the concept of greyness as defined in equations (3.2) and (3.3) suffers from the following problems:

(1) When the length $l(\otimes)$ of the grey interval approaches infinity, the degree of greyness as defined in both (3.2) and (3.3) is likely to approach infinity.
(2) Grey numbers centered at zero will not have greyness. In this case, in equation (3.2), one has $b_1 = b_2 = 0$; and in equation (3.3), one faces $\hat{\otimes} = 0$. That is, neither (3.2) nor (3.3) is meaningful.

A grey number is a way to express the behavioral characteristics of a specific grey system (Deng, 1990). The greyness of grey numbers reflects the degree to which the researcher understands the uncertainty involved in such numbers (Chen, 2001; Liu, 1989). Therefore, the magnitude of the greyness of a grey number should be closely related to the background on which the grey number is come from, or to the field of discourse within which the said number becomes grey. If this background, or field of discourse, and the characteristics of a grey system are not detailed, there is no means through which to discuss the degree of greyness of a given grey number. With this understanding in place, let $\Omega$ be the field of discourse within which grey number $\otimes$ is come from, and $\mu(\otimes)$ is the measure of the number field from which $\otimes$ takes its value. Then, the degree of greyness $g^{\circ}(\otimes)$ of grey number $\otimes$ should satisfy the axioms below.

**Axiom 3.3.1** $0 \le g^{\circ}(\otimes) \le 1$. That is, the degree of greyness of any grey number has to be within the range of 0–1.

**Axiom 3.3.2** Any $\otimes \in [\underline{a}, \overline{a}]$, $\underline{a} \le \overline{a}$ when $\underline{a} = \overline{a}$, $g^{\circ}(\otimes) = 0$. That is, white numbers contain no ambiguity, so their degree of greyness is 0.

**Axiom 3.3.3** $G^{\circ}(\Omega) = 1$. That is, because the background $\Omega$ within which grey number $\otimes$ is come from is generally known. Therefore, $\Omega$ does not contain any useful information leading to the greatest level of uncertainty.

**Axiom 3.3.4** $G^{\circ}(\otimes)$ is directly proportional to $\mu(\otimes)$ and inversely proportional to $\mu(\Omega)$.

**Definition 3.3.1** The following equation is called the degree of greyness of grey number $\otimes$:

$$g^{\circ}(\otimes) = \mu(\otimes)/\mu(\Omega) \tag{3.4}$$

$\Omega$ is the field of discourse of grey number $\otimes$, and $\mu$ is the measure of field $\Omega$ (Liu et al., 2010, 2012).

**Theorem 3.3.1**
*The degree of greyness of grey numbers satisfies the following properties:*

(1) *If* $\otimes_1 \subset \otimes_2$, *then* $g^{\circ}(\otimes_1) \leq g^{\circ}(\otimes_2)$.
(2) $g^{\circ}(\otimes_1 \cup \otimes_2) \geq g^{\circ}(\otimes_k), k = 1, 2,$ *where* $\otimes_1 \cup \otimes_2 = \{\xi | \xi \in [a, b] or \xi \in [c, d]\}$ *is the union* of grey numbers $\otimes_1 \in [a, b], a < b$ *and* $\otimes_2 \in [c, d], c < d$.
(3) $g^{\circ}(\otimes_1 \cap \otimes_2) \leq g^{\circ}(\otimes_k), k = 1, 2,$ *where* $\otimes_1 \cap \otimes_2 = \{\xi | \xi \in [a, b] and \xi \in [c, d]\}$ *is the interaction between grey numbers* $\otimes_1 \in [a, b], a < b$ *and* $\otimes_2 \in [c, d], c < d$.
(4) *If* $\otimes_1 \subset \otimes_2$, *then* $g^{\circ}(\otimes_1 \cup \otimes_2) = g^{\circ}(\otimes_2), g^{\circ}(\otimes_1 \cap \otimes_2) = g^{\circ}(\otimes_1)$.
(5) *If* $\mu(\Omega) = 1$ *and the measures of* $\otimes_1$ *and* $\otimes_2$ *are independent of* $\mu$, *then*

1 ° $g^{\circ}(\otimes_1 \cap \otimes_2) = g^{\circ}(\otimes_1) \cdot g^{\circ}(\otimes_2)$; and
2 ° $g^{\circ}(\otimes_1 \cup \otimes_2) = g^{\circ}(\otimes_1) + g^{\circ}(\otimes_2) - g^{\circ}(\otimes_1) \cdot g^{\circ}(\otimes_2)$.

***Proof*** All details of the proof of conclusions (3.1)–(3.4) are omitted because all of them can be exported directly from Definition 3.3.1.

For 5 1 (°), from $\mu(\Omega) = 1$ and the assumption that measures of $\otimes_1$ and $\otimes_2$ are independent of $\mu$, we have:

$$g^{\circ}(\otimes_1 \cap \otimes_2) = \mu(\otimes_1 \cap \otimes_2) = \mu(\otimes_1) \cdot \mu(\otimes_2) = g^{\circ}(\otimes_1) \cdot g^{\circ}(\otimes_2).$$

Similarly, for 2 °, we have:

$$\begin{aligned} g^{\circ}(\otimes_1 \cup \otimes_2) &= \mu(\otimes_1 \cup \otimes_2) = \mu(\otimes_1) + \mu(\otimes_2) - \mu(\otimes_1) \cdot \mu(\otimes_2) \\ &= g^{\circ}(\otimes_1) + g^{\circ}(\otimes_2) - g^{\circ}(\otimes_1) \cdot g^{\circ}(\otimes_2). \text{ QED.} \end{aligned}$$

The way in which grey numbers are combined affects the degree of greyness and the reliability of the resultant grey number. Generally, when grey numbers are "unioned" together, the resultant degree of greyness and reliability of the new information increase; when grey numbers are intersected together, the resultant degree of greyness drops and the reliability of the combined information decreases. When solving practical problems and processing a large amount of grey numbers, it is advisable to combine the numbers at several different levels so that useful information can be extracted at individual levels. Additionally, in the process of combining grey numbers, "union" and "intersection" operations should be done at individual and other levels in order to guarantee that the extracted information satisfies pre-determined requirements in terms of reliability and degree of greyness.

## 3.4 The Operations of Interval Grey Numbers

In what follows, let us look at the operations of interval grey numbers. Given grey numbers $\otimes_1 \in [a, b], a < b$, and $\otimes_2 \in [c, d], c < d$, let us use * to represent an operation between $\otimes_1$ and $\otimes_2$. If $\otimes_3 = \otimes_1 * \otimes_2$, then $\otimes_3$ should also be an interval grey number satisfying $\otimes_3 \in [e, f], e < f$, and for any $\tilde{\otimes}_1$ and $\tilde{\otimes}_2$, $\tilde{\otimes}_1 * \tilde{\otimes}_2 \in [e, f]$. The operation rules of interval grey numbers are discussed below (Deng, 1985).

**Rule 3.4.1** (*Additive operation*) Assume that $\otimes_1 \in [a, b], a < b; \otimes_2 \in [c, d], c < d$ then the following equation is called the sum of $\otimes_1$ and $\otimes_2$:

$$\otimes_1 + \otimes_2 \in [a + c, b + d] \tag{3.5}$$

***Example 3.4.1*** Assume that $\otimes_1 \in [3, 4]$, $\otimes_2 \in [5, 8]$, then $\otimes_1 + \otimes_2 \in [8, 12]$.

**Rule 3.4.2** (*Additive inverse*) Assume that $\otimes \in [a, b], a < b$, then the additive inverse of $\otimes$ is given by:

$$-\otimes \in [-b, -a] \tag{3.6}$$

***Example 3.4.2*** Assume that $\otimes \in [3, 4]$, then $-\otimes \in [-4, -3]$.

**Rule 3.4.3** (*Subtraction operation*). Assume that $\otimes_1 \in [a, b], a < b; \otimes_2 \in [c, d], c < d$ then the following is called the deviation $\otimes_1$ minus $\otimes_2$:

$$\otimes_1 - \otimes_2 = \otimes_1 + (-\otimes_2) \in [a - d, b - c] \tag{3.7}$$

***Example 3.4.3*** Assume that $\otimes_1 \in [3, 4]$, $\otimes_2 \in [1, 2]$, then:

$$\otimes_1 - \otimes_2 \in [3 - 2, 4 - 1] = [1, 3], \otimes_2 - \otimes_1 \in [1 - 4, 2 - 3] = [-3, -1].$$

**Rule 3.4.4** (*Multiplication operation*) Assume that $\otimes_1 \in [a, b], a < b; \otimes_2 \in [c, d], c < d$ then the following equation is called the product of $\otimes_1$ and $\otimes_2$:

$$\otimes_1 \cdot \otimes_2 \in [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \tag{3.8}$$

***Example 3.4.4*** Assume that $\otimes_1 \in [3, 4]$, $\otimes_2 \in [5, 10]$, then:

$$\otimes_1 \cdot \otimes_2 \in [\min\{15, 30, 20, 40\}, \max\{15, 30, 20, 40\}] = [15, 40].$$

**Rule 3.4.5** (*Reciprocal*) Assume that $\otimes \in [a, b], a < b, a \neq 0, b \neq 0, ab > 0$, then the following equation is called the reciprocal of $\otimes$:

$$\otimes^{-1} \in \left[\frac{1}{b}, \frac{1}{a}\right] \tag{3.9}$$

***Example 3.4.5*** Assume that $\otimes \in [2, 4]$, then $\otimes^{-1} \in [0.25, 0.5]$.

**Rule 3.4.6** (*Division*). Assume that $\otimes_1 \in [a, b], a < b; \otimes_2 \in [c, d], c < d$, and $c \neq 0, d \neq 0, cd > 0$, then the following is called the quotient of $\otimes_1$ division by $\otimes_2$:

$$\otimes_1 / \otimes_2 = \otimes_1 \times \otimes_2^{-1} \in \left[\min\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}, \max\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}\right] \tag{3.10}$$

***Example 3.4.6*** Assume that $\otimes_1 \in [3, 4], \otimes_2 \in [5, 10]$, then:

$$\otimes_1/\otimes_2 \in [\min\{\frac{3}{5}, \frac{3}{10}, \frac{4}{5}, \frac{4}{10}\}, \max\{\frac{3}{5}, \frac{3}{10}, \frac{4}{5}, \frac{4}{10}\}] = [0.3, 0.8].$$

**Rule 3.4.7** (*Scalar multiplication*). Let $\otimes \in [a, b]$, $a < b$, and $k$ a positive real number, then the following is called the product of scalar k with grey number $\otimes$:

$$k \cdot \otimes \in [ka, kb] \tag{3.11}$$

***Example 3.4.7*** Assume that $\otimes \in [2, 4]$, and k=5, then $5 \times \otimes \in [10, 20]$.

**Rule 3.4.8** (*Power*) Let $\otimes \in [a, b], a < b, k$ $k$ a positive real number, then the following equation is called the $k$th power of the grey number $\otimes$:

$$\otimes^k \in [a^k, b^k] \tag{3.12}$$

***Example 3.4.8*** Assume that $\otimes \in [2, 4]$, and k=5, then $\otimes^5 \in [32, 1024]$.

## 3.5 General Grey Numbers and Their Operations

### *3.5.1 Simplified Form of Interval Grey Numbers*

As the basis of grey system theory, grey numbers, grey number operations and grey algebraic systems have received much attention from grey system scholars over the past years. In the 1980s, we put forward the concept of mean whitenization of grey numbers (Liu, 1989), and based on this concept we developed a new algebraic system for grey numbers.

According to the standard definition of degree of greyness of grey numbers (Liu, 1996, 2006; Yang 2007, Yang & Liu, 2011), it is possible to address grey intervals after the operation of grey numbers, with the help of the concept of degree of greyness.

In this section, a definition for grey "kernel" is put forward. The axioms for operation of grey numbers and a grey algebraic system is built based on grey "kernel" and the degree of greyness of grey numbers. Also, the properties of the operation are discussed with regards to how the operation of grey numbers can be transformed to the operation of real numbers. Thus, to a certain extent the problem for setting up the operation of grey numbers and grey algebraic systems is solved.

**Definition 3.5.1** (*The "kernel" of grey number*)

(1) Suppose an interval grey number $\otimes \in [\underline{a}, \overline{a}], \underline{a} < \overline{a}$. In case of a lack of distributing information of the values of grey number $\otimes$, $\hat{\otimes} = \frac{1}{2}(\underline{a}+\overline{a})$ is called the "kernel" of grey number $\otimes$.
(2) If a grey number $\otimes$ is a discrete number and $a_i \in [\underline{a}, \overline{a}](i = 1, 2, \cdots n)$ are all the possible values for grey number $\otimes$, then $\hat{\otimes} = \frac{1}{n}\sum_{i=1}^{n} a_i$ is called the "kernel" of grey number $\otimes$.
(3) Suppose that grey number $\otimes \in [\underline{a}, \overline{a}], \underline{a} < \overline{a}$ is a random grey numbers with value distribution information. Then $\hat{\otimes} = E(\otimes)$ is called the "kernel" of grey number $\otimes$ ( Liu et al., 2010).

$\hat{\otimes}$, the "kernel" of grey number $\otimes$, is the representation of grey numbers $\otimes$, which cannot be exchangeable in the course of transforming the operation of grey numbers to operation of real numbers. In fact, the "kernel" of grey number $\otimes$, as a real number, can be completely operated by the operation of real numbers, such as plus, minus, multiplication, division, power, extract, and so on. Also, it is reasonable to take the operation results of the "kernels" as the "kernel" of operation results of grey numbers.

**Definition 3.5.2** Let $\hat{\otimes}$ and $g^\circ$ be the kernel and the degree of greyness of a grey number $\otimes$, respectively. Then $\hat{\otimes}_{(g^\circ)}$ is called the simplified form of grey number $\otimes$. The simplified form $\hat{\otimes}_{(g^\circ)}$ contains two important information, the kernel and the degree of greyness of a grey number $\otimes \in [\underline{a}, \overline{a}], \underline{a} < \overline{a}$.

**Proposition 3.5.1** *For interval grey numbers, there is an one-to-one correspondence between the simplified form $\hat{\otimes}_{(g^\circ)}$ and grey number $\otimes \in [\underline{a}, \overline{a}], \underline{a} < \overline{a}$.*

In fact, for any chosen grey number $\otimes \in [\underline{a}, \overline{a}], \underline{a} < \overline{a}$, one can compute $\hat{\otimes}_{(g^\circ)}$ through both $\hat{\otimes}$ and $g^\circ$. On the other hand, when $\hat{\otimes}_{(g^\circ)}$ is given, one can determine the position of $\otimes$ from $\hat{\otimes}$. Therefore, from the definition of degree of greyness $g^\circ$, one can compute the measure of the grey number $\otimes$ and consequently the upper and lower bounds $\overline{a}$ and $\underline{a}$, which provides detailed information for $\otimes \in [\underline{a}, \overline{a}], \underline{a} < \overline{a}$.

***Example 3.5.1*** Assume that the grey numbers $\otimes_1 = [-2, -1], \otimes_2 = [8, 18], \otimes_3 = [-2, 18]$ all on background $\Omega \in [-2, 20]$. Take the length of grey interval as the measure of grey numbers, and calculate the simplified forms of $\otimes_1, \otimes_2, \otimes_3$.

Solution: The measures of $\Omega, \otimes_1, \otimes_2, \otimes_3$ are $\mu(\Omega) = 20-(-2) = 22, \mu(\otimes_1) = 1, \mu(\otimes_2) = 10, \mu(\otimes_3) = 20$. Then we can get to the kernels and the degree of greyness of $\otimes_1, \otimes_2, \otimes_3$ as follows:

$\hat{\otimes}_1 = -1.5, \hat{\otimes}_2 = 13, \hat{\otimes}_3 = 8; g_1^\circ(\otimes_1) = 0.045, g_2^\circ(\otimes_2) = 0.45, g_3^\circ(\otimes_3) = 0.91$.
Therefore, we obtained:

$$\otimes_1 = -1.5_{(0.045)}, \otimes_2 = 13_{(0.45)}, \otimes_3 = 8_{(0.91)}.$$

### 3.5.2 General Grey Number and Its Simplified Form

**Definition 3.5.3** (*Basic element of grey number*) Together, an interval grey number and a white number are called the basic element of a grey number.

**Definition 3.5.4** (*General grey number*) Let $g^{\pm} \in \Re$ be an unknown real number within a union set of closed or open grey intervals, where:

$$g^{\pm} \in \bigcup_{i=1}^{n} \left[\underline{a}_i, \overline{a}_i\right] \tag{3.13}$$

If $i = 1, 2, \ldots, n$, $n$ is an integer and $0 < n < \infty$, $\underline{a}_i, \overline{a}_i \in \Re$ and $\overline{a}_{i-1} \le \underline{a}_i \le \overline{a}_i \le \underline{a}_{i+1}$, for any grey interval $\otimes_i \in [\underline{a}_i, \overline{a}_i] \subset \bigcup_{i=1}^{n} \left[\underline{a}_i, \overline{a}_i\right]$, then $g^{\pm}$ is called a general grey number. $g^{-} = \inf_{\underline{a}_i \in g^{\pm}} \underline{a}_i$ and $g^{+} = \sup_{\overline{a}_i \in g^{\pm}} \overline{a}_i$ are called the lower and upper limits of $g^{\pm}$( Liu, et al., 2012).

**Definition 3.5.5** (*The "kernel" of general grey number*)

(1) For a general grey number $g^{\pm} \in \bigcup_{i=1}^{n} \left[\underline{a}_i, \overline{a}_i\right]$, the following is called the "kernel" of a general grey number:

$$\hat{g} = \frac{1}{n} \sum_{i=1}^{n} \hat{a}_i \tag{3.14}$$

(2) If the probability distribution of $g^{\pm} \in \left[\underline{a}_i, \overline{a}_i\right](i = 1, 2, \ldots, n)$ is known, assume that $p_i$ is the probability for $g^{\pm} \in \left[\underline{a}_i, \overline{a}_i\right]$ $(i = 1, 2, \ldots, n)$ , $\hat{a}_i$ the "kernel" of grey interval $\otimes_i \in [\underline{a}_i, \overline{a}_i]$, and the following conditions hold:

$p_i > 0, i = 1, 2, \ldots, n$; and
$\sum_{i=1}^{n} p_i = 1.$

Then, the "kernel" $\hat{g}$ of general grey number $g^{\pm} \in \bigcup_{i=1}^{n} \left[\underline{a}_i, \overline{a}_i\right]$ can be defined as follows:

$$\hat{g} = \sum_{i=1}^{n} p_i \hat{a}_i \tag{3.15}$$

**Definition 3.5.6** (*The degree of greyness of a general grey number*) Suppose that the background which makes a general grey number $g^{\pm} \in \bigcup_{i=1}^{n} \left[\underline{a}_i, \overline{a}_i\right]$ come into being is $\Omega$, $\mu$ is the measure of $\Omega$, and $\otimes_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2, \ldots, n$ are basic elements of general grey number $g^{\pm} \in \bigcup_{i=1}^{n} \left[\underline{a}_i, \overline{a}_i\right]$. Then the following is called the degree of greyness of general grey number $g^{\pm} \in \bigcup_{i=1}^{n}\left[\underline{a}_i, \overline{a}_i\right]$, also denoted as $g^{\circ}$ for short (Liu, et al., 2012):

$$g^{\circ}(g^{\pm}) = \frac{1}{\hat{g}}\sum_{i=1}^{n}\hat{a}_i\mu(\otimes_i)/\mu(\Omega) \tag{3.16}$$

**Definition 3.5.7** (*The simplified form of general grey number*). If $\hat{g}$ is the "kernel" of a general grey number $g^{\pm} \in \bigcup_{i=1}^{n}\left[\underline{a}_i, \overline{a}_i\right]$ and $g^{\circ}$ is the degree of greyness of this general grey number, then, $\hat{g}_{(g^{\circ})}$ is called the simplified form of a general grey number.

The simplified form $\hat{g}_{(g^{\circ})}$ of a general grey number contains important information regarding the values of general grey number $g^{\pm} \in \bigcup_{i=1}^{n}\left[\underline{a}_i, \overline{a}_i\right]$. If all the $\hat{a}_i$ and $\mu(\otimes_i)$( $i = 1, 2, \ldots, n$) are known, then the simplified form of grey number $\hat{g}_{(g^{\circ})}$ contains two important information, the kernel and the degree of greyness of general grey numbers $g^{\pm} \in \bigcup_{i=1}^{n}\left[\underline{a}_i, \overline{a}_i\right]$.

***Example 3.5.2*** Let us take a mixed general grey number $g^{\pm} = \otimes_1 \cup \otimes_2 \cup 2 \cup \otimes_4 \cup 6$, where $\otimes_1 \in [1, 3], \otimes_2 \in [2, 4], \otimes_4 \in [5, 9]$. Assume that the background or field which makes general grey number $g^{\pm}$ come into being is $\Omega$=[0,32]. If we take the length of the interval as the measure of these grey numbers, try and work out the simplified forms of general grey number $g^{\pm}$.

**Solution** $\hat{\otimes}_1 = 2, \hat{\otimes}_2 = 3, \hat{\otimes}_4 = 7$, thus, the kernel of general grey number $g^{\pm}$ is as follows:

$$\hat{g} = \frac{1}{5}(\hat{\otimes}_1 + \hat{\otimes}_2 + 2 + \hat{\otimes}_4 + 6) = \frac{1}{5}(2 + 3 + 2 + 7 + 6) = 4.$$

From that, $\mu(\otimes_1) = 2$, $\mu(\otimes_2) = 2$, $\mu(\otimes_4) = 4$, $\mu(2) = \mu(6) = 0$, we have:

$$g^{\circ}(g^{\pm}) = \frac{1}{\hat{g}}\sum_{i=1}^{5}\hat{\otimes}_i\mu(\otimes_i)/\mu(\Omega) = \frac{1}{4}(2\times 2 + 3\times 2 + 2\times 0 + 7\times 4 + 6\times 0)/32 \approx 0.297.$$

Therefore, the reduced forms of general grey number $g^{\pm}$ is $4_{(0.297)}$. When the probability distribution of $g^{\pm}$ is known, assume that:

$$p_1 = 0.1, p_2 = 0.2, p_3 = 0.3, p_4 = 0.3, p_5 = 0.1.$$

Then: $\hat{g} = \sum_{i=1}^{n} p_i \cdot \hat{\otimes}_i = (0.1\cdot 2 + 0.2\cdot 3 + 0.3\cdot 2 + 0.3\cdot 7 + 0.1\cdot 6) = 4.1.$

Therefore, the simplified form of general grey number $g^{\pm}$ is $4.1_{(0.297)}$.

### 3.5.3 Synthesis of Degree of Greyness and Operations of General Grey Numbers

**Axiom 3.5.1** (*The synthesis axiom of degree of greyness*). When plus and minus are operated on $n$ general grey numbers of $g_1^{\pm}, g_2^{\pm}, \ldots, g_n^{\pm}$, then the degree of greyness $g^{\circ}$ of the operation results in $g^{\pm}$, which can be arrived at as follows:

$$g^{\circ} = \frac{1}{\sum_{i=1}^{n} \hat{g}_i} \sum_{i=1}^{n} g_i^{\circ} \hat{g}_i = \sum_{i=1}^{n} w_i g_i^{\circ} \tag{3.17}$$

where $w_i = \frac{\hat{g}_i}{\sum_{i=1}^{n} \hat{g}_i}$, $i = 1, 2, \cdots, n$, are the weights of $g_i^{\circ}$.

One can arrive the conclusion as following Proposition 3.5.1 through Axiom 3.5.1.

**Proposition 3.5.2** *When sums and subtractions are operated on n general grey numbers of* $g_1^{\pm}, g_2^{\pm}, \ldots, g_n^{\pm}$, $g^{\circ}$ *is the degree of greyness of the operation result* $g^{\pm}$; if $g_m^{\circ} = \min\limits_{1 \le i \le n} \{g_i^{\circ}\}$, $g_M^{\circ} = \max\limits_{1 \le i \le n} \{g_i^{\circ}\}$, *then:*

$$g_m^{\circ} \le g^{\circ} \le g_M^{\circ} \tag{3.18}$$

**Axiom 3.5.2** (*The unreduction axiom of degree of greyness*) When divisions and multiplications are operated on $n$ general grey numbers, the degree of greyness $g^{\circ}$ of the operation result $g^{\pm}$ is not less than $g_M^{\circ}$, the maximum value of the degree of greyness $g_1^{\circ}, g_2^{\circ}, \cdots, g_n^{\circ}$ of $n$ general grey numbers $g_1^{\pm}, g_2^{\pm}, \ldots, g_n^{\pm}$.

Usually, $g_M^{\circ}$, the maximum number of the degree of greyness of $n$ general grey numbers is taken as the degree of greyness of the operation results. One can arrive at this conclusion through Proposition 3.5.3 below.

**Proposition 3.5.3** *When divisions and multiplications are operated on n general grey numbers with the same degree of greyness, then the degree of greyness of the operation result holds the line.*

**Proposition 3.5.4** *When divisions and multiplications are operated on a white number and a general grey number, the degree of greyness of the result is equal to the degree of greyness of the general grey number*.

Suppose that $g_1^{\pm}$, $g_2^{\pm}$ are two general grey numbers; $\hat{g}_1$, $\hat{g}_2$ are their kernels, respectively, and $g_1^{\circ}$, $g_2^{\circ}$ are their degrees of greyness, respectively. Then, the following rules come into existence according to Axioms 3.5.1 and 3.5.2:

$$\textbf{Rule 1}\ \hat{g}_{1(g_1^{\circ})} + \hat{g}_{2(g_2^{\circ})} = (\hat{g}_1 + \hat{g}_2)_{(w_1 g_1^{\circ} + w_2 g_2^{\circ})} \tag{3.19}$$

$$\textbf{Rule 2}\ -\hat{g}_{1(g_1^{\circ})} = (-\hat{g}_1)_{(g_1^{\circ})} \tag{3.20}$$

$$\textbf{Rule}\quad \textbf{3}\,\hat{g}_{1(g_1^\circ)} - \hat{g}_{2(g_2^\circ)} = (\hat{g}_1 - \hat{g}_2)_{(w_1 g_1^\circ + w_2 g_2^\circ)} \tag{3.21}$$

$$\textbf{Rule}\quad \textbf{4}\,\hat{g}_{1(g_1^\circ)} \times \hat{g}_{2(g_2^\circ)} = (\hat{g}_1 \times \hat{g}_2)_{(g_1^\circ \vee g_2^\circ)} \tag{3.22}$$

$$\textbf{Rule}\quad \textbf{5}\,\text{If}\,\hat{g}_1 \neq 0,\ \text{then}\ 1/\hat{g}_{1(g_1^\circ)} = (1/\hat{g}_1)_{(g_1^\circ)} \tag{3.23}$$

$$\textbf{Rule}\quad \textbf{6}\,\text{If}\,\hat{g}_2 \neq 0,\ \text{then}\,\hat{g}_{1(g_1^\circ)} \div \hat{g}_{2(g_2^\circ)} = (\hat{g}_1 \div \hat{g}_2)_{(g_1^\circ \vee g_2^\circ)} \tag{3.24}$$

$$\textbf{Rule}\quad \textbf{7}\,\text{If}\,k\,\text{is a real number},\ \text{then}\,k \cdot \hat{g}_{(g_1^\circ)} = (k \cdot \hat{g})_{(g_1^\circ)} \tag{3.25}$$

The operations of general grey numbers can be extended to cases where many general grey numbers must be operated. In such cases, we can take the operation results of the "kernels" as the "kernel" of operation results of general grey numbers. We can then get the degree of greyness of the results according to axioms 1 or 2, and, thus, we can arrive at the reduced forms of the results.

***Example 3.5.3*** Take two mixed general grey numbers $g_1^\pm = \otimes_1 \cup \otimes_2 \cup 2 \cup \otimes_4 \cup 6$ and $g_2^\pm = \otimes_6 \cup 20 \cup \otimes_8 \cup \otimes_9$, where $\otimes_1 \in [1,3]$, $\otimes_2 \in [2,4]$, $\otimes_4 \in [5,9]$,$\otimes_6 \in [12,16]$, $\otimes_8 \in [11,15]$, $\otimes_9 \in [15,19]$. Assume that the background or field which makes general grey number $g_1^\pm$ come into being is Ω=[0,32], and the background or field which makes general grey number $g_2^\pm$ come into being is Ω=[10,60]. Try and calculate the values of $g_3^\pm = g_1^\pm + g_2^\pm$, $g_4^\pm = g_1^\pm - g_2^\pm$, $g_5^\pm = g_1^\pm \times g_2^\pm$, and $g_6^\pm = g_1^\pm \div g_2^\pm$.

**Solution** First, calculate the simplified forms of $g_1^\pm$ and $g_2^\pm$. From Example 3.5.2, we have $g_1^\pm$=$4_{(0.297)}$. From that, $\hat{\otimes}_6 = 14$, $\hat{\otimes}_8 = 13$, $\hat{\otimes}_9 = 17$, and $\mu(\otimes_6) = 4$, $\mu(\otimes_7) = 0$, $\mu(\otimes_8) = 4$, $\mu(\otimes_9) = 4$, we have:

$$\hat{g}_2 = \frac{1}{4}(\hat{\otimes}_6 + 20 + \hat{\otimes}_8 + \hat{\otimes}_9) = \frac{1}{4}(14 + 20 + 13 + 17) = 16;\ \text{and}$$

$$g_2^\circ(g^\pm) = \frac{1}{\hat{g}_2}\sum_{i=1}^{4}\hat{\otimes}_i\mu(\otimes_i)/\mu(\Omega_2) = \frac{1}{16}(14 \times 4 + 20 \times 0 + 13 \times 4 + 17 \times 4)/50 = 0.22.$$

Thus, the simplified form of general grey number $g_2^\pm$ is $16_{(0.22)}$. With the simplified forms, as well as $w_1 = \frac{4}{20} = 0.2$, $w_2 = \frac{16}{20} = 0.8$, it is possible for us to get the following results:

$$g_3^\pm = g_1^\pm + g_2^\pm = (\hat{g}_1 + \hat{g}_2)_{(w_1 g_1^\circ + w_2 g_2^\circ)} = (4 + 16)_{(0.2\times0.297+0.8\times0.22)} = 20_{0.235}$$
$$g_4^\pm = g_1^\pm - g_2^\pm = (\hat{g}_1 - \hat{g}_2)_{(g_1^\circ \vee g_2^\circ)} = (4 - 16)_{(0.2\times0.297+0.8\times0.22)} = (-12)_{0.235}$$
$$g_5^\pm = g_1^\pm \times g_2^\pm = (\hat{g}_1 \times \hat{g}_2)_{(g_1^\circ \vee g_2^\circ)} = (4 \times 16)_{(0.297\vee0.22)} = 64_{0.297}$$
$$g_6^\pm = g_1^\pm \div g_2^\pm = (\hat{g}_1 \div \hat{g}_2)_{(g_1^\circ \vee g_2^\circ)} == (4 \div 16)_{(0.297\vee0.22)} = \left(\frac{1}{4}\right)_{0.297}$$

**Definition 3.5.8** Assume that $F(g^{\pm})$ is a set of general grey numbers, and that $g_i^{\pm}, g_j^{\pm} \in F(g^{\pm})$. If $g_i^{\pm} + g_j^{\pm}$, $g_i^{\pm} - g_j^{\pm}$, $g_i^{\pm} \cdot g_j^{\pm}$, and $g_i^{\pm} \div g_j^{\pm}$ all belong to $F(g^{\pm})$ (when division is considered, the conditions in rule 6 need to be satisfied), then $F(g^{\pm})$ is called a field of general grey numbers.

**Theorem 3.5.1**
*The totality of all general grey numbers constitutes a field of general grey numbers.*

**Definition 3.5.9** Assume that $R(g^{\pm})$ is a set of general grey numbers. If for $g_i^{\pm}$, $g_j^{\pm}$ and $g_k^{\pm} \in R(g^{\pm})$, the following hold true:

(1) $g_i^{\pm} + g_j^{\pm} = g_j^{\pm} + g_i^{\pm}$;
(2) $(g_i^{\pm} + g_j^{\pm}) + g_k^{\pm} = g_i^{\pm} + (g_j^{\pm} + g_k^{\pm})$;
(3) There exists a zero element $0 \in R(g^{\pm})$, such that $g_i^{\pm} + 0 = g_i^{\pm}$;
(4) For any $g_i^{\pm} \in R(g^{\pm})$, there exists a $-g_i^{\pm} \in R(g^{\pm})$, such that $g_i^{\pm} + (-g_i^{\pm}) = 0$;
(5) $(g_i^{\pm} \cdot g_j^{\pm}) \cdot g_k^{\pm} = g_i^{\pm} \cdot (g_j^{\pm} \cdot g_k^{\pm})$;
(6) There exists a unit element $1 \in R(g^{\pm})$, such that $1 \cdot g_i^{\pm} = g_i^{\pm} \cdot 1 = g_i^{\pm}$;
(7) $(g_i^{\pm} + g_j^{\pm}) \cdot g_k^{\pm} = g_i^{\pm} \cdot g_k^{\pm} + g_j^{\pm} \cdot g_k^{\pm}$; and
(8) $g_i^{\pm} \cdot (g_j^{\pm} + g_k^{\pm}) = g_i^{\pm} \cdot g_j^{\pm} + g_i^{\pm} \cdot g_k^{\pm}$.

Thus, $R(g^{\pm})$ is called a linear space of general grey numbers.

**Theorem 3.5.2** *The totality of all synchronous general grey numbers constitutes a linear space.*

A grey number is the most elementary component of grey system theory and forms the basis for studying the quantitative relations of a grey system. The operation of grey numbers is the starting point for grey maths, and it has much significance in the development of grey system theory. On the basis of intensifying the effect and significance of the "kernel" of general grey numbers, and with the degree of greyness of general grey numbers as a link, the operation of grey numbers has been translated into the operation of real numbers. Therefore, to a certain extent the problem of operation of grey numbers has been solved, and a grey algebraic system based on this operation has been constructed. The operation of grey numbers defined in this chapter can be extended to grey algebraic equations, grey differential equations and grey matrix operations. This is a development of great significance to the study of grey input-output models and grey programming, which has been progressing slowly due to the difficulty of grey number operations.

The calculation of degree of greyness of general grey numbers relates to the field $\Omega$ of general grey numbers. Thus, the field $\Omega$ must be considered in order to translate the reduced form of general grey number to its common form. Researchers tend to pay attention only to the operation of general grey numbers and ignore the field of the results, which creates difficulties in reverting general grey numbers. However, the reduced form of a general grey number provides relevant information about the "kernel" and degree of greyness, So that we can know what we know. This is similar to the digital characteristics of a random variable such as mean and variance, which hold the distribution information of the random variable. The "kernel" and degree of greyness arising from the reduced form are very important as they allow us to learn the value information of a general grey number.

## References


Chen, S. L. (2001). Grey element and grey element space in grey system. *Journal of Kunming University of Science and Technology, 26*(2), 92–95.

Deng, J. L. (1985). *Grey control systems*. Press of Huazhong University of Science and Technology.

Deng, J. L. (1990). *Grey system theory tutorial*. Press of Huazhong University of Science and Technology.

Jiang, S. Q., Liu, S. F., Fang, Z. G., & Liu, Z. X. (2017). Study on distance measuring and sorting method of general grey number. *Grey Systems: Theory and Application, 7*(3), 320–328.

Jiang, S. Q., Liu, S. F., & Liu, Z. X. (2021). General grey number decision-making model and its application based on intuitionistic grey number set. *Grey Systems: Theory and Application, 11*(4), 556–570.

Li, L., Li, X. (2024). Some properties of generalized whiteness of interval grey number. *The Journal of Grey System, 36*(3), 25–36.

Liu, S. F. (1989). On Perron-Frobenius theorem of grey nonnegative matrix. *Journal of Grey System, 1*(2), 157–166.

Liu, S. F. (1996). Axiom on grey degree. *The Journal of Grey System, 8*(4), 397–400.

Liu, S. F., & Lin, Y. (2004). Definition and properties of grey number and grey scale. *Chinese Engineering Science, 6*(8), 91–99. In Chinese.

Liu, S. F., Fang, Z. G., & Xie, N. M. (2010). Algorithm rules of interval grey numbers based on the "Kernel" and the degree of greyness of grey numbers. *Systems Engineering and Electronics, 32*(2), 313–316.

Liu, S. F. (2024). *Grey systems theory and its applications* (10th ed.). Science Press.

Liu, S., Lin, Y. (2006). On measures of information content of grey numbers. *Kybernetes: The International Journal of Systems & Cybernetics*, *35*(6), 899–904.

Liu, S. F., Fang, Z. G., Yang, Y. J., & Forrest, J. (2012). General grey numbers and its operations. *Grey Systems: Theory and Application, 2*(3), 341–349.

Yang, Y. (2007). Extended grey numbers and their operations. In *Proceedings of 2007 IEEE International Conference on Systems, Man and Cybernetics*, Montreal, Canada, pp. 2181–2186.

Yang, Y., & Liu, S. (2011). Reliability of operations of grey numbers using kernels. *Grey Systems: Theory and Application., 1*(1), 57–71.

Yang, Y., & John, R. (2012). Grey sets and greyness. *Information Sciences, 185*(1), 249–264.

# Chapter 4
# Sequence Operators and Grey Data Mining

The totality of a series of operations applied to sequence data in a certain order is called a sequence operator. The concept of sequence operator was first proposed by the author of this book in 1991 (Liu, 1991).

The grey system theory regards the data sequence with poor information characteristics as the grey data that changes in a certain amplitude range and a certain time zone, and uses the sequence operator to eliminate the impact disturbance factors contained in the data sequence, estimate the missing data in the sequence, reduce the data noise, test whether the data sequence has the grey exponential law, and mine the hidden change law in the data sequence.

The sequence operators used to eliminate the impact disturbance factors contained in the data sequence are called buffer operators, including weakening buffer operators and strengthening buffer operators (Liu, 1991). The operator used to estimate the missing data of the sequence is called the estimation operator (Liu, 2024a, 2024b), the operator used to reduce the data noise includes the mean operator and the average smoothing denoise operator (Li et al., 2023a, 2023b; Liu, 2024a, 2024b), the operator used to test whether the data sequence has the grey index law is called the stepwise operator (Deng, 1985), the sequence operator used to mine the hidden index law of the data sequence is called the accumulation operator, and the inverse operator of the accumulation operator is called the inverse accumulation operator (Deng, 1985), etc.

## 4.1 Introduction

One of the main tasks of grey systems theory is to uncover the mathematical relationships between different system variables and the laws of change of certain system variables themselves based on the available data of characteristic behaviors of social, economic and ecological systems, for example. Grey systems theory looks at each



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_4

stochastic variable as a grey quantity that varies within a fixed region and within a certain time frame, and each stochastic process as a grey process.

When investigating the behavioral characteristics of a system, what is available is often a sequence of definite white numbers. There is no substantial difference between whether we treat the sequence as a trajectory or actualization of a stochastic process, or as whitenized values of a grey process. However, to uncover the laws of evolution of systems' behavioral characteristics, different methods are developed using different thinking logics. For instance, the theory of stochastic investigates statistical laws on the basis of probabilities borrowed from prior knowledge. This methodology generally requires large amounts of data. However, even with large amounts of data there is no guarantee that any of the desired laws can be successfully uncovered. That is because the number of basic forms of distribution considered in this methodology is very limited. It is often extremely difficult to deal with non-typical distribution processes. Nonetheless, grey systems theory uncovers laws of change by excavating and organizing the available raw data, representing an approach of finding data out of data through grey sequence operators. Grey systems theory believes that a system possesses overall functions and properties, even if the expression of such an objective system might be complicated, and its data chaotic.. Therefore, there must be internal laws governing the existence of the system and its operation. The key is to choose an appropriate method to excavate the internal laws and make use of such laws. For any given grey sequence, its implicit pattern can always be revealed through weakening the explicit randomness.

For example, the following sequence does not clearly show any regularity or pattern:

$$X^{(0)} = (1, 2, 1.5, 3) = \left(x^{(0)}(1), x^{(0)}(2), x^{(0)}(3), x^{(0)}(4)\right).$$

Now, we depict the data set with the graph in Fig. 4.1. From this graph, it can be seen that the curve of $X^{(0)}$ undulates with relatively large amplitude. If we apply the accumulating operator once to the original data set $X^{(0)}$, and denote the resultant sequence as $X^{(1)}$, then we have:

$$X^{(1)} = (1, 3, 4.5, 7.5) = \left(x^{(1)}(1), x^{(1)}(2), x^{(1)}(3), x^{(1)}(4)\right).$$

where for k = 1, 2, 3, 4, $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i)$.

Now, the processed sequence $X^{(1)}$ clearly shows a growing tendency (see Fig. 4.2 for more details).

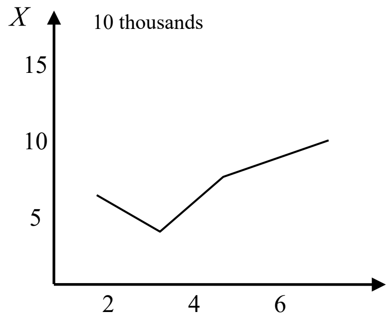



**Fig. 4.1** The curve of $X^{(0)}$

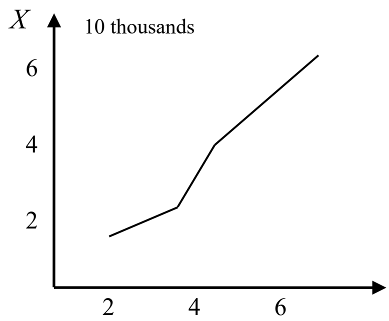



**Fig. 4.2** The curve of $X^{(1)}$

## 4.2 Systems Under Shocking Disturbances and Buffer Operators

### *4.2.1 The Trap for Shocking Disturbed System Forecasting*

Behavioral prediction of problems under the influence of shocking disturbances has always been a difficult problem. For such predictions, any theory on how to choose models would lose its validity. This is because the problem to be address here is not about which model is the best; instead, when a system is severely impacted by shocks, the available behavioral data of the past no long represent the current state of the system. In this case, the available data of the system's behavior can no longer truthfully reflect the law of change of the system.

**Definition 4.2.1** Assume that

$$X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\right)$$

stands for a sequence of a system's true behaviors. If the observed behaviors of the system are

$$X = (x(1), x(2), \ldots, x(n))$$
$$= \left(x^{(0)}(1) + \varepsilon_1, x^{(0)}(2) + \varepsilon_2, \ldots, x^{(0)}(n) + \varepsilon_n\right) = X^{(0)} + \varepsilon$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ is a term for the shocking disturbance, then X is called a shock-disturbed sequence (Liu, 1991).

To correctly uncover and recognize the true behavior sequence $X^{(0)}$ of the system from the shock-disturbed sequence $X$, one first has to go over the hurdle $\varepsilon$. If we directly established our model and made our predictions using the severely affected data $X$ without first cleaning up the disturbance, then our predictions would most likely fail. This is because the model would not have described the true state $X^{(0)}$ of change of the underlying system.

The wide spread existence of severely shocked systems often causes quantitative predictions to disagree with the outcomes of intuitive qualitative analyses. Hence, there is a need to seek an organic equilibrium between quantitative predictions and qualitative analyses, by eliminating shock wave disturbances in order to recover the true state of the systems' behavioral data. This way the accuracy of the consequent predictions can be greatly improved, which is one of the most important tasks performed by grey systems scientists. To this end, the discussion in this section is centered around the overall goal of reaching $X^{(0)}$ from $X$.

### *4.2.2 Axioms of Buffer Operators*

**Definition 4.2.2** Assume that $X = (x(1), x(2), \ldots, x(n))$ is a system's behavior data sequence.

(1) If $\forall\, k = 2, 3, \ldots, n$, $x(k) - x(k-1) > 0$, then $X$ is referred to as a monotonic increasing sequence;
(2) If the inequality sign in (1) is inversed, then $X$ is referred to as a monotonic decreasing sequence;
(3) If there are $k, k' \in \{2, 3, \ldots, n\}$ such that $x(k) - x(k-1) > 0$, $x(k') - x(k'-1) < 0$, then $X$ is referred to as a random vibrating or fluctuating sequence. If $M = \max\{x(k) | k = 1, 2, \ldots, n\}$ and $m = \min\{x(k) | k = 1, 2, \ldots, n\}$, then $M - m$ is referred to as the amplitude of sequence $X$.

**Definition 4.2.3** Assume that $X$ is a data sequence of a system's behavior, $D$ an operator to work on $X$, and after being applied by the operator $D$, $X$ becomes the following sequence:

$$XD = (x(1)d, x(2)d, \ldots, x(n)d)$$

where $D$ is referred to as a sequence operator and $XD$ the first order sequence of operator $D$ (Liu, 1991). If $D_1$, $D_2$, and $D_3$ are all sequence operators, then $D_1D_2$ is referred to as a second order sequence operator, and

$$XD_1D_2 = (x(1)d_1d_2, x(2)d_1d_2, \ldots, x(n)d_1d_2)$$

a second order sequence of $D_1D_2$. Similarly, $D_1D_2D_3$ is referred to as a third order sequence operator and

$$XD_1D_2D_3 = (x(1)d_1d_2d_3, x(2)d_1d_2d_3, \ldots, x(n)d_1d_2d_3)$$

a third order sequence of $D_1D_2D_3$.

**Axiom 4.2.1** (*Fixed Point*) Assume that $X$ is a data sequence of a system's behavior and $D$ a sequence operator. Then D satisfies $x(n)d = x(n)$.

This fixed point axiom means that under the effect of a sequence operator, data point $x(n)$ remains unchanged, and this is the last entry of the system's behavior data sequence. Based on the conclusions of relevant qualitative analysis, we can also leave several of the last entries of the data unchanged by the operator $D$, say,

$$x(j)d \neq x(j) \text{ and} x(i)d = x(i)$$

for $j = 1, 2, \ldots, k - 1; i = k, k + 1, \ldots, n.$

**Axiom 4.2.3** (*In accordance with information*) The sequence operator must be defined in accordance with information in the data sequence $X$. That is, each entry value $x(k)$, $k = 1, 2, \ldots, n,$ in the data sequence $X$ of the system's behavior should sufficiently participate in the entire process of application of the operator.

This axiom requires that any sequence operator be defined by using known information of the given sequence. It cannot be produced without referencing available raw data (Liu, 1991).

**Axiom 4.2.3** (*Expressed normality*) Each $x(k)d$, $k = 1, 2, \ldots, n$, is expressed by a uniform, elementary analytic representation of $x(1), x(2),\ldots, x(n)$ (Liu, 1991).

This last axiom requires that the procedure of applying sequence operators be clear, normalized, and uniform, so that it can be conveniently carried out on computers.

**Definition 4.2.4** Any sequence operator satisfying these three axioms is referred to as a buffer operator; the first order, second order, third order, …, sequences obtained by applying a buffer operator are referred to as the first order, second order, third order, …, buffered sequences.

**Definition 4.2.5** For a raw data sequence $X$ and a buffer operator $D$, when $X$ is respectively an increasing, decreasing, or fluctuating sequence:

(1) If the buffered sequence XD increases, decreases, or fluctuates slower or with smaller amplitude, respectively, than the original sequence X, then D is referred to as a weakening operator.
(2) If the buffered operator XD increases, decreases, or fluctuates faster or with larger amplitude, respectively, than the original sequence X, then D is referred to as a strengthening operator (Liu, 1991).

### 4.2.3 Properties of Buffer Operators

**Theorem 4.2.1** *Assume that X is a monotonic increasing sequence, then:*

(1) *If D is a weakening operator* $\Leftrightarrow x(k)d \geq x(k)$, $k = 1, 2, \ldots, n$;
(2) *If D is a strengthening operator* $\Leftrightarrow x(k)d \leq x(k)$, $k = 1, 2, \ldots, n$ (Liu, 1991).

***Proof*** Assume that

$$r(k) = \frac{x(n) - x(k)}{n - k + 1}, k = 1, 2, 3, \ldots$$

is the average increasing rate from $x(k)$ to $x(n)$ in the sequence X of raw data, and

$$r(k)d = \frac{x(n)d - x(k)d}{n - k + 1}, k = 1, 2, 3, \ldots,$$

is the average increasing rate from $x(k)d$ to $x(n)d$ in the buffered sequence XD. Given the condition that

$$x(n)d = x(n)$$

It follows that

$$r(k) - r(k)d = \frac{[x(n) - x(k)] - [x(n)d - x(k)d]}{n - k + 1} = \frac{x(k)d - x(k)}{n - k + 1}$$

If $D$ is a weakening operator, then, $r(k) \geq r(k)d$, that is $r(k) - r(k)d \geq 0$. Therefore $x(k)d - x(k) \geq 0$, that is, $x(k)d \geq x(k)$ and vice versa.

If $D$ is a strengthening operator, then $r(k) \leq r(k)d$, that is $r(k) - r(k)d \leq 0$. Therefore $x(k)d - x(k) \leq 0$, that is, $x(k) \geq x(k)d$ and vice versa.

**Theorem 4.2.2** *Assume that X is a monotonic decreasing sequence, then:*

(1) *If D is a weakening operator* $\Leftrightarrow x(k)d \leq x(k)$, $k = 1, 2, \ldots, n$;
(2) *If D is a strengthening operator* $\Leftrightarrow x(k)d \geq x(k)$, $k = 1, 2, \ldots, n$ (Liu, 1991).

**Theorem 4.2.3** *Assume that X is a fluctuating sequence and XD a buffered sequence, then:*

(1) *If D is a weakening operator, then* $\max_{1 \leq k \leq n}\{x(k)\} \geq \max_{1 \leq k \leq n}\{x(k)d\}$ *and* $\min_{1 \leq k \leq n}\{x(k)\} \leq \min_{1 \leq k \leq n}\{x(k)d\}$;
(2) *If D is a strengthening operator, then* $\max_{1 \leq k \leq n}\{x(k)\} \leq \max_{1 \leq k \leq n}\{x(k)d\}$ *and* $\min_{1 \leq k \leq n}\{x(k)\} \geq \min_{1 \leq k \leq n}\{x(k)d\}$.

For detailed proofs and relevant discussions of these theorems, please consult Liu and Lin (2006, pp. 64–67). What theorem implies is that each monotonic increasing sequence expands under the effect of a weakening operator and shrinks under a strengthening operator. What theorem indicates is that each monotonic decreasing sequence shrinks under the effect of a weakening operator and expands under a strengthening operator.

## 4.3 Construction of Practically Useful Buffer Operators

### *4.3.1 Weakening Buffer Operators*

**Theorem 4.3.1** *Given a raw data sequence* $X = (x(1), x(2), \ldots, x(n))$*, let* $XD = (x(1)d, x(2)d, \ldots, x(n)d)$*, where*

$$x(k)d = \frac{1}{n-k+1}[x(k) + x(k+1) + \cdots + x(n)], k = 1, 2, \ldots, n \tag{4.1}$$

*Then D is always a weakening operator regardless of whether X is a monotonic increasing, decreasing, or vibrating sequence. This operator is referred to as an average weakening buffer operator (AWBO)* (Liu, 1991, 2024; Liu et al., 2022).

*The weakening operator D in Theorem 4.3.1 possesses some very good properties and has been applied widely in modeling and prediction of systems with interference of uncontrollable shock waves.*

**Corollary 4.3.1** *For the weakening operator D as defined in Theorem 4.3.1, let:*

$$XD^2 = XDD = (x(1)d^2, x(2)d^2, \ldots, x(n)d^2)$$

$$x(k)d^2 = \frac{1}{n-k+1}[x(k)d + x(k+1)d + \cdots + x(n)d]; k = 1, 2, \ldots, n \tag{4.2}$$

*Then* $D^2$ *is always a second-order weakening operator for monotonic increasing, monotonic decreasing, and fluctuating* sequences.

***Example 4.3.1*** Let $X = (36.5, 54.3, 80.1, 109.8, 143.2)$ and D and $D^2$ as defined in Theorem 4.3.1 and Corollary 4.3.1 respectively, calculate the buffered sequence $XD$ and $XD^2$.

**Solution** Here $n = 5$, from formula 4.1, we have:

$$\begin{aligned} x(1)d &= \frac{1}{n-k+1}[x(k) + x(k+1) + \cdots + x(n)] = \frac{1}{5-1+1}[x(1) + x(2) + \cdots + x(5)] \\ &= \frac{1}{5-1+1}[36.5 + 54.3 + 80.1 + 109.8 + 143.2] = 84.78 \\ x(2)d &= \frac{1}{n-k+1}[x(k) + x(k+1) + \cdots + x(n)] = \frac{1}{5-2+1}[x(2) + \cdots + x(5)] \end{aligned}$$

$$= \frac{1}{4}[54.3 + 80.1 + 109.8 + 143.2] = 96.85$$

$$x(3)d = \frac{1}{5-3+1}[x(3) + x(4) + x(5)] = \frac{1}{3}[80.1 + 109.8 + 143.2] = 111.03$$

$$x(4)d = \frac{1}{5-4+1}[x(4) + x(5)] = \frac{1}{2}[109.8 + 143.2] = 126.5$$

$$x(5)d = 143.2$$

Therefore:

$$XD = (84.78, 96.85, 111.03, 126.5, 143.2).$$

Similarly, we can obtained the second-order buffered sequence $XD^2$ as follows:

$$XD^2 = (112.47, 119.4, 126.91, 134.85, 143.2).$$

**Theorem 4.3.2** *Assume that* $X = (x(1), x(2), \ldots, x(n))$ *is a sequence of raw data,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *is a weight vector, and* $\omega_i > 0, i = 1, 2, , \ldots, n$. *Let:*

$$XD = (x(1)d, x(2)d, \ldots, x(n)d)$$

*where*

$$x(k)d = \frac{\omega_k x(k) + \omega_{k+1} x(k+1) + \cdots + \omega_n x(n)}{\omega_k + \omega_{k+1} + \cdots + \omega_n} = \frac{1}{\sum_{i=k}^{n} \omega_i} \sum_{i=k}^{n} \omega_i x(i), \; (k = 1, 2, \ldots, n) \tag{4.3}$$

*Then D is always a weakening operator regardless of whether X is a monotonic increasing, decreasing, or vibrating sequence (Dang et al., 2004). This operator D is called as a weighted average (or mean) weakening buffer operator* (*WAWBO*).

**Corollary 4.3.2** *For the weighted average weakening operator D as defined in Theorem 4.3.2, let:*

$$\omega = (1, 1, \ldots, 1).$$

Then:

$$\frac{1}{\sum_{i=k}^{n} \omega_i} \sum_{i=k}^{n} \omega_i x(i) = \frac{1}{n-k+1} \sum_{i=k}^{n} x(i)$$

*That is, the average weakening buffer operator (AWBO) is a special case of the weighted average weakening buffer operator* (*WAWBO*).

**Theorem 4.3.3** *Assume that* $X = (x(1), x(2), \ldots, x(n))$ *is a sequence of raw data,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *is a weight vector, and* $\omega_i > 0, i = 1, 2, , \ldots, n$. *Let:*

$$XD = (x(1)d, x(2)d, \ldots, x(n)d)$$

*where*

$$\begin{aligned} x(k)d &= \left[x(k)^{\omega_k} \cdot x(k+1)^{\omega_{k+1}} \ldots x(n)^{\omega_n}\right]^{\frac{1}{\omega_k+\omega_{k+1}+\cdots+\omega_n}} \\ &= \left[\prod_{i=k}^{n} x(i)^{\omega_i}\right]^{\frac{1}{\sum_{i=k}^{n}\omega_i}}, (k = 1, 2, \ldots, n) \end{aligned} \tag{4.4}$$

*Then D is always a weakening operator, regardless of whether X is a monotonic increasing, decreasing, or vibrating sequence (Dang et al.,* 2004).

*This operator D is called as a weighted geometric average weakening buffer operator* (*WGAWBO*).

***Example 4.3.1*** From 1983 to 1986, the overall business revenue of private enterprises in Changge county, located in the Henan Province of The People's Republic of China, was recorded as:

$$\text{X} = (10155, 12588, 23480, 35388).$$

This showed a tendency of rapid growth. The average rate of revenue growth for these years was 51.6%, and the average rate of revenue growth from 1984 to 1986 was 67.7%. The people involved in the economic planning of the county, including politicians, scholars, policy makers, and residents, commonly believed that the overall revenue of private enterprises in this county would not be able to keep up with this record speed of growth in the coming years. If relevant data had been used to build models and make predictions, nobody would have accepted the resultant conclusions. After numerous rigorous analyses and discussions, all parties involved recognized that the reason for such a high growth rate between 1983 and 1986 was mainly a low baseline. Such a low baseline had been a consequence of the fact that, in the past, policies relevant to private enterprises had been neither existent, nor encouraged. To weaken the growth rate of the sequence of the raw data, it was necessary to artificially add all favorable environmental factors to past years' data, and such environmental factors were created based on the introduction of relevant policies for the development of private enterprise in recent years. With this goal in mind, we introduced the second-order weakening operator, as defined in Theorem 4.3.1, and obtained the following second-order buffered sequence:

$$\text{XD}^2 = (27260, 29547, 32411, 35388).$$

As a result, the consequent modeling based on $XD^2$ produced credible predictions for the county's business revenue growth between 1987 and 2000.

### 4.3.2 Strengthening Buffer Operators

**Theorem 4.3.4** *Assume that* $X = (x(1), x(2), \cdots, x(n))$ *is a sequence of raw data, and* $D_i$ *is a sequence operator defined by:*

$$x(k)d_i = \frac{x(k-1) + x(k)}{2}; k = 2, 3, \ldots, n; i = 1, 2 \tag{4.5}$$

*If* $x(1)d_1 = \alpha x(1), \alpha \in [0, 1], x(1)d_2 = (1 + \alpha)x(1), \alpha \in [0, 1]$*, and* $x(n)d_i = x(n), i = 1, 2$*, then* $D_1$ *is a strengthening buffer operator for monotonic increasing sequences, and* $D_2$ *a weakening buffer operator for monotonic decreasing sequences* (Liu, 1991).

*Both* $D_1$ *and* $D_2$ *are called even strengthening buffer operators (ESBO).*

**Theorem 4.3.5** *For a given increasing or decreasing sequence X of raw data, the operator D is defined as follows:*

$$x(k)d = \frac{(n - k + 1)[x(k)]^2}{x(k) + x(k+1) + \cdots + x(n)}, k = 1, 2, \ldots, n \tag{4.6}$$

*D is a strengthening buffer operator, and is called average strengthening buffer operator (ASBO)* (Liu, 2024a, 2024b).

**Theorem 4.3.6** *Assume that* $X = (x(1), x(2), \ldots, x(n))$ *is a sequence of raw data,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *is a weight vector, and* $\omega_i > 0, i = 1, 2, , \ldots, n$*. Let* $XD = (x(1)d, x(2)d, \ldots, x(n)d)$*, where D is defined as follows:*

$$x(k)d = \frac{(\omega_k + \omega_{k+1} + \cdots + \omega_n)(x(k))^2}{\omega_k x(k) + \omega_{k+1} x(k+1) + \cdots + \omega_n x(n)} = \frac{\sum_{i=k}^{n} \omega_i (x(k))^2}{\sum_{i=k}^{n} \omega_i x(i)}, (k = 1, 2, \ldots, n) \tag{4.7}$$

*D is a strengthening buffer operator regardless of whether the raw data sequence X is a monotonic increasing, decreasing, or vibrating sequence* (Dang et al., 2005). *D is called a weighted average strengthening buffer operator (WASBO).*

### 4.3.3 The General Form of Buffer Operator

**Theorem 4.3.6** *Assume that* $X = (x(1), x(2), \ldots, x(n))$ *is a sequence of raw data,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a *weight vector, and* $\omega_i > 0, i = 1, 2, , \ldots, n$. *Let* $XD = (x(1)d, x(2)d, \ldots, x(n)d)$, *where D is defined as follows:*

$$x(k)d = x(k) \cdot \left[ x(k) / \frac{\omega_k x(k) + \omega_{k+1} x(k+1) + \cdots + \omega_n x(n)}{\omega_k + \omega_{k+1} + \cdots + \omega_n} \right]^{\alpha}$$
$$= x(k) \cdot \left[ x(k) / \frac{1}{\sum_{i=k}^{n} \omega_i} \sum_{i=k}^{n} \omega_i x(i) \right]^{\alpha} \tag{4.8}$$

*Then:*

(1) *When* $\alpha < 0$,*D is a weakening operator regardless of whether X is a monotonic increasing or decreasing sequence.*
(2) *When* $\alpha > 0$,*D is a strengthening buffer operator regardless of whether the raw data sequence X is a monotonic increasing or decreasing sequence.*
(3) *When* $\alpha = 0$,*D is an identical operator* (Wei et al., 2011).

*D is called the general form of buffer operator (GFBO).*

**Corollary 4.3.3** *Take* $\alpha = -1$ *in Theorem 4.3.6, then formula* (4.8) *changes to* (*4.2*). *That is, the weighted average weakening buffer operator* (*WAWBO*) *is a special case of the general form of buffer operator (GFBO).*

**Corollary 4.3.4** *Take* $\alpha = 1$ in *Theorem 4.3.6*, *then formula* (*4.8*) *changes to* (*4.7*). *That is, the weighted average strengthening buffer operator (WASBO) is a special case of the general form of buffer operator (GFBO).*

*The buffer operator concept has been employed not only in grey systems modeling, but also in other kinds of model building. Generally, before building a mathematical model based on qualitative analysis and its conclusions, one applies a buffer operator on the original data sequence. This is done to soften or eliminate the effects of shock-disturbances on the behavior sequence of a given system. By doing so, expected results are often obtained.*

***Example 4.3.2*** From 1996 to 1999, the annual gross revenues produced by the agricultural, forestry, animal husbandry, and fishery sectors in the area of Nanjing were (in 0.1 billion yuan):

$$X = (91.9895, 94.2439, 96.9644, 98.9199).$$

The growth rate shown in $X$ is very slow, as it represents an average of about 2.4% annually. Such a slow growth rate was not aligned with the fast advances of the overall annual economic development of the area. If such a slow growth continued in these economic sectors, it would have caused imbalances in the development of the overall economic structure of the region and sustained regional economic growth would have been adversely affected. In 2000, Nanjing City gradually adjusted the economic structure of the countryside to counteract slow economic growth. In order to accurately control that economic development tendency in a timely fashion, there was a need to produce scientifically reasonable economic forecasts. To achieve this goal we had to address available data where slow growth was recorded. This would allow the resultant predictions to possess practical value in the realm of economic forecast and pro-growth government intervention. By applying the strengthening operator in Theorem 2.12 twice on the available data sequence, we obtained the following second order buffered data sequence:

$$XD^2 = (79.5513, 85.5446, 93.1686, 98.9199).$$

A GM(1,1) model based on this buffered sequence provided:

$$\frac{dX^{(1)}}{dt} - 0.0720X^{(1)} = 77.1389.$$

The time response function was as follows:

$$\overset{\wedge}{X}^{(1)}(k+1) = 1150.7003e^{0.0720k} - 1071.1503.$$

Based on this equation, the computational simulation results, effectiveness of the data fit, and prediction efficacy are given in Tables 4.1 and 4.2

Tables 4.1 and 4.2 show that by employing the buffered data using a strengthening operator to establish our model, the simulated results and corresponding predictions

**Table 4.1** The effectiveness of the simulation results

| Year | Strengthened data $x^{(0)}(k)$ | Simulated data $\hat{x}^{(0)}(k)$ | Error $\varepsilon(k) = \hat{x}^{(0)}(k) - x^{(0)}(k)$ | Relative error $\Delta_k = \frac{\lvert\varepsilon(k)\rvert}{x^{(0)}(k)}$ |
|---|---|---|---|---|
| 1997 | 85.5446 | 85.9245 | 0.3799 | 0.4441% |
| 1998 | 93.1686 | 92.3407 | – 0.8279 | 0.8886% |
| 1999 | 98.9199 | 99.2359 | 0.316 | 0.3195% |

**Table 4.2** The efficacy of the predictions

| Year | Actual data $x^{(0)}(k)$ | Predictions $\hat{x}^{(0)}(k)$ | Error $\varepsilon(k) = \hat{x}^{(0)}(k) - x^{(0)}(k)$ | Relative error $\Delta_k = \frac{\lvert\varepsilon(k)\rvert}{x^{(0)}(k)}$ |
|---|---|---|---|---|
| 2000 | 106.3412 | 106.6460 | 0.3048 | 0.2866% |
| 2001 | 113.29 | 114.6094 | 1.3194 | 1.1646% |
| 2005 | | 152.8703 | | |

are quite good. In particular, for 2000 and 2001, predicted values reached an accuracy rate of over 98% compared to the actual data for those years.

Over the years, research on buffer operator is pretty active and some new results have emerged. For example, Dang, Yaoguo (Dang et al., 2004) Wu Zhengpeng (Wu et al., 2009) Cui, Jie (Cui & Dang, 2009) Cui Lizhi (Cui et al., 2010) Guan Yeqing, (Guan & Liu, 2008), Hu, Xiaoli (Hu et al., 2013) Gao Yan (Gao et al., 2013), Dai Wenzhan (Dai & Su, 2012), Wang Zhengxin (Wang et al., 2009), Li Xuemei (Li et al., 2012), Wu, Lifeng (Wu et al., 2016), Li Chong (Li et al., 2019, 2023), Wang Yong (Wang et al., 2023), Liu Shuanghua (Liu, 2024), etc. constructed a variety of different weaken and strengthen buffer operators based on the three buffer operator axioms.

In 2011, Wei Yong et al. brought forth the general form of buffer operator (Wei et al., 2011). Ye Jing proposed the forecasting effect of grey model (GM)(1,1) and applicability evaluation criteria of weakening buffer operators based on systemic analysis of buffer operators working process to GM(1,1) prediction (Ye et al., 2014).

Because of the abundant shock disturbed system, the thinking methods and technology that buffer operator make the qualitative analysis results expressed quantitatively are widely applied in practice. Such as the research on radar target tracking by Liu et al. (2006), the research on the economic effects of meteorological disasters by Guo et al. (2014),the research on the analysis of transformer oil dissolved gas content by Liao et al. (2012) and the research on the grey PID forecast control by Zhu et al. (2012), etc.

Faced with actual vibration data, how to select and construct suitable buffer operator? How to determine the weight parameters and effect index of buffer operator? How the properties of the buffer operator are changed with the change of parameters and index? All of these are the problems that need further research. The answers to these questions are certainly the next step of development in this field.

## 4.4 Average Operator and Moving Average Denoise Operator

Due to various obstacles that are difficult to overcome, available data sequences may or may not contain missing entries. Nevertheless, even if data sequences are complete without any missing entries, systems' behaviors can change suddenly at any point in time, and corresponding entries in data sequences can become out of the ordinary. This can create great difficulties for the researcher. For example, if abnormal entries are removed, blank entries are created. Hence, how to effectively fill blanks in data sequences naturally becomes one of the first questions one has to address when processing available data. Data generation using averages is another frequently used method to create new data, fill a vacant entry in the available data sequence, and construct new sequences.

**Definition 4.4.1** Assume that sequence X have missing data at k, denoted as

$$X = (x(1), x(2), \ldots, x(k-1), \phi(k), x(k+1), \ldots, x(n))$$

Let

$$\text{D}: x(k)d = x^*(k) = \alpha x(k-1) + (1-\alpha)x(k+1), \alpha \in [0, 1] \tag{4.9}$$

then D is called an estimation operator, $x^*(k)$ is referred as a estimation value of $x(k)$.

**Definition 4.4.2** Assume that $X = (x(1), x(2), \ldots, x(n))$ is a the sequence operator D is defined as:

$$D: x(k)d = x^*(k) = \alpha x(k) + (1-\alpha)x(k+1), \alpha \in [0, 1] \tag{4.10}$$

D is called a 2 items weighted moving average operator.

Especially, 2 items equally weighted moving average operator is called a mean operator.

**Definition 4.4.3** For a given sequence $X = (x(1), x(2), \ldots, x(n))$, the sequence operator D is defined as:

$$x(k)d = x*(k) = 0.5x(k) + 0.5x(k-1) \tag{4.11}$$

In this case, D is referred to as a mean operator.

The sequence worked by mean operator is referred to as a mean sequence. In the process of modeling grey prediction models, it is usually necessary to apply the mean operator to the sequence acted by first-order accumulation operators to further eliminate the influence of random disturbances. Traditionally, the mean sequence is denoted as Z (Deng, 1990).

Generally, when $X = (x(1), x(2), \ldots, x(n))$ is a sequence of n items, then the mean sequence Z of X is a sequence of n-1 items. That is

$$Z = (z(2), z(3), \ldots, z(n))$$

**Definition 4.4.4** Assume that the original sequence:

$X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$, where

$x^{(0)}(k) \geq 0, k = 1, 2, \ldots, n$; let: $x^{(0)}D = \left(x^{(0)}(1)d, x^{(0)}(2)d, \ldots, x^{(0)}(n)d\right)$, where,

$$x^{(0)}(k)d = \frac{x^{(0)}(k-m) + \cdots + x^{(0)}(k-1) + x^{(0)}(k) + x^{(0)}(k+1) + \cdots + x^{(0)}(k+m)}{2m+1} \tag{4.12}$$

then D is called a moving average denoise operator.

At the case of m = 1, 2, we have

$$x^{(0)}(k)d = \frac{x^{(0)}(k-1) + x^{(0)}(k) + x^{(0)}(k+1)}{3} \tag{4.13}$$

$$x^{(0)}(k)d = \frac{x^{(0)}(k-2) + x^{(0)}(k-1) + x^{(0)}(k) + x^{(0)}(k+1) + x^{(0)}(k+2)}{5} \tag{4.14}$$

Moving average denoise operator.is a center operator. Its characteristic is that it can keep the corresponding relationship between $x^{(0)}(k)d$ and $x^{(0)}(k)$.

The moving average denoise operator has low-pass filtering effect. The low-frequency part (Evolution Law) of the data will basically remain unchanged under the action of the moving average denoise operator, and the high-frequency part (noise) will be compressed and suppressed (Lin et al, 2022; Liu et al., 2020)).

## 4.5 The Quasi-Smooth Sequence and Stepwise Ratio Operator

**Definition 4.5.1** Assume that $X = (x(1), x(2), \ldots, x(n))$, $x(k) \geq 0, k = 1, 2, \ldots, n$, then the following is referred to as the smoothness ratio of the sequence $X$ (Deng, 1985):

$$\rho(k) = \frac{x(k)}{\sum_{i=1}^{k-1} x(i)}; k = 2, 3, \ldots, n \tag{4.15}$$

The concept of smoothness ratio reflects the smoothness of a sequence from a special angle. In particular, it uses the ratio $\rho(k)$ of the $k$th data value $x(k)$ over the sum $\sum_{i=1}^{k-1} x(i)$ of the previous values to check whether or not the changes in the data points of $X$ are stable. The more stable the changes of the data points in sequence $X$ are, the smaller the smoothness ratio $\rho(k)$.

**Definition 4.5.2** If a sequence $X = (x(1), x(2), \ldots, x(n))$, $x(k) \geq 0, k = 1, 2, \ldots, n$ satisfies the following, then $X$ is referred to as a quasi-smooth sequence:

(1) $\frac{\rho(k+1)}{\rho(k)} < 1; k = 2, 3, \ldots, n-1;$
(2) $\rho(k) \in [0, \varepsilon]; k = 3, 4, \ldots, n$; and
(3) $\varepsilon < 0.5$.

Quasi-smooth conditions are very important criteria, which are employed to check whether a sequence can be used to build a grey model.

If the first entry $x(1)$ or the last entry $x(n)$ of a sequence are blank, that is, $x(1) = \emptyset(1)$ or $x(n) = \emptyset(n)$, we cannot fill these missing entries by using the method of adjacent neighbor mean generation operator. In this case, the operator of stepwise ratio is often employed.

**Definition 4.5.3** Assume that a sequence $X = (x(1), x(2), \ldots, x(n)), x(k) \geq 0, k = 1, 2, \ldots, n$, then the following is referred to as the operator of stepwise ratios of $X$ (Deng, 1985):

$$x(k)d = \sigma(k) = \frac{x(k)}{x(k-1)}; k = 2, 3, \cdots, n \tag{4.16}$$

The missing entry $x(1) = \emptyset(1)$ can be generated by using the operator of stepwise ratio of its right-hand side neighbors, and $x(n) = \emptyset(n)$ its left-hand side neighbors. The sequence obtained by filling all its missing entries using the operators of stepwise ratio is referred to as stepwise ratio generated.

**Proposition 4.5.1** *Assume that a sequence* $X = (x(1), x(2), \ldots, x(n))$, $x(k) \geq 0, k = 1, 2, \ldots, n$, *and x(1) = Ø(1) or x(n) = Ø(n). If both x(1) and x(n) are generated by operator of stepwise ratio, then:*

$$x(1) = x(2)\big/\sigma(3), x(n) = x(n-1)\sigma(n-1).$$

**Proposition 4.5.2** *Stepwise ratio* $\sigma(k+1)$ *and smoothness ratio as defined in formulas* (4.16) and (4.15)*, respectively, satisfy the relation as follows:*

$$\sigma(k+1) = \frac{\rho(k+1)}{\rho(k)}(1 + \rho(k)); k = 2, 3, \ldots, n \tag{4.17}$$

**Proposition 4.5.3** *If X = (x(1), x(2), …, x(n)) is an increasing sequence, and satisfies the following conditions:*

(1) For any $k = 2, 3, \ldots, n$, $\sigma(k) < 2$; and
(2) $\frac{\rho(k+1)}{\rho(k)} < 1$;

*then for any* $\varepsilon \in [0, 1]$ *and* $k = 2, 3, \ldots, n$, *when* $\rho(k) \in [0, \varepsilon]$, *we have* $\sigma(k+1) \in [1, 1 + \varepsilon]$.

## 4.6 Accumulation and Inverse Accumulation Operators

Accumulation operator is a method employed to mine the law implied in a grey data sequence. It plays an extremely important role in grey system modelling. Through the accumulation operator method, one can potentially uncover a development tendency existing in the process of accumulated grey quantities. This allows the characteristics and laws of integration hidden in chaotic original data to be sufficiently revealed. For instance, when looking at the financial outflows of a family, if we do our computations on a daily basis, we may not see obvious patterns. However, if our calculations are done on a monthly basis, some patterns of spending, which are somehow related to the monthly income of the family, will likely emerge.

The inverse accumulation operator is often employed to acquire additional insights from a small amount of available information. It plays the role of recovery from the acts of the accumulation operator and is its inverse operation. In particular,

**Definition 4.6.1** For an original sequence $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$, $D$ is a sequence operator defined as follows:

$X^{(0)}D = \left(x^{(0)}(1)d, x^{(0)}(2)d, \ldots, x^{(0)}(n)d\right)$, where

$$x^{(0)}(k)d = \sum_{i=1}^{k} x^{(0)}(i); \; k = 1, 2, \ldots, n \tag{4.18}$$

Here, D is called a once accumulation generation operator of $X^{(0)}$, denoted as 1-AGO. And $X^{(0)}D$, the sequence worked by accumulation operator $D$ on $X^{(0)}$, is denoted as $X^{(1)}$ for parsimony:

$$X^{(0)}D = X^{(1)} = \left(x^{(0)}(1)d, x^{(0)}(2)d, \ldots, x^{(0)}(n)d\right).$$

If the accumulation operator $D$ is applied $r$ times on $X^{(0)}$, we obtain:

$$X^{(0)}D^{r} = X^{(r)} = \left(x^{(r)}(1), x^{(r)}(2), \ldots, x^{(r)}(n)\right)$$

where

$$x^{(r)}(k) = \sum_{i=1}^{k} x^{(r-1)}(i); \; k = 1, 2, \ldots n \tag{4.19}$$

$D^{r}$ is denoted as r-AGO (Deng, 1985). Corresponding to the accumulation operator, the inverse accumulation operator $D$ is defined below.

The accumulation operator has low-pass filtering effect. The noise in the data sequence belongs to high frequency information. These information will be suppressed in the process of the accumulation operator equivalent digital filter. The evolution law of aperiodic system belongs to low-frequency signal, which can pass through or be amplified in the process of the accumulation operator equivalent digital filter. This also proves that for general non negative quasi smooth sequences, the randomness can be reduced by the action of accumulation operator, so that they show an approximate exponential variation law (Lin et al., 2021, 2022).

**Definition 4.6.2** For an original sequence $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$, $D$ is a sequence operator defined as follows:

**Table 4.3** The 1-AGO, 2-AGO and 1-IAGO of $X^{(0)}$

| | | | | | |
|---|---|---|---|---|---|
| $X^{(0)}$ | 5.3 | 7.6 | 10.4 | 13.8 | 18.1 |
| $X^{(1)}$ | 5.3 | 12.9 | 23.3 | 37.1 | 55.2 |
| $X^{(2)}$ | 5.3 | 18.2 | 41.5 | 78.6 | 133.8 |
| $\alpha^{(1)}X^{(0)}$ | 5.3 | 2.3 | 2.8 | 3.4 | 4.3 |

$X^{(0)}D = \left(x^{(0)}(1)d, x^{(0)}(2)d, \ldots, x^{(0)}(n)d\right)$, where

$$x^{(0)}(k)d = x^{(0)}(k) - x^{(0)}(k-1); k = 2, \ldots, n \tag{4.20}$$

D is called an inverse accumulation generation operator of $X^{(0)}$, denoted as 1-IAGO. In $X^{(0)}D$, the sequence works by inverse accumulation operator $D$ on $X^{(0)}$, and is denoted as $\alpha^{(1)}X^{(0)}$.

If the inverse accumulation operator $D$ is applied $r$ times on $X^{(0)}$, we write conventionally:

$$X^{(0)}D^r = \alpha^{(r)}X^{(0)} = \left(\alpha^{(r)}x^{(0)}(1), \alpha^{(r)}x^{(0)}(2), \cdots, \alpha^{(r)}x^{(0)}(n)\right)$$

where $\alpha^{(r)}x^{(0)}(k) = \alpha^{(r-1)}x^{(0)}(k) - \alpha^{(r-1)}x^{(0)}(k-1)$; $k = 1, 2, \cdots, n$ (Deng, 1985).

**Proposition 4.6.1** *For an original sequence* $X^{(0)} = (x^{(0)}(1), x^{(0)}(2),\ldots, x^{(0)}(n))$, *if both* $X^{(r)}$ and $\alpha^{(r)}$ *are defined according to Definitions 4.6.1 and 4.6.2, then:*

$$\alpha^{(r)}X^{(r)} = X^{(0)}.$$

**Example 3.6.1** If $X = (5.3, 7.6, 10.4, 13.8, 18.1)$, calculate the 1-AGO $X^{(1)}$, 2-AGO $X^{(2)}$ and 1-IAGO $\alpha^{(1)}X^{(0)}$.

**Solution** The results are shown in Table 4.3.

## 4.7 Exponentiality of Accumulation Sequence

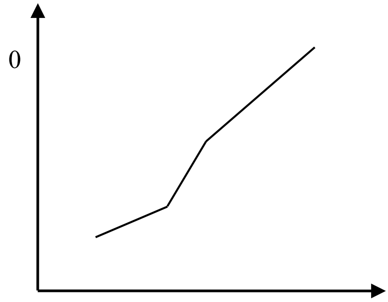
After applying the accumulation operator a few times, the general non-negative quasi-smooth sequence will show the pattern of exponential growth with decreased randomness. The smoother the original sequence is, the more obvious an exponential growth pattern in the first order accumulation sequence will appear.

***Example 4.7.1*** The sales quantity of electric cars from 2018 to 2023 in a city located in southeast of China is as follows:

$$X^{(0)} = \left\{x^{(0)}(k)\right\}_1^6 = (50810, 46110, 51177, 93775, 110574, 110524).$$

**Fig. 4.3** The curve of $X^{(0)}$


**Fig. 4.4** The curve of $X^{(1)}$




The 1-AGO sequence of $X^{(0)}$ is:

$$X^{(1)} = \left\{x^{(1)}(k)\right\}_1^6 = (50810, 96920, 148097, 241872, 352446, 462970).$$

The Figures of $X^{(0)}$ and $X^{(1)}$ are shown in Figs. 4.3 and 4.4, respectively.

For the curve shown in Fig. 4.3, it is difficult to find a simple curve as the approximation of $X^{(0)}$. However, the curve shown in Fig. 4.4 is very close to an exponential growth curve. $X^{(1)}$ can be fitted with an exponential curve.

**Definition 4.7.1** Assume that $X(t) = ce^{at} + b, c, a \neq 0$ is a continuous exponential function, then:

(1) $X(t)$ is referred to as homogeneous exponential function, if $b = 0$;
(2) $X(t)$ is referred to as non-homogeneous exponential function, if $b \neq 0$.

**Definition 4.7.2** If a sequence $X = (x(1), x(2), \ldots, x(n))$ satisfies:

(1) $x(k) = ce^{ak}$, $c, a \neq 0$, for $k = 1, 2, \ldots, n$, then $X$ is referred to as a homogeneous exponential sequence; and
(2) $x(k) = c\text{eak} + b$, $c, a, b \neq 0$, for $k = 1, 2, \ldots, n$, then $X$ is referred to as a non-homogeneous sequence.

**Theorem 4.7.1** *A sequence* $X = (x(1), x(2), \ldots, x(n))$ *is a homogeneous exponential sequence if, and only if, for* $k = 1, 2, \ldots, n$, $\sigma(k)$ *is a constant.*

***Proof***

(1) (1) Assume that $\forall\, k = 1, 2, \ldots, n$, $x(k) = ce^{ak}$, $c, a \neq 0$,then:

$$\sigma(k) = \frac{x(k)}{x(k-1)} = \frac{ce^{ak}}{ce^{a(k-1)}} = e^a = const$$

(2) Assume that $\forall\, k = 1, 2, \ldots, n$, $\sigma(k) = const = e^a$, then:

$$x(k) = e^a x(k-1) = e^{2a} x(k-2) = \cdots = x(1)e^{a(k-1)}$$

**Definition 4.7.3** For the given sequence $X = (x(1), x(2), \ldots, x(n))$,

(1) if $\forall\, k$, $\sigma(k) \in (0, 1)$, then $X$ is referred to as satisfying the law of negative grey exponent;
(2) if $\forall\, k$, $\sigma(k) \in (1, b)$, for some $b > 1$, then $X$ is referred to as satisfying the law of positive grey exponent;
(3) if $\forall\, k$, $\sigma(k) \in [a, b]$, $b - a = \delta$, then $X$ is referred to as satisfying the law of grey exponent with the absolute degree of greyness $\delta$; and
(4) if $\delta < 0.5$, then $X$ is referred to as satisfying the law of quasi-exponent.

**Theorem 4.7.2** *Assume that $X^{(0)}$ is a non-negative quasi-smooth sequence. Then, the sequence $X^{(1)}$, generated by applying accumulating operator once on $X^{(0)}$, satisfies the law of quasi-exponent.*

***Proof*** According to the definition of quasi-smooth sequence and.

$$\sigma^{(1)}(k) = \frac{x^{(1)}(k)}{x^{(1)}(k-1)} = \frac{x^{(0)}(k) + x^{(1)}(k-1)}{x^{(1)}(k-1)} = 1 + \rho(k)$$

We have

$$\forall k, \rho(k) < 0.5$$

Therefore

$$\sigma^{(1)}(k) \in [1, 1.5), \delta < 0.5.$$

Thus, $X^{(1)}$ is a sequence that satisfies the law of quasi-exponent.

Theorem 4.7.2 is the theoretical foundation of grey systems modeling. In fact, because economic, ecological and agricultural systems (among others) can be seen as energy systems, and given that the accumulation and release of energy generally satisfy an exponential law, this explains why exponential modeling of grey systems theory has found an extremely wide range of applications.

**Theorem 4.7.3** *Assume that $X^{(0)}$ is a non-negative sequence. If $X^{(r)}$ satisfies a law of exponent, and the stepwise ratio of $X^{(r)}$ is given by $\sigma^{(r)}(k) = \sigma$, then according to* Deng (1985):

(1) $\sigma^{(r+1)}(k) = \frac{1-\sigma^k}{1-\sigma^{k-1}}$;
(2) *When* $\sigma \in (0, 1)$, $\lim_{k\to\infty} \sigma^{(r+1)}(k) = 1$; *and for each* $k$, $\sigma^{(r+1)}(k) \in (1, 1+\sigma]$;
(3) *When* $\sigma > 1$, $\lim_{k\to\infty} \sigma^{(r+1)}(k) = \sigma$; *and for each* $k$, $\sigma^{(r+1)}(k) \in (\sigma, 1+\sigma]$.

***Proof***

(1) Assume that $X^{(r)}$ satisfies a law of exponent, and $\forall\, k$, $\sigma^{(r)}(k) = \frac{x^{(r)}(k)}{x^{(r)}(k-1)} = \sigma$, then $\forall\, k$,

$$x^{(r)}(k) = \sigma x^{(r)}(k-1) = \sigma^2 x^{(r)}(k-2) = \cdots = \sigma^{(k-1)} x^{(r)}(1)$$

$$X^{(r)} = (x^{(r)}(1), \sigma x^{(r)}(1), \sigma^2 x^{(r)}(1), \ldots, \sigma^{(n-1)} x^{(r)}(1))$$

$$\begin{aligned} X^{(r+1)} &= (x^{(r)}(1), (1+\sigma)x^{(r)}(1), (1+\sigma+\sigma^2) \\ & x^{(r)}(1), \ldots, (1+\sigma+\cdots+\sigma^{(n-1)})x^{(r)}(1)) \end{aligned}$$

Therefore

$$\sigma^{(r+1)}(k) = \frac{x^{(r+1)}(k)}{x^{(r+1)}(k-1)} = \frac{(1+\sigma+\cdots+\sigma^{k-1})x^{(r)}(1)}{(1+\sigma+\cdots+\sigma^{k-2})x^{(r)}(1)} = \frac{\frac{1-\sigma^k}{1-\sigma}}{\frac{1-\sigma^{k-1}}{1-\sigma}} = \frac{1-\sigma^k}{1-\sigma^{k-1}}$$

(2) When $\sigma \in (0, 1)$,$\sigma^{(r+1)}(k)$ will decrease as $k$ increases.

$k = 2$

$$\sigma^{(r+1)}(2) = \frac{x^{(r+1)}(2)}{x^{(r+1)}(1)} = 1+\sigma$$

$k \to \infty$

$$\sigma^{(r+1)}(k) = \frac{1-\sigma^k}{1-\sigma^{k-1}} \to 1$$

Therefore $\forall\, k$,

$$\sigma^{(r+1)}(k) \in [1, 1+\sigma].$$

(3) When $\sigma > 1$, $\sigma^{(r+1)}(k)$ will decrease as $k$ increases.

$k = 2$

$$\sigma^{(r+1)}(2) = 1+\sigma$$

$k \to \infty$

$$\sigma^{(r+1)}(k) = \frac{1-\sigma^k}{1-\sigma^{k-1}} \to \sigma$$

Therefore $\forall\, k$,

$$\sigma^{(r+1)}(k) \in (\sigma, 1+\sigma]$$

The Theorem 4.7.3 says that if the $r$th accumulating sequence of $X^{(0)}$ satisfies an obvious law of exponent, additional application of the accumulating operator will destroy the pattern of exponent. In practical applications, if the $r$th accumulating sequence of $X^{(0)}$ satisfies the law of quasi-exponent, we generally stop applying the accumulating operator. To this end, Theorem 4.7.2 implies that only one application of the accumulating operator is needed for a non-negative quasi-smooth sequence before establishing an exponential model.

## References

Cui, J., & Dang, Y. G (2009). Study on prediction accuracy of GM(1,1) based on a kind novel strengthening buffer operator. *Control and Decision, 24*(1), 44–48 (In Chinese).

Cui, L. Z., Liu, S. F., Wu, Z. P. (2010). On the construction of a new strengthening buffer operator and its application. *Systems Engineering Theory and Practice, 30*(3), 484–489 (In Chinese).

Dai, W. Z. & Su, Y. (2012). On the construction of a strengthening buffer operator of new information priority and its application. *Systems Engineering Theory and Practice, 38*(8), 1329–1334 (In Chinese).

Dang, Y. G., Liu, S. F., Liu, B., & Tang, X. W. (2004). Research on weakening buffer operator. *Chinese Journal of Management Science, 12*(2), 108–111. In Chinese.

Dang, Y. G., Liu, B., & Guang, Y. Q. (2005). On strengthening buffer operators. *Control and Decision, 20*(12), 1332–1336. In Chinese.

Deng, J. L. (1985). *Grey control systems*. Press of Huazhong University of Science and Technology.

Deng, J. L. (1990). Grey System Theory Tutorial. Wuhan: Press of Huazhong University of Science and Technology.

Gao, Y., Zhou, D. Q., Liu, C. S., et al. (2013). On Construction method and their internal connection of variable weight buffer operators. *Systems Engineering Theory and Practice, 33*(2), 489–497. In Chinese.

Guan, Y. Q., & Liu, S. F. (2008). On matrix of linear buffer operator and its application. *Journal of Applied Mathematics of Universities, 23*(3), 357–362.

Guo, C., Xu, X., & Gong, Z. (2014). Co-integration analysis between gdp and meteorological catastrophic factors of nanjing city based on the buffer operator. *Natural Hazards, 71*(2), 1091–1105.

Hu, X. L., Wu, Z. P., Han, R. (2013). Analysis on the strengthening buffer operator based on the strictly monotone function. *International Journal of Applied Physics and Mathematics, 3*(2):132–136.

Li, C., Yang, Y., & Liu, S. (2019). Comparative analysis of properties of weakening buffer operators in time series prediction models. *Communications in Nonlinear Science and Numerical Simulation, 68*, 257–285.

Li, C., Yang, Y., & Zhu, X. (2023a). Stability of time series models based on fractional-order weakening buffer operators. *Fractal and Fractional., 7*(7), 554.

Li, Q., Liu, S., & Lin, C. (2023), A moving average denoise operator and grey discrete production process quality prediction model. In *Grey systems: Theory and application* (Vol. 13 No. 1, pp. 34.57).

Li, X. M., Dang, Y. G., Wang, Z. X. (2012). Harmonic buffer operators with variable weight and effect strength comparison. *Systems Engineering Theory and Practice, 32*(11), 2486–2492.

Liao R. J., Yang J. P., & Grzybowski, S., et al. (2012). Forecasting dissolved gases content in power transformer oil based on weakening buffer operator and least square support vector machine-Markov. *IET Generation, Transmission & Distribution, 6*(2), 142–151.

Lin, C. H., Song, Z. Y., & Liu, S. F. (2021). Study on mechanism and filter efficacy of AGO/IAGO. *Grey Systems: Theory and Application, 11*(1), 1–21.

Lin, C. H., Liu, S. F., Fang, Z. G., & Yang, Y. J. (2022). Spectrum analysis of moving average operator and construction of time-frequency hybrid sequence operator. *Grey Systems: Theory and Application, 12*(1), 101–116.

Liu, S. H. (2024). A new grey buffer operator and its application. *Journal of Intelligent and Fuzzy Systems, 46*(3), 5723–5731.

Liu, S. F. (1991). The three axioms of buffer operator and their application. *Journal of Grey System, 3*(1), 39–48.

Liu, S. F., Lin, C. H., & Tao, L. Y. (2020). On spectral analysis and new research directions in greysystem theory. *The Journal of Grey System, 32*(1), 108–117.

Liu, S. F. (2024). *Grey system theory and its application* (10th ed.). Science Press. In Chinese.

Liu, S. F., & Lin, Y. (2006). *Grey information: Theory and practical applications*. Springer.

Liu, S. F., Yang, Y. J., & Forrest, J. (2017). *Grey data analysis*. Springer.

Liu, S. F., Yang, Y. J., & Forrest, J. (2022). *Grey systems analysis*. Springer.

Liu, Y. A., Chen, S. C., Zhang, M. J. et al. (2006). The application in target tracking of buffer operator and data fusion technology. *Journal of Applied Sciences, 24*(2), 154–158.

Wang, Y., Chi, P., & Nie, W. G. B. (2023). A novel fractional discrete grey model with variable weight buffer operator and its applications in renewable energy prediction. *Soft Computing: A Fusion of Foundations, Methodologies and Applications, 27*(14), 9321–9345.

Wang, Z. X., Dang, Y. G., & Liu, S. F. (2009). On buffer operators with variable weight and supplement to buffer operator axioms. *System Engineering, 27*(1), 113–117.

Wei, Y., Kong, X., & Hu, D. (2011). A kind of universal constructor method for buffer operators. *Grey systems: Theory and Application, 1*(2), 178–185.

Wu, L. F., Liu, S. F., & Yang, Y. J. et al. (2016). Multi-variable weakening buffer operator and its application. *Information Sciences, 339,* 98–107

Wu, Z. P., Liu, S. F., & Cui, L. Z. (2009). Study on new weakening buffer operators. *Control and Decision, 24*(12), 1805–1809, 1815

Ye, J., Li, B.J., & Liu, F. (2014). Forecasting effect and applicability of weakening buffer operators on GM(1, 1). *Systems Engineering-Theory and Prectace, 34*(9), 2364–2371.

Zhu, J. M., Huang, Z. W., Zhai, D. T., et al. (2012). Research on grey forecasting PID control simulation based on strengthening buffer operator. *Journal of Shanghai University of Science and Technology, 34*(4), 327–332.

# Chapter 5
# Grey Relational Analysis Models

As one of the most active branches of grey system theory, grey relation analysis (GRA) holds the basic idea of judging the closeness between sequences depending on the geometrical shape of their curves. The primary method is to turn the observed values of discrete behaviors of systematic factors to piecewise continuous lines through linear interpolation and further construct models to measure the degree of relation according to the geometrical characteristic of the lines. On the basis of GRA model proposed by Professor Deng (Deng, 1985), a great number of scholars conduct meaningful explorations focusing on the construction and properties of GRA models and achieving many valuable results. The process of the research develops from the early GRA models relying on relation coefficients of particular points to the generalized models based on integral or overall perspective (Liu, 1992) and from the GRA models which measure similarity based on nearness to the models which consider similarity and nearness (Liu et al., 2011), respectively. Besides, the research objects extend from the analysis of relationship among curves to that among curved surfaces and further to the analysis of relationship in three-dimensional space (Zhang & Liu, 2009) and even the relationship among sequences of vectors, complex numbers, interval numbers, fuzzy numbers, general grey numbers and tensor sequences, respectively. (Gui et al., 2004; Zhou et al., 2005; Xiong, 2000; Jiang et al., 2019).

## 5.1 Introduction

Any given system, such as a social, economic, agricultural, ecological, and educational system, will encompass different kinds of factors. It is the result of the mutual interactions of these factors that determines the development tendency and behavior of the system. It is often the case that, among all the factors, investigators will need to know which ones are primary and which ones are secondary. Primary factors have



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_5

dominant effects on the development of systems. Such factors drive the development of systems positively and must be strengthened. Conversely, secondary factors exert less influence on the development of systems. They tend to pose obstacles for the development of systems and, therefore, must be weakened. For instance, there are generally many influencing factors on the overall performance of an economic system. In order to realize the production of additional output with less input, systems analysis must be conducted prudently and a key part of this analysis is the identification of primary and secondary factors.

Regression analysis, variance analysis, and main component analysis are the most commonly employed methods for conducting systems analysis. However, these methods suffer from the following weaknesses:

(1) Large samples are needed in order to produce reliable conclusions.
(2) Available data need to satisfy some typical types of probability distribution; linear relationships between factors and system behaviors are assumed, while no interactions can be found between factors. Generally, these requirements are difficult to satisfy.
(3) The amount of computation is large and generally done by using computers.
(4) At times quantitative conclusions do not resonate with qualitative analysis outcomes so that the laws governing system development are distorted or misunderstood.

In fact, when available data are small it is extremely difficult to apply such traditional methods of statistics to analyze such data. This is because small data do not satisfy the modelling conditions of traditional methods; they contain relatively large amounts of grey information and do not follow any conventional probability distribution.

The Grey Relational Analysis (GRA) model is a new method to analyze systems where statistical methods do not seem appropriate. It can be applied to large or small samples and does not have conventional distribution requirements. Additionally, the amount of computation involved is small and can be carried out conveniently, without issues of disagreement between quantitative and qualitative conclusions.

The basic idea of grey relational analysis is to use the degree of similarity of the geometric curves of available data sequences to determine whether or not their connections are close. The more similar the curves, the closer the relational between sequences, and vice versa.

A number of scholars have conducted meaningful research focused on the construction and properties of GRA models, and such researchers have achieved valuable results. For example, Zhang et al. (1996) has analyzed the predominant point trend of Deng's (1985) GRA model. They has introduced grey relation entropy to improve the traditional model, and has proposed a new method to calculate degree of grey relational. Xiao and colleagues (2006) have constructed a weighted degree of grey relational through the weighted compound of relational coefficient of each point. Zhao and colleagues (1998a) have introduced Euclid nearness into grey relational analysis, and have established the Euclid relational degree model based on the measurement of nearness of factor points through calculating nearness. Furthermore, Zhao et al. (1998b) have defined a GRA model according to upper and lower boundaries of distances between grey factor points. The authors have also demonstrated that their GRA model as well as Deng's (1985) GRA model through weighted relational analysis and the Euclid relational degree model are three special types of GRA model. Shi (1995) has proposed extreme difference relation according to the difference between distance of maximum value and distance of sequences, complementing Deng's (1985) relational coefficient. Zhang et al. (2007) have integrated the method of discrimination coefficient correction, the entropy weight method and the projection method to advance Deng's GRA model. Zhao and colleagues () has introduced variant coefficient to relational analysis, and improved Deng's GRA model through weighted values of variant coefficient and relational coefficient. Further, Zhou et al. (2005) defined relational coefficient with the application of generalized distance in fuzzy math to measure the difference between reference sequence and compared sequence. Peng (2008) has extended Deng's GRA model to second-order trend relational analysis model through second-order difference. Finally, Wang (1989) has proposed the B-type relational degree model, Tang (1995) has developed the T-type relational degree model, and Dang and Liu (2004) has proposed the gradient relational degree model as well as its improved version. In 2005, Olson et al. proposed the grey incidence analysis method to solve the interval multiple attribute decision problems. In 2006, they simulated analyzed different kinds of fuzzy multiple attribute decision-making model using the grey incidence analysis (Olson & Wu, 2006). In 2010, Wu et al. proposed the DEA model based on the grey incidence fuzzy set to solve the location problem. Amanna et al. (2011) used GM model and GIA model comprehensively to study cognitive inference engine and automatic adjustment algorithm in wireless communication. Liu (2013) came up with a kind of generalized grey interval number incidence model and clarified the calculating process and feasibility through the examples. Among these models, the GRA model proposed by Professor Deng (1985) is the most influential one.

In 1992, the author of this book put forward the grey relational analysis model based on the overall perspective, including the grey absolute relational degree, grey relative relational degree and grey comprehensive relational degree model (Liu, 1992). Then, proposed the grey relational analysis models based on visual angle of similarity and nearness respectively after 20 years (Liu et al., 2011; Liu et al., 2013; Liu et al., 2016). In 2022, after 40 years of unremitting exploration, the problem of reverse sequence correlation analysis was finally solved, and a variety of negative grey correlation analysis models were proposed (Liu et al., 2022). After joining Northwestern Polytechnic University in 2023, Liu Sifeng broke through the problem of cross sequences relational analysis with his students (Liu et al., 2024; Lu et al., 2023).

The grey relational analysis model has been widely used, successfully solving a large number of scientific and practical problems in various fields, and the application results are numerous.

For example, Xie et al. (2004) solved the aircraft top-level design scheme selection decision problems using the grey incidence analysis model. Huang (2006) put forward the new grey incidence geological evaluation model and the principle of maximum entropy to evaluate Apricot River oil field in Shaanxi Gansu Ningxia basin. Zhang et al. (2014) studied the aircraft customized solutions using the grey incidence analysis model. Xiao and Zhang (2009) researched drone crashed fault using grey incidence analysis and fault tree method comprehensively, which provided a theoretical basis to diagnose the cause of drone crashed fault, to reduce the fault and improving the reliability of the system.

Shi et al. (2008) researched the main influence factors of U type steel encased concrete composite beam ductility using the grey incidence analysis model and built up the calculation formula of U-type steel encased concrete composite beam displacement ductility coefficient. Tan et al. (2011) combined the grey incidence analysis model with GM(1,1) model and put forward the effective method to predict the force state of cablestayed bridge in cold area.

Chen et al. (2011) used the grey incidence analysis model and studied two kinds of the sky optical measurement method based on ASD spectroradiometer and clearfied different methods applicable scenarios. Wang et al. (2011) studied cosmic ray μ sub imaging applying the grey incidence cluster analysis method and improved efficiency of material sorting.

Wang et al. (2013) predicted Zn and Au polymetallic deposit scientifically by the grey incidence analysis method. Rajesh and Ravi (2015) solved the problem of supplier selection in resilient supply chains using a grey relational analysis (GRA) approach. Xie, et al optimized the soil dissolved organic matter extraction by GRA model (Xie et al., 2020). Scarlat, et al. and Delcea, et al. analyzed the financial sector in Europe by GRA model (Scarlat & Delcea, 2011; Delcea et al., 2012; Delcea et al., 2013). Ejnioui, et al studied the prioritisation of software requirements using GRA model. Skrinjari dynamic portfolio optimization based on GRA method (Skrinjari, 2020). Yan, et al analyzed the blood lipids and hematological parameters by GRA method (Yan et al., 2019). Zhang, et al studied the customization model of aircraft based on GIA method (Zhang et al,. 2014). Zhou and Peng studied the customization model of aircraft using grey model (Zhou & Peng, 2008).

Lin et al. (2009) solved the main influential factors of explosively formed projectile (EFP) velocity using the grey incidence analysis method, which gained results that has important reference value on the EFP cover design of explosive type and explosive charges structure design. Zhao et al., (2007a, 2007b) established the assessment model on ship antimissile incoming missile threat, which provided the decision basis on timely judgment of shipborne system target threat assessment.

Chen et al. (2005) studied well logging, drilling and coring, oil testing and related geological data using the grey system theory. Then he divided lithology, physical property, oil bearing on the statistical analysis characteristic value and its accuracy, resolution through matching, fitting and extracting parameters, which provided the geological basis for oil field exploration and development. Liang et al. (2014) set up multiple index grey incidence degree optimization model.

Sarikaya and Güllü's research on multi-response optimization of minimum quantity lubrication parameters using Taguchi-based grey relational analysis in turning of difficult-to-cut alloy Haynes (2015), Ghosh and Banerjee's research of Iot-based freezing of gait detection (2019), the research on harnessing heterogeneous social networks for better recommendations by Wang et al. (2021), Optimization of Mixing Parameters in Nanosilica Toughened Cement Mortar (Vasanthi & Selvan, 2022), the research on Chinese medicine chromatographic fingerprint pattern recognition (Wei et al., 2013). Research on multi-objective optimization based grey relational analysis and investigation of using the waste animal fat biodiesel on the engine characteristics (Gad et al., 2023), evaluation of healthcare service quality factors (Aydemir & Sahin, 2019), water quality assessment (Gai & Guo, 2023), the research on portfolio optimization and dynamic portfolio optimization (Mehlawat et al., 2023).

The research on the formation reaction of high-temperature sulfur retention phase calcium sulphoaluminate (Liu et al., 2007) and study on the relationship between the rolling bearings machining quality and vibration (Xia et al., 2005).

Analysis of the double tooth difference of swing movable teeth transmission failure (Zhang et al., 2012), research on in deep drawing robust design of square box conservatism (Xie et al., 2007), researched the formation reaction of high-temperature sulfur retention phase calcium sulphoaluminate using the grey incidence analysis and prediction model. Xia et al. (2005) studied the relationship between the rolling bearings machining quality and vibration using the grey incidence analysis and found that the structure size error parameters had a larger effect on vibration of bearing. Zhang et al. (2012) analyzed the double tooth difference of swing movable teeth transmission failure using the grey incidence analysis model, which provided a scientific basis to improve the reliability of the double tooth difference of swing movable teeth transmission system. Xie et al. (2007) obtained the optimum parameters of each factor square box conservatism according to the outcome of variance analysis of the degree of grey incidence of the target sequence and every factors.

Research on state-of-health estimation for liion batteries (Li et al., 2019), study on Tribological Properties of Al 7075 Composite Reinforced with ZrB2 (Karumuri et al., 2022), Wire-Electrochemical Discharge Machining of SiC Reinforced Z-Pinned Polymer Matrix Composite (Kumar et al., 2021), Improved bag-of-features for classification of histology images (Pal et al., 2021), study on a micromixer with cantor fractal obstacle (Lv et al., 2022), Multi-response optimisation for turning of magnesium alloy with untreated and cryogenic treated carbide inserts (Ravikumar et al., 2023), optimization of electrochemical machining processes (Das & Chakraborty, 2024), Single-Sample Retinal Vessel Segmentation Method (Wang & Li, 2024), sucrose anaerobic hydrogen production prediction (Wang et al., 2024), Bi-objective optimization of an EDM process with Cu-MWCNT composite tool (Mandal et al., 2024), Parametric optimisation of 3D-printed PLA/wood dust composite for load-bearing (Mishra et al., 2024), etc.

## 5.2 Grey Relational Factors and Set of Grey Relational Operators

When analyzing a system, one must choose the mapping variable to reflect the characteristics of such a system, and determine the factors that influence the behavior of the system. If a quantitative analysis is considered, one needs to process the chosen mapping variable and the effective factors using sequence operators so that the available data are converted to their relevant non-dimensional values of roughly equal magnitudes.

**Definition 5.2.1** Assume that $X_i$ is a system factor and its observation value at the ordinal position $k$ is $x_i(k)$, $k = 1, 2, \ldots, n$, then $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is referred to as the behavioral sequence of factor $X_i$.

If $k$ stands for the time order, then $x_i(k)$ is referred to as the observational value of factor $X_i$ at time moment $k$, and $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is the behavioral time sequence (or series) of $X_i$.

If $k$ stands for an index ordinal number and $x_i(k)$ the observational value of the $k$th index of factor $X_i$, then $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is referred to as the behavioral index sequence of factor $X_i$.

If $k$ stands for the ordinal number of the observed object and $x_i(k)$ is the observed value of the $k$th object of factor $X_i$, then $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is referred to as the horizontal sequence of factor $X_i$'s behavior.

For example, if $X_i$ represents an economic factor, $k$ time, and $x_i(k)$ the observed value of factor $X_i$ at time moment k, then $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is a time series of economic behaviors. If $k$ is the ordinal number of an index, then $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is the index sequence of an economic behavior. If $k$ represents the ordinal number of different economic regions or departments, then $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is a horizontal sequence of an economic behavior. No matter what kinds of sequence data are available, they can be employed in relational analysis.

**Definition 5.2.2** Let $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ be the behavioral sequence of factor $X_i$, and $D_1$ a sequence operator such that $X_i D_1 = (x_i(1)d_1, x_i(2)d_1, \ldots, x_i(n)d_1)$, where:

$$x_i(k)d_1 = x_i(k)/x_i(1),\ x_i(1) \neq 0,\ k = 1, 2, \ldots, n \tag{5.1}$$

Then $D_1$ is referred to as an initialing operator and $X_i D_1$ is its image, called initial image of $X_i$. (Deng, 1985).

***Example 5.2.1*** Let $X = (3.2, 3.7, 4.5, 4.9, 5.6)$, and calculate the initial image of $X$.

**Solution**: From formula 5.1, we have:

$$x(1)d_1 = x(1)/x(1) = 1,\ x(2)d_1 = x(2)/x(1) = 3.7 \div 3.2 = 1.15625.$$

Similarly,

$$x(3)d_1 = 1.40625, x(4)d_1 = 1.53125, x(5)d_1 = 1.75.$$

Therefore:

$$XD_1 = (x(1)d_1, x(2)d_1, x(3)d_1, x(4)d_1, x(5)d_1) = (1, 1.15625, 1.40625, 1.53125, 1.75).$$

**Definition 5.2.3** Let $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ be the behavioral sequence of factor $X_i$. Sequence operator $D_2$ satisfies $X_iD_2 = (x_i(1)d_2, x_i(2)d_2, \ldots, x_i(n)d_2)$, and:

$$x_i(k)d_2 = \frac{x_i(k)}{\overline{X_i}},\ \overline{X_i} = \frac{1}{n}\sum_{k=1}^{n} x_i(k),\ k = 1, 2, \ldots, n \tag{5.2}$$

Here, $D_2$ is referred to as an averaging operator and $X_iD_2$ is its image, called the average image of $X_i$. (Deng, 1985).

***Example 5.2.2*** Let $X$ be the same as Example 5.2.1 and calculate the average image of $X$.

**Solution**: From formula 5.2, we have:

$$\overline{X} = \frac{1}{5}\sum_{k=1}^{5} x(k) = 4.38,\ x(1)d_2 = x(1)/\overline{X} = 0.73,\ x(2)d_2 = x(2)/\overline{X} = 0.84.$$

Similarly:

$$x(3)d_2 = 1.03,\ x(4)d_2 = 1.12,\ x(5)d_2 = 1.28.$$

Therefore:

$$XD_2 = (x(1)d_2, x(2)d_2, x(3)d_2, x(4)d_2, x(5)d_2) = (0.73, 0.84, 1.03, 1.12, 1.28).$$

**Definition 5.2.4** Let $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ be the behavioral sequence of factor $X_i$. Sequence operator $D_3$ satisfies $X_iD_3 = (x_i(1)d_3, x_i(2)d_3, \ldots, x_i(n)d_3)$, and:

$$x_i(k)d_3 = \frac{x_i(k) - \min\limits_k x_i(k)}{\max\limits_k x_i(k) - \min\limits_k x_i(k)};\ k = 1, 2, \ldots, n \tag{5.3}$$

$D_3$ is referred to as an interval operator and $X_iD_3$ is its image, called the interval image of $X_i$. (Deng, 1985).

***Example 5.2.3*** Let $X$ be the same as Example 5.2.1, and calculate the interval image of $X$.

**Solution**: $\min_k x(k) = 3.2$, $\max_k x(k) = 5.6$. From formula 5.3, we have:

$$x(1)d_3 = 0,\ x(2)d_3 = 0.208$$
$$x(3)d_3 = 0.542,\ x(4)d_3 = 0.708,\ x(5)d_3 = 1.$$

Therefore:

$$XD_3 = (x(1)d_3, x(2)d_3, x(3)d_3, x(4)d_3, x(5)d_3) = (0, 0.208, 0.542, 0.708, 1).$$

As usual, $D_1$, $D_2$, $D_3$ should not be mixed or overlapped. Only one of them can be selected according to a particular situation.

**Definition 5.2.4** The set $D = \{D_i | i = 1, 2, 3\}$ is referred to as the set of grey relational operators.

**Definition 5.2.5** If $X$ stands for the set of all system factors and $D$ the set of grey relational operators, then $(X, D)$ is referred to as the space of grey relational factors of a system.

## 5.3 Grey Relational Axioms and Deng's Grey Relational Analysis Model

Given the sequence $X = (x(1), x(2), \ldots, x(n))$, we can image the corresponding zigzagged line of the plane $X = \{x(k) + (t - k)(x(k + 1) - x(k)) | k = 1, 2, \ldots, n - 1; t \in [k, k + 1]\}$. Without causing confusion, the same symbol is used for both the sequence and its zigzagged line. For parsimony, we will not distinguish between the two in our discussions.

**Definition 5.3.1** The given sequence $X = (x(1), x(2), \ldots, x(n))$, $\alpha = \frac{x(s)-x(k)}{s-k}$, $s > k$, $k = 1, 2, \ldots, n - 1$, (5.4) is referred to as the slope of $X$ on interval $[k, s]$, and $\alpha = \frac{1}{n-1}(x(n) - x(1))$ (5.5) the average slope of $X$.

**Theorem 5.3.1** Assume that $X_i$ and $X_j$ are non-negative increasing sequences such that $X_j = X_i + c$, where $c$ is a nonzero constant. Let $D_1$ be an initialing operator, $Y_i = X_i D_1$ and $Y_j = X_j D_1$. If $\alpha_i$ and $\alpha_j$ are respectively the average slopes of $X_i$ and $X_j$, and $\beta_i$ and $\beta_j$ the average slopes of $Y_i$ and $Y_j$, then, the following must be true: $\alpha_i = \alpha_j$; when $c < 0$, $\beta_i < \beta_j$; and when $c > 0$, $\beta_i > \beta_j$.

What is meant here is that when the absolute amount of increase of two increasing sequences are the same, the sequence with the smallest initial value will increase faster than the other. To maintain the same relative rate of increase, the absolute amount of increase of the sequence with the greatest initial value must be greater than that of the sequence with the smallest initial value.

**Definition 5.3.2** Let $X_0 = (x_0(1), x_0(2), \ldots, x_0(n))$ be a data sequence of a system's behavioral characteristic and the following are relevant factor sequences:

$$\begin{aligned} X_1 &= (x_1(1), x_1(2), \ldots, x_1(n)) \\ &\cdots\cdots\cdots\cdots \\ X_i &= (x_i(1), x_i(2), \ldots, x_i(n)) \\ &\cdots\cdots\cdots\cdots \\ X_m &= (x_m(1), x_m(2), \ldots, x_m(n)) \end{aligned}$$

Given real numbers $\gamma(x_0(k), x_i(k))$, $i = 1, 2, ..., m$, and $k = 1, 2, ..., n$, if the following satisfies conditions of normality (1) and closeness (2) below:

$$\gamma(X_0, X_i) = \frac{1}{n}\sum_{k=1}^{n} \gamma(x_0(k), x_i(k)).$$

(1) Normality: $0 < \gamma(X_0, X_i) \leq 1$, $\gamma(X_0, X_i) = 1 \Leftrightarrow X_0 = X_i$; and
(2) Closeness: the smaller $|x_0(k) - x_i(k)|$, the greater $\gamma(x_0(k), x_i(k))$.

In this case, $\gamma(X_0, X_i)$ is referred to as the Deng's grey relational degree between $X_i$ and $X_0$, $\gamma(x_0(k), x_i(k))$ as the Deng's grey relational coefficient of $X_i$ and $X_0$ at point $k$ (Deng, 1985).

**Theorem 5.3.2** Given a system's behavioral sequences $X_0 = (x_0(1), x_0(2), \ldots, x_0(n))$ and $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$, $i = 1, 2, \ldots, m$, for $\xi \in (0, 1)$, it is possible to define:

$$\gamma(x_0(k), x_i(k)) = \frac{\min\limits_i \min\limits_k |x_0(k) - x_i(k)| + \xi \max\limits_i \max\limits_k |x_0(k) - x_i(k)|}{|x_0(k) - x_i(k)| + \xi \max\limits_i \max\limits_k |x_0(k) - x_i(k)|} \tag{5.6}$$

And:

$$\gamma(X_0, X_i) = \frac{1}{n}\sum_{k=1}^{n} \gamma(x_0(k), x_i(k)) \tag{5.7}$$

In this case, $\gamma(X_0, X_i)$ is the Deng's grey relational degree between $X_0$ and $X_i$, where $\xi$ is known as the distinguishing coefficient (Deng, 1985).

The Deng's grey relational degree of $\gamma(X_0, X_i)$ is commonly written as $\gamma_{0i}$, and the Deng's grey relational coefficient of $\gamma(x_0(k), x_i(k))$ as $\gamma_{0i}(k)$.

Based on Theorem 5.3.1, the computation steps of the Deng's grey relational degree between $X_0$ and $X_i$ can be accomplished as explained below.

Step 1: Calculate the initial image (or average image) of $X_0$ and $X_i$, $i = 1, 2, ..., m$, where:

$$X_i' = X_i / x_i(1) = \left(x_i'(1), x_i'(2), \ldots, x_i'(n)\right) i = 0, 1, 2, \ldots, m.$$

Step 2: Compute the difference sequences of $X_0'$ and $X_i'$,$i = 1, 2, ..., m$, and write as:

$$\Delta_i(k) = \left|x_0'(k) - x_i'(k)\right|, \quad \Delta_i = (\Delta_i(1), \Delta_i(2), \ldots, \Delta_i(n)) i = 1, 2, \ldots, m.$$

Step 3: Find the maximum and minimum differences, and denote as:

$$M = \max_i \max_k \Delta_i(k), m = \min_i \min_k \Delta_i(k).$$

Step 4: Calculate the Deng's grey relational coefficients:

$$\gamma_{0i}(k) = \frac{m + \xi M}{\Delta_i(k) + \xi M}, \quad \xi \in (0, 1) \quad k = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, m.$$

Step 5: Compute the Deng's grey relational degree:

$$\gamma_{0i} = \frac{1}{n} \sum_{k=1}^{n} \gamma_{0i}(k); \quad i = 1, 2, \ldots, m.$$

***Example 5.3.1*** Let.

$$X_0 = (x_0(1), x_0(2), x_0(3), x_0(4), x_0(5)) = (12011.65, 7568.15, 3969.87, 2630.42, 2933.20)$$
$$X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5)) = (127467, 73378, 47472, 28728, 24063)$$
$$X_2 = (x_2(1), x_2(2), x_2(3), x_2(4), x_2(5)) = (281.02, 197.78, 97.88, 55.50, 62.02)$$
$$X_3 = (x_3(1), x_3(2), x_3(3), x_3(4), x_3(5)) = (2.50, 2.65, 2.50, 2.31, 2.05)$$
$$X_4 = (x_4(1), x_4(2), x_4(3), x_4(4), x_4(5)) = (391, 423, 262, 497, 104)$$

where $X_0$ is the sequence of the regional GDP of the Suzhou, Wuxi, Changzhou, Zhenjiang and Yangzhou in Jiangsu Province in 2012, unit: 100 million yuan. And

$X_1$ is the sequence of the number of people engaged in R&D activities of the above five cities, unit: number of people.

$X_2$ is the sequence of the R&D expenditure of the above five cities, unit: 100 million yuan.

$X_3$ is the sequence of the R&D expenditure/regional GDP of the above five cities, unit: %.

$X_4$ is the sequence of the number of invention patents authorized of the above five cities, unit: number of items.

Data sources: China Statistical Yearbook 2013.

Calculate the Deng's grey relational degree between $X_i$, i = 1, 2, 3, 4 and $X_0$. (Liu, 2021).

**Solution**: Take $X_0$ as the system's behavioral characteristics sequence.

Step 1: Calculate the mean image of $X_i$, i = 0, 1, 2, 3, 4

From $X_i' = X_i/\overline{X}_i = (x_i'(1), x_i'(2), x_i'(3), x_i'(4), x_i'(5)); i = 0, 1, 2, 3, 4$, we have:

$$X_0' = X_0/\overline{X}_0 = (2.0629, 1.2998, 0.6818, 0.4518, 0.5038)$$
$$X_1' = X_1/\overline{X}_1 = (2.1166, 1.2185, 0.7883, 0.4770, 0.3996)$$
$$X_2' = X_2/\overline{X}_2 = (2.0241, 1.4245, 0.7050, 0.3997, 0.4467)$$
$$X_3' = X_3/\overline{X}_3 = (1.0408, 1.1032, 1.0408, 0.9617, 0.8535)$$
$$X_4' = X_4/\overline{X}_4 = (1.1658, 1.2612, 0.7812, 1.4818, 0.3101)$$

Step 2: Compute the difference sequences.

From $\Delta_i(k) = \left|x_0'(k) - x_i'(k)\right|$;i = 1, 2, 3, 4, it follows that:

$$\Delta_1 = (0.0531, 0.0813, 0.1065, 0.0252, 0.1042)$$
$$\Delta_2 = (0.0388, 0.1247, 0.0232, 0.0521, 0.0571)$$
$$\Delta_3 = (1.0221, 0.1966, 0.3590, 0.5099, 0.3497)$$
$$\Delta_4 = (0.8971, 0.0386, 0.0994, 1.0300, 0.1937)$$

Step 3: Find the maximum and minimum differences.

$$M = \max_i \max_k \Delta_i(k) = 1.0300$$
$$m = \min_i \min_k \Delta_i(k) = 0.0232$$

Step 4: Calculate the Deng's relational coefficients.

Let $\xi = 0.5$, it follows that:

$$\gamma_{0i}(k) = \frac{m + \xi M}{\Delta_i(k) + \xi M} = \frac{0.5382}{\Delta_i(k) + 0.5150}; i = 1, 2, 3, 4; k = 1, 2, \ldots, 5$$

Therefore:

$$\gamma_{01}(1) = 0.9474, \gamma_{01}(2) = 0.9026, \gamma_{01}(3) = 0.8660, \gamma_{01}(4) = 0.9963, \gamma_{01}(5) = 0.8692$$
$$\gamma_{02}(1) = 0.9718, \gamma_{02}(2) = 0.8413, \gamma_{02}(3) = 1.0000, \gamma_{02}(4) = 0.9490, \gamma_{02}(5) = 0.9407$$
$$\gamma_{03}(1) = 0.3501, \gamma_{03}(2) = 0.7563, \gamma_{03}(3) = 0.6158, \gamma_{03}(4) = 0.5251, \gamma_{03}(5) = 0.6224$$
$$\gamma_{04}(1) = 0.3811, \gamma_{04}(2) = 0.9722, \gamma_{04}(3) = 0.8760, \gamma_{04}(4) = 0.3483, \gamma_{04}(5) = 0.7594$$

Step 5: Compute the Deng's grey relational degrees.

$$\gamma_{01} = \frac{1}{5}\sum_{k=1}^{5}\gamma_{01}(k) = 0.9163$$

$$\gamma_{02} = \frac{1}{5}\sum_{k=1}^{5}\gamma_{02}(k) = 0.9406$$

$$\gamma_{03} = \frac{1}{5}\sum_{k=1}^{5}\gamma_{03}(k) = 0.5739$$

$$\gamma_{04} = \frac{1}{5}\sum_{k=1}^{5}\gamma_{04}(k) = 0.6674$$

According to the calculation results based on the data of five cities in Jiangsu Province, both of the R&D expenditure of $X_2$ and the number of people engaged in R&D activities of $X_1$ have great impact on the regional GDP of $X_0$. Note that both $X_2$ and $X_1$ are input factors of R&D, it can be seen that scientific and technological funds and personnel investment play an important role in regional economic development.

## 5.4 Grey Absolute Relational Degree

**Proposition 5.4.1** Let $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ be the data sequence of a system's behavior, $X_i - x_i(1)$ denote the zigzagged line $(x_i(1) - x_i(1), x_i(2) - x_i(1), \ldots, x_i(n) - x_i(1))$, and let.

$$s_i = \int_{1}^{n} (X_i - x_i(1))dt \tag{5.8}$$

Then, when $X_i$ increases, $s_i \geq 0$; when $X_i$ decreases, $s_i \leq 0$; and when $X_i$ vibrates, the sign of $s_i$ varies.

The results of Proposition 5.4.1 are represented in Fig. 5.1, where (a) shows the case where the sequence increases; (b) the situation where $X_i$ decreases; and (c) the scenario where $X_i$ vibrates.

**Definition 5.4.1** Let $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ be the data sequence of a system's behavior and $D$ the sequence operator which satisfies $X_iD = (x_i(1)d, x_i(2)d, \ldots, x_i(n)d)$ and $x_i(k)d = x_i(k) - x_i(1)$, $k = 1, 2, \ldots, n$. Then $D$ is referred to as a zero-starting point operator and $X_iD$ is the image of $X_i$. $X_iD$ is often written as $X_iD = X_i^0 = (x_i^0(1), x_i^0(2), \ldots, x_i^0(n))$.

**Proposition 5.4.2** Assume that the images of the zero-starting point of two behavioral sequences $X_i$ and $X_j$ are respectively $X_i^0 = (x_i^0(1), x_i^0(2), \ldots, x_i^0(n))$ and $X_j^0 = (x_j^0(1), x_j^0(2), \ldots, x_j^0(n))$. Let:

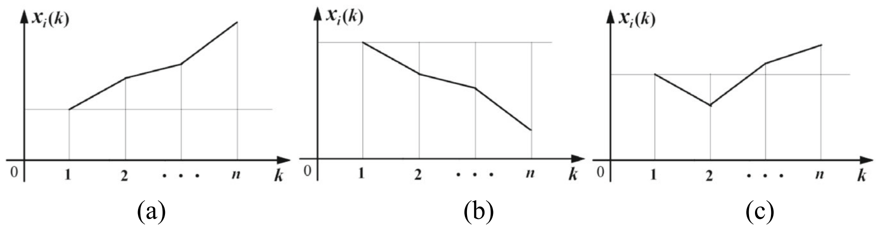



**Fig. 5.1** The zigzagged line of Proposition 5.1

$$s_i - s_j = \int_1^n (X_i^0 - X_j^0)dt; \quad (5.9)$$

and

$$S_i - S_j = \int_1^n \left(X_i - X_j\right)dt \quad (5.10)$$

Then, when $X_i^0$ is entirely located above $X_j^0$, $s_i - s_j \geq 0$; when $X_i^0$ is entirely underneath $X_j^0$, $s_i - s_j \leq 0$; and when $X_i^0$ and $X_j^0$ alternate their positions, the sign of $s_i - s_j$ is not fixed.

As shown in Fig. 5.2, when $X_i^0$ is entirely located above $X_j^0$ (Fig. 5.2a), the shaded area is positive so that $s_i - s_j \geq 0$. When $X_i^0$ and $X_j^0$ alternate their positions (Fig. 5.2b), the sign of $s_i - s_j$ is not fixed. Similarly, We can discuss the sign of $X_i$ as $s_i - s_j$.

**Definition 5.4.2** The sum of time intervals between consecutive observation values of a sequence $r_{ij}$ is called the length of $r_{ij}$.It should be noted that two sequences with the same length may not have the same number of data. For example:

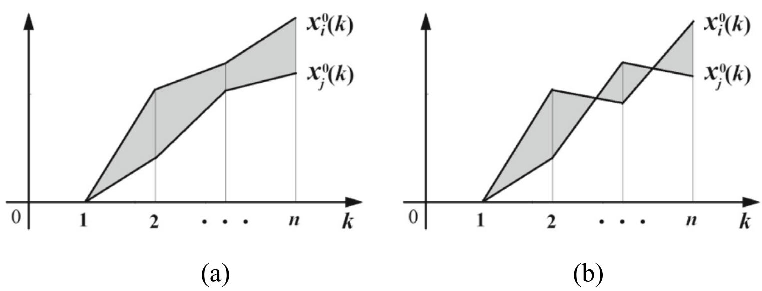



**Fig. 5.2** A description of the relationship between $X_i^0$ and $X_j^0$

$$X_1 = (x_1(1), x_1(3), x_1(6))$$
$$X_2 = (x_2(1), x_2(3), x_2(5), x_2(6))$$
$$X_3 = (x_3(1), x_3(2), x_3(3), x_3(4), x_3(5), x_3(6))$$

The lengths of $X_1, X_2, X_3$ are all 6, but $X_1$ has 3 data, $X_2$ has 4 data, and $X_3$ has 6 data.

**Definition 5.4.3** Let $X_i$ and $X_j$ be two sequences of the same length, and $s_i$ and $s_j$ are defined as above. Then, the following is referred to as the grey absolute relational degree between $X_i$ and $X_j$, or absolute relational degree for short (Liu, 1992):

$$\varepsilon_{ij} = \frac{1 + |s_i| + |s_j|}{1 + |s_i| + |s_j| + |s_i - s_j|} \tag{5.11}$$

As for sequences of different lengths, the concept of absolute relational degree can be defined by either shortening the longest sequence or by prolonging the shortest sequence using appropriate methods. This procedure will ensure that the sequences have the same length. However, by doing so, the ultimate value of the absolute relational degree will be affected.

**Proposition 5.4.3** Assume that $X_i$ and $X_j$ are two sequences with the same length. Let $X_i' = X_i - a$, $X_j' = X_j - b$, where $a, b$ are real numbers. Denote $\varepsilon_{0i}'$ as the grey absolute relational degree between $X_i'$ and $X_j'$, then $\varepsilon_{0i}' = \varepsilon_{0i}$. In fact, when $X_i$ and $X_j$ have been transformed, the values of $s_i$, $s_j$, and $\mathrm{S_i}$ - $\mathrm{S_j}$ are not changed. Therefore, the value of absolute relational degree does not change.

**Definition 5.4.4** If the time intervals of any two consecutive observation values of a sequence $X_i$ with the same length, then $X_i$ is called an equal- time-interval sequence.

**Lemma 5.4.1** Assume that $X_i$ is an equal-time-interval sequence. If the length of time-interval $l \neq 1$, then following can transform $X_i$ into an 1-time-interval sequence:

$$t : T \to T$$
$$t \mapsto t/l$$

**Lemma 5.4.2** Assume that $X_i$ and $X_j$ are 1-time-interval sequences of the same length, and the following are zero-starting point images of $X_i$ and $X_j$:

$$X_i^0 = \left(x_i^0(1), x_i^0(2), \ldots, x_i^0(n)\right)$$
$$X_j^0 = \left(x_j^0(1), x_j^0(2), \ldots, x_j^0(n)\right)$$

Then, according to Liu and Guo (1991):

$$|s_i| = \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_i^0(n)\right|$$

$$\left|s_j\right| = \left|\sum_{k=2}^{n-1} x_j^0(k) + \frac{1}{2}x_j^0(n)\right|$$

$$\left|s_i - s_j\right| = \left|\sum_{k=2}^{n-1} \left(x_i^0(k) - x_j^0(k)\right) + \frac{1}{2}\left(x_i^0(n) - x_j^0(n)\right)\right|.$$

**Theorem 5.4.1** Assume that $X_i$ and $X_j$ are two sequences with the same length, same time distances from one moment to another, and equal time moment intervals. Then, the grey absolute relational degree can also be computed as follows (Liu & Guo, 1991):

$$\begin{aligned}\varepsilon_{ij} = [1 + \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_j^0(n)\right| + \left|\sum_{k=2}^{n-1} x_j^0(k) + \frac{1}{2}x_j^0(n)\right|] \\ \times [1 + \left|\sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2}x_i^0(n)\right| + \left|\sum_{k=2}^{n-1} x_j^0(k) + \frac{1}{2}x_j^0(n)\right| \\ + \left|\sum_{k=2}^{n-1} (x_i^0(k) - x_j^0(k)) + \frac{1}{2}(x_i^0(n) - x_j^0(n))\right|]^{-1}\end{aligned}$$

***Example 5.4.1*** Calculate the absolute relational degree $X_j$ of sequences $X_0$ and $X_1$. Let sequences $X_0$ and $X_1$ be as follows:

$$X_0 = (x_0(1), x_0(2), x_0(3), x_0(4), x_0(5), x_0(7)) = (10, 9, 15, 14, 14, 16)$$
$$X_1 = (x_1(1), x_1(3), x_1(7)) = (46, 70, 98)$$

**Solution**

Step 1: Transform $X_1$ into a sequence with the same corresponding time-intervals as $X_0$.

$$x_1(2) = \frac{1}{2}(x_1(1) + x_1(3)) = \frac{1}{2}(46 + 70) = 58$$
$$x_1(5) = \frac{1}{2}(x_1(3) + x_1(7)) = \frac{1}{2}(70 + 98) = 84$$
$$x_1(4) = \frac{1}{2}(x_1(3) + x_1(5)) = \frac{1}{2}(70 + 84) = 77$$

Thus, we have a new sequence $X_1$ in place of the original $X_1$:

$$X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(7)) = (46, 58, 70, 77, 84, 98)$$

Step 2: Transform $X_0$ and $X_1$ into equal- time-interval sequences:

$$x_0(6) = \frac{1}{2}(x_0(5) + x_0(7)) = \frac{1}{2}(14 + 16) = 15$$
$$x_1(6) = \frac{1}{2}(x_1(5) + x_1(7)) = \frac{1}{2}(84 + 98) = 91$$

We have:

$$X_0 = (x_0(1), x_0(2), x_0(3), x_0(4), x_0(5), x_0(6), x_0(7)) = (10, 9, 15, 14, 14, 15, 16)$$
$$X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(6), x_1(7)) = (46, 58, 70, 77, 84, 91, 98)$$

where $X_0$ and $X_1$ are 1- time-interval sequences.

Step 3: Compute the zero-starting point images of sequences $X_0$ and $X_1$.

$$X_0^0 = \left(x_0^0(1), x_0^0(2), x_0^0(3), x_0^0(4), x_0^0(5), x_0^0(6), , x_0^0(7)\right) = (0, -1, 5, 4, 4, 5, 6)$$
$$X_1^0 = \left(x_1^0(1), x_1^0(2), x_1^0(3), x_1^0(4), x_1^0(5), x_1^0(6), , x_1^0(7)\right) = (0, 12, 24, 31, 38, 45, 52)$$

Step 4: Calculate $|s_0|$, $|s_1|$, $|s_1 - s_0||s_0|$, $|s_1|$, $|s_1 - s_0|$

$$|s_0| = \left|\sum_{k=2}^{6} x_0^0(k) + \frac{1}{2}x_0^0(7)\right| = 20$$
$$|s_1| = \left|\sum_{k=2}^{6} x_1^0(k) + \frac{1}{2}x_1^0(7)\right| = 176$$
$$|s_1 - s_0| = \left|\sum_{k=2}^{6} \left(x_1^0(k) - x_0^0(k)\right) + \frac{1}{2}\left(x_1^0(7) - x_0^0(7)\right)\right| = 156$$

Step 5: Compute the grey absolute relational degree $\varepsilon_{01}$ of sequences $X_0$ and $X_1$.

$$\varepsilon_{01} = \frac{1 + |s_0| + |s_1|}{1 + |s_0| + |s_1| + |s_1 - s_0|} = \frac{197}{353} \approx 0.5581.$$

**Theorem 5.4.2** The grey absolute relational degree $\varepsilon_{ij}$ satisfies the following properties:

(1) $0 < \varepsilon_{ij} \leq 1$;
(2) $\varepsilon_{ij}$ is only related to the geometric shapes of $X_i$ and $X_j$, and has no relationship with the spatial positions of these sequences;
(3) Any two sequences are not absolutely unrelated. That is, $\varepsilon_{ij}$ never equals zero;
(4) The more $X_i$ and $X_j$ are geometrically similar, the greater $\varepsilon_{ij}$ is;
(5) If $X_i$ and $X_j$ are parallel or $X_i^0$ fluctuates around $X_j^0$, with the area of the parts of $X_i^0$ located above $X_j^0$ equal to that of the parts with $X_i^0$ located underneath $X_j^0$, then $\varepsilon_{ij} = 1$;

(6) When one of the observed values of $X_i$ and $X_j$ change, $\varepsilon_{ij}$ also changes accordingly;
(7) When the lengths of $X_i$ and $X_j$ change, $\varepsilon_{ij}$ also changes;
(8) $\varepsilon_{jj} = \varepsilon_{ii} = 1$; and
(9) $\varepsilon_{ij} = \varepsilon_{ji}$.

## 5.5 Grey Relative and Synthetic Relational Degree

### *5.5.1 Relative Grey Relational Degree*

**Definition 5.5.1** Let $X_i$ and $X_j$ be sequences of the same length with non-zero initial values, and $X_i'$ and $X_j'$ the initial images of $X_i$ and $X_j$, respectively. The grey absolute relational degree of $X_i'$ and $X_j'$ is referred to as the relative grey relational degree of $X_i$ and $X_j$, denoted $r_{ij}$ (Liu, 1992). This relative relational degree is a quantitative representation of the relationship between the rates of change of sequences $X_i$ and $X_j$, relative to their initial values. The closer the rates of change of $X_i$ and $X_j$ are, the greater $r_{ij}$ is, and vice versa.

**Proposition 5.5.1** Let $X_i$ be a sequence with a non-zero initial value. If $X_j = cX_i$. If $c > 0$ is a constant, then $r_{ij} = 1$.

***Proof*** Assume that $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$,then:

$$X_j = (x_j(1), x_j(2), \ldots, x_j(n)) = (cx_i(1), cx_i(2), \ldots, cx_i(n)).$$

The initial images of $X_i$ and $X_j$ are as follows:

$$\begin{aligned}
X_i' &= X_i/x_i(1) = (\frac{x_i(1)}{x_i(1)}, \frac{x_i(2)}{x_i(1)}, \ldots, \frac{x_i(n)}{x_i(1)}) \\
X_j' &= X_j/x_j(1) = (\frac{x_j(1)}{x_j(1)}, \frac{x_j(2)}{x_j(1)}, \ldots, \frac{x_j(n)}{x_j(1)}) \\
&= (\frac{cx_i(1)}{cx_i(1)}, \frac{cx_i(2)}{cx_i(1)}, \ldots, \frac{cx_i(n)}{cx_i(1)} = (\frac{x_i(1)}{x_i(1)}, \frac{x_i(2)}{x_i(1)}, \ldots, \frac{x_i(n)}{x_i(1)}).
\end{aligned}$$

Therefore, $X_j' = X_i'$, so $r_{ij} = 1$.

**Proposition 5.5.2** Let $X_i$ and $X_j$ be two sequences of the same length with non-zero initial values. Additionally, the relative grey relational degree of $r_{ij}$ and the grey absolute relational degree of $\varepsilon_{ij}$ do not have any connections. When $\varepsilon_{ij}$ is relatively large, $r_{ij}$ can be very small; when $\varepsilon_{ij}$ is very small, $r_{ij}$ can also be very large.

**Proposition 5.5.3** Let $X_i$ and $X_j$ be two sequences of the same length with non-zero initial values. Then, for any non-zero constants $a$ and $b$, the relative grey relational degree $r_{ij}'$ between $aX_i$ and $bX_j$ is the same as the $r_{ij}$ of $X_i$ and $X_j$.

In fact, the initial images of a $X_i$ and b $X_j$ are equal to those of $X_i$ and $X_j$, respectively. Thus, scalar multiplication does not act in any way under the function of initialing operators. Hence, $r'_{ij} = r_{ij}$.

***Example 5.4.2*** Calculate the relative grey relational degree $r_{01}$ for sequences $X_0$ and $X_1$ of Example 5.4.1.

**Solution**

Step1: Transform $X_1$ and $X_0$ into the same 1-time-interval sequences.

$$X_0 = (x_0(1), x_0(2), x_0(3), x_0(4), x_0(5), x_0(6), x_0(7)) = (10, 9, 15, 14, 14, 15, 16)$$
$$X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(6), x_1(7)) = (46, 58, 70, 77, 84, 91, 98)$$

Step 2: Calculate the initial images of sequences $X_0$ and $X_1$.

$$X'_0 = (1, 0.9, 1.5, 1.4, 1.4, 1.5, 1.6)$$
$$X'_1 = (1, 1.26, 1.52, 1.67, 1.83, 1.98, 2.13)$$

Step 3: Compute the zero-starting point images of sequences $X'_0$ and $X_1\prime$.

$$X_0^{\prime 0} = (x_0^{\prime 0}(1), x_0^{\prime 0}(2), x_0^{\prime 0}(3), x_0^{\prime 0}(4), x_0^{\prime 0}(5), x_0^{\prime 0}(6), x_0^{\prime 0}(7)) = (0, -0.1, 0.5, 0.4, 0.4, 0.5, 0.6)$$
$$X_1^{\prime 0} = (x_1^{\prime 0}(1), x_1^{\prime 0}(2), x_1^{\prime 0}(3), x_1^{\prime 0}(4), x_1^{\prime 0}(5), x_1^{\prime 0}(6), x_1^{\prime 0}(7)) = (0, 0.26, 0.52, 0.67, 0.83, 0.98, 1.13)$$

Step 4: Calculate $|s'_0|, |s'_1|, |s'_1 - s'_0|$.

$$|s'_0| = \left| \sum_{k=2}^{6} x_0^{\prime 0}(k) + \frac{1}{2} x_0^{\prime 0}(7) \right| = 2$$
$$|s'_1| = \left| \sum_{k=2}^{6} x_1^{\prime 0}(k) + \frac{1}{2} x_1^{\prime 0}(7) \right| = 3.828$$
$$|s'_1 - s'_0| = \left| \sum_{k=2}^{6} (x_1^{\prime 0}(k) - x_0^{\prime 0}(k)) + \frac{1}{2} (x_1^{\prime 0}(7) - x_0^{\prime 0}(7)) \right| = 1.925$$

Step 5: Calculate the relative grey relational degree of $r_{01}$.

$$r_{01} = \frac{1 + |s'_0| + |s'_1|}{1 + |s'_0| + |s'_1| + |s'_1 - s'_0|} = \frac{6.825}{8.75} \approx 0.78$$

**Theorem 5.5.1** The relative grey relational degree of $r_{ij}$ satisfies the following properties:

(1) $0 < r_{ij} \le 1$;

(2) The value of $r_{ij}$ relates only the rates of change of the sequences $X_i$ and $X_j$ with respect to their individual initial values. It does not relate to the magnitudes of other entries. In other words, scalar multiplication does not change the value of relative grey relational degree;
(3) The rates of change of any two sequences are somehow related. That is, $r_{ij}$ is never zero;
(4) The closer the individual rates of change of $X_i$ and $X_j$ with respect to their initial values, the greater the $r_{ij}$;
(5) If $X_j = aX_i$, or when the images of zero initial points of the initial images of $X_i$ and $X_j$ satisfy that $X_i^{'0}$ fluctuates around $X_j^{'0}$, and if the area of the parts where $X_i^{'0}$ is located above $X_j^{'0}$ equals that of the parts where $X_i^{'0}$ is located underneath $X_j^{'0}$, then $r_{ij} = 1$;
(6) When an entry in $X_i$ or $X_j$ is changed, $r_{ij}$ will change accordingly;
(7) When the length of $X_i$ or $X_j$ is changed, $r_{ij}$ also changes;
(8) $r_{jj} = r_{ii} = 1$; and
(9) $r_{ij} = r_{ji}$.

### *5.5.2 Grey Synthetic Relational Degree*

**Definition 5.5.2** Let $X_i$ and $X_j$ be sequences of the same length with non-zero initial entries, $\varepsilon_{ij}$ and $r_{ij}$ be respectively the absolute and relative relational degrees between $X_i$ and $X_j$, and $\theta \in [0, 1]$. Then the following is referred to as the grey synthetic relational degree between $X_i$ and $X_j$ (Liu, 1992):

$$\rho_{ij} = \theta\varepsilon_{ij} + (1 - \theta)r_{ij}. \tag{5.12}$$

The concept of grey synthetic relational degree reflects the degree of similarity between the zigzagged lines of $X_i$ and $X_j$, and the closeness between the rates of change of $X_i$ and $X_j$ with respect to their individual initial values. It is an index that describes relatively completely the closeness relationship between sequences. In general, we take $\theta = 0.5$. If the focus of a study is the relationship between relevant absolute quantities, $\theta$ can take a greater value than 0.5. On the other hand, if the focus is more on comparison between rates of change, then $\theta$ can take a smaller value than 0.5.

***Example 5.4.3*** Calculate the synthetic grey relational degree of $\rho_{01}$ for sequences $X_0$ and $X_1$ of Example 5.4.1.

**Solution** From Examples 5.4.1 and 5.4.2, we have $X_i$ and $X_j$. If $\rho_{ij}$:

$$\rho_{01} = \theta\varepsilon_{01} + (1 - \theta)r_{01} = 0.5 \times 0.5581 + 0.5 \times 0.78 \approx 0.669.$$

We can obtain different $\rho_{01}$ values if we take $\theta = 0.2, 0.3, 0.4, 0.6, 0.8$, respectively (see Table 5.1).

**Table 5.1 The** values of $\rho_{01}$ with different $\theta$

| $\theta$ | 0.2 | 0.3 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| $\rho_{01}$ | 0.73562 | 0.71343 | 0.69124 | 0.64686 | 0.60248 |

**Theorem 5.5.2** The grey synthetic relational degree of $\rho_{ij}$ satisfies the following properties:

(1) $0 < \rho_{ij} \leq 1$;
(2) The value of $\rho_{ij}$ relates to the individual observed values of sequences $X_i$ and $X_j$,as well as to the rates of change of these values with respect to their initial values;
(3) $\rho_{ij}$ will never be zero;
(4) $\rho_{ij}$ changes along with the values in $X_i$ and $X_j$;
(5) When the lengths of $X_i$ and $X_j$ change, so does $\rho_{ij}$;
(6) With different $\theta$ value, $\rho_{ij}$ also varies;
(7) When $\theta = 1$, $X_j$; when $\theta = 0$, $\rho_{ij} = r_{ij}$;
(8) $\rho_{jj} = \rho_{ii} = 1$; and
(9) $\rho_{ij} = \rho_{ji}$.

## 5.6 Grey Similarity, Closeness and Three-Dimensional Relational Degree

This section focuses on the new models which measure mutual influences and connections between sequences from two different angles: similarity and closeness. These new models are much easier to apply to practical problems than original model. Also, three-dimensional grey relational degree can be used to analyze the relationship among curved surfaces in three-dimensional space and this is discussed next.

### *5.6.1 Grey Relational Analysis Models Based on Similarity and Closeness*

**Definition 5.6.1** Let $X_i$ and $X_j$ be sequences of the same length, and $s_i - s_j$ the same as defined in Proposition 5.4.2. Then, the following formula (5.13) is referred to as the grey similitude relational degree between $X_i$ and $X_j$ (Liu et al., 2011):

$$\varepsilon_{ij} = \frac{1}{1 + |s_i - s_j|} \tag{5.13}$$

The concept of similitude relational degree is employed to measure the geometric similarity of the shapes of sequences $X_i$ and $X_j$. The more similar the geometric shapes of $X_i$ and $X_j$, the greater the value of $\varepsilon_{ij}$, and vice versa.

**Definition 5.6.2** Let $X_i$ and $X_j$ be sequences of the same length, and $S_i - S_j$ the same as defined in Proposition 5.4.2. Then, the following formula (5.14) is referred to as the grey closeness relational degree between $X_i$ and $X_j$(Liu et al., 2011):

$$\rho_{ij} = \frac{1}{1 + |S_i - S_j|} \tag{5.14}$$

The concept of the grey closeness relational degree is employed to measure the spatial closeness of sequences $X_i$ and $X_j$. The closer the $X_i$ and $X_j$ sequences, the greater the value of $\rho_{ij}$, and vice versa.

**Proposition 5.6.1** Let $X_i$ and $X_j$ be sequences of 1-time-intervals with the same length. Then:

$$\left|S_i - S_j\right| = \left|\frac{1}{2}\left[x_i(1) - x_j(1)\right] + \sum_{k=2}^{n-1}\left[x_i(k) - x_j(k)\right] + \frac{1}{2}\left[x_i(n) - x_j(n)\right]\right| \tag{5.15}$$

It should be noted that the concept of the grey closeness relational degree is only meaningful when sequences $X_i$ and $X_j$ possess similar meanings and identical units. Otherwise, it does not stand for any practical significance.

**Theorem 5.6.1** The grey similitude relational degree of $\varepsilon_{ij}$ satisfies the following properties:

(1) $0 < \varepsilon_{ij} \leq 1$;
(2) The value of $\varepsilon_{ij}$ is determined only by the geometric shape of sequences $X_i$ and $X_j$ without any relationship with their relative spatial positions. In other words, the transform translation of $X_i$ and $X_j$ will not change the value of $\varepsilon_{ij}$;
(3) The more geometrically similar the sequences $X_i$ and $X_j$, the greater the value of $\varepsilon_{ij}$, and vice versa;
(4) If $X_i$ and $X_j$ are parallel, or when $X_i^0$ fluctuates around $X_j^0$, and the area of the parts where $X_i^0$ is located above $X_j^0$ equals that of the parts where $X_i^0$ is located beneath $X_j^0$, then $\varepsilon_{ij} = 1$;
(5) $\varepsilon_{ii} = 1$, $\varepsilon_{jj} = 1$; and
(6) $\varepsilon_{ij} = \varepsilon_{ji}$.

**Theorem 5.6.2** The grey closeness relational degree of $\rho_{ij}$ satisfies the following properties:

(1) $0 < \rho_{ij} \leq 1$;
(2) The value of $\rho_{ij}$ is determined not only by the geometric shape of sequences $X_i$ and $X_j$, but also by their relative spatial positions. In other words, the transform translation of $X_i$ and $X_j$ will change the value of $\rho_{ij}$;

(3) The closer the sequences $X_i$ and $X_j$, the greater the $\rho_{ij}$ value, and vice versa;
(4) If $X_i$ and $X_j$ coincide, or $X_i$ fluctuates around $X_j$, and the area of the parts where $X_i$ is located above $X_j$ equals that of the parts where $X_i$ is located beneath $X_j$, then $\rho_{ij} = 1$;
(5) $\rho_{ii} = 1$, $\rho_{jj} = 1$; and
(6) $\rho_{ij} = \rho_{ji}$.

***Example 5.6.1*** Compute the grey similitude relational degrees of $\varepsilon_{12}, \varepsilon_{13}$ and the grey closeness relational degrees of $\rho_{12}, \rho_{13}$ between $X_1$ and $X_2$, $X_3$, respectively, given the sequences below:

$$X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(7)) = (0.91, 0.97, 0.90, 0.93, 0.91, 0.95)$$
$$X_2 = (x_2(1), x_2(2), x_2(3), x_2(5), x_2(7)) = (0.60, 0.68, 0.61, 0.63, 0.65)$$
$$X_3 = (x_3(1), x_3(3), x_3(7)) = (0.82, 0.90, 0.86)$$

**Solution**

Step 1: Let us translate both $X_2$ and $X_3$ into sequences with the same time intervals as $X_1$. To this end, consider the following:

$$x_2(4) = \frac{1}{2}(x_2(3) + x_2(5)) = \frac{1}{2}(0.61 + 0.63) = 0.62$$
$$x_3(2) = \frac{1}{2}(x_3(1) + x_3(3) = \frac{1}{2}(0.82 + 0.90) = 0.86$$
$$x_3(5) = \frac{1}{2}(x_3(3) + x_3(7)) = \frac{1}{2}(0.90 + 0.86) = 0.88$$
$$x_3(4) = \frac{1}{2}(x_3(3) + x_3(5)) = \frac{1}{2}(0.90 + 0.88) = 0.89$$

Thus, we have:

$$X_2 = (x_2(1), x_2(2), x_2(3), x_2(4), x_2(5), x_2(7)) = (0.60, 0.68, 0.61, 0.62, 0.63, 0.65)$$
$$X_3 = (x_3(1), x_3(2), x_3(3), x_3(4), x_3(5), x_3(7)) = (0.82, 0.86, 0.90, 0.89, 0.88, 0.86)$$

Step 2: Let us translate $X_1$, $X_2$, and $X_3$ into sequences of equal time distance. To this end:

$$x_1(6) = \frac{1}{2}(x_1(5) + x_1(7)) = \frac{1}{2}(0.91 + 0.95) = 0.93$$
$$x_2(6) = \frac{1}{2}(x_2(5) + x_2(7)) = \frac{1}{2}(0.63 + 0.65) = 0.64$$
$$x_3(6) = \frac{1}{2}(x_3(5) + x_3(7)) = \frac{1}{2}(0.88 + 0.86) = 0.87$$

Therefore, the following sequences are all 1-time distance, which means that the time distances between consecutive entries are all 1.

$$\begin{aligned} X_1 &= (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5), x_1(7)) \\ &= (0.91, 0.97, 0.90, 0.93, 0.91, 0.93, 0.95) \\ X_2 &= (x_2(1), x_2(2), x_2(3), x_2(4), x_2(5), x_2(7)) \\ &= (0.60, 0.68, 0.61, 0.62, 0.63, 0.64, 0.65) \\ X_3 &= (x_3(1), x_3(2), x_3(3), x_3(4), x_3(5), x_3(7)) \\ &= (0.82, 0.86, 0.90, 0.89, 0.88, 0.87, 0.86) \end{aligned}$$

Step 3: Compute the images of zero-starting points provided below.

$$\begin{aligned} X_1^0 &= (x_1^0(1), x_1^0(2), x_1^0(3), x_1^0(4), x_1^0(5), x_1^0(6), x_1^0(7)) \\ &= (0, 0.06, -0.01, 0.02, 0, 0.02, 0.04) \\ X_2^0 &= (x_2^0(1), x_2^0(2), x_2^0(3), x_2^0(4), x_2^0(5), x_2^0(6), x_2^0(7)) \\ &= (0, 0.08, 0.01, 0.02, 0.03, 0.04, 0.05) \\ X_3^0 &= (x_3^0(1), x_3^0(2), x_3^0(3), x_3^0(4), x_3^0(5), x_3^0(6), x_3^0(7)) \\ &= (0, 0.04, 0.08, 0.07, 0.06, 0.05, 0.04) \end{aligned}$$

Step 4: Compute $|s_1 - s_2|$, $|s_1 - s_3|$ and $|S_1 - S_2|$, $|S_1 - S_3|$ as follows.

$$|s_1 - s_2| = \left| \sum_{k=2}^{6} (x_1^0(k) - x_2^0(k)) + \frac{1}{2}(x_1^0(7) - x_2^0(7)) \right| = 0.095$$

$$|s_1 - s_3| = \left| \sum_{k=2}^{6} (x_1^0(k) - x_3^0(k)) + \frac{1}{2}(x_1^0(7) - x_3^0(7)) \right| = 0.21$$

$$|S_1 - S_2| = \left| \sum_{k=2}^{6} (x_1(k) - x_2(k)) + \frac{1}{2}(x_1(7) - x_2(7)) \right| = 1.91$$

$$|S_1 - S_3| = \left| \sum_{k=2}^{6} (x_1(k) - x_3(k)) + \frac{1}{2}(x_1(7) - x_3(7)) \right| = 0.375$$

Step 5: Calculate the similitude relational degrees of $\varepsilon_{12}, \varepsilon_{13}$ and closeness relational degrees of $\rho_{12}, \rho_{13}$.

$$\varepsilon_{12} = \frac{1}{1 + |s_1 - s_2|} = 0.91, \ \varepsilon_{13} = \frac{1}{1 + |s_1 - s_3|} = 0.83$$

$$\rho_{12} = \frac{1}{1 + |S_1 - S_2|} = 0.34, \ \rho_{13} = \frac{1}{1 + |S_1 - S_3|} = 0.73$$

Because $\varepsilon_{12} > \varepsilon_{13}$, it follows that $X_2$ is more similar to $X_1$ than $X_3$. Because $\rho_{12} < \rho_{13}$, it follows that $X_3$ is closer to $X_1$ than $X_2$.

Please note that the grey relational analysis focus on relevant order relationship and influence between sequences rather than the value of the grey relational degree. For instance, let us assume that one must compute the similitude relational degrees or closeness relational degrees based on Eqs. (5.13) or (5.14). When the absolute values of the sequence data are relatively large, the values of both $\left|s_i - s_j\right|$ and $\left|S_i - S_j\right|$ might be large, too, which in turn leads to the resultant similitude and closeness relational degrees being relatively small. This scenario does not have any substantial impact on the analysis of order relationships. If a particular problem demands relatively large numerical magnitudes in the value of grey relational degree, one can replace the number 1 appearing in the numerators or denominators of Eqs. (5.13) and (5.14) by a relevant constant, or use the grey absolute relational degree, or use other appropriate models.

## *5.6.2 Three Dimensional Grey Relational Analysis Models*

The above GRA models can be generalized to three-dimensional space based on geometric descriptions of a behavior matrix.

**Definition 5.6.3** Assume that $X$ is a two-dimensional system factor, and $a_{ij}$ is an observation value of the system's behavior at two-dimensional point ($i, j$), where $1 \leq i \leq m; 1 \leq j \leq n$. Then the following expression is called the behavior matrix of system factor $X$:

$$A = (a_{ij})_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

For example, if the prices (e.g., opening prices, closing prices, maximum prices, or minimum prices) of a share have been recorded on different dates, we can obtain the behavior matrix of the different prices $X$ of the share. The behavior matrix will reduce to a behavior sequence if only one share price has been recorded on different dates.

The scatter diagram in behavior matrix A and the corresponding behavioral curved surface in three-dimensional space are shown in Figs. 5.3 and 5.4.

**Definition 5.6.4** Assume the following behavior matrix of system factor $X$.

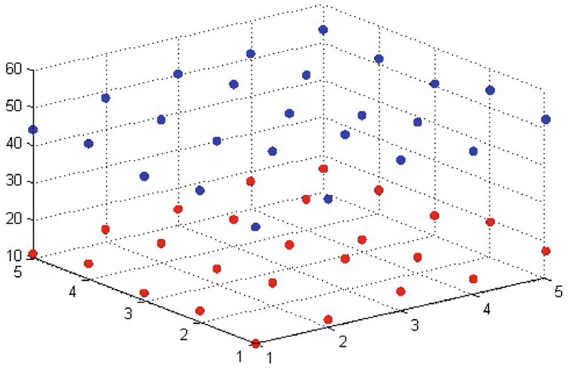



**Fig. 5.3** The scatter diagram as behavior matrix

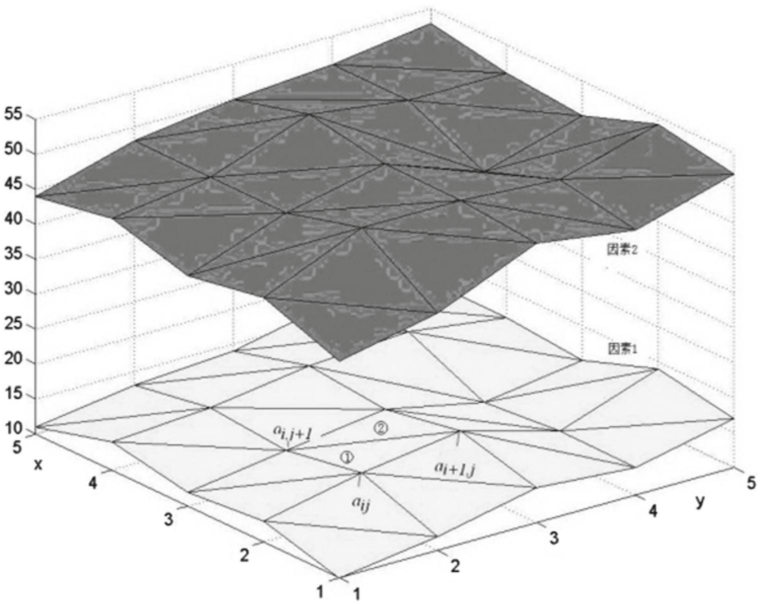



**Fig. 5.4** The corresponding behavioral curved surface

$$A = (a_{ij})_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$AD = (a_{ij}d)_{m\times n}$, where D is a matrix operator, $a_{ij}d = a_{ij} - a_{1j}$, then D is called a zero-starting edge operator, $AD$ is called the zero-starting edge image of A, and they are denoted as $AD = A^0 = (a^0_{ij})_{m\times n}$.

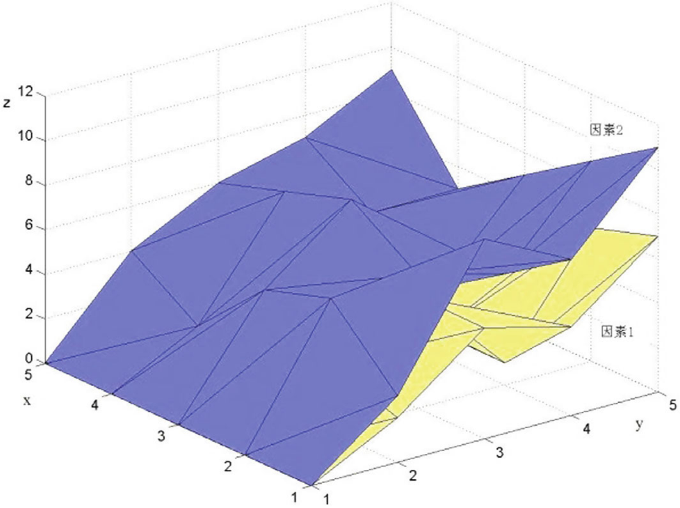



**Fig. 5.5** The zero-starting edge image of a behavior curved surface

The zero-starting edge curved surface of $A$ is shown in Fig. 5.5.

**Definition 5.6.5** Assume that behavior matrices $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$ are matrices of the same type. Then the following formula is called the three-dimensional grey absolute relational degree between A and B (Zhang & Liu, 2009):

$$\varepsilon_{ab} = \frac{1 + |s_a| + |s_b|}{1 + |s_a| + |s_b| + |s_a - s_b|} \tag{5.16}$$

This occurs when $s_a = {}_{D_a} A^0 dxdy$, $s_b = {}_{D_b} B^0 dxdy$, $s_a - s_b = {}_{D_{ab}} (A^0 - B^{0)} dxdy . s_a = {}_{D_a} A^0 dxdy$, $s_b = {}_{D_b} B^0 dxdy$, $s_a - s_b = {}_{D_{ab}} (A^0 - B^{0)} dxdy.$

Formula (5.16) looks similar to the absolute GRA model shown in formula (5.11). However, the meaning is different. The meaning of $|s_i|$, $|s_j|$, $|s_i - s_j|$ in formula (5.11) is the area of curved edge trapezoids surrounded by axis $X_i^0$, $X_j^0$, the zero-starting point curves, and the area of curved edge trapezoid surrounded by $X_i^0$ and $X_j^0$. However, the meaning of $|s_p|$, $|s_q|$, $|s_p - s_q|$ in formula (5.16) is the volume of curved roof cylinders surrounded by the axis plane and $A^0$, $B^0$, the curved surface of zero-starting edge, and the volume of curved roof cylinders surrounded by $A^0$ and $B^0$.

The three-dimensional grey relational analysis model can truly reflect the relational degree between system behavior matrices. The analysis results are objective, reliable and easy to implement on computer. The three-dimensional GRA model is seen to have expansive application prospects in many fields such as multi-criterion decision-making, panel data analysis, image processing, among others, which include matrices as objects of study.

## 5.7 Negative Grey Relational Analysis Models

In the past 40 years, driven by the realistic demand of measuring the relationship between the reverse sequences of the system, many scholars have made unremitting attempts and exploration around the construction of negative grey relational analysis model. In 2008, Shi Hongxing, Liu Sifeng, and Fang Zhigeng proposed a kind of grey relational analysis model referring to the grey absolute relational analysis model. In this paper, the positive and negative sign of the grey relational degree is determined according to the concave and convex direction of the periodic waveform to describe the inverse relationship between the periodic factors (Shi et al., 2008). In 2015, based on dissolved gas analysis (DGA), Song Bin et al. studied the latent fault diagnosis of power transformer. In order to correctly describe the reverse change relationship between different fault types, a calculation method of negative grey relational degree is proposed (Song et al., 2015). In 2019, Saad Ahmed Javed and Sifeng Liu proposed a bidirectional gabsolute GRA model for uncertain systems. The proposed model can be used to evaluate both positive and negative relation of different sequences (Javed & Liu, 2019).

Firstly, the definition of inverse sequence will be given in this section. Then, several different negative grey relational analysis models, such as negative grey similarity relational analysis model, negative grey absolute relational analysis model, negative relative grey relational analysis model, negative grey synthetic relational analysis model, and negative Deng's grey relational analysis model will be put forward based on the corresponding common grey relational analysis models. The properties of the new models will be studied.

In order to build a negative grey relational model, it is necessary to give the definition of inverse sequence at first.

**Definition 5.7.1** Assume that $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$.
is a system's behavior data sequence,

(1) If $\forall k = 2, 3, \ldots, n, x_i(k) - x_i(k-1) > 0$, then $X_i$ is referred to as a monotonic increasing sequence;
(2) If the inequality sign in (1) is inversed, then $X_i$ is referred to as a monotonic decreasing sequence.

Monotonic increasing sequence and monotonic decreasing sequence are collectively referred to as monotone sequence. Please see Fig. 5.6 for the curves of monotonic increasing sequence and monotonic decreasing sequences.

**Definition 5.7.2** Assume that $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$.

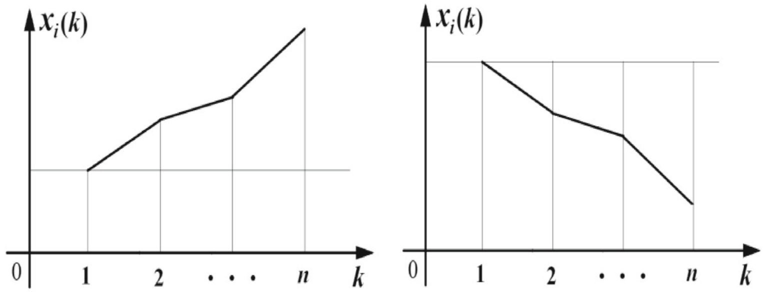



**Fig. 5.6** Monotone sequence curves

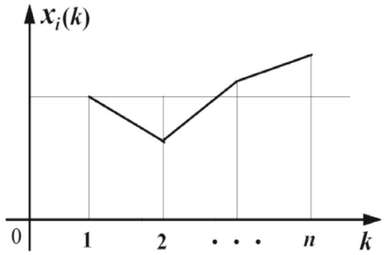



**Fig. 5.7** Oscillation sequence curve

is a system's behavior data sequence, if there are $k$, $k' \in \{2, 3, \ldots, n\}$ such that $x(k) - x(k-1) > 0, x(k') - x(k'-1) < 0$, then $X$ is referred to as an oscillation sequence.

Figure 5.7 shows the case of a curve of oscillation sequence.

**Definition 5.7.3** Assume that $X_i = (x_i(1), x_i(2), \ldots, x_i(n))$ is a system's behavior data sequence, $X_i^0 = X_i - x_i(1)$ is the zero-starting point sequence of $X_i$, let $s_i = \int_1^n (X_i - x_i(1))dt$, then.

(1) If $s_i > 0$, then $X_i$ is referred to as an increasing sequence;
(2) If $s_i < 0$, then $X_i$ is referred to as a decreasing sequence;
(3) If $s_i = 0$, then $X_i$ is referred to as a horizontal sequence.

Obviously, monotonic increasing sequence is a special case of increasing sequence and monotonic decreasing sequence is a special case of decreasing sequence. An oscillation sequence can be an increasing sequence, decreasing sequence, or a horizontal sequence. And stationary sequence is a special case of horizontal sequence.

**Definition 5.7.4** Assume that.

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences.

(1) When both $X_i, X_j$ are increasing sequences or decreasing sequences, then $X_i$ and $X_j$ are called sequences with the same direction;
(2) When one of $X_i$ and $X_j$ is an increasing sequence and the other is a decreasing sequence, then $X_i$ and $X_j$ are called reverse sequences (Liu, 2022).

The relationship between two sequences with the same direction can be measured by positive grey relational analysis model. The relationship between two reverse sequences needs to be measured by negative grey relational analysis model.

**Proposition 5.7.1** Assume that.

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences. The zero-starting point sequences of $X_i$ and $X_j$ as follows,

$$X_i^0 = (x_i^0(1), x_i^0(2), \ldots, x_i^0(n))$$
$$X_j^0 = (x_j^0(1), x_j^0(2), \ldots, x_j^0(n))$$

Let

$$s_i = \int_1^n (X_i - x_i(1))dt \tag{5.17}$$

$$s_i - s_j = \int_1^n (X_i^0 - X_j^0)dt \tag{5.18}$$

then

$$|s_i| = \left| \sum_{k=2}^{n-1} x_i^0(k) + \frac{1}{2} x_i^0(n) \right| \tag{5.19}$$

$$\left|s_i - s_j\right| = \left| \sum_{k=2}^{n-1} (x_i^0(k) - x_j^0(k)) + \frac{1}{2}(x_i^0(n) - x_j^0(n)) \right| \tag{5.20}$$

***Proof*** $|s_i|$ and $\left|s_i - s_j\right|$ are determined by areas of the following curved triangles, respectively.

$$\mathrm{X} = 0,\ \mathrm{X} = X_i^0,\ \text{and}\, \mathrm{t} = \mathrm{n}$$
$$\mathrm{X} = X_i^0,\ \mathrm{X} = X_j^0,\ \mathrm{t} = \mathrm{n}$$

They are sums of little areas of n-1 small trapezoids of height 1. Note the length of the bottom edges of the small trapezoids, and it is easy to know that the conclusion is true (Liu, 2022).

**Proposition 5.7.2** Assume that $X_i$, $X_j$, $X_i^0$, $X_j^0$, and $|s_i|$,$\left|s_i - s_j\right|$ as shown in Proposition 1, then.

(1) When $X_i$ and $X_j$ are with the same direction, and $X_i^0$, $X_j^0$ intersect only at the starting point, then $\left|s_i - s_j\right| = \left||s_i| - \left|s_j\right|\right|$;
(2) When $X_i$ and $X_j$ are two reverse sequences, then $\left|s_i - s_j\right| = |s_i| + \left|s_j\right|$;
(3) If $X_i^0$ fluctuates around $X_j^0$, then $\left|s_i - s_j\right|$ is the absolute value of algebraic sum of area enclosed by $X_i^0$ and $X_j^0$. The parts where $X_i^0$ are above $X_j^0$ take positive sign, and the parts where $X_i^0$ are underneath $X_j^0$ take negative sign.

Positional relationship of $X_i^0$ and $X_j^0$ can be clearly seen from Fig. 5.3. Figure 5.3a shows the case where $X_i$ and $X_j$ are both increasing sequences, Fig. 5.3b shows the case where $X_i$ and $X_j$ are both increasing sequences, Fig. 5.3c shows the case where $X_i$ and $X_j$ are two reverse sequences, and Fig. 5.3d shows the case where $X_i^0$ fluctuates around $X_j^0$ (Liu, 2022).

It can be seen from Fig. 5.8c, when $X_i$ and $X_j$ are reverse sequences, the value of $\left|s_i - s_j\right|$ is large. At this time, the value of grey relational degree calculated by positive grey relational model will be very small. Before the negative grey relational analysis model was proposed, people usually convert the inverse sequence into the same direction sequence through inverse operator or reciprocal operator, then calculate the positive grey relational degree of the sequences with the same direction, but the results are not completely reasonable.

Therefore, for the measurement of the relationship between reverse sequences, the construction of negative grey relational analysis model has become an inevitable choice.

Corresponding to the normalization and proximity axioms of the grey relational analysis model, the negative grey relational degree $\phi_{ij}^N$ shall meet the following axioms.

**Axiom 5.7.1 Normalization**

$$-1 < \phi_{ij}^N \leq 0, \phi_{ij}^N = 0 \Leftarrow X_i = X_j$$

The value of $\phi_{ij}^N$ is negative. The minimum value is—1 and the maximum value is 0 (Liu, 2022).

**Axiom 5.7.2 Reversibility** The stronger the inverse relation between $X_i$ and $X_j$,the smaller the value of $\phi_{ij}^N$.

Note that the value of negative grey relational degree belongs to interval (− 1,0], the smaller the value of $\phi_{ij}^N$,the greater the absolute value of $\phi_{ij}^N$ (Liu, 2022).

**Definition 5.7.5** Suppose the following system's behavior data sequences.

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$

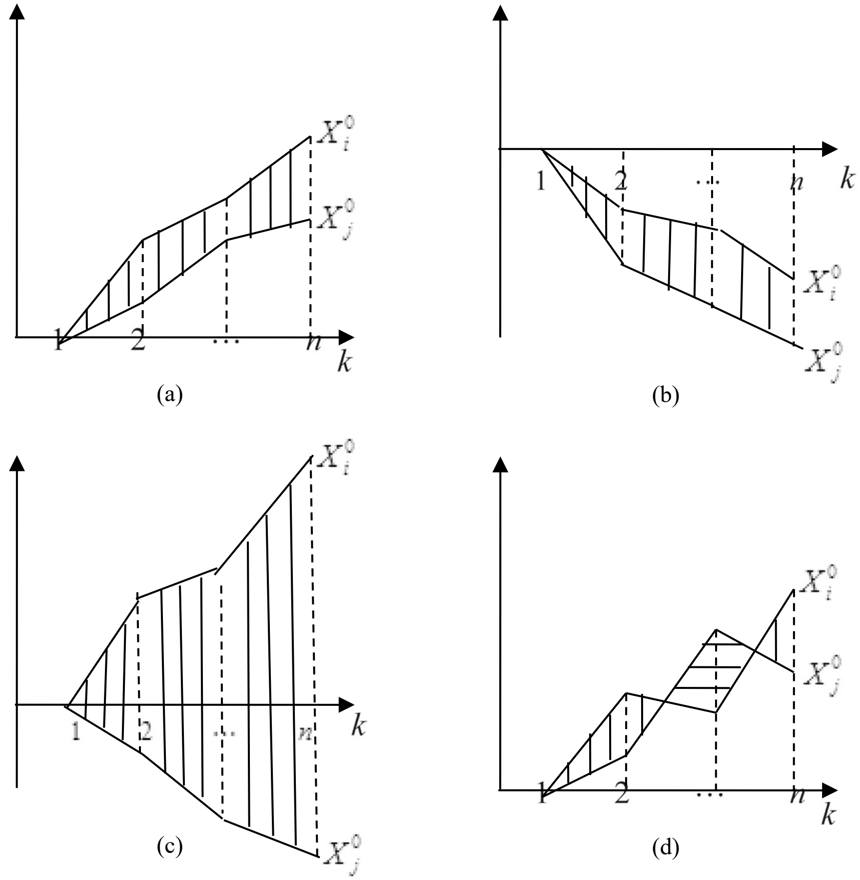



**Fig. 5.8 Positional relationship of** $X_i^0$ and $X_j^0$

$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are reverse sequences, then

$$\phi_{ij}^N = -\frac{|s_i - s_j|}{1 + |s_i - s_j|} \tag{5.21}$$

is called the negative grey similarity relational degree of $X_i$ and $X_j$ (Liu, 2022).

It can be easily proved that the negative grey similarity relational degree defined by formula (5.21) satisfies the axioms of normalization and reversibility, and has the following properties.

**Theorem 5.7.1** The negative grey similarity relational degree $\phi_{ij}^N$ satisfies the following properties:

(1) $-1 < \phi_{ij}^N < 0$.
(2) $\phi_{ij}^N$ is only related to the geometric shapes of $X_i$ and $X_j$,and has no relationship with the spatial positions of these sequences. In other words, the translation transformation does not change the value of negative grey similarity relational degree.
(3) The stronger the reverse relation between $X_i$ and $X_j$, the closer $\phi_{ij}^N$ is to $-1$;The weaker the reverse relation between $X_i$ and $X_j$, the closer $\phi_{ij}^N$ is to 0.
(4) If $X_i$ and $X_j$ are parallel or $X_i^0$ fluctuates around $X_j^0$, with the area of the parts of $X_i^0$ located above $X_j^0$ equal to that of the parts with $X_i^0$ located underneath $X_j^0$, then $\phi_{ij}^N = 0$.
(5) $\phi_{ii}^N = \phi_{jj}^N = 0$.
(6) $\phi_{ij}^N = \phi_{ji}^N$.

(Liu, 2022).

***Example 1*** Let $X_1 = (x_1(1), x_1(2), x_1(3), x_1(4), x_1(5)) = (1, 2, 3, 3, 5)$.
and $X_2 = (x_2(1), x_2(2), x_2(3), x_2(4), x_2(5)) = (5, 4, 2, 2, 1)$.
Then the zero-starting point sequences of $X_1$ and $X_2$ as follows

$$X_1^0 = (x_1^0(1), x_1^0(2), x_1^0(3), x_1^0(4), x_1^0(5)) = (0, 1, 2, 2, 4)$$
$$X_2^0 = (x_2^0(1), x_2^0(2), x_2^0(3), x_2^0(4), x_2^0(5)) = (0, -1, -3, -3, -4)$$

We have $\mathbf{s_1 = 7, s_2} = -9$, therefore, $X_1$ is an increasing sequence, and $X_2$ is a decreasing sequence. That is, $X_1$ and $X_2$ are reverse sequences. From formula(5)

$$\phi_{12}^N = -\frac{|s_1 - s_2|}{1 + |s_1 - s_2|} = -\frac{16}{1 + 16} \approx -0.9412$$

It shows that there is a strong inverse correlation between $X_1$ and $X_2$.

Similarly, the definitions of negative grey absolute relational degree, negative grey relative relational degree and negative grey comprehensive relational degree can be given as follows.

**Definition 5.7.6** Assume that $X_i$ and $X_j$ are system's behavior data sequences,

(1) If $X_i$ and $X_j$ are reverse sequences, then

$$\varepsilon_{ij}^N = -\frac{|s_i - s_j|}{1 + |s_i| + |s_j| + |s_i - s_j|} \tag{5.22}$$

Is called negative grey absolute relational degree of $X_i$ and $X_j$.

(2) If the initial valued sequences of $X_i$ and $X_j$ are reverse sequences, then

$$r_{ij}^N = -\frac{\left|s_i' - s_j'\right|}{1 + \left|s_i'\right| + \left|s_j'\right| + \left|s_i' - s_j'\right|} \tag{5.23}$$

Is called negative relative grey relational degree of $X_i$ and $X_j$.

(3) If both of $X_i$ and $X_j$, and the initial valued sequences of $X_i$ and $X_j$ are all reverse sequences, then

$$\rho_{ij}^N = \theta \varepsilon_{ij}^N + (1 - \theta) r_{ij}^N \tag{5.24}$$

is called negative grey synthetic relational degree of $X_i$ and $X_j$.Where $\theta \in [0, 1]$ (Liu, 2022).

It should be noted that the grey proximity relational analysis model have been constructed to measure the spatial relative position relationship of the sequences. The grey proximity relational degree does not consider the change direction of the sequences and does not pay attention to the same or reverse relationship between the two sequences. Therefore, it is not necessary to define the corresponding "negative grey proximity relational analysis model".

**Definition 5.7.7** Let $X_0 = (x_0(1), x_0(2), \ldots, x_0(n))$ be a data sequence of a system's behavioral characteristic and the following are relevant factor sequences:

$$\begin{aligned} X_1 &= (x_1(1), x_1(2), \ldots, x_1(n)) \\ &\ldots\ldots\ldots \\ X_i &= (x_i(1), x_i(2), \ldots, x_i(n)) \\ &\ldots\ldots\ldots \\ X_m &= (x_m(1), x_m(2), \ldots, x_m(n)) \end{aligned}$$

If $X_i$ is a reverse sequence of $X_0$, for $\xi \in (0, 1)$,let

$$\gamma_{0i}^N(k) = \frac{\min_i \min_k |x_0(k) - x_i(k)| - |x_0(k) - x_i(k)|}{|x_0(k) - x_i(k)| + \xi \max_i \max_k |x_0(k) - x_i(k)|} \tag{5.25}$$

$$\gamma_{0i}^N = \frac{1}{n} \sum_{k=1}^{n} \gamma_{0i}^N(k) \tag{5.26}$$

Then $\gamma_{0i}^N$ is called negative Deng's grey relational degree of $X_i$ and $X_0$, and $\gamma_{0i}^N$ (k) is called the negative Deng's grey relational coefficient of relevant factor sequence $X_i$ and the system's behavioral characteristic sequence $X_0$ at point k (Liu, 2022).

It is easy to show that the negative grey absolute relational degree, negative relative grey relational degree, negative grey synthetic relational degree, and negative Deng's grey relational degree are all satisfy the axioms of normalization and Reversibility.

## 5.8 Grey Relational Analysis Model for Cross-Sequences

### *5.8.1 Measure of Difference in Cross Sequences*

All the grey relational analysis models from a global perspective based on integral elements of $s_i, s_j, s_i', s_j', s_i - s_j, s_i' - s_j', S_i - S_j$. As can be seen from previous sections of this chapter that all the integral elements are the algebraic sum of the corresponding data in sequences of $X_i, X_j, X_i^0, X_j^0, X_i^{'0}, X_j^{'0}, X_i - X_j, X_i^0 - X_j^0, X_i^{'0} - X_j^{'0}$.In the case of intersecting sequences or sequences intersecting with coordinate axes, data with opposite symbols cancel out each other, reducing the absolute value of the integral element and affecting the correlation calculation results. Especially when the value of integral elements of $s_i - s_j$ or $s_i' - s_j'$ is equal to 0, the grey relational degree reaches its maximum value of 1. This result is obviously not entirely reasonable.

In 1993, Guorong Xu discovered the problem of positive and negative offset in the calculation process of the grey absolute relational model and proposed an improved model (Xu, 1993). Lu et al. (2023) proposed a grey relational analysis model for cross-sequences based on both the angle variations within a time interval and between time intervals in 2023.

To further analyze the relationship between cross sequences and propose a new definition of grey relational analysis model for cross-sequences based on them, we will first provide the definitions of cross sequences and area elements.

**Definition 5.8.1** Assume that.

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences.

If the fold line corresponding to $X_i$ and $X_j$ has at least one intersection point other than the starting or ending point, then $X_i$ and $X_j$ are referred to as cross sequences (Liu et al., 2024).

**Definition 5.8.2** Assume that.

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences. Then

$$\|s_i\| = \int_1^n \left|X_i^0\right| dt,\ \|s_j\| = \int_1^n \left|X_j^0\right| dt,\ \|s_i - s_j\| = \int_1^n \left|X_i^0 - X_j^0\right| dt,$$

$$\|s_i'\| = \int_1^n |X_i'^0| dt,\ \|s_j'\| = \int_1^n |X_j'^0| dt,\ \|s_i' - s_j'\| = \int_1^n |X_i'^0 - X_j'^0| dt,$$

are referred to as area elements (Liu et al., 2024).

**Proposition 5.8.1** For the area elements of $\|s_i\|$, $\|s_j\|$, $\|s_i - s_j\|$, we have.

$$\|s_i\| = \sum_{k=2}^{n-1} |x_i^0(k)| + \frac{1}{2}|x_i^0(n)|$$

$$\|s_j\| = \sum_{k=2}^{n-1} |x_j^0(k)| + \frac{1}{2}|x_j^0(n)|$$

$$\|s_i - s_j\| = \sum_{k=2}^{n-1} |x_i^0(k) - x_j^0(k)| + \frac{1}{2}|x_i^0(n) - x_j^0(n)|$$

(Liu et al., 2024).

**Proposition 5.8.2** For the area elements of $\|s_i'\|$, $\|s_j'\|$, $\|s_{i'} - s_{j'}\|$, we have.

$$\|s_i'\| = \sum_{k=2}^{n-1} |x_i'^0(k)| + \frac{1}{2}|x_i'^0(n)|$$

$$\|s_j'\| = \sum_{k=2}^{n-1} |x_j'^0(k)| + \frac{1}{2}|x_j'^0(n)|$$

$$\|s_i' - s_j'\| = \sum_{k=2}^{n-1} |x_i'^0(k) - x_j'^0(k)| + \frac{1}{2}|x_i'^0(n) - x_j'^0(n)|$$

(Liu et al., 2024).

From Propositions 5.8.1 and 5.8.2, it can be seen that in the process of calculating area elements, the absolute value of the difference between the data in the sequence and the corresponding data in different sequences is first calculated, and then the sum is calculated. There is no problem of positive or negative offset.

**Proposition 5.8.3** Assume that.

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences. If

$$|s_i - s_j| \neq \|s_i - s_j\|$$

then $X_i^0$ and $X_j^0$ are cross sequences (Liu et al., 2024).

**Definition 5.8.3** Assume that $X_i^0$ and $X_j^0$ are cross sequences. And $\|s_i\|$, $\left\|s_j\right\|$, $\left\|s_i - s_j\right\|$ are the area elements as shown in Proposition 1, then

$$\Delta_{ij} = \frac{\left\|s_i - s_j\right\|}{1 + \|s_i\| + \left\|s_j\right\| + \left\|s_i - s_j\right\|} \tag{5.27}$$

is referred to as the degree of difference between sequences $X_i^0$ and $X_j^0$ (Liu et al., 2024).

**Theorem 5.8.1** Assume that $X_i^0$ and $X_j^0$ are cross sequences. Then the degree of difference $\Delta_{ij}$ between sequences $X_i^0$ and $X_j^0$ having the following properties (Liu et al., 2024):

(1) $0 \leq \Delta_{ij} \leq 1$;
(2) The value of $\Delta_{ij}$ related to the curved triangle areas of $\|s_i\|$, $\left\|s_j\right\|$ and $\left\|s_i - s_j\right\|$;The larger the value of $\|s_i\|$, $\left\|s_j\right\|$, the smaller the value of $\Delta_{ij}$ is, The greater the value of $\left\|s_i - s_j\right\|$, The larger the value of $\Delta_{ij}$ is;
(3) $\Delta_{ii} = \Delta_{jj} = 0$;
(4) $\Delta_{ij} = \Delta_{ji}$

**Definition 5.8.4** Assume that $X_i^{'0}$ and $X_j^{'0}$ are cross sequences. $\left\|s_i'\right\|$, $\left\|s_j'\right\|$, $\left\|s_i' - s_j'\right\|$ are the area elements as shown in Proposition 5.8.2, then.

$$\Psi_{ij} = \frac{\left\|s_i' - s_j'\right\|}{1 + \left\|s_i'\right\| + \left\|s_j'\right\| + \left\|s_i' - s_j'\right\|} \tag{5.28}$$

is referred to as the degree of difference between sequences of $X_i^{'0}$ and $X_j^{'0}$.(Liu, et al., 2024)

Similarly, we can discuss the properties of the degree of difference between sequences of $X_i^{'0}$ and $X_j^{'0}$.

***Example 5.8.1*** Let

$$X_1 = (1, 1.2, 0.8, 1.2, 0.8, 1)$$
$$X_2 = (1.5, 1.3, 1.7, 1.3, 1.7, 1.5)$$

Calculate the grey similitude relational degree between sequences $X_1$ and $X_2$, and the degree of difference between sequences $X_1^0$ and $X_2^0$.

**Solution**:

The zero-starting point sequences of $X_1$, $X_2$ are

$$X_1^0 = (0, 0.2, -0.2, 0.2, -0.2, 0)$$
$$X_2^0 = (0, -0.2, 0.2, -0.2, 0.2, 0)$$

So,

$$s_1 - s_2 = \sum_{k=2}^{5} [x_1^0(k) - x_2^0(k)] + \frac{1}{2}[x_1^0(6) - x_2^0(6)] = 0$$
$$\|s_1\| = \sum_{k=2}^{5} |x_1^0(k)| + \frac{1}{2}|x_1^0(6)| = 0.8$$
$$\|s_2\| = \sum_{k=2}^{5} |x_2^0(k)| + \frac{1}{2}|x_2^0(6)| = 0.8$$
$$\|s_1 - s_2\| = \sum_{k=2}^{5} |x_1^0(k) - x_2^0(k)| + \frac{1}{2}|x_1^0(6) - x_2^0(6)| = 1.6$$

Therefore, we have

$$\delta_{12} = \frac{1}{1 + |s_1 - s_2|} = 1$$
$$\Delta_{12} = \frac{\|s_1 - s_2\|}{1 + \|s_1\| + \|s_2\| + \|s_1 - s_2\|} = 0.38$$

In Example 5.8.1, $X_1$ and $X_2$ have a high degree of similarity in shape. After translation,$X_1^0$ can completely coincides with $X_2^0$.But from a spatial perspective, there is a significant difference between $X_1$ and $X_2$. The degree of difference between sequences $X_1^0$ and $X_2^0$ is $\Delta_{12} = 0.38$.

From Fig. 5.9, it can be seen that the overall trend of changes in the line corresponding to sequences $X_1^0$ and $X_2^0$ remains basically consistent.$X_2^0$ can be regarded as a lagging variable of $X_1^0$.In the time intervals of [0, 1], [1, 2], [2, 3], [3, 4], [4, 5], the direction of change between $X_1^0$ and $X_2^0$ are opposite.

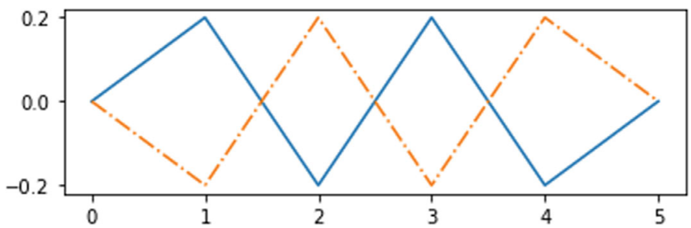



**Fig. 5.9** The fold line of $X_1$ and $X_2$

### 5.8.2 The Modified Grey Relational Models

For the sequences with same or reverse direction that do not cross, the calculated results by the grey relational analysis models from a global perspective based on integral elements of $s_i$, $s_j$, $s'_i$, $s'_j$, $s_i - s_j$, $s'_i - s'_j$ can accurately reflect the relationship between sequences. For cross sequences, it is necessary to use the degree of difference to modify the calculation results.

**Definition 5.8.5** Assume that

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences, then

$$\varepsilon_{ij}^{EC} = \varepsilon_{ij} - \Delta_{ij} = \frac{1 + |s_i| + |s_j|}{1 + |s_i| + |s_j| + |s_i - s_j|} - \frac{\|s_i - s_j\|}{1 + \|s_i\| + \|s_j\| + \|s_i - s_j\|} \tag{5.29}$$

is referred to as the modified grey absolute relational degree (Liu et al., 2024)

**Definition 5.8.6** Assume that

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences, then

$$\gamma_{ij}^{EC} = \gamma_{ij} - \Psi_{ij} = \frac{1 + |s'_i| + |s'_j|}{1 + |s'_i| + |s'_j| + |s'_i - s'_j|} - \frac{\|s'_i - s'_j\|}{1 + \|s'_i\| + \|s'_j\| + \|s'_i - s'_j\|} \tag{5.30}$$

is called the modified grey relative relational degree (Liu et al., 2024).

**Definition 5.8.7** Assume that

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences, and $\varepsilon_{ij}^{EC}$, $\gamma_{ij}^{EC}$ are the modified grey absolute relational degree and the modified grey relative relational degree respectively, then

$$\rho_{ij}^{EC} = \theta\varepsilon_{ij}^{EC} + (1-\theta)\gamma_{ij}^{EC}, \theta \in [0, 1] \tag{5.31}$$

is referred to as the modified grey synthetical relational degree (Liu et al., 2024).

**Definition 5.8.8** Assume that

$$X_i = (x_i(1), x_i(2), \ldots, x_i(n))$$
$$X_j = (x_j(1), x_j(2), \ldots, x_j(n))$$

are two system's behavior data sequences, then

$$\delta_{ij}^{EC} = \delta_{ij} - \Delta_{ij} = \frac{1}{1 + |s_i - s_j|} - \frac{\|s_i - s_i\|}{1 + \|s_i\| + \|s_j\| + \|s_i - s_j\|} \tag{5.32}$$

is called the modified grey similitude relational degree (Liu et al., 2024).

***Example 5.8.2*** Let $X_1$ and $X_2$ as shown in Example 1, please calculate the modified grey similitude relational degree.

**Solution**: From Example 1, we have $\delta_{12} = 1$,$\Delta_{12} = 0.38$,therefore

$$\delta_{12}^{EC} = \delta_{12} - \Delta_{12} = 0.62$$

This result better reflects the difference between $X_1$ and $X_2$ and the consistency of their overall trend of change.

***Example 5.8.3*** Let

$$X_1 = (1, 1.2, 0.8, 1.2, 0.8, 1)$$
$$X_3 = (1.2, 0.6, 1.8, 1.2, 2.4, 1.2)$$

Calculate the modified grey absolute relational degree, the modified grey relative relational degree and the modified grey synthetical relational degree of $X_1$ and $X_3$.

**Solution**:

(1) Calculation of the modified grey absolute relational degree

The zero-starting point sequences of $X_1$ and $X_3$ are

$$X_1^0 = (0, 0.2, -0.2, 0.2, -0.2, 0)$$
$$X_3^0 = (0, -0.6, 0.6, 0, 1.2, 0)$$

From

$$s_1 = \sum_{k=2}^{5} x_1^0(k) + \frac{1}{2}x_1^0(n) = 0, s_3 = \sum_{k=2}^{5} x_3^0(k) + \frac{1}{2}x_3^0(n) = 1.2$$

$$s_1 - s_3 = \sum_{k=2}^{5} [x_1^0(k) - x_3^0(k)] + \frac{1}{2}[x_1^0(6) - x_3^0(6)] = -1.2$$

$$\|s_1\| = \sum_{k=2}^{5} \left|x_1^0(k)\right| + \frac{1}{2}\left|x_1^0(6)\right| = 0.8$$

$$\|s_3\| = \sum_{k=2}^{5} \left|x_3^0(k)\right| + \frac{1}{2}\left|x_3^0(6)\right| = 2.4$$

$$\|s_1 - s_3\| = \sum_{k=2}^{5} \left|x_1^0(k) - x_3^0(k)\right| + \frac{1}{2}\left|x_1^0(6) - x_3^0(6)\right| = 3.2$$

We have

$$\varepsilon_{13} = \frac{1 + |s_1| + |s_3|}{1 + |s_1| + |s_3| + |s_1 - s_3|} = 0.65$$

$$\Delta_{13} = \frac{\|s_1 - s_3\|}{1 + \|s_1\| + \|s_3\| + \|s_1 - s_3\|} = 0.43$$

$$\varepsilon_{13}^{EC} = \varepsilon_{13} - \Delta_{13} = 0.22$$

(2) Calculation of the modified grey relative relational degree

The initial value sequences and the zero-starting point sequences of initial value sequences of $X_1$ and $X_3$ are as follows

$$X_1' = (1, 1.2, 0.8, 1.2, 0.8, 1)$$

$$X_3' = (1, 0.5, 1.5, 1, 2, 1)$$

$$X_1'^0 = (0, 0.2, -0.2, 0.2, -0.2, 0)$$

$$X_3'^0 = (0, -0.5, 0.5, 0, 1, 0)$$

From

$$s_1' = \sum_{k=2}^{5} x_1'^0(k) + \frac{1}{2}x_1'^0(6) = 0$$

$$s_3' = \sum_{k=2}^{5} x_3'^0(k) + \frac{1}{2}x_3'^0(6) = 1$$

$$s_1' - s_3' = \sum_{k=2}^{5} [x_1'^0(k) - x_3'^0(k)] + \frac{1}{2}[x_1'^0(6) - x_3'^0(6)] = 1$$

$$\|s_1'\| = \sum_{k=2}^{5} \left|x_1'^0(k)\right| + \frac{1}{2}\left|x_1'^0(6)\right| = 0.8$$

$$\|s_3'\| = \sum_{k=2}^{5} \left|x_3'^0(k)\right| + \frac{1}{2}\left|x_3'^0(6)\right| = 2$$

$$\|s_1' - s_3'\| = \sum_{k=2}^{5} \left|x_1'^0(k) - x_3'^0(k)\right| + \frac{1}{2}\left|x_1'^0(6) - x_3'^0(6)\right| = 2.8$$

We have

$$\gamma_{13} = \frac{1 + |s_1'| + |s_3'|}{1 + |s_1'| + |s_3'| + |s_1' - s_3'|} = 0.67$$

$$\Psi_{13} = \frac{\|s_1' - s_3'\|}{1 + \|s_1'\| + \|s_3'\| + \|s_1' - s_3'\|} = 0.42$$

$$\gamma_{13}^{EC} = \gamma_{13} - \Psi_{13} = 0.25$$

(3) Calculation of the modified grey synthetical relational degree

Let $\theta = 0.5$, then

$$\rho_{13}^{EC} = 0.5 \cdot \varepsilon_{13}^{EC} + 0.5 \cdot \gamma_{13}^{EC} = 0.235$$

## 5.9 Superiority Analysis

**Definition 5.9.1** Assume that $Y_1, Y_2, \ldots, Y_s$ are a system's characteristic behavioral sequences, and $X_1, X_2, \ldots, X_m$ are behavioral sequences of relevant factors with the same length. Let $\gamma_{ij}$ be the grey relational degree between $Y_i$ and $X_j$, $i = 1, 2, \ldots, s$, and $j = 1, 2, \ldots, m$. Then:

$$\Gamma = \left(\gamma_{ij}\right)_{s \times m} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{s1} & \gamma_{s2} & \cdots & \gamma_{sm} \end{bmatrix}.$$

This formula is referred to as the grey relational matrix of the system, where the $i$th row is made up of the grey relational degree between the characteristic sequence $Y_i (i = 1, 2, \ldots, s)$ and each of the factor sequences $X_1, X_2, \ldots, X_m$; and the $j$th column consists of the grey relational degree between each of the characteristic sequences $Y_1, Y_2, \ldots, Y_s$ and $X_j (j = 1, 2, \ldots, m)$. We can analyze both the superiority of a system's characteristic behavioral variables or the behavioral variables of relevant factors.

**Definition 5.9.2** Assume that $Y_1, Y_2, \ldots, Y_s$ are a system's characteristic behavioral sequences, $X_1, X_2, \ldots, X_m$ are behavioral sequences of relevant factors, and $\Gamma = (\gamma_{ij})_{s\times m}$ is the grey relational matrix. If there are $k, i \in \{1, 2, \ldots, s\}$ such that $\gamma_{kj} \geq \gamma_{ij}, j = 1, 2, \ldots, m$, then the system's characteristic variable $Y_k$ is said to be more favorable than the system's characteristic variable $Y_i$, written as $Y_k \succ Y_i$.

If $\forall i = 1, 2, \ldots, s, i \neq k$, $Y_k \succ Y_i$ always holds true, then $Y_k$ is said to be the most favorable characteristic variable.

**Definition 5.9.3** Assume that $Y_1, Y_2, \ldots, Y_s$ are a system's characteristic behavioral sequences, $X_1, X_2, \ldots, X_m$ are behavioral sequences of relevant factors, and $\Gamma = (\gamma_{ij})_{s\times m}$ is the grey relational matrix. If there are $l, j \in \{1, 2, \ldots, m\}$ such that $\gamma_{il} \geq \gamma_{ij}$, $i = 1, 2, \ldots, s$, then we say that the system's factor $X_l$ is more favorable than factor $X_j$, written as $X_l \succ X_j$.

If $\forall j = 1, 2, \ldots, m, j \neq l$, $X_l \succ X_j$ always holds true, then $X_l$ is said to be the most favorable factor.

**Definition 5.9.4** Assume that $Y_1, Y_2, \ldots, Y_s$ are a system's characteristic behavioral sequences, $X_1, X_2, \ldots, X_m$ are behavioral sequences of relevant factors, and $\Gamma = (\gamma_{ij})_{s\times m}$ is the grey relational matrix.

(1) If there are $k, i \in \{1, 2, \ldots, s\}$ satisfying $\sum_{j=1}^{m} \gamma_{kj} \geq \sum_{j=1}^{m} \gamma_{ij}$, then the system's characteristic variable $Y_k$ is said to be more quasi-favorable than $Y_i$, which is denoted as $Y_k \succcurlyeq Y_i$.
(2) If there are $l, j \in \{1, 2, \ldots, m\}$ satisfying $\sum_{i=1}^{m} \gamma_{il} \geq \sum_{i=1}^{m} \gamma_{ij}$, then the system's factor $X_l$ is more quasi-favorable than $X_j$, which is denoted as $X_l \succcurlyeq X_j$.

**Definition 5.9.5** Assume that $Y_1, Y_2, \ldots, Y_s$ are a system's characteristic behavioral sequences, $X_1, X_2, \ldots, X_m$ are behavioral sequences of relevant factors, and $\Gamma = (\gamma_{ij})_{s\times m}$ is the grey relational matrix.

(1) If there is $k \in \{1, 2, \ldots, s\}$ such that $\forall i = 1, 2, \ldots, s, i \neq k$, $Y_k \succcurlyeq Y_i$, then the system's characteristic variable $Y_k$ is said to be quasi-preferred.
(2) If there is $l \in \{1, 2, \ldots, m\}$ such that $\forall j = 1, 2, \ldots, m, j \neq l$, $X_l \succcurlyeq X_j$, then the system's factor $X_l$ is said to be quasi-preferred.

**Proposition 5.9.1** In a system of $S$ characteristic variables and $m$ relevant factors, there may not be a most favorable characteristic variable and a most favorable factor. However, there must be quasi-preferred characteristic variable and factor.

***Example 5.9.1*** The formulas below are system's characteristic behavioral sequences.

$$Y_1 = (170, 174, 197, 216.4, 235.8)$$
$$Y_2 = (57.55, 70.74, 76.8, 80.7, 89.85)$$
$$Y_3 = (68.56, 70, 85.38, 99.83, 103.4)$$

The formulas below are behavioral sequences of relevant factors.

$$X_1 = (308.58, 310, 295, 346, 367)$$
$$X_2 = (195.4, 189.9, 189.2, 205, 222.7)$$
$$X_3 = (24.6, 21, 12.2, 15.1, 14.57)$$
$$X_4 = (20, 25.6, 23.3, 29.2, 30)$$
$$X_5 = (18.98, 19, 22.3, 23.5, 27.655)$$

Try and analyze the superiority of the system's characteristic behavioral variables and the superiority of the behavioral variables of relevant factors.

**Solution**

We analyze the superiority of the system's characteristic behavioral sequences and the behavioral sequences of relevant factors by absolute degree of GRA model.

(1) Find the matrix of the grey absolute relational degree. Calculate the images of zero-starting point for all the system's characteristic behavioral sequences as well as the behavioral sequences of relevant factors as follows:

$$Y_1^0 = (0, 4, 27, 46.4, 65.8)$$
$$Y_2^0 = (0, 13.19, 19.25, 23.15, 32.3)$$
$$Y_3^0 = (0, 1.44, 16.82, 31.27, 34.84)$$
$$X_1^0 = (0, 1.42, -13.58, 37.42, 58.42)$$
$$X_2^0 = (0, -5.5, -8.2, 9.6, 27.3)$$
$$X_3^0 = (0, -3.6, -12.4, , -9.5, -10.03)$$
$$X_4^0 = (0, 5.6, 3.3, 9.2, 10)$$
$$X_5^0 = (0, 0.02, 3.32, 4.52, 8.675)$$

For the system's characteristic behavioral variable $\mathrm{Y}_1$, we have:

$$|s_{y_1}| = \left|\sum_{k=2}^{4} y_1^0(k) + \frac{1}{2} y_1^0(5)\right| = \left|4 + 27 + 46.4 + \frac{1}{2} \times 65.8\right| = 110.3$$

$$|s_{x_1}| = \left|\sum_{k=2}^{4} x_1^0(k) + \frac{1}{2} x_1^0(5)\right| = \left|1.42 + (-13.58) + 37.42 + \frac{1}{2} \times 58.42\right| = 54.47$$

$$|s_{y_1} - s_{x_1}| = \left|\sum_{k=2}^{4} (y_1^0(k) - x_1^0(k)) + \frac{1}{2}(y_1^0(5) - x_1^0(5))\right| = 55.9$$

$$\varepsilon_{11} = \frac{1 + |s_{y_1}| + |s_{x_1}|}{1 + |s_{y_1}| + |s_{x_1}| + |s_{y_1} - s_{x_1}|} = \frac{1 + 110.3 + 54.47}{1 + 110.3 + 54.47 + 55.9} = 0.748$$

$$\left|s_{x_2}\right| = \left|\sum_{k=2}^{4} x_2^0(k) + \frac{1}{2}x_2^0(5)\right| = \left|(-5.5) + (-8.2) + 9.6 + \frac{1}{2} \times 27.3\right| = 9.55$$

$$\left|s_{y_1} - s_{x_2}\right| = \left|\sum_{k=2}^{4} (y_1^0(k) - x_2^0(k)) + \frac{1}{2}(y_1^0(5) - x_2^0(5))\right| = 100.75$$

$$\varepsilon_{12} = \frac{1 + \left|s_{y_1}\right| + \left|s_{x_2}\right|}{1 + \left|s_{y_1}\right| + \left|s_{x_2}\right| + \left|s_{y_1} - s_{x_2}\right|} = \frac{1 + 110.3 + 9.55}{1 + 110.3 + 9.55 + 100.75} = 0.545$$

Similarly:

$$\varepsilon_{13} = \frac{1 + \left|s_{y_1}\right| + \left|s_{x_3}\right|}{1 + \left|s_{y_1}\right| + \left|s_{x_3}\right| + \left|s_{y_1} - s_{x_3}\right|} = 0.502$$

$$\varepsilon_{14} = \frac{1 + \left|s_{y_1}\right| + \left|s_{x_4}\right|}{1 + \left|s_{y_1}\right| + \left|s_{x_4}\right| + \left|s_{y_1} - s_{x_4}\right|} = 0.606$$

$$\varepsilon_{15} = \frac{1 + \left|s_{y_1}\right| + \left|s_{x_5}\right|}{1 + \left|s_{y_1}\right| + \left|s_{x_5}\right| + \left|s_{y_1} - s_{x_5}\right|} = 0.557$$

For the system's characteristic behavioral variable $Y_2$, $Y_3$, we have:

$$\varepsilon_{21} = 0.880, \varepsilon_{22} = 0.570, \varepsilon_{23} = 0.502, \varepsilon_{24} = 0.663, \varepsilon_{25} = 0.588$$
$$\varepsilon_{31} = 0.907, \varepsilon_{32} = 0.574, \varepsilon_{33} = 0.503, \varepsilon_{34} = 0.675, \varepsilon_{35} = 0.594$$

Therefore, we have the grey absolute relational degree matrix as follows:

$$A = (\varepsilon_{ij}) = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \varepsilon_{14} & \varepsilon_{15} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} & \varepsilon_{24} & \varepsilon_{25} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} & \varepsilon_{34} & \varepsilon_{35} \end{bmatrix} = \begin{bmatrix} 0.748 & 0.545 & 0.502 & 0.606 & 0.557 \\ 0.880 & 0.570 & 0.502 & 0.663 & 0.588 \\ 0.907 & 0.574 & 0.503 & 0.675 & 0.594 \end{bmatrix}$$

(2) Calculate the relative grey relational degree matrix. Calculate the initial images:

$$Y_i'(i = 1, 2, 3) \text{ and } X_j'(j = 1, 2, 3, 4, 5) \text{ of } Y_i(i = 1, 2, 3) \text{ and } X_j(j = 1, 2, 3, 4, 5).$$

Then find the images of zero-starting point for all system's characteristic behavioral sequences $Y_i(i = 1, 2, 3)$ and the behavioral sequences of relevant factors $X_j(j = 1, 2, 3, 4, 5)$.

$Y_i'^0(i = 1, 2, 3)$ and $X_j'^0(j = 1, 2, 3, 4, 5)$ of $Y_i'(i = 1, 2, 3)$ and $X_j'(j = 1, 2, 3, 4, 5)$.

From:

$$\left|s_{y_i}'\right| = \left|\sum_{k=2}^{4} y_i'^0(k) + \frac{1}{2}y_i'^0(5)\right|; \text{i} = 1, 2, 3$$

$$\left|s'_{x_j}\right| = \left|\sum_{k=2}^{4} x_j'^0(k) + \frac{1}{2}x_j'^0(5)\right|; \text{j} = 1, 2, 3, 4, 5$$

$$\left|s'_{y_i} - s'_{x_j}\right| = \left|\sum_{k=2}^{4} (y_i'^0(k) - x_j'^0(k)) + \frac{1}{2}(y_i'^0(5) - x_j'^0(5))\right| ;\text{i} = 1,2,3,;\text{j} = 1, 2, 3, 4, 5$$

$$r_{ij} = \frac{1 + \left|s'_{y_i}\right| + \left|s'_{x_j}\right|}{1 + \left|s'_{y_i}\right| + \left|s'_{x_j}\right| + \left|s'_{y_i} - s'_{x_j}\right|}; i = 1, 2, 3; j = 1, 2, 3, 4, 5,$$

we have:

$$r_{11} = 0.7945, r_{12} = 0.7389, r_{13} = 0.6046, r_{14} = 0.8471, r_{15} = 0.9973$$
$$r_{21} = 0.6937, r_{22} = 0.6571, r_{23} = 0.5837, r_{24} = 0.9738, r_{25} = 0.8271$$
$$r_{31} = 0.7300, r_{32} = 0.6866, r_{33} = 0.6101, r_{34} = 0.9444, r_{35} = 0.8884$$

Therefore, we have the relative grey relational degree matrix as follows:

$$B = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} \end{bmatrix} = \begin{bmatrix} 0.7945 & 0.7389 & 0.6046 & 0.8471 & 0.9973 \\ 0.6937 & 0.6571 & 0.5837 & 0.9738 & 0.8271 \\ 0.7300 & 0.6866 & 0.6101 & 0.9444 & 0.8884 \end{bmatrix}$$

(3) Compute the grey synthetic relational degree matrix. If $\theta = 0.5$, we have:

$$C = \theta A + (1 - \theta)B = (\theta\varepsilon_{ij} + (1 - \theta)r_{ij}) = (\rho_{ij})$$
$$= \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} \end{bmatrix}$$
$$= \begin{bmatrix} 0.7713 & 0.6420 & 0.5533 & 0.7266 & 0.7772 \\ 0.7869 & 0.6136 & 0.5429 & 0.8184 & 0.7076 \\ 0.8185 & 0.6303 & 0.5566 & 0.8097 & 0.7412 \end{bmatrix}$$

(4) Analysis and discussion. In matrix A of the grey absolute relational degree, the rows of A satisfy the following formula:

$$\varepsilon_{3j} > \varepsilon_{2j} \geq \varepsilon_{1j}; \quad \text{j} = 1, 2, 3, 4, 5.$$

Therefore, we have $Y_3 \succ Y_2 \succ Y_1$. That is, $Y_3$ is the most favorable characteristic variable, $Y_2$ is the second, and $Y_1$ the least favorable characteristic variable. All columns of A satisfy:

$$\varepsilon_{i1} > \varepsilon_{i4} > \varepsilon_{i5} > \varepsilon_{i2} > \varepsilon_{i3}; \quad i = 1, 2, 3.$$

Therefore, we have:

$$X_1 \succ X_4 \succ X_5 \succ X_2 \succ X_3.$$

That is, $X_1$ is the most favorable factor, $X_4$ the second, $X_5$ the third, $X_2$ the fourth, and $X_3$ the least.

From the matrix $B$ of relative degree of relational, it can be seen that because the elements of $B$ satisfy

$$r_{i4} > r_{i1} > r_{i2} > r_{i3}; \quad i = 1, 2, 3$$
$$r_{i5} > r_{i1} > r_{i2} > r_{i3}; \quad i = 1, 2, 3$$

Thus, we can conclude that:

$$X_4 \succ X_1 \succ X_2 \succ X_3, X_5 \succ X_1 \succ X_2 \succ X_3.$$

Hence, $X_3$ is the most unfavorable factor of the system. Further, let us consider the following:

$$\sum_{j=1}^{5} r_{1j} = 3.9824 > \sum_{j=1}^{5} r_{3j} = 3.8595 > \sum_{j=1}^{5} r_{2j} = 3.7354.$$

Thus, we can conclude that $Y_1 \succeq Y_3 \succeq Y_2$, that is, $Y_1$ is the quasi-preferred characteristic. Also, given that:

$$\sum_{i=1}^{3} r_{i4} = 2.7653 > \sum_{i=1}^{3} r_{i5} = 2.7128 > \sum_{i=1}^{3} r_{i1} = 2.2182$$
$$> \sum_{i=1}^{3} r_{i2} = 2.0826 > \sum_{i=1}^{3} r_{i3} = 1.7984,$$

we have:

$$X_4 \succeq X_5 \succeq X_1 \succeq X_2 \succeq X_3.$$

That is, $X_4$ is the quasi-preferred factor, $X_5$ the next, and $X_3$ the most unfavorable factor.

On matrix $C$ of the grey synthetic relational degree, it can be seen that the elements of $C$ satisfy:

$$\rho_{i1} > \rho_{i2} > \rho_{i3}, \rho_{i4} > \rho_{i2} > \rho_{i3}, \rho_{i5} > \rho_{i2} > \rho_{i3}, i = 1, 2, 3.$$

Therefore, we have:

$$X_1 \succeq X_2 \succeq X_3,\ X_4 \succeq X_2 \succeq X_3,\ X_5 \succeq X_2 \succeq X_3.$$

That is, $X_3$ is the least preferred factor. We further consider the following:

$$\sum_{j=1}^{5} \rho_{3j} = 3.5563 > \sum_{j=1}^{5} \rho_{1j} = 3.4704 > \sum_{j=1}^{5} \rho_{2j} = 3.4694.$$

Thus,

$$Y_3 \succeq Y_1 \succeq Y_2.$$

That is, $Y_3$ is the quasi-preferred characteristic variable. Also, based on:

$$\sum_{i=1}^{3} \rho_{i1} = 2.3767 > \sum_{i=1}^{3} \rho_{i4} = 2.3547 > \sum_{i=1}^{3} \rho_{i5} = 2.226 > \sum_{i=1}^{3} \rho_{i2} = 1.8859 > \sum_{i=1}^{3} \rho_{i3} = 1.6528,$$

it follows that:

$$X_1 \succeq X_4 \succeq X_5 \succeq X_2 \succeq X_3.$$

Therefore, $X_1$ is the quasi-preferred factor, $X_4$ the next, $X_5$ is more favorable than $X_2$, and $X_3$ is the most unfavorable factor.

When investigating practical problems, the analyses of the three relational orders may not provide cohesive conclusions. This is because the absolute relational order looks at the relationship between absolute quantities, the relative relational order focuses on the rates of change with respect to the initial values of the observed sequences, while the synthetic relational order combines both the relationships between absolute quantities and rates of change. When considering the background of the problem of concern, we can choose one of the relational orders. For parsimony purposes, after a particular grey relational operator is applied to the system's characteristic behavioral sequences and relevant factor sequences, one only needs to employ the absolute relational order to the processed data.

## 5.10 Practical Application

Through the example below, we look at how to apply GRA models to analyze the time difference of economic indices.

***Example 5109.1*** In order to effectively monitor the performance of macro-economic systems and provide timely warnings, there is a need to investigate the time relationship of various economic indices with respect to economic cycles in terms of their peaks and valleys. In order to do so, questions such as the following must be addressed: Which indices can provide warning ahead of time? Which indices would be synchronic with the evolution of economic systems? And which indices tend to lag behind economic development? In other words, there is a need to divide economic indices into three classes: leading indicators, synchronic indices, and stagnant representations. To this end, grey relational analysis is an effective method for classifying economic indices (Chen & Liu, 2005).

Through careful research and analysis, we selected the following 8 major classes and 17 criteria as indices for economic performance:

(1) The Energy and raw materials class: the total production of energy;
(2) The investments class: the total investment in real estate;
(3) The production class: increase in industry output, increase in light industry output, increase in heavy industry output;
(4) The revenue class: national income, national expenditure;
(5) The currency and credit class: currency in circulation, savings at various financial institutions, amount of loans issued by financial institutions, cash payout in the form of salary and wages, net amount of currency in circulation;
(6) The consumption class: the gross retail amount of the society;
(7) The foreign trade class: gross amount of imports, gross amount of exports, direct investments by foreign entities; and
(8) The commodity prices class: the consumer price index.

By applying the following standards, we classify the previous criteria into three classes: leading indicators, synchronic indices, and stagnant representations. The standards for determining leading indicators are as follows:

(1) The indicated appearance of economic cyclic peaks needs to be at least three months ahead of their actual occurrence. Such leading relationship must be relatively stable with few exceptions;
(2) Indicated cycles and historical cycles are nearly one-to-one corresponded to each other. Also, for the most recent three economic cycles, the indicated cycles must be at least two times ahead of the actual occurrences with at least 3 months of lead time; and
(3) The economic characteristics of the indices provide relatively definite and clear leading relationships with respect to the background economic cycles.

The standards for determining both synchronic indices and stagnant representations are similar to those outlined above. However, for synchronic indices the time differences between the indicated appearances and the actual occurrences of economic cycles must be within plus and minus 3 months, while for stagnant representations the indicated appearances of economic cycles are behind the actual occurrences by at least 3 months.

In practice, it is almost impossible to find an index that meets all the stated standards. Therefore, based on the recorded reference cycles, we look for the statistical indices that meet the previously stated standards as closely as possible. In reality, a leading indicator can sometimes lag behind actual economic development, while an identified stagnant representation can also provide good lead-time in its forecast of a specific economic evolution. Similar scenarios also occur with regard to synchronic indices. However, theoretically, if the index is leading the actual occurrences among the one-to-one correspondences between an index and the actually recorded cycles over 2/3 of times, then we treat such an index as leading. Similar treatments are applied to synchronic indices and stagnant representations.

Given that the increase in industry output has played a significant role in the Chinese economy, as a synchronic index it has high quality. Therefore, it can be employed as the basic index in our grey relational analysis. We will compute not only the grey absolute relational degree between each criterion and the increase in industry output, but also the grey absolute relational degree of the other 16 criteria with their data translated 1 – 12 months along the time axis either left or right. When data are translated to the left, the months will take negative values; when translated to the right, the months will take positive values. The amount of horizontal translation is denoted by $L$. That is, we compute the grey absolute relational degree between all 16 individual criteria, excluding that of increase in industry output, and that of increase in industry output for $L = -12, \ldots, 12$. For each $L$-value, we order the obtained the grey absolute relational degree from the smallest to the largest, with the criterion listed in the front chosen as candidate criterion for that specific $L$-value. For instance, when $L = 0$, the grey absolute relational degree of the criteria are listed in Table 5.2.

Synchronic indices should be selected from those with large grey absolute relational degree, because large degrees of relational indicate that these criteria have greater similarities in comparison with that of increase in industry output, which we employ as the basic standard of the Chinese economic cycles. However, we still do not have theoretical evidence to support that an index with large grey absolute relational degree must be synchronic. To this end, we also need to consider whether or not the related grey absolute relational degree will be even greater when $L \neq 0$. If when $L = 0$ the value of the grey absolute relational degree of a certain index is ranked in the front, and if when $L = -4$ its value is even greater, it means that after this index is translated to four months earlier, it is more similar to the pattern of the increase in industry output. Thus, in this case, this specific index can be seen as one leading the economic cycle by as much as about four months. By using these two standards, we can not only classify indices as synchronic, leading, or stagnant, but also specify the amount of leading or staggering time.

When $L = 0$, the index of "cash payout as salaries" is ranked relatively in the front. Therefore, it is a natural candidate for being a synchronic indicator. When the $L$–value changes, the relevant changes in its absolute degrees of grey relational are given in Table 5.3.

From Table 5.3, it follows that when $L = 1$, the grey absolute relational degree reaches its maximum. Therefore, this specific index should be seen as one that is

**Table 5.2** The absolute degrees of grey relational of the criteria when $L = 0$

| Index | Absolute degree of grey relational | Index | Absolute degree of grey relational |
|---|---|---|---|
| Increase in heavy industry output | 0.979810 | National income | 0.559540 |
| Increase in light industry output | 0.972655 | Gross amount of exports | 0.544870 |
| Gross retail amount | 0.862105 | Total production of energy | 0.541044 |
| Cash payout as salaries | 0.789278 | Net amount of currency in circulation | 0.525936 |
| Currency in circulation | 0.753681 | Loans issued by financial institutions | 0.507958 |
| Total investment in real estate | 0.726366 | Savings at financial institutions | 0.505226 |
| Gross amount of imports | 0.598248 | Consumer price index | 0.500173 |
| National expenditure | 0.566914 | Direct investments by foreign entities | 0.500002 |

**Table 5.3** The grey absolute relational degrees of "cash payout as salaries" when $L \neq 0$

| L | The grey absolute relational degree | L | The grey absolute relational degree |
|---|---|---|---|
| – 12 | 0.664615 | 1 | 0.877090 |
| – 11 | 0.705983 | 2 | 0.867859 |
| – 10 | 0.733564 | 3 | 0.857366 |
| – 9 | 0.752740 | 4 | 0.832260 |
| – 8 | 0.753598 | 5 | 0.825027 |
| – 7 | 0.732221 | 6 | 0.806787 |
| – 6 | 0.723942 | 7 | 0.806782 |
| – 5 | 0.731232 | 8 | 0.820384 |
| – 4 | 0.742249 | 9 | 0.803771 |
| – 3 | 0.752628 | 10 | 0.806649 |
| – 2 | 0.770216 | 11 | 0.805679 |
| – 1 | 0.800838 | 12 | 0.836308 |
| 0 | 0.789278 | | |

lagging the economic cycle by as much as one month. An index which is leading or lagging no more than two months is usually seen as synchronic. However, if it exceeds this range of time it will be treated as either a leading or staggering index.

As a second example, the computational results for the index of "gross retail amount" are provided in Table 5.4.

**Table 5.4** The grey absolute relational degrees of "gross retail amount"

| L | The grey absolute relational degree | L | The grey absolute relational degree |
|---|---|---|---|
| − 12 | 0.914466 | 1 | 0.856944 |
| − 11 | 0.915117 | 2 | 0.866789 |
| − 10 | 0.918527 | 3 | 0.876758 |
| − 9 | 0.887243 | 4 | 0.882430 |
| − 8 | 0.888258 | 5 | 0.889590 |
| − 7 | 0.928151 | 6 | 0.895899 |
| − 6 | 0.948684 | 7 | 0.900899 |
| − 5 | 0.939351 | 8 | 0.900130 |
| − 4 | 0.923900 | 9 | 0.895977 |
| − 3 | 0.909621 | 10 | 0.894374 |
| − 2 | 0.884610 | 11 | 0.892662 |
| − 1 | 0.846814 | 12 | 0.889532 |
| 0 | 0.862105 | | |

From Table 5.4, it can be seen that when $L = -6$ the grey absolute relational degree of the particular index reaches its maximum. Therefore, it can be seen as a leading indicator. By using this method, we can compute the $L$–values corresponding to the maximum grey absolute relational degree of each of the indices of our interest. The results are listed in Table 5.5.

**Table 5.5** $L$–values corresponding to maximum grey absolute relational degree of the indices of our interest

| Index | L | Absolute degree | Index | L | Absolute degree |
|---|---|---|---|---|---|
| Currency in circulation | − 6 | 0.983452 | National income | + 12 | 0.718998 |
| Increase in heavy industry output | 0 | 0.979810 | Gross amount of imports | − 9 | 0.606556 |
| Increase in light industry output | 0 | 0.972655 | Gross amount of exports | + 10 | 0.560054 |
| Gross retail amount | − 6 | 0.948684 | Total production of energy | − 6 | 0.555035 |
| Cash payout as salaries | + 1 | 0.877090 | Direct investments by foreign entities | − 11 | 0.510016 |
| National expenditure | + 12 | 0.800533 | Loans issued by financial institutions | − 5 | 0.508375 |
| Net amount of currency in circulation | + 8 | 0.796688 | Savings at financial institutions | − 6 | 0.505588 |
| Total investment in real estate | − 11 | 0.769778 | Consumer price index | + 11 | 0.503235 |

Table 5.5 indicates that we can classify the 16 indices of the eight major classes into three classes as leading, synchronic, and stagnant indices, as shown in Table 5.6.

***Example 5.10.2*** **Measurement of Reverse Incentive Effect of Fields Medal (Liu, 2022).**

Most people agree that knowledge production can promote long-term economic growth. Yet little is known about how knowledge is produced (Borjas & Doran, 2015). For example, it is difficult for the author to explain clearly how the models of the negative grey similarity relational degree, the negative grey absolute relational degree, negative grey relative relational degree, negative grey comprehensive relational degree, and negative Deng's grey relational degree are finally proposed in this paper after 40 years of thinking. People try to motivate knowledge producers through awards. Hundreds of scientific prizes are awarded throughout the world and across all scientific disciplines. Although these prizes are frequently awarded with the explicit goal of inspiring more and better scientific work (Scotchmer, 2006). But

**Table 5.6** Classifications of leading, synchronic, and stagnant indices

| | Leading index | Synchronic index | Stagnant index |
|---|---|---|---|
| Energy and raw materials | Total production of energy (− 6)* | | |
| Investment | Total investment in real estate (− 11) | | |
| Production | | Increase in light industry output (0)<br>Increase in heavy industry output (0) | |
| Finance | | | National income (+ 12)<br>National expenditure (+ 12) |
| Currency and credit | Currency in circulation (− 6)<br>Savings at financial institutions (− 6)<br>Loans issued by financial institutions (− 5) | Cash payout as salaries (+1) | Net amount of currency in circulation (+ 8) |
| Consumption | Gross retail amount (− 6) | | |
| Foreign trade | Gross amount of imports (− 9)<br>Direct investments by foreign entities (− 11) | | Gross amount of exports (+ 10) |
| Commodity price | | | Consumer price index (+ 11) |

* Numbers in parentheses stand for the time difference between indicated cycles and reference cycles

a question remains: what kind of incentive effect does these prizes have produced (Rosen, 1986)?

Fields Medal is recognized internationally as the highest academic award project in the field of mathematics. Mathematicians all over the world are proud to win the Fields Medal. Because there is no mathematics award in the Nobel Prize for natural science, Fields Medal is also known as the "Nobel Prize in mathematics".

In 1932, according to the proposal of the Canadian mathematician John Charles Fields, the 9th International Conference of mathematicians held in Zurich decided to establish an international mathematics award named after his surname—Fields Medal.

Fields Medal is awarded every four years. The award ceremony is held at the Quadrennial International Conference of mathematicians hosted by the International Mathematical Federation. Each time, it is awarded to 2 to 4 young mathematicians with outstanding contributions. Winners will receive a bonus of 15,000 Canadian dollars and a gold medal. According to the award rules, Fields Medal is only awarded to mathematicians under the age of 40 on January 1, the year of award.

Fields Medal was first awarded in 1936. By 2018, a total of 60 mathematicians in the world had won Fields Medal.

As a prestigious World Award, Fields Medal has played an important role in attracting a large number of talented young scholars to participate in mathematical research and solve the world's mathematical problems.

Unlike the Nobel Prize, Fields Medal is awarded only to mathematicians under the age of 40. Mathematicians over the age of 40, no matter how much academic achievements they have made, are not eligible for Fields Medal. If there are a large number of mathematicians who have made greater contributions than the winners and can not win the prize only because of their age, the fairness of such a "grand prize" is obviously debatable.

For those scholars who won Fields Medal, what effect does the award have on their research work?

In 2015, George J. Borjas and Kirk B. Doran with University of Notre Dame conducted an in-depth study on the effect of Fields Medal. They selected 142 mathematicians at first, including all 56 Fields Medal winners (Medalists) at that time and 86 mathematicians in the control group (Contenders). Then collected the data of the published academic papers and other relevant data every year from the beginning of their academic career to the age of 60. Trying to analyze the impact of Fields Medal on the research output of the winners according to the actual data (Borjas & Doran, 2015).

The 86 mathematicians of contenders are all the winners of other prestigious mathematics awards. Such as the Abel Prize and the Wolf Prize. Other important awards are issued by the American Mathematical Society which including the Cole Prize for algebra, the Bôcher Prize for mathematical analysis, the Veblen Prize for Geometry, and Salem Prize for Fourier Series. Most of the winners of Fields Medal were won the above awards at first, and then won their Fields Medal. Therefore, it can be said that the 86 mathematicians in the control group are scholars who have the strength to participate competition for Fields Medal and finally fail to win Fields Medal.

We divide the sequences of annual average number of papers published by the Medalists and the Contenders into two parts: 16 years before the award and 20 years after the award. The data sequences of annual average number of papers published by Medalists and Contenders for 16 years before the award are denoted by $X_M, X_C$ respectively. And the data sequences of annual average number of papers published by the Medalists and the Contenders for 20 years after the award are denoted by $Y_M, Y_C$ respectively.

Calculate the three term center moving average smoothing sequence of $X_M, X_C; Y_M, Y_C$. Still denoted by $X_M, X_C; Y_M, Y_C$ as before.

$$\begin{aligned} X_{\mathrm{M}} &= (x_M(1), x_M(2), \ldots, x_M(16)) \\ &= (2.64, 2.63, 2.70, 2.69, 2.80, 2.91, 3.01, 3.09, 3.38, \\ &\quad 3.63, 3.81, 3.83, 3.95, 3.75, 3.85, 3.62) \end{aligned}$$

$$\begin{aligned} X_{\mathrm{C}} &= (x_C(1), x_C(2), \ldots, x_C(16)) \\ &= (1.95, 2.11, 2.32, 2.72, 3.03, 3.48, 3.51, 3.61, 3.60, \\ &\quad 3.50, 3.59, 3.75, 4.02, 4.10, 4.02, 4.00) \end{aligned}$$

$$\begin{aligned} Y_{\mathrm{M}} &= (y_M(1), y_M(2), \ldots, y_M(20)) \\ &= (3.72, 3.50, 3.51, 3.35, 3.15, 2.90, 2.95, 2.92, 3.02, 3.01, 3.02, 3.10, \\ &\quad 3.25, 3.30, 3.40, 3.35, 3.33, 2.96, 2.72, 2.60) \end{aligned}$$

$$\begin{aligned} Y_{\mathrm{C}} &= (y_C(1), y_C(2), \ldots, y_C(20)) \\ &= (3.95, 3.90, 4.20, 4.40, 4.50, 4.53, 4.48, 4.46, 4.01, 4.54, 4.75, 4.72, 4.49, 4.23, 4.50, \\ &\quad 4.62, 4.91, 4.95, 5.24, 5.49) \end{aligned}$$

Calculate the zero-starting point sequences of $X_M, X_C; Y_M, Y_C$,

$$\begin{aligned} X_M^0 &= (0, -0.01, 0.06, 0.05, 0.16, 0.27, 0.37, 0.45, 0.74, \\ &\quad 0.99, 1.17, 1.19, 1.31, 1.11, 1.21, 0.98) \\ X_C^0 &= (0, 0.16, 0.37, 0.77, 1.08, 1.53, 1.56, 1.66, 1.65, 1.55, \\ &\quad 1.64, 1.80, 2.07, 2.15, 2.07, 2.05) \\ Y_M^0 &= (0, -0.22, -0.21, -0.37, -0.57, -0.82, -0.77, -0.80, \\ &\quad -0.70, -0.71, -0.70, -0.62, -0.47, -0.42, -0.32, \\ &\quad -0.37, -0.39, -0.76, -1.00, -1.12) \end{aligned}$$

$$\begin{aligned} Y_C^0 &= (0, -0.05, 0.25, 0.45, 0.55, 0.58, 0.53, 0.51, 0.06, 0.59, 0.80, 0.77, 0.54, 0.28, \\ &\quad 0.55, 0.67, 0.96, 1.00, 1.29, 1.07) \end{aligned}$$

From definition 2, we have

$$s_{X_M} = 9.56, s_{X_C} = 21.085$$

and

$$s_{Y_M} = -10.78, s_{Y_C} = 10.865$$

Therefore, both $X_M$ and $X_C$ are all increasing sequences. $Y_M$ and $Y_C$ are reverse sequences.

By formula (5), we have

$$\phi^N_{Y_M Y_C} = -\frac{|s_{Y_M} - s_{Y_C}|}{1 + |s_{Y_M} - s_{Y_C}|} \approx -0.96$$

Before the award, the average annual number of papers published by both the Medalists and the Contenders are all showed increasing trend, and the number was roughly the same. After winning the award, the average annual number of papers published by the Contenders is still an increase sequence, while the average annual number of papers published by the Medalists is an attenuation sequence, and the two show a strong inverse relation. The results clearly reveals that the “highest award” won by researchers in their prime of life has a significant reverse incentive effect on their research output.

At the same time, George J. Borjas and Kirk B. Doran are collected and analyzed the relevant data that can reflect the research “quality” of the Medalists and the Contenders. They found that from the data such as the citation of the papers, the quality of research work of the winners of Fields Medal were also significantly reduced (Borjas & Doran, 2015).

Borjas and Doran’s research further shows that, compared with the Contenders, more the Medalists changed their research direction after winning the prize. The Medalists are usually not as worried about the “failure” of the research as before. Therefore, the proportion of those who change the research direction in the Medalists is significantly higher than that in the Contenders (Borjas & Doran, 2015).

The Fields Medal not only won the winners social reputation and respect, but also produced a huge wealth effect. Many academic institutions have hired or hope to hire the winners of Fields Medal with high salaries, giving them more opportunities and choices. After winning the prize, some people began to “Revel in being sought after “ and “To play the game of life”, giving up their previous academic pursuit (Borjas & Doran, 2015). This may be one of the reasons for the reverse incentive effect.

Some people say, “small awards inspire people to forge ahead, and big awards stop people.” maybe it’s not unreasonable.

***Example 5.10.3*** **Grey relational analysis of consumer price indices of various foods in Shaanxi Province, China** (Liu et al., 2024)

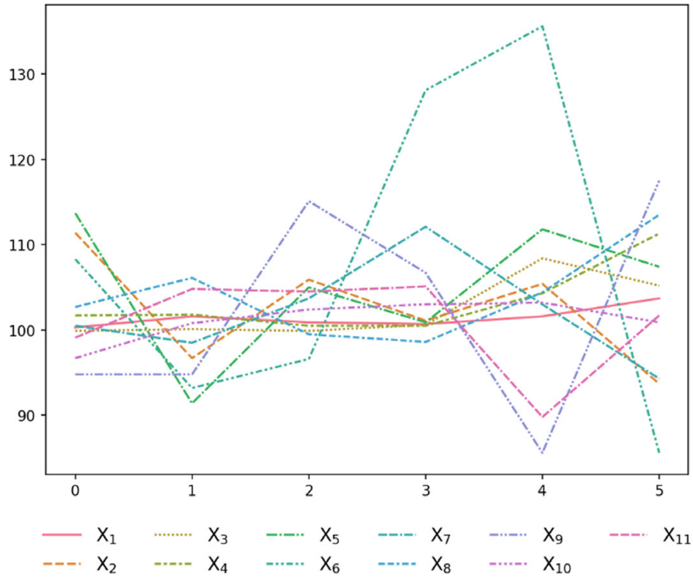



**Fig. 5.10** The line graphs of the consumer price indices of 11 main foods

An in-depth analysis will be conducted on the relationship between various food consumer price indices in Shaanxi Province, China. Here, the consumer price indices of 11 main foods including Grain, Potato, Beans, Edible oil, Vegetables, Livestock meat, Poultry, Aquatic products, Eggs, Milk, and Fruits, etc. are considered.

The data sequences of the consumer price indices of Grain, Potato, …, and Fruits are recorded as $X_1$, $X_2$, …, and $X_{11}$ respectively. Based on the actual data from 2016 to 2021, the line graphs of $X_1$, $X_2$, …, and $X_{11}$ can be drawn as Fig. 5.10.

From Fig. 5.10, it can be seen that the relationship between the consumer price index curves of 11 major foods is very complex. Especially since most curves intersect with each other, traditional grey relational analysis models cannot be used to calculate the correlation degree of the consumer price index sequences of 11 major foods.

For the convenience of discussion, we will refer to the classification of substitutes and complements in economics and divide the 11 main foods into 5 groups as follows:

(1) Group 1: Grain category including Grain, Potato, Beans;
(2) Group 2: Meat category including Livestock meat, Poultry, and Aquatic products;
(3) Group 3: Livestock meat, Eggs, and Milk;
(4) Group 4: Grain, Vegetables, and Livestock meat;
(5) Group 5: Vegetables, Edible oil, and Fruits.

According to the conclusions of economic analysis, there is a mutual substitution relationship between the foods in group 1, group 2 and group 3. And there is a complementary relationship between the foods in group 4 and group 5.

Based on the actual data from 2012 to 2021, the line graphs of the data sequences in group 1, group 2 and group 3, group 4 and group 5 can be drawn as Figs. 5.11, 5.12, 5.13, 5.14 and 5.15.

According to economic theory, there is a positive correlation between the price indices of substitutes, while the price indices of complementary products have a reverse correlation. But it's very difficult to find the positive or reverse correlation from Figs. 5.11, 5.12, 5.13, 5.14 and 5.15. We need to use the grey relational analysis model of cross sequences and reverse sequences (Liu, 2023) for in-depth quantitative research.

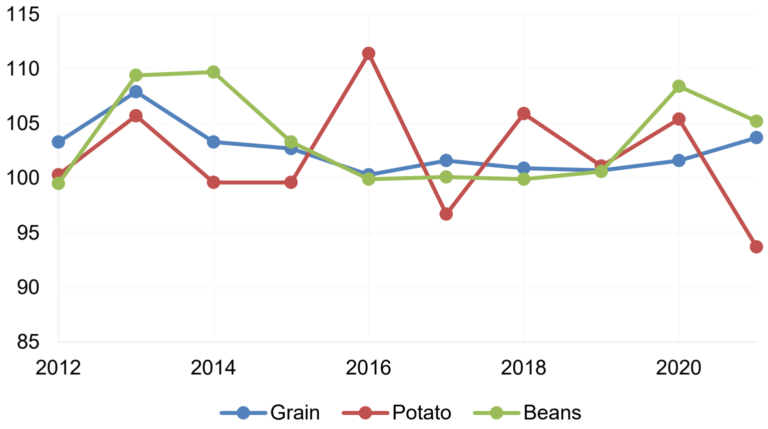



**Fig. 5.11** The line graphs of the data sequences in group 1

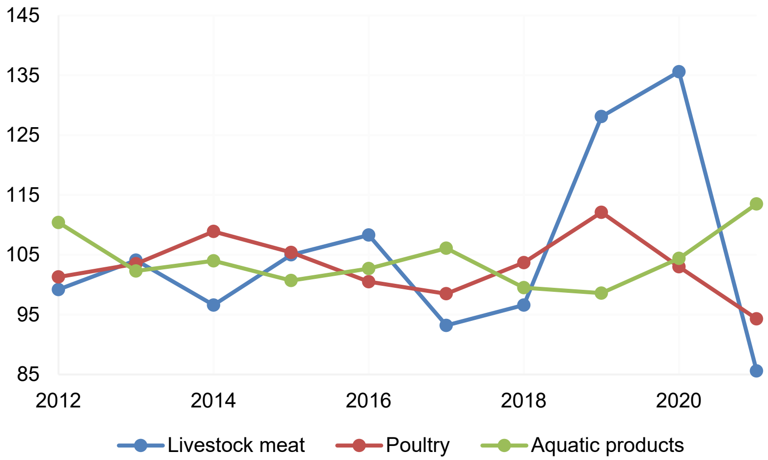



**Fig. 5.12** The line graphs of the data sequences in group 2

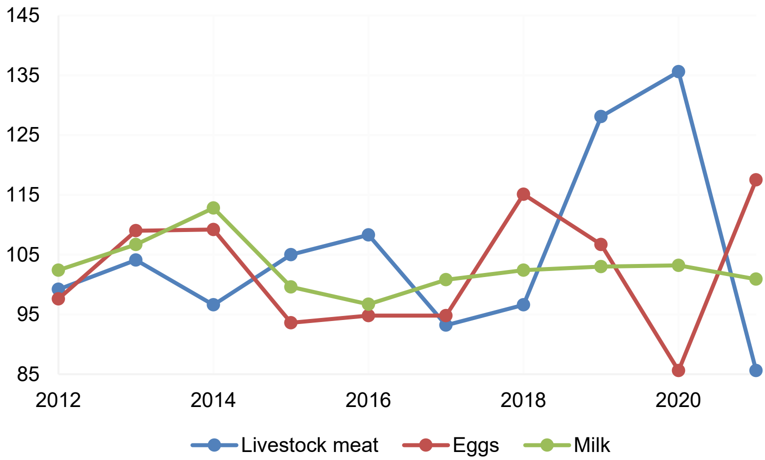



**Fig. 5.13** The line graphs of the data sequences in group 3

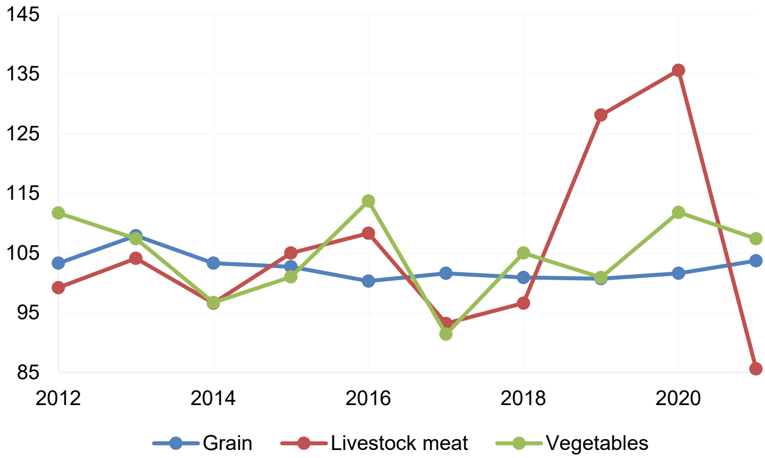



**Fig. 5.14** The line graphs of the data sequences in group 4

For group 1, from the Statistical Yearbooks of Shaanxi Province of 2013–2022, we can find out the consumer price index sequences of Grain, Potato, and Beans as follows:

$$\text{Grain } X_1 = (103.3, 107.9, 103.3, 102.7, 100.3, 101.6, \\ 100.9, 100.7, 101.6, 103.7)102.6.$$

$$\text{Potato } X_2 = (100.3, 105.7, 99.6, 99.6, 111.4, 96.7, \\ 105.9, 101.1, 105.4, 93.7)101.94.$$

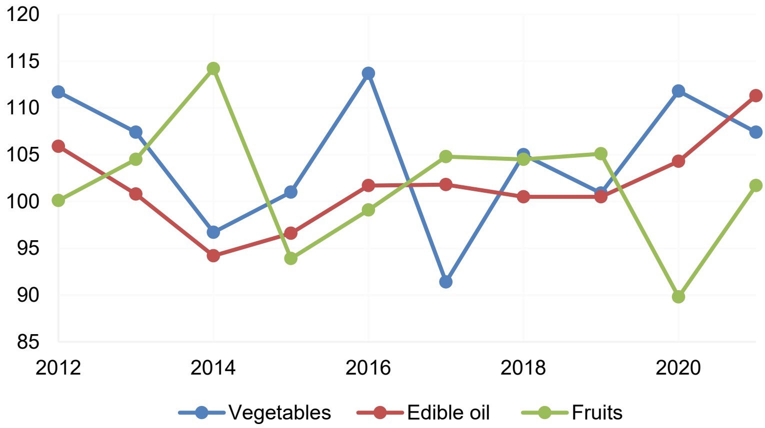



**Fig. 5.15** The line graphs of the data sequences in group 5

$$\text{Beans } X_3 = (99.5, 109.4, 109.7, 103.3, 99.9, 100.1, \\ 99.9, 100.6, 108.4, 105.2)103.6.$$

The mean sequences of $X_1$, $X_2$, and $X_3$ as follows:

$$Y_1 = (1.01, 1.05, 1.01, 1, 0.98, 0.99, 0.98, 0.98, 0.99, 1.01)$$
$$Y_2 = (0.98, 1.04, 0.98, 0.98, 1.09, 0.95, 1.04, 0.99, 1.03, 0.92)$$
$$Y_3 = (0.96, 1.06, 1.06, 1, 0.96, 0.97, 0.96, 0.97, 1.05, 1.02)$$

The starting point zero sequences of $Y_1$, $Y_2$, and $Y_3$ as follows:

$$Y_1^0 = (0, 0.04, 0, -0.01, -0.03, -0.02, -0.03, -0.03, -0.02, 0)$$
$$Y_2^0 = (0, 0.06, 0, 0, 0.11, -0.03, 0.06, 0.01, 0.05, -0.06)$$
$$Y_3^0 = (0, 0.1, 0.1, 0.04, 0, 0.01, 0, 0.01, 0.09, 0.06)$$

Therefore, we can obtain the following results

$$s_1 = -0.1\ \|s_1\| = 0.15$$
$$s_2 = 0.23\ \|s_2\| = 0.35$$
$$s_3 = 0.38\ \|s_3\| = 0.38$$

The calculation results are different from the conclusions of traditional economic research. In traditional economic theory, Potato and Beans are substitute for grain. But the opposite sign between $s_1$ and $s_2$,and between $s_1$ and $s_3$ indicates a complementary relationship between Potato and grain, and Beans and grain. In fact, the results obtained in this article are reasonable. Because in China, people usually make potatoes into dishes, and make soybeans into tofu, bean sprouts, etc. They are indeed to a large extent complementary to grain as sideline foods.

From

$$s_1 - s_2 = -0.33 \; \|s_1 - s_2\| = 0.41$$
$$s_1 - s_3 = 0.48 \; \|s_1 - s_3\| = 0.48$$
$$s_2 - s_3 = -0.15 \; \|s_2 - s_3\| = 0.49$$

we have

$$\varepsilon_{12} - \Delta_{12} = 0.75 - 0.21 = 0.54 \quad \phi_{12}^N = -\frac{|s_1 - s_2|}{1 + |s_1 - s_2|} = -0.25$$
$$\varepsilon_{13} - \Delta_{13} = 0.68 - 0.24 = 0.44 \quad \phi_{13}^N = -\frac{|s_1 - s_3|}{1 + |s_1 - s_3|} = -0.32$$

And

$$\varepsilon_{23} - \Delta_{23} = 0.87 - 0.22 = 0.65$$

Here, $\phi_{12}^N = -0.25 < \phi_{13}^N = -0.32$

This indicates that the complementarity between soybeans and grains is stronger than that between potatoes and grains. And there is a significant substitution relationship between soybeans and potatoes.

Similarly, we can obtain the results for group 2-group 5 as follows:

For group 2, we have

$$s_6 = 0.675 \quad \|s_6\| = 0.985$$
$$s_7 = 0.235 \quad \|s_7\| = 0.365$$
$$s_8 = -0.605 \|s_8\| = 0.635$$

$$s_6 - s_7 = 0.44 \; \|s_6 - s_7\| = 0.86$$
$$s_6 - s_8 = 1.2 \; \|s_6 - s_8\| = 1.46$$
$$s_7 - s_8 = 0.84 \; \|s_7 - s_8\| = 0.94$$

$$\varepsilon_{67} - \Delta_{67} = 0.69 \text{ - } 0.27 = 0.42$$
$$\varepsilon_{68} - \Delta_{68} = 0.45 \text{ - } 0.36 = 0.09$$

$$\phi_{68}^N = -\frac{|s_6 - s_8|}{1 + |s_6 - s_8|} = -0.55$$
$$\varepsilon_{78} - \Delta_{78} = 0.54 - 0.32 = 0.22$$

$$\phi_{78}^N = -\frac{|s_7 - s_8|}{1 + |s_7 - s_8|} = -0.46$$

The opposite sign between $s_6$ and $s_8$, and between $s_7$ and $s_8$ indicates a complementary relationship between Livestock meat and Aquatic products, and Poultry and Aquatic products. As an inland province, The aquatic products' consumption of residents in Shaanxi Province is much lower than that of coastal areas. Here, in people's daily lives, aquatic products are difficult to replace meat.

For group 3, we have

$$\begin{aligned} s_6 &= 0.675 \; \|s_6\| = 0.985 \\ s_9 &= 0.4 \; \|s_9\| = 0.78 \\ s_{10} &= 0.02 \; \|s_{10}\| = 0.26 \end{aligned}$$

$$\begin{aligned} s_6 - s_9 &= 0.365 \quad \|s_6 - s_9\| = 1.355 \\ s_6 - s_{10} &= 0.655 \; \|s_6 - s_{10}\| = 1.105 \\ s_9 - s_{10} &= 0.29 \quad \|s_9 - s_{10}\| = 0.53 \end{aligned}$$

$$\begin{aligned} \varepsilon_{69} - \Delta_{69} &= 0.73 - 0.33 = 0.40 \\ \varepsilon_{610} - \Delta_{610} &= 0.60 - 0.33 = 0.27 \\ \varepsilon_{910} - \Delta_{910} &= 0.78 - 0.21 = 0.57 \end{aligned}$$

The sign of $s_6$, $s_9$, and $s_{10}$ are the same. The results indicate that Livestock meat, eggs and milk are substitutes for each other. The calculation results both of grey similarity relational degrees and the modified grey similarity relational degrees indicate that the substitution relationship between eggs and milk is the strongest, and compared to the substitution relationship between Livestock meat and eggs, the substitution relationship between Livestock meat and milk is weaker.

For group 4, we have,

$$\begin{aligned} s_1 &= -0.1 \; \|s_1\| = 0.15 \\ s_5 &= -0.5 \; \|s_5\| = 0.54 \\ s_6 &= 0.675 \; \|s_6\| = 0.985 \end{aligned}$$

$$\begin{aligned} s_1 - s_5 &= 0.58 \quad \|s_1 - s_5\| = 0.72 \\ s_1 - s_6 &= -0.775 \; \|s_1 - s_6\| = 1.005 \\ s_5 - s_6 &= 1.355 \quad \|s_5 - s_6\| = 1.445 \end{aligned}$$

$$\varepsilon_{15} - \Delta_{15} = 0.63 - 0.30 = 0.33$$

$$\varepsilon_{16} - \Delta_{16} = 0.56 - 0.40 = 0.16 \quad \phi_{16}^N = -\frac{|s_1 - s_6|}{1 + |s_1 - s_6|} = -0.44$$

$$\varepsilon_{56} - \Delta_{56} = 0.42 - 0.36 = 0.06 \quad \phi_{56}^N = -\frac{|s_5 - s_6|}{1 + |s_5 - s_6|} = -0.58$$

The sign of $s_1$ and $s_5$ are the same. This result tells us that the relationship between grains and vegetables is not entirely complementary, and the two can sometimes be substitutes for each other. In fact, the total food intake of a person is relatively stable. As people's vegetable consumption increases, the amount of grain consumed will naturally decrease. The calculation results of negative grey similarity relational degrees $\phi_{56}^N = -0.58 < \phi_{16}^N = -0.44$ indicate that compared to the complementary relationship between Livestock meat and grain, the complementary relationship between Livestock meat and vegetables is stronger.

For group 5, we have,

$$\begin{aligned} s_4 &= -0.425 \quad \|s_4\| = 0.475 \\ s_5 &= -0.5 \quad \|s_5\| = 0.54 \\ s_{11} &= 0.18 \quad \|s_{11}\| = 0.52 \end{aligned}$$

$$\begin{aligned} s_4 - s_5 &= 0.255 \quad \|s_4 - s_5\| = 0.435 \\ s_4 - s_{11} &= -0.605 \ \|s_4 - s_{11}\| = 0.795 \\ s_5 - s_{11} &= -0.86 \quad \|s_5 - s_{11}\| = 1.08 \end{aligned}$$

$$\begin{aligned} \varepsilon_{45} - \Delta_{45} &= 0.80 - 0.18 = 0.62 \\ \varepsilon_{4,11} - \Delta_{4,11} &= 0.62 - 0.28 = 0.34 \\ \phi_{4,11}^N &= -\frac{|s_4 - s_{11}|}{1 + |s_4 - s_{11}|} = -0.38 \\ \varepsilon_{5,11} - \Delta_{5,11} &= 0.54 - 0.34 = 0.20 \\ \phi_{5,11}^N &= -\frac{|s_5 - s_{11}|}{1 + |s_5 - s_{11}|} = -0.46 \end{aligned}$$

The sign of $s_4$ and $s_5$ are the same. This result tells us that the relationship between edible oil and vegetables is not entirely complementary, and the two can sometimes be substitutes for each other. In northern China, residents prefer fried foods. Some people seek convenience by eating fried foods to avoid the trouble of cooking vegetables. The calculation results of negative grey similarity relational degrees $\phi_{5,11}^N = -0.46 < \phi_{4,11}^N = -0.38$ indicate that compared to the complementary relationship between edible oil and fruits, the complementary relationship between vegetables and fruits is stronger.

The grey relational analysis models of cross sequences and reverse sequences was used to analyze the consumer price indices of various foods in Shaanxi Province. We have obtained some interesting results that are not completely consistent with general economic theory. The group analysis results of the foods consumer price indies in Shaanxi Province indicate that there is also a certain degree of substitution relationship between foods that are considered complementary by people. There is also a certain degree of complementarity between foods that are considered to be substitutable. There is no complete substitution or complementary relationship between different types of food. In the event of a shortage of supply in a certain food market, complementary foods may also replace each other. With the improvement of people's living standards, the diversity of food has become the norm, and the complementary relationship between alternative foods will gradually strengthen. To some extent, it may be possible to consider using the complementary relationship between alternative foods as a criterion to determine the level of living standards of people in different regions.

## References


Amanna, A., Price, M. J., & Thamvichai, R. (2011). Grey systems theory applications to wireless communications. *Analog Integrated Circuits and Signal Processing, 69*, (2–3), SI:259–269.

Aydemir, E., & Sahin, Y. (2019). Evaluation of healthcare service quality factors using grey relational analysis in a dialysis center. *Grey Systems Theory & Application, 9*(4), 432–448.

Borjas, G. J., & Doran, K. B. (2015). Prizes and productivity: How winning the fields medal affects scientific output. *Journal of Human Resources, 50*(3), 728–758.

Chen, K. J., & Liu, S. F. (2005). Time difference analysis of economic indicators based on grey relational model. In *Proceedings of IEEE international conference on networking, sensing and control* (pp. 311–342).

Chen, L., Qin, Y., Deng, R. R. (2011). Comparative analysis of the two common methods for skylight measurement based on the ASD spectroradiometer. *Tropical Geography, 31*(2), 182–186.

Chen R. H., Song Z. Q., & Kang L. M. (2005). Application of grey system to reservoir evaluation in the seven district of Xinjiang Karamay Oilfield. *Inner Mongolia Petroleum Chemical Industry, 7*, 110–113 (In Chinese).

Dang, Y. G., & Liu, S. F. (2004). The GM models that x(n) is taken as initial value. *Kybernetes: The International Journal of Systems & Cybernetics, 33*(2), 247–254.

Das, P. P., & Chakraborty, S. (2024). Dynamic grey relational analysis-based optimization of electrochemical machining processes. *World Journal of Engineering*, published online. https://doi.org/10.1108/WJE-10-2023-0436

Delcea, C., Scarlat, E., MaracineV. (2012). Grey relational analysis between firm's current situation and its possible causes: A bankruptcy syndrome approach. *Grey Systems: Theory and Application, 2*(2), 229–239.

Delcea, C., Bradea, I., & Scarlat, E. (2013). A computational grey based model for companies risk forecasting. *The Journal of Grey System, 25*(3), 70–83.

Deng, J. L. (1985). *Grey control systems*. Press of Huazhong University of Science and Technology.

Huang, S. Y. (2006). Entropy model of grey systems and its application in oil-bearing area evaluation. *Journal of Geomechanics, 12*(1), 77–83 (In Chinese).

Ejnioui, A., Otero, C. E., & Otero, L. D. (2013). Prioritisation of software requirements using grey relational analysis. *International Journal of Computer Applications in Technology, 47*(2–3), 100–109.

Gad, M., El-Shafay, A., Alqsair, U., Abulut, & Attia, E. A. (2023). Multi-objective optimization based grey relational analysis and investigation of using the waste animal fat biodiesel on the engine characteristics. *Fuel*, 127950.

Gai, R. L., & Guo, Z. B. (2023). A water quality assessment method based on an improved grey relational analysis and particle swarm optimization multi-classification support vector machine. *Frontiers in Plant Science, 14*, 1099668.

Ghosh, N., & Banerjee, I. (2019). IoT-based freezing of gait detection using grey relational analysis. *Internet of Things*, 100068.
Gui, Y. F., Xia, G. F., & Deng, L. C. (2004). Grey Incident Degree in Linear Norm Space[J]. *Journal of Wuhan University of Technology*, *28*(3), 399–403.
Jabbar, M., Shaker, K., Umair, M., & Nawab, Y. (2017). Optimizing the performance of woven protective gloves using grey relational analysis. *The Journal of the Textile Institute, 108*(10), 1715–1719.
Javed, S. A., & Liu, S. F. (2019). Bidirectional Absolute GRA/GIA model for Uncertain Systems: Application in Project Management. *IEEE Access*, *7*(1), 60885–60896.
Jiang, S. Q., Liu, S. F., & Liu, Z. X. (2019). General grey relational decision model based on moment estimation theory and its application. *Statistics and Decision Making, 35*(24), 29–33.
Karumuri, S., Das, V. C., & Mallarapu, G. K. (2022). Tribological properties of Al 7075 composite reinforced with ZrB2 using grey relational. *Advances Progress in Science and Technology-Research Journals, 16*(4), 22–28.
Kumar, U., Sing, M., & Singh, S. (2021). Wire-electrochemical discharge machining of SiC reinforced Z-pinned polymer matrix composite using grey relational analysis. *Silicon, 13*(3), 777–786.
Liu, S. F. & Guo, T. B. (1991). Grey System Theory and Its Applications. Kaifeng: Henan University Press.
Liu, S. F., Yang, Y. J., & Forrest, J. (2022). Grey Systems Analysis. Springer-Verlag, Singapore.
Li, X., Wang, Z., Zhang, L., Zou, C., & Dorrell, D. D. (2019). State-of-health estimation for Li-ion batteries by combing the incremental capacity analysis method with grey relational analysis. *Journal of Power Sources, 410–411*, 106–114.
Liang, B., Dai, Y. Y., Chen, T. Y., Sun, W. J., & Qin, B. (2014). Grey correlation optimization for shale gas exploration and development areas of complicated geological parameter features. *Journal of China Coal Society, 39*(3):524–530.
Lin, J. J., Ren, H. Q., & Shen, Z. W. (2009). Study on primary influence factors for application of grey system theong to velocity of explosive forming prjectile. *Journal of Projectiles, Rockets, Missiles and Guidance, 29*(3), 112–116.
Liu, S. F. (1992). Generalized degree of grey incidence. *Zhang Shengkai, Information and systems* (pp. 113–116). DMU Publishing House.
Liu, S. F., Xie, N. M., & Forrest, J. (2011). Novel models of grey relational analysis based on visual angle of similarity and nearness. *Grey Systems: Theory and Application, 1*(1), 8–18.
Liu, S. F. (2024). *Grey system theory and its application* (10th ed.). Science Press.
Liu, S. F. (2023). Negative grey relational model and measurement of the reverse incentive effect of Fields Medal. *Grey Systems: Theory and Application, 13*(1), 1–13.
Liu, S. F., Yang, Y. J., Cao, Y., et al. (2013). A summary on the research of GRA models. *Grey Systems: Theory and Application, 3*(1), 7–15.
Liu, S. F., Yang, Y. J., Xie, N. M., & Forrest, J. (2016). New progress of grey system theory in the new millennium. *Grey Systems Theory and Application, 6*(1), 2–31.
Liu, S., Lu, N., Shang, Z., & Rathnayaka, R. M. K. T. (2024). A new grey relational analysis model of cross-sequences. *Grey Systems: Theory and Application, 14*(2), 299–317.
Liu, W. F. (2013). Generalized incidence analysis model of interval grey number. *Journal of Zhengzhou University, 45*(2), 41–44, 89 (In Chinese).
Yaoxin, L., Tianhua, Y., & Rundong, L. (2007). Grey correlation analysis and prediction model for formation reaction of high-temperature sulphur-fixing phase sulphocal ciumaluminate. *Thermal Power Generation, 6*, 37–40 (In Chinese).
Lu, N., Liu, S., Du, J., Fang, Z., Dong, W., Tao, L., & Yang, Y. (2023). Grey relational analysis model with cross-sequences and its application in evaluating air quality index. *Expert Systems with Applications, 233*, 120910.
Lv, H., Chen, X., Wang, X., Zeng, X., & Ma, Y. (2022). A novel study on a micromixer with cantor fractal obstacle through grey relational analysis. *International Journal of Heat and Mass Transfer, 183*, 122159.

Mandal, P., Chatterjee, S., & Chakraborty, S. (2024). Bi-objective optimization of an EDM process with Cu-MWCNT composite tool using single-valued neutrosophic grey relational analysis. *World Journal of Engineering*, published online. https://doi.org/10.1108/WJE-10-2023-0443

Mehlawat, M. K., Gupta, P. & Khan, A. Z. (2023). An integrated fuzzy-grey relational analysis approach to portfolio optimization. *Applied Intelligence, 53*, 3804–3835.

Mishra, V., Kumar, A., Bhatt, N. K., Negi, S. (2024). Parametric optimisation of 3D-printed PLA/wood dust composite for load-bearing application using grey relational analysis. *Proceedings of the Institution of Mechanical Engineers, Part E: Journal of Process Mechanical Engineering.* https://doi.org/10.1177/09544089241241460

Olson, D. L., & Wu, D. S. (2006). Simulation of fuzzy multiattribute models for grey relationships. *European Journal of Operational Research, 175*(1), 111–120.

Pal, R., Saraswat, M., & Mittal, H. (2021). Improved bag-of-features using grey relational analysis for classification of histology images. *Complex & Intelligent Systems, 7*(3), 1429–1443.

Rajesh, R.; & Ravi, V. (2015). Supplier selection in resilient supply chains: a grey relational analysis approach. *Journal of Cleaner Production*, *86*, 343–359.

Ravikumar, N., Vijayan, R., & Viswanathan, R. (2023). Multi-response optimisation for turning of magnesium alloy with untreated and cryogenic treated carbide inserts by grey relational analysis. *Journal of Ceramic Processing Research, 24*(1), 142–152.

Rosen, S. (1986). Prizes and incentives in elimination tournaments. *American Economic Review, 76*, 701–715.

Sarikaya, M., & Güllü, A. (2015). Multi-response optimization of minimum quantity lubrication parameters using Taguchi-based grey relational analysis in turning of difficult-to-cut alloy Haynes 25. *Journal of Cleaner Production, 91*, 347–357.

Scarlat, E., & Delcea, C. (2011). Complete analysis of bankruptcy syndrome using grey systems theory. *Grey Systems: Theory and Application, 1*(1), 19–32.

Scotchmer, S. (2006). *Innovation and incentives*. MIT Press.

Shi, B. Z. (1995). Presentation of range incidence quantity. *Journal of China University of Petroleum, 19*(5), 114–116.

Shi, Q. Y., Huang, Z. Y., Li, A. Q. (2008). Experimental investigation on ductility of U-section steel encased concrete composite beams. *Journal of Southwest Jiaotong University, 43*(2), 206–212.

Skrinjari, T. (2020). Dynamic portfolio optimization based on grey relational analysis approach. *Expert Systems with Applications, 147*, 113207.

Tang, W. X. (1995). The Concept and the Computation Method of T's Correlation Degree. *Application of Statistics and Management*, 1, 34–37.

Tan, X. R., Deng, J. L. (1995). Grey Incidence Analysis: A New Method of Multi-factor Statistics[J]. *Statistical Research*, *3*, 43–45.

Tan, G. J., Wang, L. L., & Cheng, Y. C. (2011). Prediction method for cable tension state of cable-stayed bridges based on grey system theory in could areas. *Journal of Jilin University (Engineering and Technology Edition), 41*(2), 170–173 (In Chinese).

Twala, B. (2014). Extracting grey relational systems from incomplete road traffic accidents data: The case of Gauteng Province in South Africa. *Expert Systems, 31*(3). 220–231.

Vasanthi, P., & Selvan, S. S. (2022). Optimization of mixing parameters in nanosilica toughened cement mortar using Taguchi-grey relational analysis. *Silicon, 14*(1), 127–133.

Wang, Q. Y. (1989). Analysis of B's Correlation Degree [J]. *Journal of Huazhong University of Technology*, *17*(6), 77–81.

Wang, X. Z., Wang, Y. F., Tseng, Y. Y., et al. (2024). Integration of the grey relational analysis with machine learning for sucrose anaerobic hydrogen production prediction. *International Journal of Hydrogen Energy, 68*, 388–397.

Wang, Y., Cheng, Z. H., Wang, H. Y., et al. (2011). Application of gray relational cluster method in muon tomography. *Nuclear Electronics & Detection Technology., 31*(8), 871–873.

Wang, Y. T., & Li, H. J. (2024). A novel single-sample retinal vessel segmentation method based on grey relational analysis. Sensors, published online. https://doi.org/10.3390/s24134326

Wang, Y., Zhou, T., Zhang, M., Li, X. H., Ding, Y., & Qian, F. C (2013). Application of grey relational analysis in Yaojialing Zn–Au polymetallic deposit prediction. *Journal of Hefei University of Technology, 36*(10), 1236–1241.

Wei, H., Lin, L., Zhang, Y., Wang, L., & Chen, Q. (2013). Research on the application of grey system theory in the pattern recognition for chromatographic fingerprints of traditional Chinese medicine. *Chinese Journal of Chromatography, 31*(2), 127–132.

Weng, L. J., Zhang, Q. S., Zhibin Lin, Z. B., & Wu, L. (2021). Harnessing heterogeneous social networks for better recommendations: A grey relational analysis approach. *Expert Systems with Applications, 174*, 114771.

Xia, X. T., Wang, Z. Y., & Chang, H. (2005). Degree of grey incidence for the quality of processing and vibration of rolling bearing. *Journal of Aerospace Power, 20*(2), 250–254.

Xiao, J., & Zhang, W. W. (2009). Grey incidence analysis applied to fault diagnosis of drone crash. *Journal of Sichuan Ordnance Engineering, 30*(9), 112–115.

Xie, W. M., et al. (2020). Optimizing soil dissolved organic matter extraction by grey relational analysis. *Pedosphere, 30*(5), 589–596.

Xie, Y. M., Yu, L. P., & Chen J. (2007). Application of grey theory in deep drawing robust design. *Chinese Journal of Mechanical Engineering, 43*(3), 54–59.

Xiong, H. J (2000). Some Extensions of Grey Relational Grade Formula. *Systems Engineering and Electronics*, *22*(5), 75–77

Xu, G. R. (1993). Improvement of Generalized Degree of Grey Incidence and Standard of Multi-target Grey Correlation Decision[C]. Liu Sifeng: Grey System Theory and Applications Conference Proceedings. Kaifeng: Hennan University Press, 96–100.

Yan, J., Zhu, J. X., Lu, N., et al. (2019). Superior grey relational analysis on blood lipids and hematological parameters. *Grey Systems Theory & Application, 9*(2), 207–212.

Zhao, Y. L, Wei, S. Y, & Mei ,Z. X. (1998a). Grey Euclid Relation Grade. *Journal of Guangxi University*, *1*, 10–13.

Zhao, Y. L., Wei, S. Y., & Mei, Z. X. (1998b). A new theoretical model for analysis of grey relation. *Systems Engineering and Electronics*, *9*(10), 34–36.

Zhao, H., Ma, L. Y., & Jia, Q. (2007a). The theory and application of grey coincidence analysis model based on variant coefficient. *Heilongjiang Science and Technology of Water Conservancy, 35*(2), 26–27.

Zhang, C., Ding, S. B., & Wang, B. (2014). Study on customization model of aircraft based on grey incidence analysis. *Traffic Information and Safety, 32*(4), 131–136.

Zhang, J., Liang, S., Zhou, R., Yao, L., & Luo, P. (2012). Fault tree analysis of two teeth difference swing movable teeth transmission based on grey correlation. *Machinery Design & Manufacture, 6*, 183–185.

Zhang, K., & Liu, S. F. (2009). A novel algorithm of image edge detection based on matrix degree of grey incidence. *The Journal of Grey System, 19*(3), 265–276.

Zhang, Q. S., Guo, X. J., & Deng, J. L. (1996). The entropy analysis method of grey incidences. *Systems Engineering: Theory and Practice, 16*(8), 7–11.

Zhang, Q. Y., Zhou, X. H., & Wang, W. T. (2007). Evaluation of defending efficiency in engineering based on improved grey interrelated analysis method. *Journal of PLA University of Science and Technology (Natural Science Edition), 8*(3), 283–287.

Zhou, J., Chen, Y. P., & Liang, J. (2005). Application of Gray Vector Relational An alysis to Location of the Fault-phase on Transmission Line[J]. *High Voltage Engineering*, *31*(3), 12–14.

Zhao, G. G., Sun, Y. K., & Xu, Y. J. (2007). Gray decision analysis of threat estimation in ant-missile combat of SurfaceWarship. *Tactical Missile Technology, 3*, 32–35.

Zhou, T., & Peng, C. H. (2008). Application of grey model on analyzing the passive natural circulation residual heat removal system of HTR-10. *Nuclear Science and Techniques, 19*(5), 308–313.

# Chapter 6
# Grey Clustering Evaluation Models

## 6.1 Introduction

Grey clustering is a method developed for classifying observation indices or observation objects into definable classes using grey incidence matrices or grey possibility functions. Each cluster can be seen as a set consisting of all the observational objects of a kind. When investigating practical problems, it is often the case that each observational object possesses quite a few characteristic indices, which are difficult to accurately classify. Depending on the objects to be clustered, grey clustering can be based on two methods: clustering using GRA models, and clustering using grey possibility functions. The first method is mainly applied to group the same kinds of factors into their individual categories, so that a complicated system can be simplified. By using the clustering method of grey relational analysis, we can examine whether or not some of the factors under consideration really belong to the same kind. This allows a synthetic index of these factors, or one of these factors, to be used to represent all factors without losing any part of the available information carried by such factors. This problem regards the selection of variables to be used in the study of a system. Before conducting a large-scale survey, which generally costs a lot of money and man power, by using the clustering method of grey relational analysis on a typical sample data, one can reduce the amount of data collection to a minimal level by eliminating the unnecessary variables so that tangible savings can be achieved.

The clustering method based on grey possibility functions is mainly used for checking whether or not the observational objects belong to pre-determined classes so that they can be treated differently. In practice, we need to set the possibility functions and the weights for different criterion according to the corresponding clustering index and the grey classes we intend to partition if using the clustering method based on grey possibility functions.

Grey clustering evaluation models using possibility functions are used widely for uncertain systems analysis. For the past four decades, much research on modeling



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_6

techniques has been done, and new research results emerge constantly. For example, Professor Julong Deng has proposed the variable weight grey clustering model (Deng, 1985), while Professor Liu et al. has proposed the fixed weight grey clustering evaluation model (Liu, 1993), the grey clustering evaluation model using end-point triangular possibility functions (Liu & Guo, 1991; Liu & Zhu, 1993), the grey cluster evaluation model using center-point triangular possibility functions (Liu & Xie, 2011),the grey cluster evaluation model using mixed possibility functions (Liu et al., 2015a, 2015b), among others. Grey variable weight clustering model is applicable to the problems with criteria that have the same meanings and dimensions. When the criteria for clustering involve different meanings and dimensions, the fixed weight grey clustering evaluation model and grey clustering evaluation model using triangular possibility functions are suitable. In particular, compared with the variable weight grey clustering and fixed weight grey clustering models, the grey clustering evaluation model using triangular possibility functions is more suitable for problems of poor information clustering evaluation. The grey clustering evaluation model using mixed end-point triangular possibility functions is suitable for situations where all grey boundaries are clear, but where the most likely points belonging to each grey class are unknown. Conversely, the grey clustering evaluation model using mixed center-point triangular possibility functions is suitable for problems where it is easy to judge the most likely points belonging to each grey class, but where the grey boundaries are unclear (Liu et al., 2015a, 2015b). Additionally, both of the last two grey clustering evaluation models based on mixed possibility function which composed by the possibility function of moderate measure, the possibility function of lower measure, and the possibility function of upper measure.

Further, Xiao (1997), Xiong and Chen (1999), Dong et al. (2010), Pei et al. (2012), Xu0 (2006), Zhou et al.(2013), Liu et al. (2013), Dang et al.(2017), Qian et al. (2016), Sun et al. (2022), Zhang and Dang (2023), Li and Li(2023), and others are improved and optimized grey clustering evaluation models from different perspectives. Furthermore, Zhang (2002) has studied the measurement problem of Grey Characteristics of Grey Clustering Result. He has investigated the relation between a grey clustering analysis result and the entropy of the weight sequence, and proposed a measure method for the grey characteristics of a grey clustering analysis result.

Grey clustering models based on mixed possibility functions are used widely. For example, Caixin Sun and his group used to online monitoring and diagnosis technology of the state of power transmission and transformation equipment (Sun, 2005; Yuan et al., 2005). Liang, et al. applied in torpedo life cycle cost effectiveness evaluation (Liang et al., 2007). Zhang et al. applied in safety evaluation of carier aircraft system (Zhang et al., 2010). Yao and Hu used to analysis the operational efficiency of OTH Ground-Wave Radar (Yao & Hu, 2008). Han et al. used to evaluate the development scheme of warhead in antimissile missile of air defense (Han et al., 2014).Chen, et al. used in evaluation of radar netting operational effectiveness (Chen et al., 2019). Su and Xie applied in safety evaluation of civil aircraft (2018). Delcea, et al applied to reverse pyramid boarding method (Delcea et al., 2022)

Fang Peng applied in evaluation of oil and gas cap layer (Peng et al., 2005). Fang, et al. applied in Coal and Gas Outburst Prediction (Fang et al., 2012). Jiskani et al. applied in safety evaluation in mines (2021).

Maleki and Taghavi Fard used in scheduling evaluation (2015). Ţilică et al. used in portfolio management under capital market frictions (Ţilică et al., 2024). Liu and his group used in disabled elders evaluation (2019). Delgado and Romero applied in environmental conflict analysis (Delgado & Romero, 2016), etc.

In this chapter, two novel grey cluster evaluation models based on mixed center-point triangular possibility functions and mixed end-point triangular possibility functions are put forward. These new grey clustering models based on mixed possibility functions are especially applicable to evaluation and classification of poor information objects, and have broad application prospects.

## 6.2 Grey Relational Clustering Model

**Definition 6.2.1** Assume that there are $n$ observational objects. For each object the data of $m$ attribute indexes are collected, producing the following sequences:

$$\begin{aligned} X_1 &= (x_1(1), x_1(2), \ldots, x_1(n)) \\ X_2 &= (x_2(1), x_2(2), \ldots, x_2(n)) \\ &\cdots\cdots\cdots\cdots \\ X_m &= (x_m(1), x_m(2), \ldots, x_m(n)) \end{aligned}$$

Then, for all $i < j$, $i, j = 1, 2, ..., m$, calculate $\varepsilon_{ij}$, the absolute grey relational degree between $X_i$ and $X_j$, so that we have the following upper triangular matrix A:

$$A = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \cdots & \varepsilon_{11} \\ & \varepsilon_{22} & \cdots & \varepsilon_{2m} \\ & & \ddots & \vdots \\ & & & \varepsilon_{mm} \end{pmatrix}$$

A is referred to as the grey relational matrix of the attribute indexes, where $\varepsilon_{ii} = 1$, $i = 1, 2, ..., m$. For a chosen threshold value $r \in [0, 1]$, which in general satisfies $r > 0.5$, if $\varepsilon_{ij} \geq r$, $i \neq j$, the variables $X_j$ and $X_i$ are seen as the same attribute.

**Definition 6.2.2** The classification of the attribute indexes with the chosen value $r$ is referred to as the $r$- classification by grey relational degree.

When studying a specific problem, the particular value $r$ is determined based on the circumstances involved. The closer the $r$ is to 1, the finer the classification and the fewer the variables in each class. Conversely, the smaller the $r$, the coarser the classification and the greater the number of variables in each class.

***Example 6.2.1*** The talent search committee of a firm has proposed 15 candidate recruitment criteria as follows:

1. Impression of overall application package;
2. Academic abilities;
3. Likability by others;
4. Level of self-confidence;
5. Intelligence;
6. Honesty;
7. Ability to sell;
8. Experience;
9. Motivation;
10. Ambition;
11. Presentation skills;
12. Ability to comprehend instructions;
13. Potential for future growth;
14. Interpersonal skills; and
15. Adaptability.

Members of the committee admit that some of these 15 criteria can overlap and hope that through the study of a sample of a few data points, these 15 criteria can be classified into fewer categories. By using the scoring method to quantify the criteria, 9 observational objects have been scored according to each of the criteria. Table 6.1 gives the scores, where Oi stands for the $i$th object, $i = 1, 2, \ldots, 9$.

To calculate the absolute grey relational degree of $\varepsilon_{ij}$ of $X_i$ and $X_j$ for all $i \leq j$, $i, j = 1, 2 \ldots, 15$, we obtained the upper triangular matrix A as shown in Table 6.2.

We divided the 15 criteria into different classes based on Table 6.2, where the value of threshold $r$ can be different based on the requirements involved. For example, if we take $r = 1$, all 15 criteria above belong to their own classes with each in its own class. If we take $r = 0.80$, then we check the values in Table 6.2, row by row, and pick out all the values of $\varepsilon_{ij}$ which are greater than 0.80. Thus, we have:

$$\varepsilon_{1,3} = 0.88,\ \varepsilon_{1,11} = 0.90,\ \varepsilon_{1,12} = 0.88,\ \varepsilon_{1,13} = 0.80,\ \varepsilon_{2,8} = 0.99$$

$$\varepsilon_{3,11} = 0.80,\ \varepsilon_{3,13} = 0.90,\ \varepsilon_{6,11} = 0.84,\ \varepsilon_{6,12} = 0.86,\ \varepsilon_{6,14} = 0.81$$

$$\varepsilon_{7,10} = 0.83,\ \varepsilon_{7,15} = 0.89,\ \varepsilon_{9,10} = 0.81,\ \varepsilon_{10,15} = 0.92,\ \varepsilon_{11,12} = 0.97$$

Therefore, we know that $X_3$, $X_{11}$, $X_{12}$, and $X_{13}$ belong to the same class as $X_1$; $X_8$ belong to the same class as $X_2$; $X_{11}$ and $X_{13}$ belong to the same class as $X_3$; $X_{11}$, $X_{12}$, and $X_{14}$ belong to the same class as $X_6$; $X_{10}$ and $X_{15}$ belong to the same class as $X_7$; $X_{10}$ belong to the same class as $X_9$; $X_{15}$ belong to the same class as $X_{10}$; and $X_{12}$ belong to the same class as $X_{11}$.

**Table 6.1** The scores of 9 observational objects

| | Objects | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Attributes | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $X_1$ | 6 | 9 | 7 | 5 | 6 | 7 | 9 | 9 | 9 |
| $X_2$ | 2 | 5 | 3 | 8 | 8 | 7 | 8 | 9 | 7 |
| $X_3$ | 5 | 8 | 6 | 5 | 8 | 6 | 8 | 8 | 8 |
| $X_4$ | 8 | 10 | 9 | 6 | 4 | 8 | 8 | 9 | 8 |
| $X_5$ | 7 | 9 | 8 | 5 | 4 | 7 | 8 | 9 | 8 |
| $X_6$ | 8 | 9 | 9 | 9 | 9 | 10 | 8 | 8 | 8 |
| $X_7$ | 8 | 10 | 7 | 2 | 2 | 5 | 8 | 8 | 5 |
| $X_8$ | 3 | 5 | 4 | 8 | 8 | 9 | 10 | 10 | 9 |
| $X_9$ | 8 | 9 | 9 | 4 | 5 | 6 | 8 | 9 | 8 |
| $X_{10}$ | 9 | 9 | 9 | 5 | 5 | 5 | 10 | 10 | 9 |
| $X_{11}$ | 7 | 10 | 8 | 6 | 8 | 7 | 9 | 9 | 9 |
| $X_{12}$ | 7 | 8 | 8 | 8 | 8 | 8 | 8 | 9 | 8 |
| $X_{13}$ | 5 | 8 | 6 | 7 | 8 | 6 | 9 | 9 | 8 |
| $X_{14}$ | 7 | 8 | 8 | 6 | 7 | 6 | 8 | 9 | 8 |
| $X_{15}$ | 10 | 10 | 10 | 5 | 7 | 6 | 10 | 10 | 10 |

**Table 6.2** The grey relational matrix of attribute indexes

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{14}$ | $X_{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | 1 | 0.66 | 0.88 | 0.52 | 0.58 | 0.77 | 0.51 | 0.66 | 0.51 | 0.51 | 0.9 | 0.88 | 0.8 | 0.67 | 0.51 |
| $X_2$ | | 1 | 0.72 | 0.51 | 0.53 | 0.59 | 0.5 | 0.99 | 0.51 | 0.51 | 0.63 | 0.62 | 0.77 | 0.55 | 0.51 |
| $X_3$ | | | 1 | 0.56 | 0.7 | 0.51 | 0.72 | 0.51 | 0.51 | 0.51 | 0.8 | 0.78 | 0.9 | 0.63 | 0.51 |
| $X_4$ | | | | 1 | 0.56 | 0.53 | 0.58 | 0.51 | 0.69 | 0.62 | 0.52 | 0.52 | 0.51 | 0.54 | 0.6 |
| $X_5$ | | | | | 1 | 0.65 | 0.51 | 0.53 | 0.53 | 0.52 | 0.61 | 0.61 | 0.55 | 0.75 | 0.52 |
| $X_6$ | | | | | | 1 | 0.51 | 0.59 | 0.52 | 0.52 | 0.84 | 0.86 | 0.66 | 0.81 | 0.51 |
| $X_7$ | | | | | | | 1 | 0.5 | 0.7 | 0.83 | 0.51 | 0.51 | 0.51 | 0.51 | 0.89 |
| $X_8$ | | | | | | | | 1 | 0.51 | 0.51 | 0.63 | 0.62 | 0.77 | 0.55 | 0.51 |
| $X_9$ | | | | | | | | | 1 | 0.81 | 0.0.52 | 0.52 | 0.51 | 0.53 | 0.76 |
| $X_{10}$ | | | | | | | | | | 1 | 0.51 | 0.51 | 0.51 | 0.52 | 0.92 |
| $X_{11}$ | | | | | | | | | | | 1 | 0.97 | 0.74 | 0.71 | 0.51 |
| $X_{12}$ | | | | | | | | | | | | 1 | 0.73 | 0.72 | 0.51 |
| $X_{13}$ | | | | | | | | | | | | | 1 | 0.6 | 0.51 |
| $X_{14}$ | | | | | | | | | | | | | | 1 | 0.52 |
| $X_{15}$ | | | | | | | | | | | | | | | 1 |

Let each class be represented with the criterion with the minimum index contained in the class, and combine the classes containing $X_6$ and $X_{11}$, respectively, with the class containing $X_{1.}$ Put $X_9$ and $X_{10}$ into the class containing $X_7$, and treat $X_4$ and $X_5$ as individual classes. Then, we have obtained a classification of the 15 attribute criteria for our shortened list as follows:

$$\{X_1, X_3, X_6, X_{11}, X_{12}, X_{13}, X_{14}\}, \{X_2, X_8\}, \{X_4\}, \{X_5\}, \\ \{X_7, X_9, X_{10}, X_{15}\}$$

Here, the class of $\{X_1, X_3, X_6, X_{11}, X_{12}, X_{13}, X_{14}\}$ including the attribute criteria such as impression of overall application package, likability by others, honesty, presentation skills, ability to comprehend instructions, potential for future growth, and interpersonal skills, all of which direct impression, can be obtained through the application form or interviews. These attribute criteria can be replaced by one synthetic impression attribute criterion because all these attribute criteria correlate and it is difficult to be separate them completely. The class of $\{X_2, X_8\}$ includes two attribute criteria, namely academic abilities and experience, which can be evaluated through investigation and understanding of the academic research and practical work accomplished by the candidate. The class of $\{X_7, X_9, X_{10}, X_{15}\}$ includes four attribute criteria, namely ability to sell, motivation, ambition, and adaptability, which can be judged synthetically by investigating the learning and working background of the candidate. Special investigation is required for assessment of the attribute criterion level of self-confidence of $\{X_4\}$, and the attribute criterion intelligence of $\{X_5\}$.

## 6.3 Common Possibility Functions

The variable weight grey clustering evaluation model, the fixed weight grey clustering evaluation model, the grey clustering evaluation model using end-point and center-point triangular possibility functions, and the grey clustering evaluation model based on mixed possibility functions are all grey clustering evaluation models based on different possibility functions. Therefore, the four kinds of common possibility functions are explained in this section.

The possibility function of the $j$th criterion about the $k$th class is denoted by $f_j^k(\cdot)$, $j = 1, 2, \ldots m, k = 1, 2, \ldots s$.

**Definition 6.3.1** Assume that the possibility function $f_j^k(\cdot)$ of the $j$th criterion about $k$th class is a trapezoidal function shown in Fig. 6.1. Then $f_j^k(\cdot)$ is referred to as possibility function of typical form, and $x_j^k(1)$, $x_j^k(2)$, $x_j^k(3)$, and $x_j^k(4)$ are referred to as turning points of $f_j^k(\cdot)$.

The possibility function of typical form is denoted by $f_j^k\left[x_j^k(1),\ x_j^k(2),\ x_j^k(3)x_j^k(4)\right]$.

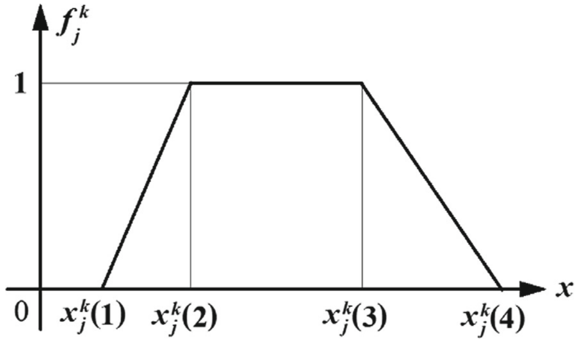



**Fig. 6.1** The possibility function of typical form

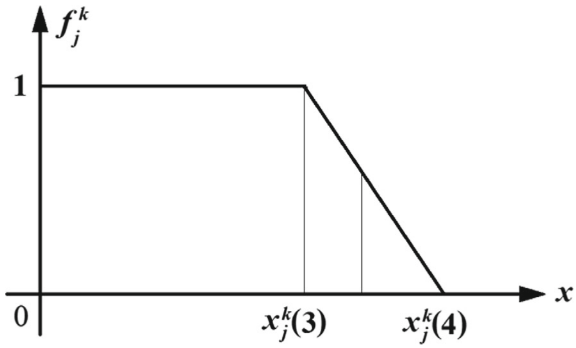



**Fig. 6.2** The possibility function of lower measure

**Definition 6.3.2** Assume that the possibility function $f_j^k(\cdot)$ of the $j$th criterion about $k$th class does not have the first and second turning points $x_j^k(1)$ and $x_j^k(2)$, as shown in Fig. 6.2. Then $f_j^k(\cdot)$ is referred to as the possibility function of lower measure.

The possibility function of lower measure is denoted by, $f_j^k\left[-, -, x_j^k(3), x_j^k(4)\right]$.

**Definition 6.3.3** Assume that the possibility function $f_j^k(\cdot)$ of the $j$th criterion about $k$th class does not have the third turning point $x_j^k(3)$, or that the second and third turning points $x_j^k(2)$ and $x_j^k(3)$ of $f_j^k(\cdot)$ coincide, as shown in Fig. 6.3. In this case, $f_j^k(\cdot)$ is referred to as a possibility function of moderate measure, or a triangular possibility function. The possibility function of moderate measure, or triangular possibility function, is denoted by $f_j^k\left[x_j^k(1), x_j^k(2), -, x_j^k(4)\right]$.

**Definition 6.3.4** Assume that the possibility function $f_j^k(\cdot)$ of the $j$th criterion about $k$th class does not have turning points $x_j^k(3)$ and $x_j^k(4)$, as shown in Fig. 6.4. Function $f_j^k(\cdot)$ is then referred to as a possibility function of upper measure. The possibility function of upper measure is denoted by $f_j^k\left[x_j^k(1), x_j^k(2), -, -\right]$.

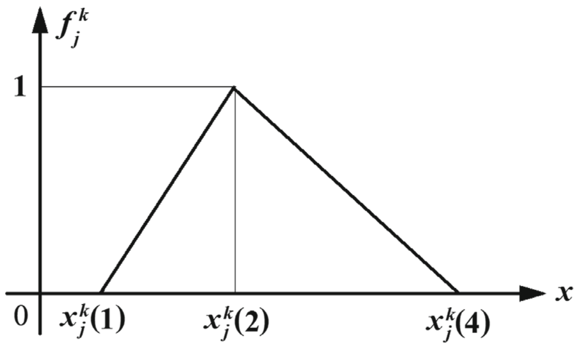



**Fig. 6.3** The possibility function of moderate measure

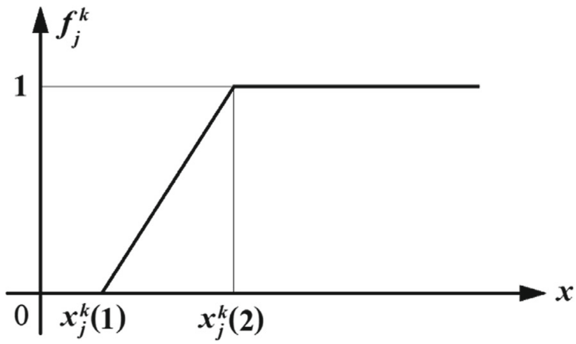



**Fig. 6.4** The possibility function of upper measure

**Proposition 6.3.1**

(1) For the possibility function of typical form as shown in Fig. 6.1, we have:

$$f_j^k(x) = \begin{cases} 0 & x \notin \left[x_j^k(1), x_j^k(4)\right] \\ \frac{x - x_j^k(1)}{x_j^k(2) - x_j^k(1)} & x \in \left[x_j^k(1), x_j^k(2)\right] \\ 1 & x \in \left[x_j^k(2), x_j^k(3)\right] \\ \frac{x_j^k(4) - x}{x_j^k(4) - x_j^k(3)} & x \in \left[x_j^k(3), x_j^k(4)\right] \end{cases} \tag{6.1}$$

(2) For the possibility function of lower measure as shown in Fig. 6.2, we have:

$$f_j^k(x) = \begin{cases} 0 & x \notin \left[0, x_j^k(4)\right] \\ 1 & x \in \left[0, x_j^k(3)\right] \\ \frac{x_j^k(4) - x}{x_j^k(4) - x_j^k(3)} & x \in \left[x_j^k(3), x_j^k(4)\right] \end{cases} \tag{6.2}$$

(3) For the possibility function of moderate measure as shown in Fig. 6.3, we have:

$$f_j^k(x) = \begin{cases} 0 & x \notin \left[x_j^k(1), x_j^k(4)\right] \\ \frac{x - x_j^k(1)}{x_j^k(2) - x_j^k(1)} & x \in \left[x_j^k(1), x_j^k(2)\right] \\ \frac{x_j^k(4) - x}{x_j^k(4) - x_j^k(2)} & x \in \left[x_j^k(2), x_j^k(4)\right] \end{cases} \tag{6.3}$$

(4) For the possibility function of upper measure as shown in Fig. 6.4, we have:

$$f_j^k(x) = \begin{cases} 0, & \mathrm{x} < x_j^k(1) \\ \frac{x - x_j^k(1)}{x_j^k(2) - x_j^k(1)}, & x \in \left[x_j^k(1), x_j^k(2)\right] \\ 1, & x \geq x_j^k(2) \end{cases} \tag{6.4}$$

## 6.4 Variable Weight Grey Clustering Model

**Definition 6.4.1** Assume that there are $n$ objects to be classified according to $m$ criteria into $s$ different grey classes. Classifying the $i$th object into the $k$th grey class according to the observed value of the $i$th object judged against the $j$th criterion, $x_{ij}, i = 1, 2, \ldots n, j = 1, 2, \ldots m$, is called grey clustering (Deng, 1985).

**Definition 6.4.2**

(1) For the possibility function of typical form as shown in Fig. 6.1, let $\lambda_j^k = \frac{1}{2}\left(x_j^k(2) + x_j^k(3)\right)$.
(2) For the possibility function of lower measure as shown in Fig. 6.2, let $\lambda_j^k = x_j^k(3))$.
(3) For the possibility function of moderate measure as shown in Fig. 6.3 and the possibility function of upper measure as shown in Fig. 6.4, let $\lambda_j^k = x_j^k(2)$.

Then $\lambda_j^k$ is referred to as the basic value of the $j$th criterion about the $k$th class.

**Definition 6.4.3** Assume that $\lambda_j^k$ is the basic value of the $j$th criterion about the $k$th class. Then formula is referred to as the weight of the $j$th criterion about $k$th class (Deng, 1985):

$$\eta_j^k = \lambda_j^k \Big/ \sum_{j=1}^{m} \lambda_j^k \tag{6.5}$$

**Definition 6.4.4** Assume that $x_{ij}, i = 1, 2, \ldots n, j = 1, 2, \ldots m$ is the observed value of object $i$ with regard to the $j$th criterion, $f_j^k(\cdot)$ the possibility function and $\eta_j^k$ the weight of the $j$th criterion about the $k$th class, with $j = 1, 2, \ldots m$, $k = 1, 2, \ldots s$. Then formula (6.6) is referred to as the grey clustering coefficient of variable weight for object $i$ to belong to the $k$th grey class (Deng, 1985):

$$\sigma_i^k = \sum_{j=1}^{m} f_j^k\left(x_{ij}\right)\cdot \tag{6.6}$$

**Definition 6.4.5**

(1) The following formula is referred to as the clustering coefficient vector of object $i$:

$$\sigma_i = \left(\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^s\right) = \left(\sum_{j=1}^{m} f_j^1\left(x_{ij}\right)\cdot\eta_j^1, \sum_{j=1}^{m} f_j^2\left(x_{ij}\right)\cdot\eta_j^2, \ldots, \sum_{j=1}^{m} f_j^s\left(x_{ij}\right)\cdot\eta_j^s\right)$$

(2) The following matrix is referred to as the cluster coefficient matrix:

$$\sum = \left(\sigma_i^k\right) = \begin{bmatrix} \sigma_1^1 & \sigma_1^2 & \cdots & \sigma_1^s \\ \sigma_2^1 & \sigma_2^2 & \cdots & \sigma_2^s \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_n^1 & \sigma_n^2 & \cdots & \sigma_n^s \end{bmatrix}$$

**Definition 6.4.6** If $\max\limits_{1\le k\le s}\left\{\sigma_i^k\right\} = \sigma_i^{k^*}$, then it is called object $i$ belongs to grey class $k^*$.

The variable weight clustering method is used to study problems with criteria that have the same meanings and units. Otherwise, it is not appropriate to employ this method. Also, if the numbers of observed values of individual criteria are greatly different from each other, this clustering method should not be applied.

**Example 6.4.1** Assume that we are interested in the study of three economic districts with the added value by the primary, secondary and tertiary industries as the three cluster criteria. The observational values $x_{ij}$, $i = 1, 2, 3; j = 1, 2, 3$, of the $i$th economic district with respect to the $j$th criterion is given in the following matrix A, where the unit of the three criteria is same as a hundred million RMB:

$$A = \left(x_{ij}\right)\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 80 & 20 & 100 \\ 40 & 30 & 30 \\ 10 & 90 & 60 \end{bmatrix}$$

Please try to perform a synthetic clustering based on high, medium, and low added values.

**Solution**: Assume that the possibility functions $f_j^k(\cdot)$ for the $j$th criterion about the $k$th class are as follows:

$$f_1^1[30, 80, -, -], \quad f_1^2[10, 40, -, 70], \quad f_1^3[-, -, 10, 30]$$

$$f_2^1[30, 90, -, -], \quad f_2^2[20, 50, -, 90], \quad f_2^3[-, -, 20, 40]$$

$$f_2^1[40, 100, -, -], \quad f_3^2[30, 60, -, 90], \quad f_3^3[-, -, 30, 50]$$

It follows that:

$$f_1^1(x) = \begin{cases} 0, & x < 30 \\ \frac{x-30}{80-30}, & 30 \le x < 80 \\ 1, & x > 80 \end{cases}; \quad f_1^2(x) = \begin{cases} 0, & x \notin [10, 70] \\ \frac{x-10}{40-10}, & 10 \le x < 40 \\ \frac{70-x}{70-40}, & 40 \le x < 70 \end{cases}$$

$$f_1^3(x) = \begin{cases} 0, & x \notin [0, 30] \\ 1, & 0 \le x < 10 \\ \frac{30-x}{30-10}, & 10 \le x < 30 \end{cases}; \quad f_2^1(x) = \begin{cases} 0, & x < 30 \\ \frac{x-30}{90-30}, & 30 \le x < 90 \\ 1, & x > 90 \end{cases}$$

$$f_2^2(x) = \begin{cases} 0, & x \notin [20, 90] \\ \frac{x-20}{50-20}, & 20 \le x < 50 \\ \frac{90-x}{90-50}, & 50 \le x < 90 \end{cases}; \quad f_2^3(x) = \begin{cases} 0, & x \notin [0, 40] \\ 1, & 0 \le x < 20 \\ \frac{40-x}{40-20}, & 20 \le x < 40 \end{cases}$$

$$f_3^1(x) = \begin{cases} 0, & x < 40 \\ \frac{x-40}{100-40}, & 40 \le x < 100 \\ 1, & x > 100 \end{cases}; \quad f_3^2(x) = \begin{cases} 0, & x \notin [30, 90] \\ \frac{x-30}{50-30}, & 30 \le x < 50 \\ \frac{90-x}{90-50}, & 50 \le x < 90 \end{cases}$$

$$f_3^3(x) = \begin{cases} 0, & x \notin [0, 50] \\ 1, & 0 \le x < 30 \\ \frac{50-x}{50-30}, & 30 \le x < 50 \end{cases}$$

Therefore:

$$\lambda_1^1 = 80,\ \lambda_2^1 = 90,\ \lambda_3^1 = 100,\ \lambda_1^2 = 40,$$

$$\lambda_2^2 = 50,\ \lambda_3^2 = 60,\ \lambda_1^3 = 10,\ \lambda_2^3 = 20,\ \lambda_3^3 = 30$$

By $\eta_j^k = \frac{\lambda_j^k}{\sum_{j=1}^3 \lambda_j^k}$ we have:

$$\eta_1^1 = \frac{80}{270}, \eta_2^1 = \frac{90}{270}, \eta_3^1 = \frac{100}{270}, \eta_1^2 = \frac{40}{150}, \eta_2^2 = \frac{50}{150},$$
$$\eta_3^2 = \frac{60}{150}, \eta_1^3 = \frac{10}{60}, \eta_2^3 = \frac{20}{60}, \eta_3^3 = \frac{30}{60}$$

Thus, from $\sigma_i^k = \sum_{j=1}^m f_j^k\left(x_{ij}\right) \cdot \eta_j^k$, when $i = 1$ for economic district 1, we have:

$$\sigma_1^1 = \sum_{j=1}^3 f_j^1\left(x_{1j}\right) \cdot \eta_j^1 = f_1^1(80) \times \frac{80}{270} + f_2^1(20) \times \frac{90}{270} + f_3^1(100) \times \frac{100}{270} = 0.6667$$

Similarly, we obtained the following:

$$\sigma_1^2 = 0, \sigma_1^3 = 0.3333$$

Therefore, $\sigma_1 = (\sigma_1^1, \sigma_1^2, \sigma_1^3) = (0.6667, 0, 0.3333)$.

Similarly, we can calculate the clustering coefficient vector for economic districts 2 and 3 as done for economic district 1.

When $i = 2$, $\sigma_2 = \left(\sigma_2^1, \sigma_2^2, \sigma_2^3\right) = (0.0593, 0.3778, 0.6667)$.

When $i = 3$, $\sigma_3 = \left(\sigma_3^1, \sigma_3^2, \sigma_3^3\right) = (0.4667, 0.4, 0.1667)$.

The clustering coefficient matrix is as follows:

$$\sum = \left(\sigma_i^k\right) = \begin{bmatrix} \sigma_1^1 & \sigma_1^2 & \sigma_1^3 \\ \sigma_2^1 & \sigma_2^2 & \sigma_2^3 \\ \sigma_3^1 & \sigma_3^2 & \sigma_3^3 \end{bmatrix} = \begin{bmatrix} 0.6667 & 0 & 0.3333 \\ 0.0593 & 0.3778 & 0.6667 \\ 0.4667 & 0.4 & 0.1667 \end{bmatrix}$$

From $\max_{1 \le k \le 3}\left\{\sigma_1^k n\right\} = \sigma_1^1 = 0.6667$, $\max_{1 \le k \le 3}\left\{\sigma_2^k\right\} = \sigma_2^3 = 0.6667$, $\max_{1 \le k \le 3}\left\{\sigma_3^k\right\} = \sigma_3^1 = 0.4667$, it follows that the second economic district belongs to the low grey class of added value, and the first and third economic districts belong to the high grey class of added value. Furthermore, from the cluster coefficients $\sigma_1^1 = 0.6667$ and $\sigma_3^1 = 0.4667$, it follows that there still exists some differences between the first and third districts, even though both belong to the high grey class of added value. If the grey classes of added value are refined, that is, if we use five grey classes such as high, mid-high, medium, mid-low, and low added value, then different results can be obtained.

Furthermore, to determine the possibility function for the $j$th criterion about the $k$th class, it is generally possible to use the background information of the problem at hand. When resolving practical problems, one can determine the possibility functions from either the angle of the objects that are to be clustered or by looking at all the same type objects in the whole system, not just the ones involved in the clustering. For example, in Example 6.4.1, we could determine the possibility functions not only from the three economic districts in question, but also from the same level of economic districts in a city, a province, or from around the nation. Therefore, the results of grey clustering evaluation can only be applied to a certain range, which is the same as the one used in the determination of relevant possibility functions.

## 6.5 The Fixed Weight Grey Clustering Model

When the criteria for clustering have different meanings, dimensions (units), and drastically different numbers of observed data points, the variable weight clustering method will fail. Because the indexes with different meanings and dimensions do not meet the additivity. There are two ways to get around this problem. The first is to transform the sample of data values of all the criteria into non-dimensional values by applying either the initiating operator or the averaging operator, and then clustering the transformed data. When employing this method, all the criteria are treated equally so that no difference played by the criteria in the process of clustering is reflected. The second way to solve non additive problem is to assign each clustering criterion a weight ahead of the clustering process. In this section, we address this second method.

**Definition 6.5.1** Assume that $x_{ij}$ is the observed value of object $i$ with regard to criterion $j$, $i = 1, 2, \ldots n$, $j = 1, 2, \ldots m$, and the possibility function $f_j^k(\cdot)$ of the $j$th criterion about $k$th class, $j = 1, 2, \ldots m,\ k = 1, 2, \ldots, s$. If the weight $\eta_j^k$ of the $j$th criterion about the $k$th class is not a function of $k, j = 1, 2, \ldots m,\ k = 1, 2, \ldots, s$. That is, if for any $k_1, k_2 \in \{1, 2, \ldots, s\}$ we always have $\eta_j^{k_1} = \eta_j^{k_2}$, then the symbol $\eta_j^k$ can be written as $\eta_j$, $j = 1, 2, \ldots m$, with the superscript $k$ removed. In this case, formula (6.7) is referred to as the fixed weight clustering coefficient for object $i$ to belong to the $k$th grey class (Liu, 1993).

$$\sigma_i^k = \sum_{j=1}^{m} f_j^k(x_{ij})\eta_j \tag{6.7}$$

**Definition 6.5.2** In formula (6.7), if $\eta_j = \frac{1}{m}$, for $j = 1, 2, \ldots, m$, then the following formula is referred to as the equal weight clustering coefficient for object $i$ to belong to the $k$th grey class:

$$\sigma_i^k = \sum_{j=1}^{m} f_j^k(x_{ij}) \cdot \eta_j = \frac{1}{m}\sum_{j=1}^{m} f_j^k(x_{ij})$$

The method of clustering objects by using grey fixed weight clustering coefficients is known as grey fixed weight clustering. The method which uses grey equal weight clustering coefficients is known as grey equal weight clustering.

Grey fixed weight clustering can be carried out according to the following steps:

Step 1: Determine the possibility function $f_j^k(\cdot)$ for the $j$th criterion about the $k$th class, $j = 1, 2, \ldots m,\ k = 1, 2, \ldots, s$.

Step 2: Determine a clustering weight $\eta_j$ for each criterion $j = 1, 2, \ldots m$.

Step 3: Based on the possibility functions $f_j^k(\cdot)$ obtained in step 1, the clustering weights $\eta_j$ obtained in step 2, and the observed data value $x_{ij}$ of object $i$

with respect to criterion $j$, calculate the fixed weight clustering coefficients $\sigma_i^k = \sum_{j=1}^m f_j^k(x_{ij})\eta_j,\ i = 1, 2, \ldots n,\ j = 1, 2, \ldots m,\ k = 1, 2, \ldots, s$

Step 4: If $\max_{1 \le k \le s}\{\sigma_i^k\} = \sigma_i^{k^*}$, then it is called object $i$ belongs to grey class $k^*$.

***Example 6.5.1*** Let us perform a grey clustering for the ecological adaptation of major strains of trees commercially used in China (Li et al., 1994). China is a huge country with a very diverse ecological environment, and different strains of trees obviously require different growing conditions. The area where a certain strain of trees has been growing reflects the adaptability of the strain to that particular ecological environment. We now classify ecological environmental conditions into four main quantification criteria:

(1) Geographical measure;
(2) Temperature measure;
(3) Precipitation measure; and
(4) Arid measure.

Here, geographical measure is an index representing the geographical width of the region in which the strain of trees grows. The numerical value of this measure is given by the product of differences of longitudes in the directions of east and west, and latitudes in the directions of south and north. The temperature measure indicates the adaptability of the strain of trees to various temperatures. Its numerical value is computed by using the difference of annual average temperatures of the southern and the northern bounds of the growing region. The precipitation measure is the adaptability of the trees to precipitation conditions. Its numerical value is recorded as the difference between the maximum and minimum annual average precipitation in all areas of the growing region. The arid measure is selected to describe a strain's adaptability to arid conditions in the atmosphere. Its value is the difference between the maximum and minimum annual average aridities in different areas of the growing region.

Statistics regarding the four measures for the 17 main strains of trees planted in China are given in Table 6.3.

With such data it is possible to carry out grey clustering based on wide adaptability, medium adaptability, and narrow adaptability.

**Solution**: Because the meanings of the criteria are different and there exists much difference among the values observed, we must apply the fixed weight clustering method.

Step 1: Assume that the possibility functions $f_j^k(\cdot)(j = 1, 2, 3, 4;\ k = 1, 2, 3)$ for the $j$th criterion about the $k$th class are as follows:

$$f_1^1[100, 300, -, -], f_1^2[50, 150, -, 250], f_1^3[-, -, 50, 100]$$

$$f_2^1[3, 10, -, -],\ f_2^2[2, 6, -, 10],\quad f_2^3[-, -, 15, 30]$$

**Table 6.3** The four measures of the 17 main strains of trees in China

| Measure trees | Geo. Eco measure | Temp. eco measure | Prec. eco measure | Arid eco measure |
|---|---|---|---|---|
| 1 Camphor pine | 22.50 | 4 | 0 | 0 |
| 2 Korean pine | 79.37 | 6 | 600 | 0.75 |
| 3 Northeast China ash | 144.00 | 7 | 300 | 0.75 |
| 4 Diversiform-leaved poplar | 300.00 | 6.1 | 189 | 12.00 |
| 5 Sacsaoul | 456.00 | 12 | 250 | 12.00 |
| 6 Chinese pine | 189.00 | 8 | 700 | 1.5 |
| 7 Oriental arborvitae | 369.00 | 8 | 1300 | 2.25 |
| 8 White elm | 1127.11 | 16.2 | 550 | 3.00 |
| 9 Dryland willow | 260.00 | 11 | 600 | 1.00 |
| 10 Chinese white poplar | 200.00 | 8 | 600 | 1.25 |
| 11 Oak | 475.00 | 10 | 1000 | 0.75 |
| 12 Huashan pine | 314.10 | 8 | 900 | 0.75 |
| 13 Masson pine | 282.80 | 7.4 | 1300 | 0.5 |
| 14 China fir | 240.00 | 8 | 1200 | 0.5 |
| 15 Bamboo | 160.00 | 5 | 1000 | 0.25 |
| 16 Camphor tree | 270.00 | 8 | 1200 | 0.25 |
| 17 Southern Asian pine | 9.00 | 1 | 200 | 0 |

$$f_3^1[200, 1000, -, -],\ f_3^2[100, 600, -, 1100],\ f_3^3[-, -, 300, 600]$$

$$f_4^1[0.25, 1.25, -, -],\ f_4^2[0, 0.5, -, 1],\ f_4^3[-, -, 0.25, 0.5]$$

Step 2: Let the weights for the geographical, temperature, precipitation, and aridity measures be:

$$\eta_1 = 0.3,\ \eta_2 = 0.25,\ \eta_3 = 0.25,\ \eta_4 = 0.2$$

Step 3: Based on $\sigma_i^k = \sum_{j=1}^m f_j^k(x_{ij}) \cdot \eta_j$ ; $i = 1, 2, \ldots, 17$; $k = 1, 2, 3$ and Table 6.3, when i = 1,

$$\sigma_1^1 = \sum_{j=1}^4 f_j^1(x_{ij}) \cdot \eta_j = f_1^1(22.5) \times 0.3 + f_2^1(4) \times 0.25 + f_3^1(0) \times 0.25 + f_4^1(0) \times 0.2 = 0.0357$$

and

$$\sigma_1^2 = \sum_{j=1}^{m} f_j^2(x_{ij}) \cdot \eta_j = 0.125,\ \sigma_1^3 = \sum_{j=1}^{m} f_j^3(x_{ij}) \cdot \eta_j = 1$$

Therefore,

$$\sigma_1 = \left(\sigma_1^1, \sigma_1^2, \sigma_1^3\right) = (0.0357, 0.125, 1)$$

Similarly, we can calculate and obtain:

$$\sigma_2 = (\sigma_2^1, \sigma_2^2, \sigma_2^3) = (0.3321{,}0.6881, 0.2488)$$

$$\sigma_3 = (\sigma_3^1, \sigma_3^2, \sigma_3^3) = (0.3401{,}0.6695, 0.3125)$$

$$\sigma_4 = (\sigma_4^1, \sigma_4^2, \sigma_4^3) = (0.6107, 0.2883, 0.3688)$$

$$\sigma_5 = (\sigma_5^1, \sigma_5^2, \sigma_5^3) = (0\ .7656, 0.075\ {,}0\ .25)$$

$$\sigma_6 = (\sigma_6^1, \sigma_6^2, \sigma_6^3) = (0.6683, 0.508, 0)$$

$$\sigma_7 = (\sigma_7^1, \sigma_7^2, \sigma_7^3) = (0.9286{,}0.125, 0)$$

$$\sigma_8 = (\sigma_8^1, \sigma_8^2, \sigma_8^3) = (0.8594{,}0.225, 0.0417)$$

$$\sigma_9 = (\sigma_9^1, \sigma_9^2, \sigma_9^3) = (0.765{,}0.25, 0)$$

$$\sigma_{10} = (\sigma_{10}^1, \sigma_{10}^2, \sigma_{10}^3) = (0.6536, 0.525, 0)$$

$$\sigma_{11} = (\sigma_{11}^1, \sigma_{11}^2, \sigma_{11}^3) = (0.9, 0.15, 0)$$

$$\sigma_{12} = (\sigma_{12}^1, \sigma_{12}^2, \sigma_{12}^3) = (0.7973{,}0.325, 0)$$

$$\sigma_{13} = (\sigma_{13}^1, \sigma_{13}^2, \sigma_3^3) = (0.7313{,}0.3625, 0.0375)$$

$$\sigma_{14} = (\sigma_{14}^1, \sigma_{14}^2, \sigma_4^3) = (0.6886{,}0.355, 0)$$

$$\sigma_{15} = (\sigma_{15}^1, \sigma_5^2, \sigma_{15}^3) = (0.4114, 0.6075, 0.3875\ )$$

$$\sigma_{16} = (\sigma_{16}^1, \sigma_{16}^2, \sigma_{16}^3) = (0.6836, 0.225, 0.2)$$

Furthermore,

$$\sigma_{17} = (\sigma_{17}^1, \sigma_{17}^2, \sigma_7^3) = (0, 0.05, 1)$$

Step 4: Based on the following facts, it follows that trees with numberings 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, are strains with wide adaptability:

$$\max_{1\le k\le 3}\{\sigma_1^k\} = \sigma_1^3 = 1,\ \max_{1\le k\le 3}\{\sigma_2^k\} = \sigma_2^2 = 0.6881,\ \max_{1\le k\le 3}\{\sigma_3^k\} = \sigma_3^2 = 0.6695$$

$$\max_{1\le k\le 3}\{\sigma_4^k\} = \sigma_4^1 = 0.6107,\ \max_{1\le k\le 3}\{\sigma_5^k\} = \sigma_5^1 = 0.7656,$$
$$\max_{1\le k\le 3}\{\sigma_6^k\} = \sigma_6^1 = 0.6683$$

$$\max_{1\le k\le 3}\{\sigma_7^k\} = \sigma_7^1 = 0.9286,\ \max_{1\le k\le 3}\{\sigma_8^k\} = \sigma_8^1 = 0.8594,$$
$$\max_{1\le k\le 3}\{\sigma_9^k\} = \sigma_9^1 = 0.765$$

$$\max_{1\le k\le 3}\{\sigma_{10}^k\} = \sigma_{10}^1 = 0.6536,\ \max_{1\le k\le 3}\{\sigma_{11}^k\} = \sigma_{11}^1 = 0.9,$$
$$\max_{1\le k\le 3}\{\sigma_{12}^k\} = \sigma_{12}^1 = 0.91$$

$$\max_{1\le k\le 3}\{\sigma_{13}^k\} = \sigma_{13}^1 = 0.82,\ \max_{1\le k\le 3}\{\sigma_{14}^k\} = \sigma_{14}^1 = 0.6886,$$
$$\max_{1\le k\le 3}\{\sigma_{15}^k\} = \sigma_{15}^2 = 0.6075$$

$$\max_{1\le k\le 3}\{\sigma_{16}^k\} = \sigma_{16}^1 = 0.6836,\ \max_{1\le k\le 3}\{\sigma_{17}^k\} = \sigma_{17}^3 = 1$$

Such strains are diversiform-leaved poplars, sacsaouls, Chinese pines, oriental arborvitaes, white elms, dryland willows, Chinese white poplars, oaks, Huashan pines, masson pines, China firs, and camphor trees. These trees have an extremely strong ability to adapt themselves to natural ecological environments, can grow well in most parts of China, and should be widely introduced. The trees named Korean pine, Northeast China Ash, and bamboo with numberings 2, 3, and 15, respectively, belong to the grey class of medium adaptability, and can be introduced to a relatively large area in China. Finally, trees with the names camphor pine and South Asian pine, and numberings 1 and 17, respectively, belong to the grey class of narrow adaptability, where camphor pines are found near the Northern border of China and South Asian pines are mainly located near the Southern border of China.

## 6.6 Grey Clustering Evaluation Models Based on Mixed Possibility Functions

The grey clustering evaluation model based on the mixed possibility function defines the specific steps of constructing the possibility function and solve the problem of constructing the possibility function, which is especially suitable for clustering evaluation under the background of poor information (Liu et al., 2015a).

### *6.6.1 Grey Clustering Evaluation Model Based on End-Point Mixed Possibility Functions*

The grey clustering evaluation model based on end-point mixed possibility functions is suitable for situations where the grey boundaries of all sub-intervals are clear, but the most likely points belonging to each grey class are unknown(Liu et al., 2015a). The modeling steps of the grey clustering evaluation based on end-point mixed possibility functions are explained below (Liu et al., 2015a).

Step 1: Assume that according to the assessment requirements, the number of grey classes to be divided is s. Then the value range of each index is also divided into $s$ classes. For example, the value range $\left[a_1, a_{s+1}\right]$ of index $j$ can be divided into $s$ small intervals:

$$[a_1, a_2], \ldots, \left[a_{k-1}, a_k\right], \ldots, \left[a_{s-1}, a_s\right], \left[a_s, a_{s+1}\right]$$

The value of $a_k (k = 2, \ldots, s)$ can be determined by the actual assessment requirements or the qualitative research results.

Step 2: Determine the turning point $\lambda_j^1$ and $\lambda_j^s$ of $[a_1, a_2]$ and $[a_s, a_{s+1}]$ that correspond to grey classes 1 and s. At the same time, calculate the geometric center-point $\lambda_k = (a_k + a_{k+1})/2$ for each small interval $[a_k, a_{k+1}]$, $k = 2, \ldots, s-1$.

Step 3: For grey class 1 and grey class s, construct the corresponding possibility function of lower measure $f_j^1\left[-, -, \lambda_j^1, \lambda_j^2\right]$ and the possibility function of upper measure $f_j^s\left[\lambda_j^{s-1}, \lambda_j^s, -, -\right]$.

Assume that x is an observation of index j, when $x \in \left[a_1, \lambda_j^2\right]$ or $x \in \left[\lambda_j^{s-1}, a_{s+1}\right]$, using formulas (6.8) or (6.9), respectively:

$$f_j^1(x) = \begin{cases} 0 & x \notin \left[a_1, \lambda_j^2\right] \\ 1 & x \in \left[a_1, \lambda_j^1\right] \\ \frac{\lambda_j^2 - x}{\lambda_j^2 - \lambda_j^1} & x \in \left[\lambda_j^1, \lambda_j^2\right] \end{cases} \tag{6.8}$$

or

$$f_j^s(x) = \begin{cases} 0 & x \notin \left[\lambda_j^{s-1}, a_{s+1}\right] \\ \frac{x-\lambda_j^{s-1}}{\lambda_j^s-\lambda_j^{s-1}} & x \in \left[\lambda_j^{s-1}, \lambda_j^s\right] \\ 1 & x \in \left[\lambda_j^s, a_{s+1}\right] \end{cases} \tag{6.9}$$

By using these formulas, the possibility degree of $f_j^1(x)$ or $f_j^s(x)$ regarding grey class 1 and grey class s can be calculated.

Step 4: For grey class $k(k \in \{2, 3, \ldots, s-1\})$, connecting point $(\lambda_j^k, 1)$ with center-point $(\lambda_j^{k-1}, 0)$ of grey class $k-1$ (or turning point $(\lambda_j^1, 0)$ of grey class 1), and connecting $(\lambda_j^k, 1)$ with center-point $(\lambda_j^{k+1}, 0)$ of grey class $k+1$ (or turning point $(\lambda_j^s, 0)$ of grey class $s$), we can get the triangular possibility function $f_j^k[\lambda_j^{k-1}, \lambda_j^k, -, \lambda_j^{k+1}]$, $j = 1, 2, \ldots; m$; $k = 2, 3, \ldots, s-1$ of index $j$ regarding grey class $k$ (shown in Fig. 6.5).

For index $j$, $x$ is an observation of it when $k = 2, 3, \ldots, s-1$, according to formula (6.10):

$$f_j^k(x) = \begin{cases} 0 & x \notin \left[\lambda_j^{k-1}, \lambda_j^{k+1}\right] \\ \frac{x-\lambda_j^{k-1}}{\lambda_j^k-\lambda_j^{k-1}} & x \in \left[\lambda_j^{k-1}, \lambda_j^k\right] \\ \frac{\lambda_j^{k+1}-x}{\lambda_j^{k+1}-\lambda_j^k} & x \in \left[\lambda_j^k, \lambda_j^{k+1}\right] \end{cases} \tag{6.10}$$

This formula allows the possibility degree of $f_j^k(x)$ regarding grey class $k(k \in \{2, 3, \ldots, s-1\})$ to be calculated.

Step 5: Determine the weight $w_j, j = 1, 2, \ldots, m$ of each index.

Step 6: Calculate the clustering coefficient $\sigma_i^k$ of object $i(i = 1, 2, \ldots, n)$ regarding grey class $k(k = 1, 2, \ldots, s)$:

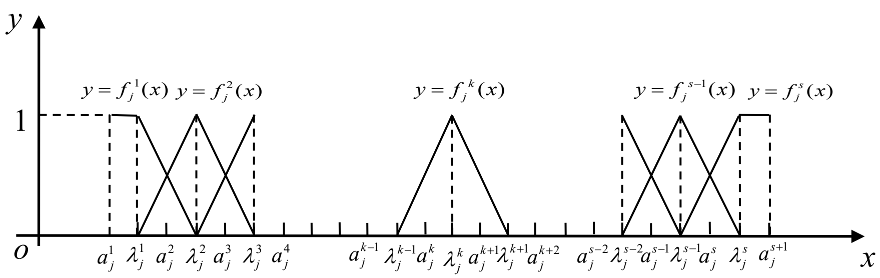



**Fig. 6.5** The end-point mixed possibility function

$$\sigma_i^k = \sum_{j=1}^{m} f_j^k(x_{ij}) \cdot w_j \tag{6.11}$$

$f_j^k(x_{ij})$ is the possibility function of index j about grey class k, $w_j$ is the weight of index j among comprehensive clustering.

Step 7: By $\max_{1 \le k \le s} \{\sigma_i^k\} = \sigma_i^{k^*}$, determine that object $i$ belongs to grey class $k^*$. When there are multiple objects belonging to the same grey class $k^*$, we can further determine individual objects' precedence in grey class $k^*$ on the basis of the size of integrate clustering coefficients.

### *6.6.2 Grey Clustering Evaluation Model Based on Center-Point Mixed Possibility Functions*

The srey clustering evaluation model based on center-point mixed possibility functions is suitable for situations where the most likely points belonging to each grey class are clear, but the grey boundaries of all sub-intervals are unknown (Liu et al., 2015a).

**Definition 6.6.1** For grey class $k(k \in \{2, 3, \ldots, s-1\})$, the point which most likely belongs to grey class $k$ is called the center-point of grey class $k$ (Liu et al., 2015a).

The center-point may or may not be the midpoint. This is determined by the maximum likelihood of such a point to belong to the grey class.

The modeling steps of the grey cluster evaluation using center-point mixed possibility functions are as follows (Liu et al., 2015a).

Step 1: Assume that $[a_j, b_j]$ is the range of index $j$. According to the evaluation requirements, we divide $[a_j, b_j]$ into s small intervals. Then we determine the turning point $\lambda_j^1, \lambda_j^s$ of grey classes 1 and s, and the center-point $\lambda_j^2, \lambda_j^3, \ldots, \lambda_j^{s-1}$ of grey class $k(k \in \{2, 3, \ldots, s-1\})$, respectively.

Step 2: Construct the corresponding lower measure possibility function $f_j^1[-, -, \lambda_j^1, \lambda_j^2]$, and the upper measure possibility function $f_j^s[\lambda_j^{s-1}, \lambda_j^s, -, -]$ for grey classes 1 and $s$ (see Fig. 6.6).

Assume $x$ is an observation value of index $j$. When $x \in [a_j, \lambda_j^2]$ or $x \in [\lambda_j^{s-1}, b_j]$, the possibility degree of $f_j^1(x)$ or $f_j^s(x)$ regarding grey classes 1 and $s$ can be calculated by using formulas (6.12) and (6.13), respectively.

$$f_j^1(x) = \begin{cases} 0 & x \notin \left[a_j, \lambda_j^2\right] \\ 1 & x \in \left[a_j, \lambda_j^1\right] \\ \frac{\lambda_j^2 - x}{\lambda_j^2 - \lambda_j^1} & x \in \left[\lambda_j^1, \lambda_j^2\right] \end{cases} \tag{6.12}$$

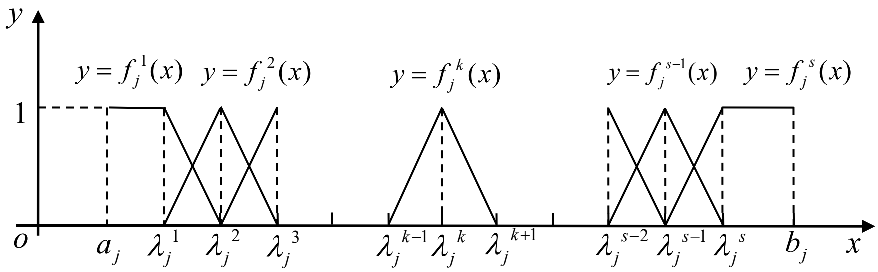



**Fig. 6.6** Center-point mixed possibility function

$$f_j^s(x) = \begin{cases} 0 & x \notin \left[\lambda_j^{s-1}, b_j\right] \\ \frac{x-\lambda_j^{s-1}}{\lambda_j^s-\lambda_j^{s-1}} & x \in \left[\lambda_j^{s-1}, \lambda_j^s\right] \\ 1 & x \in \left[\lambda_j^s, b_j\right] \end{cases} \tag{6.13}$$

Step 3: For grey class $k(k \in \{2, 3, \ldots, s-1\})$, by connecting point $(\lambda_j^k, 1)$ with center-point $(\lambda_j^{k-1}, 0)$ of grey class $k-1$ (or turning point $(\lambda_j^1, 0)$ of grey class 1), and by connecting $(\lambda_j^k, 1)$ with center-point $(\lambda_j^{k+1}, 0)$ of grey class $k+1$ (or turning point $(\lambda_j^s, 0)$ of grey class $s$), we get triangular possibility function $f_j^k\left[\lambda_j^{k-1}, \lambda_j^k, -, \lambda_j^{k+1}\right]$, $j = 1, 2, \ldots; m$; $k = 2, 3, \ldots, s-1$ of index $j$ regarding grey class $k$ (see Fig. 6.6).

Assume that $x$ is an observation value of index $j$. The degree of membership $f_j^k(x)$ regarding grey class $k(k \in \{2, 3, \ldots, s-1\})$ can be calculated by using formula (6–14).

$$f_j^k(x) = \begin{cases} 0 & x \notin \left[\lambda_j^{k-1}, \lambda_j^{k+1}\right] \\ \frac{x-\lambda_j^{k-1}}{\lambda_j^k-\lambda_j^{k-1}} & x \in \left[\lambda_j^{k-1}, \lambda_j^k\right] \\ \frac{\lambda_j^{k+1}-x}{\lambda_j^{k+1}-\lambda_j^k} & x \in \left[\lambda_j^k, \lambda_j^{k+1}\right] \end{cases} \tag{6.14}$$

Step 4: Determine the weight $w_j, j = 1, 2, \ldots, m$ of each index.

Step 5: Compute clustering coefficient $\max\limits_{1 \le k \le s}\left\{\sigma_i^k\right\} = \sigma_i^{k*}$ of object $i(i =$ $x_6$ $1, 2, \ldots, n)$ regarding grey class $k(k = 1, 2, \ldots, s)$, as seen in Eq. (6.15).

$$\sigma_i^k = \sum_{j=1}^{m} f_j^k(x_{ij}) \cdot w_j \tag{6.15}$$

$f_j^k(x_{ij})$ is the possibility function of index $j$ about class $k$, while $w_j$ is the weight of comprehensive clustering of index $j$.

Step 6: By $\max\limits_{1 \le k \le s}\left\{\sigma_i^k\right\} = \sigma_i^{k*}$, determine that object $i$ belongs to grey class $k^*$; when there are multiple objects that belong to the same grey class $k^*$, we can further determine the precedence of individual objects in grey class $k^*$ on the basis of the size of integrate clustering coefficients.

## 6.7 Practical Applications

***Example 6.7.1*** Five suppliers A, B, C, D, E who undertake the development of the C919 body component for Commercial Aircraft Corporation of China Ltd (COMAC) are evaluated on their performance and are divided into four classes including "excellent", "good", "medium" and "poor".

Step 1: Set the evaluation index system for supplier performance.

The evaluation index system for supplier performance reflects the specific requirements of the main manufacturers to supplier. It is an important basis for the main manufacturers to comprehensively evaluate the supplier and make final management decisions.

Factors that affect supplier performance are very complex. Main manufactures' foci on supplier performance are also not the same across different stages in C919 development. After four rounds of expert investigation, six first-grade evaluation indexes are determined, including quality, cost, delivery, cooperation, technology and service.

At development stage, the four second-grade indexes of quality are pass rate of product, quality control system, airworthiness certification ability and control of sub-supplier. The three second-grade indexes of cost are price, logistics costs and price stability. The two second-grade indexes of delivery are punctuality and flexibility. The three second-grade indexes of cooperation are credit, information communication and cooperation intention. The five second-grade indexes of technology are professional R&D staff, R&D investment, number of invention patents, market share and technology level. The four second-grade indexes of service are quick response, spare part support, training and technology support.

Among those indexes, pass rate and market share are shown in percentage. Price and logistics costs are quantitative indexes and the unit is ten thousand yuan. The smaller the indexes, the better. The unit of professional R&D staff is person, R&D investment is ten thousand yuan, and the unit of patent number is an item. The bigger the indexes, the better.

Other indexes like quality control system, price stability, delivery punctuality, flexibility, credit, information communication, technology level, quick response, spare part support, training and technology support are all qualitative indexes. They are usually quantified by expert grade. Here, the grade is a 10-point scale score and decimal points are allowed.

If supplier who have different tasks are evaluated together, most quantitative indexes such as price and logistics costs cannot be compared. Therefore, at this point we need to invite experienced experts to make qualitative assessments of quantitative indexes by grading them as a 10-point scale score. The evaluation index system for supplier performance and its weight at development stage of C919 are shown in Table 6.4.

For the evaluation of supplier performance at development stage, we use the index system shown in Table 6.4.

**Table 6.4** The evaluation index system for supplier performance and its weight at development stage

| First-grade index and its weight | Second-grade index | Code | Unit | Weight |
|---|---|---|---|---|
| Quality (22%) | Pass rate | $x_1$ | % | 6 |
| | Quality control system | $x_2$ | Qualitative | 6 |
| | Airworthiness certification ability | $x_3$ | Qualitative | 5 |
| | Control of sub-supplier | $x_4$ | Qualitative | 5 |
| Cost (18%) | Price | $x_5$ | Ten thousand yuan | 8 |
| | Logistic cost | $x_6$ | Ten thousand yuan | 4 |
| | Price stability | $x_7$ | Qualitative | 6 |
| Delivery (17%) | Punctuality | $x_8$ | Qualitative | 12 |
| | Flexibility | $x_9$ | Qualitative | 5 |
| Cooperation (13%) | Credit | $x_{10}$ | Qualitative | 6 |
| | Information communication | $x_{11}$ | Qualitative | 4 |
| | Cooperation intention | $x_{12}$ | Qualitative | 3 |
| Technology (16%) | Professional R&D staff | $x_{13}$ | Person | 3 |
| | R&D investment | $x_{14}$ | Ten thousand yuan | 3 |
| | Number of invention patent | $x_{15}$ | Item | 3 |
| | Market share | $x_{16}$ | % | 3 |
| | Technology level | $x_{17}$ | Qualitative | 4 |
| Service (14%) | Quick response | $x_{18}$ | Qualitative | 4 |
| | Spare part support | $x_{19}$ | Qualitative | 4 |
| | Training | $x_{20}$ | Qualitative | 3 |
| | Technology support | $x_{21}$ | Q ualitative | 3 |

Step 2: According to the evaluation results, the value range of each index is divided into four grey classes. The value of second-grade indexes are usually divided into four small sections based on the sample value. Considering the opinion of COMAC, the effect sample matrix of second-grade index is omitted. Here are the actual values of six first-grade indexes that are obtained by weighted integration of the second-grade indexes as 10-point scale scores. The values are $y_{ij}$, $(i = 1, 2 \ldots, 5; j = 1, 2, \ldots 6)$ as shown in Table 6.5.

The six first-grade indexes are all in 10-point scores, and the value range is [0,10]. Interval [0,10] is sub-divided into 4 small intervals as [0,6), [6,7.5), [7.5,9), [9,10], which correspond to "poor", "medium", "good" and "excellent".

Step 3: Determine the turning point $\lambda_j^1 = 5, \lambda_j^4 = 9.5$ of [0,6) and [9,10] that correspond to grey class 1 and grey class 4. At the same time, calculate the center-point of [6,7.5) and [7.5,9), $\lambda_j^2 = 6.75$, $\lambda_j^3 = 8.25$.

Step 4: By using formulas (6.8)–(6.10), the possibility functions of index j regarding grey class k (k = 1, 2, 3, 4) can be obtained as follows:

$$f_j^1(x) = \begin{cases} 0 & x \notin [0, 6.75] \\ 1 & x \in [0, 5) \\ \frac{6.75-x}{1.75} & x \in [5, 6.75] \end{cases} \quad f_j^2(x) = \begin{cases} 0 & x \notin [5, 8.25] \\ \frac{x-5}{1.75} & x \in [5, 6.75) \\ \frac{8.25-x}{1.5} & x \in [6.75, 8.25] \end{cases}$$

$$f_j^3(x) = \begin{cases} 0 & x \notin [6.75, 9.5] \\ \frac{x-6.75}{1.5} & x \in [6.75, 8.25) \\ \frac{9.5-x}{1.25} & x \in [8.25, 9.5] \end{cases} \quad f_j^4(x) = \begin{cases} 0 & x \notin [8.25, 10] \\ \frac{x-8.25}{1.25} & x \in [8.25, 9.5) \\ 1 & x \in [9.5, 10] \end{cases}$$

where $j = 1, 2, \ldots 6$.

Step 5: The weight of each index $w_j, j = 1, 2, 3, 4, 5, 6$ is shown in Table 6.4.

Step 6: According to formulas (6.6)–(6.11), the clustering coefficient regarding the grey class of five suppliers can be calculated $(i = 1, 2, 3, 4, 5; k = 1, 2, 3, 4)$, as shown in Table 6.6.

**Table 6.5** The actual values of first-grade index of five suppliers

| Supplier | Actual value $y_{ij}$ | | | | | |
|---|---|---|---|---|---|---|
| | Quality | Cost | Delivery | Technology | Cooperation | Service |
| A | 9.1 | 7.8 | 8.4 | 9 | 9.5 | 9.3 |
| B | 9.3 | 7.5 | 9 | 9.2 | 9 | 9 |
| C | 9 | 8.6 | 8.7 | 9 | 9.1 | 9.1 |
| D | 8.9 | 8.5 | 9 | 9.1 | 9.6 | 9.2 |
| E | 8.6 | 9 | 8.6 | 9 | 9.7 | 9.5 |

**Table 6.6** The clustering coefficient regarding to each grey class of five suppliers

| The clustering objects | The clustering coefficient | | | |
|---|---|---|---|---|
| | $\sigma_i^1$ | $\sigma_i^2$ | $\sigma_i^3$ | $\sigma_i^4$ |
| A | 0 | 5.4 | 42.04 | 52.56 |
| B | 0 | 9 | 34.44 | 56.56 |
| C | 0 | 0 | 47.44 | 52.56 |
| D | 0 | 0 | 39.28 | 60.72 |
| E | 0 | 0 | 40.48 | 59.52 |

Step 7: As can be seen from the results of $\max\limits_{1\le k\le 4}\{\sigma_A^k\} = 52.56 = \sigma_A^4$, $\max\limits_{1\le k\le 4}\{\sigma_B^k\} = 56.56 = \sigma_B^4$, $\max\limits_{1\le k\le 4}\{\sigma_C^k\} = 52.56 = \sigma_C^4$, $\max\limits_{1\le k\le 4}\{\sigma_D^k\} = 60.72 = \sigma_D^4 \max\limits_{1\le k\le 4}\{\sigma_E^k\} = 59.52 = \sigma_E^4$, the performance of five suppliers A, B, C, D, E at development stage all reach the level of "excellent". Among those supplier, the clustering coefficient of supplier D regarding grey class "excellent" is the highest and supplier E takes the second place. However, the difference between D and E is very small, so the two supplier belong to the same level. Then comes supplier B. The coefficient of supplier A and C regarding grey class "excellent" is the smallest.

Further investigation reveals that the indexes belonging to class "excellent" of supplier D and E are technology, cooperation and service. There is much room for improvement in terms of quality and cost for D, and in terms of quality and delivery for E.. The main problem for supplier B is its high cost. Although the evaluation on cooperation and service is quite good, the value is still on the low side compared with other supplier. For supplier A and C, the main problems are cost and delivery. The management department of COMAC can focus on each supplier according to their own problems and improve their whole performance level promptly.

In this example, the value range of each index as well as its turning point and center-point $\lambda_j^1, \lambda_j^2, \lambda_j^3, \lambda_j^4, j = 1, 2, \ldots, 6$ regarding different grey classes are determined according to the expert evaluation results of supplier A, B, C, D, and E. Also, the conclusion only applies to the current situation of those supplier. The results of grey clustering evaluation can be used with a certain scope: the scope used when determining the possibility function is the one that can be used in the evaluation results. The so called "excellent", "good", "medium" and "poor" classes are also relative. Supplier A, B, C, D, and E are all prominent enterprises in China. Although they are very strong, there is a big gap between their performance and that of similar manufacturers around the world.

***Example 6.7.2*** The evaluation of a project for discipline development at a university will be used to illustrate the application of the grey clustering evaluation models, which are based on mixed center-point triangular possibility functions (Jian, et al., 2007; Liu, et al., 2017; Liu, 2024).

Based on extensive surveys, there are 6 primary indicators to reflect the performance of a discipline development project, including faculty, scientific research, student cultivation, discipline platform development, conditions for development and academic communication. The corresponding weights are 0.21, 0.24, 0.23, 0.14, 0.1, and 0.08, respectively (see Fig. 6.7).

We convert the evaluation scores of each indicator to centesimal system for convenience. The evaluation results are divided into four grey classes including "excellent", "good", "medium" and "poor", according to requirements of the university authorities. 41 projects for discipline development have been conducted from 2016 to 2020. All the evaluation scores of the 6 indexes of these 41 projects for discipline development are laid in the interval of [40, 100]. We set up the turning point $\lambda_j^4 = 90$ for grey class "excellent" and the turning point $\lambda_j^1 = 60$ for grey class "poor", as well as the most likely points $\lambda_j^3 = 80$, $\lambda_j^2 = 70$, which belong to grey classes "good" and "medium".

Since the evaluation scores of each indicator are converted to centesimal system, the possibility function of all 6 indicators on four grey classes of "poor", "medium", "good", and "excellent" are the same:

$$f_j^1(x) = \begin{cases} 0 & x \notin [40, 70] \\ 1 & x \in [40, 60] \\ \frac{70-x}{70-60} & x \in [60, 70] \end{cases}, \quad f_j^2(x) = \begin{cases} 0 & x \notin [60, 80] \\ \frac{x-60}{70-60} & x \in [60, 70] \\ \frac{80-x}{80-70} & x \in [70, 80] \end{cases}$$

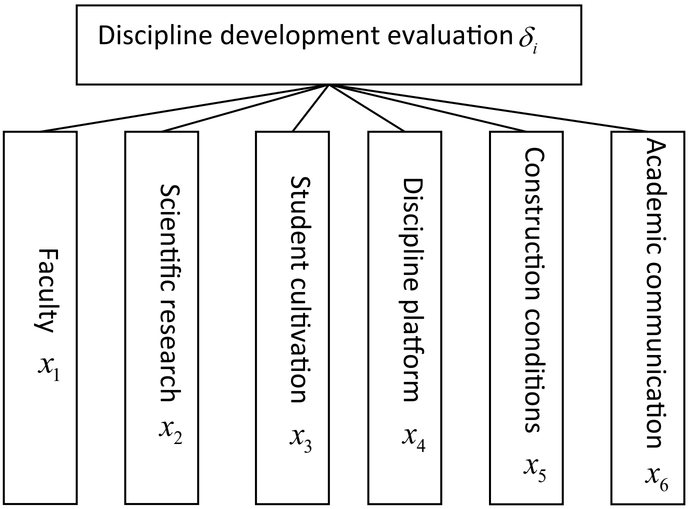



**Fig. 6.7** Evaluation indicator system of project for discipline construction

$$f_j^3(x) = \begin{cases} 0 & x \notin [70, 90] \\ \frac{x-70}{80-70} & x \in [70, 80] \\ \frac{90-x}{90-80} & x \in [80, 90] \end{cases} \quad f_j^4(x) = \begin{cases} 0 & x \notin [80, 100] \\ \frac{x-80}{90-80} & x \in [80, 90] \\ 1 & x \in [90, 100] \end{cases}$$

where the possibility function of each indicator for grey class "poor" is a possibility function of lower measure, each indicator for grey class "excellent" is a possibility function of upper measure, and each indicator for grey classes "medium" and "good" are triangular possibility functions. The values of the 6 indicators for a university's discipline development project are shown in Table 6.7.

The values of possibility functions for the different grey classes of each indicator can be calculated by using $f_j^1(x) f_j^2(x) f_j^3(x) f_j^4(x)$, $j = 1, 2, \ldots, 6$. The grey clustering coefficient $\delta_i$ can be calculated by using formulas (6.6) and (6.7). The outcomes are shown in Table 6.8.

Based on the results in Table 6.8, we can confirm that the project belongs to grey class "excellent" according to $\max_{1 \le k \le 4} \{\delta_i^k\} = \delta_i^4 = 0.419$. Therefore, the effect of the project for discipline development is remarkable. But the grey clustering coefficient which suggests that the project belongs to grey class "good" is $\delta_i^3 = 0.413$. This result is very close to $\delta_i^4$. It also shows that the execution effect of the project for discipline development is situated between grey classes "excellent" and "good". As for the sub-indicators, the indicator on student cultivation belongs to grey class "excellent", and reached a high level. The indicator on scientific research is situated between grey classes "good" and "excellent", but close to grey class "excellent". The indicators on faculty and discipline platform development basically belong to grey class "good", which indicates that the implementation effect of these two indicators are satisfactory. The indicator on development conditions is situated between grey classes "good" and "medium", but closer to grey class "medium". The indicator on academic communication belongs to grey class "poor ", which suggests that there are still significant shortcomings in development conditions and academic communication that require further strengthening.

**Table 6.7** The values of 6 indicators of a project for discipline development

| Indicator | Faculty | Scientific research | Student cultivation | Discipline platform | Development conditions | Academic communication |
|---|---|---|---|---|---|---|
| value | 81 | 87 | 92 | 78 | 74 | 53 |

**Table 6.8** Grey clustering coefficients of each indicator for different grey classes

| Grey class | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $\delta_i$ |
|---|---|---|---|---|---|---|---|
| Excellent | 0.1 | 0.7 | 1.0 | 0 | 0 | 0 | 0.419 |
| Good | 0.9 | 0.3 | 0 | 0.8 | 0.4 | 0 | 0.413 |
| Medium | 0 | 0 | 0 | 0.2 | 0.6 | 0 | 0.088 |
| Poor | 0 | 0 | 0 | 0 | 0 | 1.0 | 0.080 |

The grey cluster evaluation model based on mixed possibility function is more suitable to solve problems of poor information clustering evaluation. On the other hand, grey cluster evaluation model using center-point mixed possibility functions is suitable for problems where it is relatively easy to judge the most likely points belonging to each grey class, but the grey boundaries of all sub-intervals are not clear. Finally, grey cluster evaluation model using end-point mixed possibility functions is suitable for situations where all grey boundaries of all sub-intervals are clear, but the most likely points belonging to each grey class are unknown.

## References

Chen, D. J., & Wang, J., & Zhao, C. C. (2019). Grey clustering evaluation model of radar netting operational effectiveness. *Fire Control & Command Control, 44*(5), pp. 26–30.

Dang, Y. G., Feng, Y., Ding, S., & Wei, L. (2017). Grey clustering model for interval grey numbers based on kernel and degree of greyness. *Control and Decision*, *32*(10), 1844–1848.

Delcea, C., Cotfus, L. A., Mieriwiak, R., & Ioanfăş, C. (2022). Grey clustering of the variations in reverse pyramid boarding method considering pandemic restrictions. *Journal of Grey System*, *34*(1), 53–69.

Delgado, A., & Romero, I. (2016). Environmental conflict analysis using an integrated grey clustering and entropy-weight method: A case study of a mining project in Peru. *Environmental Modelling & Software, 77*, 108–121.

Deng, J. L. (1985). *Grey control systems*. Press of Huazhong University of Science and Technology.

Dong, F. Y., Liu, J. J., Liu, B., & Li, B. J. (2010). Improved grey integrated clustering method and its application in the evaluation to rural economic development of Henan province. *System Sciences and Comprehensive Studies in Agriculture, 26*(4), 478–483.

Fang, X., Chen, Y., & Li, S. (2012). Application of multidimensional grey evaluation methods in coal and gas outburst prediction. *Industrial Safety and Environmental Protection*, *38*(12), 81–83.

Han, X., Nan, H., Chen, J., & Jiang, K. (2014). Grey cluster model used to evaluate the development scheme of warhead in antimissile missile of air defense. *Journal of Air Force Engineering University*, *1*(1), 29–33.

Jian, L. R., Liu, S. F., & Fang, Z. G. (2007). Performance evaluation of discipline construction projects in research universities. *Journal of Management Engineering, 21*(3), 132–137.

Jiskani, I. M., Han, S., Rehman, A. U., Shahani, N. M., Tariq, M., & Brohi, M. A. (2021). An integrated entropy weight and grey clustering method–based evaluation to improve safety in mines. *Mining, Metallurgy & Exploration*, *38*(4), 1773–1787.

Li, L., & Li, X. (2023). The method and its applications of determining possibility function of grey set. *Grey Systems: Theory and Application*, *13*(1), 172–192.

Li, S. R., Zhao, Y., Liu, S. F., & Wang, T. C. (1994). Classification and stability analysis of forest ecosystem in Henan Province. *Journal of Henan Agricultural University, 28*(2), 111–118.

Liang, Q. W., Song, B. W., & Jia, Y. (2007). Grey clustering decision making for torpedo life cycle cost effectiveness evaluation. *Computer Simulation, 2007*(03), 28–30 (In Chinese).

Liu, S. F., & Guo, T. B. (1991). *Grey system theory and its application*. Henan University Press.

Liu, S. F. (1993). Fixed weight grey clustering evaluation analysis. In *New methods of grey systems* (pp. 178–184). Agriculture Press.

Liu, S. F. (2024). *Grey system theory and its application* (10th ed.). Science Press.

Liu, S. F., & Zhu, Y. D. (1993). Models based on criteria's triangular membership functions for the evaluation of regional economics. *Journal of Agricultural Engineering, 9*(2), 8–13. In Chinese.

Liu, S. F., & Xie, N. M. (2011). A new grey evaluation method based on reformative triangular whitenization weight function. *Journal of Systems Engineering, 26*(2), 244–250. In Chinese.

Liu, S. F., Fang, Z. G., & Yang, Y. J. (2014). Two stages decision model with grey synthetic measure and a betterment of triangular whitenization weight function. *Control and Decision*, *29*(7), 1232–1238. In Chinese

Liu, S. F., Yang, Y. J., Fang, Z. G., et al. (2015a). Grey cluster evaluation models based on mixed triangular whitenization weight functions. *Grey Systems: Theory and Application, 5*(3), 410–418.

Liu, S. F., Zhang, S. G., Jian, L. R., et al. (2015b). Performance evaluation of large commercial aircraft vendors. *Journal of Grey System, 27*(1), 1–11.

Liu, S. F., Yang, Y. J., & Forrest, J. (2017). *Grey data analysis*. Springer-Verlag.

Liu, et al. (2019). Research on index system for disabled elders evaluation and grey clustering model based on end-point mixed possibility functions. *Journal of Grey System*, *31*(4), 1–12.

Maleki, H., & Taghavi Fard, M. T. (2015). Evaluation of scheduling using triangular whitenization functions. *Journal of Manufacturing Technology Management, 26*(2), 297–310.

Pei, L. L., Chen, W. M., & Shen, C. G. (2012). Study on optimization of grey clustering evaluation model. *Journal of Inner Mongolia Normal University, 41*(5), 462–466.

Fang, P., Guoping, W., & Min, F. (2005). Grey programming cluster and application in evaluation of oil and gas cap layer. *Journal of Hunan University of Science and Technology*, *20*(2), 5–10.

Qian, L. L., Liu, S. F., & Xie, N. M. (2016). Grey clustering model based on entropy-weight and grey numbers. *Systems Engineering and Electronics*, *38*(2), 352–356.

Su, B. T., & Xie, N. M. (2018). Research on safety evaluation of civil aircraft based on the grey clustering model. *Grey Systems: Theory and Application, 8*(1), 110–120.

Sun, C. X. (2005). Present situation and prospect of online monitoring and diagnosis technology of the state of power transmission and transformation equipment. *China Power, 38*(2), 1–7.

Sun, L., Li, W., Ma, J., & Feng, D. (2022). A clustering evaluation models for grey panel data based on the possibility function. *Journal of Grey System*, *34*(3), 1–20.

Ţilică, Ţilică, E. V., Dragotă, V., Delcea, C., & Tătaru, R. I. (2024). Portfolio management under capital market frictions: a grey clustering approach. *Financial Innovation*, *10*(1), 110. Published online. https://doi.org/10.1186/s40854-024-00634-2

Xiao, X. P. (1997). Theoretical research and remarks on quantified models of grey degrees of incidence. *Systems Engineering: Theory and Practice, 17*(8), 76–81.

Xiong, H. J., & Chen, M. Y. (1999). *Some Problems on Grey Clustering Systems Engineering and Electronics, 21*(5), 6–91.

Xu, W. G., Zhang, Q. Y., Guo, H., & He, A. B. (2006). Improvement and application of grey clustering model in atmospheric quality comprehensive evaluation. *Mathematics in Practice and Theory, 36*(6), 200–205.

Yao, J. B., & Hu, W. W. (2008). Gray evaluation of operational efficiency of OTH ground-wave radar. *Armament Automation*, *27*(4), 12–14.

Yong, L., & Lirong, J. (2013). Probabilistic decision method of hybrid grey cluster, variable precision rough sets and fuzzy set with application. *Journal of Management Engineering*, *27*(3), 110–115.

Yuan, Z., Sun, C., Yuan, Z., Li, J., & Liao, R. (2005). Method of grey clustering decision-making to state assessment of power transformer. *Journal of Chongqing University*, *28*(3), 22–25.

Zhang, F., Wang, P., Xiao, Z., & Sun, N. (2010). Application of grey theory in safety evaluation of carier aircraft system. *Aircraft Design*, *30*(3), 56–61.

Zhang, M. M., & Luo, D. (2023). Variable stage self-correcting grey clustering model. *Applied Mathematical Modelling, 120*, 558–574.

Zhang, Q. S. (2002). Measure of grey characteristics of grey clustering result. *Chinese Journal of Management Science, 10*(1), 54–561.

Zhou, W., Dang, Y., Xiong, P., & Liu, H. S. (2013). Grey clustering model for interval grey number with variable and fixed weights. *Systems Engineering-Theory & Practice*, *33*(10), 2590–2595.

# Chapter 7
# Series of GM Models

## 7.1 Introduction

Model GM(1,1) is the basic model of grey prediction theory and has been used widely since its development in the early 1980s (Deng, 1982). Grey system theory is a new methodology that focuses on uncertain problems involving small data and poor information. Incomplete and inaccurate information is the basic characteristic of uncertainty systems. In the case of incomplete information and inaccurate data, it is impossible to pursue a refined model (Liu et al., 2012). Professor Zadeh's incompatibility principle also clearly states that, when the complexity of the system grows, our ability to make an accurate and significant description of a system's characteristics decreases until it reaches a threshold value that, if it exceeded, accuracy and significance will become mutually exclusive characteristics (Zadeh, 1994). The incompatibility principle tells us that pursuing a refinement model one-sidedly will reduce the feasibility and significance of the results. A refined model is not an effective means address complex systems.

In the last 40 years, much research has been carried out on the practical applications of Model GM(1,1), and new research results emerge continuously.

The grey prediction model is one type of the GM with most active research and used widely. In 2005, Xie and Liu proposed the discrete grey model (DGM) first and studied its properties (Xie & Liu, 2005, 2009). Later, Wu et al. (2013b, 2015) came up with a kind of the fractional accumulation DGM and completed perturbation problem of GM. Chen et al. (2009) set up the DHGM (2,2) coupled equations combining grey differential equation and selfmemory principle based on power system self-memory principle. Guo et al. (2014b) proposed the interval grey number self-memory prediction model based on the degree of greyness of synthesis grey number, then studied self-memory prediction model from different views.

Various forms of developments and derived model emerged in an endless stream.

Dang and Liu (2004) came up with GM (1,1) model based on x(n) as the initial condition. In 2014d, Liu et al. determined four kinds of GM(1,1) basic models that are



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_7

even GM(1,1) model, discrete GM(1,1) model, even difference GM(1,1) model and original difference GM(1,1) model through the experiments of simulation, then made clear of the suitable type of sequences of the different model. Li et al. (2014) proposed GM(1,1, β) model, studied the content type and parameter set form of the model and analyzed several properties of the GM(1,1, β) model, then gave its optimization algorithm. Wang (2013) provided several kinds of forms of GM(1,1) power model and studied the characteristics of their time response function. Xiao et al. (2013) studied generalized accumulation GM and proposed a combined optimization method. Qian et al. (2012) came up with the grey GM(1,1, tα) model with the time power item and studied the process of modeling and parameter estimation method. Tang (2006) proposed a new prediction model based on grey supporting vector machine. Zhang (2014) put forward the multi variable DGM based on driving control. Zeng and Liu (2014) came up with the random oscillation sequence prediction model taking smooth operator compress random oscillation amplitude.Wu et al. proposed a time power-based grey model with conformable fractional derivative (Wu et al., 2022). Zeng et al. proposed a variable-structure grey model (Zeng et al., 2023). Öztürk et al., proposed a optimization continuous fractional grey model (Öztürk et al., 2022). Saxena proposed a optimization fractional overhead power term polynomial grey model (Saxena, 2023).

Zhang (2007) used particle swarm algorithm and provided a new method of increasing the grey GM(1,1) precision through the optimization of background value interpolation coefficient and boundary value. Yao et al. (2010) studied the parameter characteristics of the new information discrete GM(1,1) model and fitting properties of the geometric sequence, then put forward a new information discrete GM(1,1) model with the sectional correction. Wu et al. (2013a) constructed the twice time-varying parameter DGM with the features of the white index law coincidence, linear law coincidence, twice law coincidence and stretching transformation consistency. Benitez et al. (2013) improved the GM model and forecasted long-term trend of American air transport industry passenger flow using improved model, then get satisfied results.Evans (2014) proposed a more general grey Verhulst model and forecasted changes of the steel strength in British used this model. Xie et al. (2014) studied the prediction problems of grey number sequence.

Liu and Deng (2000) studied the range suitable for GM(1,1) based on simulated test.The area of validity, the area to be used carefully, the area not suitable for use and the prohibited area of GM(1,1) have been divided clearly according to the threshold of the developing coefficients. Xiao and Wang (2014) studied the influences of model relative error made by the change of the background value of grey GM(1,1,α) model based on analysis of the modeling mechanism. Liu et al. (2003) utilized the method of "the least square estimate" to determine the constant number c in the time response sequence of whiterization equation of GM(1,1), then got the optimum time response sequence of whiterization equation for GM(1,1). Song et al. (2002) given a new method to handle derivative signal and background value and derived the adjusting GM. Ji et al. (2001) analyzed the characteristics of the deviation of the model. Then clarified the essence of the error of GM(1,1) model. Tong et al. (2002) shown that accumulated generating operation (AGO) of the GM can "strengthen" the law and reform randomicity of numbers, so it has nice anti-interference. Wang et al. (2001) put forward a GM(1,1) modeling method by taking the optimum weighted averages of ahead difference quotient and back difference quotient as the grey derivative whiting values, and proved that the new method have the linear transformation consistency. Mao et al. (2015) built a time-lag GM(1, N,τ) model, and provided its least squares parameter estimation formula and analytical solution. Liu et al. (2014a) analyzed the solution errors of a whitenization GM(1,1) model and a connotation GM(1,1) model, then present the condition that a connotation GM(1,1) model can be replaced by a whitenization GM(1,1) model. Tien (2003) proposed deterministic grey dynamic model with multiple inputs, DGDMMI(1,1,1) which with high prediction accuracy. Xia et al. (2015) proposed a real-time rolling grey forecasting method to provide efficient and accurate machine health prediction, while effects of influencing factors such as operating load are considered and analyzed.

The application results of the grey prediction model are numerous.

Such as Kose and Tasci (2015) predicted the vertical displacement of the Crest of Keban Dam in Turkey by grey prediction method. Their results indicate that the grey prediction method produces better results, more in-keeping with true values. Gurden et al. (2001) built a spectroscopic batch process data model using GMs to incorporate external information. In their paper, different approaches to building GMs are described and some of their properties discussed. Chirwa and Mao (2006) used GM(1,1) model to estimate the accident risk based on data of UK and USA. Cempel (2008) monitored mechanical vibration state using the grey prediction model. Hsu and Yeh (2000) developed a new methodology for lossy image compression based on GM. Hao et al. (2012) analyzed and predicted hydrological process in Karst River Basin using the grey prediction model and gained the higher precision. Then they studied human activities effect on hydrological process in Karst River Basin using sectioned GM (Hao et al., 2013). Yang and Wong (2014) made the further improvement about the unbiased GM and forecasted the amount of some city's gas supplement. Tabaszewski and Cempel (2015) developed a methodology of predicting values of vibration symptoms of fan mills in a combined heat and power plant based on grey system theory and GM(1,1) prognostic models.

Wang and Nie (2008) forecasted mechanical fatigue life using the grey system model, which made prediction error greatly reduced. Bo et al. (2012) applied BP neural network method and grey system model to predict Tianjin Qinhuangdao passenger dedicated line Luqiao transition section roadbed settlement. Wang and Yihua (2010) adopted the grey neural network method and set up a nonlinear prediction model of China civil aviation operation risk. Li et al. (2011) and Yang et al. (2008) forecasted spacecraft fault using the grey system model and obtained the high accuracy. Zhang et al. (2006) applied GM(1,1) model to study variation rule of the robot emotional state and achieved emotional robot interaction system. Liet al. (2010) measured fatigue crack propagation rate using the grey prediction model. Lin et al. (2005) established the grey prediction model about the slope rock mass deformation according to the test data of Three Gorges site slope. Then they drew the fitting and prediction curves of slope deformation, which provided reliable guarantee and theoretical basis for its prediction. Benítez et al. predicted the damp trend of the airline industry (Benítez et al., 2013). Xie et al. evaluated and forecasted the niche fitness of regional innovation ecosystems (Xie et al., 2023). Duman et al. Estimated the electronic waste using optimized multivariate grey models (2019). Özdemir and Özdagoglu predicted product demand based on small-sized data (Özdemir & Özdagoglu, 2017). Comert et al. Built a grey models for short-term queue length predictions for adaptive traffic signal control (Comert et al., 2021). Manickam et al. predicted the volatile price of gold by grey model (Manickam et al., 2023). Zeng et al. built a novel-structured multivariable grey model with various orders to forecast the bending strength of concrete (Zeng et al., 2023). Zhang et al. predicted the demand for staple food and feed grain by a novel hybrid fractional discrete multivariate grey model (Zhang et al., 2024).

In big data area, the grey system prediction method based on small data mining as a new force suddenly rises, which becomes an effective tool for valuable information extracted from a mass of data. It is a very meaningful job to build more normal model testing standards based on the grey system prediction model testing method and statistical testing theory. The investigation on the potential of grey prediction models in Big Data is certainly a future direction in this field.

## 7.2 The Four Basic Forms of GM(1,1)

In this section we present definitions of four basic forms of model GM(1,1), including Even Grey Model (EGM), Original Difference Grey Model (ODGM), Even Difference Grey Model (EDGM) and Discrete Grey Model (DGM). The properties and characteristics of different models are discussed in-depth (Liu et al., 2015).

### *7.2.1 The Basic Forms of Model GM(1,1)*

**Definition 7.2.1** Let $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)), x^{(0)}(k) \geq 0, X^{(1)}$ be the 1-AGO sequence of $X^{(0)}$; that is.

$$X^{(1)} = \left[x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n)\right]$$

where $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i), k = 1, 2, \ldots n$ Then

$$x^{(0)}(k) + ax^{(1)}(k) = b \tag{7.1}$$

is referred to as the original form of model GM(1,1), which is a difference equation (Deng, 1985).

The parameter vector $\hat{a} = [a, b]^T$ of formula (7.1) can be estimated using the least square method, which satisfies

$$\hat{a} = (B^T B)^{-1} B^T Y \tag{7.2}$$

where

$$B = \begin{bmatrix} -x^{(1)}(2) & 1 \\ -x^{(1)}(3) & 1 \\ \vdots & \vdots \\ -x^{(1)}(n) & 1 \end{bmatrix}, Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix} \tag{7.3}$$

**Definition 7.2.2** Based on the original form of model GM(1,1) and formula (7.2), which is used to estimate the model's parameters, then the model that takes the solution of the original difference Eq. (7.1) as the time response formula is called the original difference form of model GM(1,1), and is referred to as Original Difference Grey Model(ODGM) for short (Liu et al., 2015).

**Definition 7.2.3** Let $X^{(0)}$, $X^{(1)}$ and, just like Definition 7.2.1, let.

$$Z^{(1)} = \left(z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n)\right),$$

where $z^{(1)}(k) = \frac{1}{2}(x^{(1)}(k) + x^{(1)}(k-1))$, then

$$x^{(0)}(k) + az^{(1)}(k) = b \tag{7.4}$$

is referred to as the even form of model GM(1,1)(Deng, 1985).

The even form of model GM(1,1) is also essentially a difference equation. The parameter vector of formula (7.4) can also be estimated with formula (7.2), but it should be noted that the elements of matrix B are different from those in formula (7.3), which is

$$B = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{bmatrix} \tag{7.5}$$

**Definition 7.2.4** The following differential equation.

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b \tag{7.6}$$

is called a shadow equation of the even form $x^{(0)}(k) + az^{(1)}(k) = b$ of model GM(1,1) (Deng, 1985).

**Definition 7.2.5** Replace matrix B of formula (7.2) with (7.5), according to parameter vector $\hat{a} = [a, b]^T$ of the least squares estimator of (6) and the solution of whitenization Eq. (7.6), and model the difference, differential hybrid model of the time response formula of GM(1,1). This is called the even hybrid form of model GM(1,1), and is referred to as Even Grey Model (EGM) for short (Deng, 1985).

**Definition 7.2.6** The parameter $-a$ of Even GM(1,1) is called development index and b is called grey actuating quantity. The development index reflects the trend of $\hat{x}^{(1)}$ and $\hat{x}^{(0)}$.

Even Model GM(1,1) is the grey prediction model proposed firstly by Professor Deng Julong, and is currently the most influential, widely used form. When researchers mention model GM(1,1) they are often referring to EGM.

**Definition 7.2.7** Based on the even form of model GM(1,1) and the estimated model parameters, then the model that takes the solution of the even difference Eq. (7.4) as the time response formula is called the even difference form of model GM(1,1), and is referred to as Even Difference Grey Model(EDGM) for short (Liu et al., 2015).

**Definition 7.2.8** The difference equation as follows.

$$x^{(1)}(k+1) = \beta_1 x^{(1)}(k) + \beta_2 \tag{7.7}$$

is called a discrete form of model GM(1,1), and is referred to as Discrete Grey Model (DGM) for short (Xie & Liu, 2005).

The parameter vector $\hat{\beta} = [\beta_1, \beta_2]^T$ in Eq. (7.7) is similar to formula (7.2), where

$$B = \begin{bmatrix} x^{(1)}(1) & 1 \\ x^{(1)}(2) & 1 \\ \vdots & \vdots \\ x^{(1)}(n-1) & 1 \end{bmatrix}, Y = \begin{bmatrix} x^{(1)}(2) \\ x^{(1)}(3) \\ \vdots \\ x^{(1)}(n) \end{bmatrix}$$

The four different models of GM(1,1) use only the system's behavior data sequence to model the predictive models and belong to the simple and practical modeling method with a single sequence. In the case of time series data, only a regular time variables are involved; In the case of horizontal sequence data, only a regular object sequence number variables are involved, and other explanatory variables are not involved. GM(1,1) model is a modeling method which is relatively simple to apply and can mine valuable development and change information, so it is widely used.

### 7.2.2 Properties and Characteristics of the Basic Model

**Theorem 7.2.1** The time response sequence of the Even Model GM(1,1) is as follows:

$$\hat{x}^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-a(k-1)} + \frac{b}{a}, k = 1, 2, \ldots n \tag{7.8}$$

***Proof*** The solution of whitenization or shadow equation $\frac{dx^{(1)}}{dt} + ax^{(1)} = b$ is

$$x^{(1)}(t) = \mathrm{C}e^{-at} + \frac{b}{a}. \tag{7.9}$$

When t = 1, we let $x^{(1)}(1) = x^{(0)}(1)$, and feed into Eq. (7.9); we can obtain $\mathrm{C} = \left[\mathrm{x}^{(0)}(1) - \frac{b}{a}\right]e^{a}$. After that we take C into Eq. (7.9) and can get

$$\hat{x}^{(1)}(\mathrm{t}) = \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-a(t-1)} + \frac{b}{a} \tag{7.10}$$

Equation (7.8) is the discrete form of Eq. (7.10). From Eq. (7.8)'s regressive reduction formula

$$\hat{x}^{(0)}(k) = \alpha^{(1)}\hat{x}^{(1)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1), k = 1, 2, \ldots n,$$

we can obtain the time response formula of $X^{(0)}$, that is

$$\hat{x}^{(0)}(k) = \left(1 - e^{a}\right)\left(x^{(0)}(1) - \frac{b}{a}\right)e^{-a(k-1)},$$
$$k = 1, 2, \ldots n, \tag{7.11}$$

**Theorem 7.2.2** The time response formula of formula (7.7) of the Discrete Model GM(1,1) is.

$$\hat{x}^{(1)}(k) = \left[\mathrm{x}^{(0)}(1) - \frac{\beta_2}{1-\beta_1}\right]\beta_1^k + \frac{\beta_2}{1-\beta_1} \tag{7.12}$$

***Proof*** The general solution of the difference Eq. (7.13)

$$x^{(1)}(k+1) = \mathrm{A}x^{(1)}(k) + \mathrm{B} \tag{7.13}$$

is

$$x^{(1)}(k) = \mathrm{C}A^k + \frac{B}{1-A}, \tag{7.14}$$

where C is an arbitrary constant and can be defined by the initial conditions.

Formulas (7.7) and (7.14) are exactly the same difference equation. Let $A = \beta_1, B = \beta_2$, then

$$x^{(1)}(k) = \mathrm{C}\beta_1^k + \frac{\beta_2}{1-\beta_1}. \tag{7.15}$$

When k = 0, let $x^{(1)}(0) = x^{(0)}(1)$, and feed into formula (7.15), we can get $\mathrm{C} = \left[\mathrm{x}^{(0)}(1) - \frac{\beta_2}{1-\beta_1}\right]$. Then take C into formula (7.15) and we can obtain formula (7.12).

From formula (7.12)'s regressive reduction formula

$$\hat{x}^{(0)}(k) = \alpha^{(1)}\hat{x}^{(1)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1), k = 1, 2, \ldots n,$$

we can obtain the time response formula of $\mathrm{X}^{(0)}$, that is

$$\hat{x}^{(0)}(k) = (\beta_1 - 1)\left[\mathrm{x}^{(0)}(1) - \frac{\beta_2}{1-\beta_1}\right]\beta_1^{k-1}. \tag{7.16}$$

**Theorem 7.2.3** The time response formula of Original Difference Model GM(1,1) is.

$$\hat{x}^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1}{1+a}\right)^k + \frac{b}{a} \tag{7.17}$$

***Proof*** From the original form (7.1) of model GM(1,1) we can get

$$x^{(1)}(k+1) - x^{(1)}(k) + ax^{(1)}(k+1) = b. \tag{7.18}$$

After transposition, we obtain

$$x^{(1)}(k+1) = \left(\frac{1}{1+a}\right)x^{(1)}(k) + \frac{b}{1+a}$$

Contrast with the difference Eq. (7.13), when we feed $A = \frac{1}{1+a}, B = \frac{b}{1+a}$ into Eq. (7.14), we can obtain

$$x^{(1)}(k) = \mathrm{C}\left(\frac{1}{1+a}\right)^k + \frac{b}{a} \tag{7.19}$$

When k = 0, let $x^{(1)}(0) = x^{(0)}(1)$, feed into formula (7.18) and get $\mathrm{C} = \left[\mathrm{x}^{(0)}(1) - \frac{b}{a}\right]$. Then we feed C into formula (7.19) and can obtain formula (7.17).

From formula (7.17)'s regressive reduction formula

$$\hat{x}^{(0)}(k) = \alpha^{(1)}\hat{x}^{(1)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1), k = 1, 2, \ldots n,$$

we can obtain the time response formula of $X^{(0)}$, which is

$$\hat{x}^{(0)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1}{1+a}\right)^{k} + \frac{b}{a} - \left[\left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1}{1+a}\right)^{k-1} + \frac{b}{a}\right].$$

That is

$$\hat{x}^{(0)}(k) = (-a)\left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1}{1+a}\right)^{k} \tag{7.20}$$

**Theorem 7.2.4** The time response formula of Even Difference Model GM(1,1) is.

$$x^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1-0.5a}{1+0.5a}\right)^{k} + \frac{b}{a} \tag{7.21}$$

***Proof*** From the even form (7.4) of model GM(1,1) we can get

$$x^{(1)}(k+1) - x^{(1)}(k) + a\left(\frac{x^{(1)}(k+1) + x^{(1)}(k)}{2}\right) = b.$$

After transposition, we obtain

$$x^{(1)}(k+1) = \left(\frac{1-0.5a}{1+0.5a}\right)x^{(1)}(k) + \frac{b}{1+0.5a}$$

Contrast with the difference Eq. (7.13), and feed $A = \frac{1-0.5a}{1+0.5a}, B = \frac{b}{1+0.5a}$ into formula (7.14). We can obtain

$$x^{(1)}(k) = \mathrm{C}\left(\frac{2-a}{2+a}\right)^{k} + \frac{b}{a} \tag{7.22}$$

When k = 0, let $x^{(1)}(0) = x^{(0)}(1)$, feed it into formula (7.22) and get $\mathrm{C} = \left[\mathrm{x}^{(0)}(1) - \frac{b}{a}\right]$. Then feed C into formula (7.22) and we can obtain formula (7.21).

From formula (7.21)'s regressive reduction formula

$$\hat{x}^{(0)}(k) = \alpha^{(1)}\hat{x}^{(1)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1), k = 1, 2, \ldots n,$$

we can obtain the time response formula of $X^{(0)}$, which is

$$\hat{x}^{(0)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1-0.5a}{1+0.5a}\right)^{k} + \frac{b}{a} - \left[\left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1-0.5a}{1+0.5a}\right)^{k-1} + \frac{b}{a}\right].$$

That is

$$\hat{x}^{(0)}(k) = \left(\frac{-a}{1-0.5a}\right)\left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1-0.5a}{1+0.5a}\right)^k \tag{7.23}$$

**Lemma 7.2.1** When $-a \to 0^+$, $\frac{1-0.5a}{1+0.5a} \approx e^{-a}$.

***Proof*** The Maclaurin expansions of $e^{-a}$ and $\frac{1-0.5a}{1+0.5a}$ are as follows:

$$e^{-a} = 1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots + (-1)^n \frac{a^n}{n!} + o(a^n)$$

$$\frac{1-0.5a}{1+0.5a} = 1 - a + \frac{a^2}{2} - \frac{a^3}{2^2} + \cdots + (-1)^{n+1} \frac{a^{n+1}}{2^n} + o(a^{n+1})$$

As n = 3, then there is $\Delta = e^{-a} - \frac{1-0.5a}{1+0.5a} = -\frac{a^3}{6} + \frac{a^3}{4} = \frac{a^3}{12}$, therefore, when $-a \to 0^+$, $\frac{1-0.5a}{1+0.5a} \approx e^{-a}$.

**Theorem 7.2.5** When $-a \to 0^+$, Even Model GM(1,1) and Discrete Model GM(1,1) are equivalent.

***Proof*** From the even form (7.4) of Model GM(1,1).

$$x^{(1)}(k+1) = \left(\frac{1-0.5a}{1+0.5a}\right)x^{(1)}(k) + \frac{b}{1+0.5a}.$$

and contrast with the discrete form (7.7), we can obtain $\beta_1 = \frac{1-0.5a}{1+0.5a}$, $\beta_2 = \frac{b}{1+0.5a}$ and

$$a = \frac{2(1-\beta_1)}{1+\beta_1}, b = \frac{2\beta_2}{1+\beta_1}, \frac{b}{a} = \frac{\beta_2}{1-\beta_1}. \tag{7.24}$$

Take $\frac{b}{a} = \frac{\beta_2}{1-\beta_1}$ into formula (7.8), we can get

$$\hat{x}^{(1)}(k) = \left[\mathrm{x}^{(0)}(1) - \frac{\beta_2}{1-\beta_1}\right]e^{-a(k-1)} + \frac{\beta_2}{1-\beta_1}, k = 1, 2, \ldots n \tag{7.25}$$

It is known from Lemma 1 that when $-a \to 0^+$, therefore, Even Model GM(1,1) and Discrete Model GM(1,1) are equivalent.

Analogously, we can prove that when $-a \to 0^+$, the four basic forms of model GM(1,1), namely Even Model GM(1,1) (EGM), Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM) are pairwise equivalent. However, the degree of approximation between different forms is a difference. This difference leads to different forms of Model GM(1,1) being suitable for different situations, and it also offers a variety of possible options for the actual modeling process.

**Theorem 7.2.6** Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1) (EDGM) and Discrete Model GM(1,1) (DGM) can all accurately simulate homogeneous exponential sequences.

Since the time response formulas of Original Difference Model GM(1,1)(ODGM), Even Difference GM(1,1) model (EDGM) and Discrete GM (1,1) model (DGM) are all geometric sequences, they can accurately simulate homogeneous exponential sequences.

In the basic forms of GM(1,1), the development coefficient ($-a$) reflects the development states of $\hat{x}^{(1)}$ and $\hat{x}^{(0)}$. In general, the variables that act upon the system of interest should be external or pre-defined. Because GM(1,1) is a kind of model constructed on a single sequence, it uses only the behavioral sequence (also referred to as output sequence or background values) of the system without considering any externally acting sequences (also referred to as input sequences, or driving quantities). The grey action quantity $b$ in the basic forms of GM(1,1) is a value derived from the background values. It reflects changes contained in the data and its exact intension is grey. This quantity realizes the extension of the relevant intension. Its existence distinguishes grey systems modeling from the general input–output (or black-box) modeling. It is also an important test stone of separating the thoughts of grey systems and those of grey boxes.

## 7.3 Suitable Ranges of Different GM(1,1)

The suitable sequences of different basic models of GM(1,1) (Liu et al., 2015) and the applicable ranges of EGM (Liu & Deng, 2000) are studied by simulation and analysis with homogeneous exponential sequences, non-exponential increasing sequences, and vibration sequences. It can provide reference and a basis for people to choose the correct model in the actual modeling process.

### *7.3.1 Suitable Sequences of Different GM(1,1)*

For further study of the suitable sequences of four basic forms of model GM(1,1), we let

$$\begin{aligned} -a = {} & 0.01, 0.02, 0.03, 0.04, 0.05, 0.1, 0.15, 0.2, 0.25, \\ & 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.8, 0.9, \\ & 1.0, 1.1, 1.2, 1.5, 1.8 \end{aligned}$$

and conduct simulation analysis, respectively. Let k = 1, 2, 3, 4, 5, with the homogeneous exponential function $x_i^{(0)}(k) = e^{-ak}$, and accurate to six decimal places. Then we can get the corresponding sequences as follows:

$$-a = 0.01, X_1^{(0)} = \left(x_1^{(0)}(1), x_1^{(0)}(2), x_1^{(0)}(3), x_1^{(0)}(4), x_1^{(0)}(5)\right) = (1.010050, 1.020201, 1.030455, 1.040811, 1.051271)$$

$$-a = 0.02, X_2^{(0)} = \left(x_2^{(0)}(1), x_2^{(0)}(2), x_2^{(0)}(3), x_2^{(0)}(4), x_2^{(0)}(5)\right) = (1.020201, 1.040811, 1.061837, 1.083287, 1.105171)$$

..........................

$$-a = 1.8, X_{25}^{(0)} = \left(x_{25}^{(0)}(1), x_{25}^{(0)}(2), x_{25}^{(0)}(3), x_{25}^{(0)}(4), x_{25}^{(0)}(5)\right) = (6.049647, 36.59823, 221.4064, 1339.431, 8103.084).$$

We use $X_1^{(0)}$, $X_2^{(0)}$, …, $X_{25}^{(0)}$ as the original data to establish Even Model GM(1,1)(EGM), Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM), respectively. We can find that Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM) can accurately simulate homogeneous exponential sequence, which confirms the conclusions of Theorem 7.2.6 once again. Using Even Model GM(1,1)(EGM) to simulate $X_1^{(0)}$, $X_2^{(0)}$, …, $X_{25}^{(0)}$, it is fond that with the increasing of $-a$, the error will also increase. Table 7.1 shows the average relative error using four kinds of model GM(1,1) to simulate the homogeneous exponential sequence $X_1^{(0)}, X_2^{(0)}, \ldots, X_{25}^{(0)}$.

In Table 7.1, we can see that the small errors of Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM) which simulate the homogeneous exponential sequence are all caused by round-off errors. In fact, the three models can all accurately simulate the homogeneous exponential sequence.

Then, we limited the range of random numbers at first, and got the non-exponential increasing sequence $Y_1^{(0)}$, $Y_2^{(0)}$, …, $Y_{25}^{(0)}$ randomly generated by the homogeneous exponential sequence $X_1^{(0)}$, $X_2^{(0)}$, …, $X_{25}^{(0)}$, along with the vibration sequence $Z_1^{(0)}$, $Z_2^{(0)}, \ldots, Z_{25}^{(0)}$. With that, when $k = 2, 3, \ldots, 5, z_i^{(0)}(k) < z_i^{(0)}(k-1), i = 1, 2, \ldots, 25$ will arise in the sequence data but there is a growth trend as a whole, both are equally accurate to six decimal places. Then we build Even Model GM(1,1)(EGM), Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM) using sequences $Y_1^{(0)}$, $Y_2^{(0)}$, …, $Y_{25}^{(0)}$ and $Z_1^{(0)}$, $Z_2^{(0)}$,

**Table 7.1** The simulation errors of the homogeneous exponential sequence of four kinds of GM(1,1) (%)

| Code | $-a$ | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|---|
| $X_1^{(0)}$ | 0.01 | 0.000849 | 0.000027 | 0.000027 | 0.000027 |
| $X_2^{(0)}$ | 0.02 | 0.003468 | 0.000013 | 0.000013 | 0.000013 |
| $X_3^{(0)}$ | 0.03 | 0.007951 | 0.000018 | 0.000018 | 0.000018 |
| $X_4^{(0)}$ | 0.04 | 0.014403 | 0.000004 | 0.000004 | 0.000003 |
| $X_5^{(0)}$ | 0.05 | 0.022922 | 0.000016 | 0.000016 | 0.000016 |
| $X_6^{(0)}$ | 0.1 | 0.100058 | 0.000008 | 0.000008 | 0.000008 |
| $X_7^{(0)}$ | 0.15 | 0.244034 | 0.000009 | 0.000009 | 0.000009 |
| $X_8^{(0)}$ | 0.2 | 0.467588 | 0.000003 | 0.000003 | 0.000007 |
| $X_9^{(0)}$ | 0.25 | 0.78359 | 0.000005 | 0.000005 | 0.000006 |
| $X_{10}^{(0)}$ | 0.3 | 1.205144 | 0.000004 | 0.000004 | 0.00001 |
| $X_{11}^{(0)}$ | 0.35 | 1.74561 | 0.000006 | 0.000006 | 0.00001 |
| $X_{12}^{(0)}$ | 0.4 | 2.418758 | 0.000004 | 0.000004 | 0.00001 |
| $X_{13}^{(0)}$ | 0.45 | 3.238864 | 0.000007 | 0.000007 | 0.000008 |
| $X_{14}^{(0)}$ | 0.5 | 4.220851 | 0.000011 | 0.000011 | 0.000008 |
| $X_{15}^{(0)}$ | 0.55 | 5.380507 | 0.000003 | 0.000003 | 0.000003 |
| $X_{16}^{(0)}$ | 0.6 | 6.734574 | 0.000016 | 0.000016 | 0.000011 |
| $X_{17}^{(0)}$ | 0.65 | 8.30104 | 0.000009 | 0.000009 | 0.000006 |
| $X_{18}^{(0)}$ | 0.7 | 10.09936 | 0.000021 | 0.000021 | 0.000021 |
| $X_{19}^{(0)}$ | 0.8 | 14.47851 | 0.000015 | 0.000015 | 0.000015 |
| $X_{20}^{(0)}$ | 0.9 | 20.06845 | 0.000016 | 0.000016 | 0.000022 |
| $X_{21}^{(0)}$ | 1 | 27.11084 | 0.000047 | 0.000047 | 0.000047 |
| $X_{22}^{(0)}$ | 1.1 | 35.90812 | 0.00004 | 0.00004 | 0.000035 |
| $X_{23}^{(0)}$ | 1.2 | 46.84484 | 0.000105 | 0.000105 | 0.000105 |
| $X_{24}^{(0)}$ | 1.5 | 98.1885 | 0.000129 | 0.000129 | 0.000129 |
| $X_{25}^{(0)}$ | 1.8 | – | 0.000433 | 0.000433 | 0.000433 |

…, $Z_{25}^{(0)}$ respectively. The errors we can see are in Tables 7.2 and 7.4. Due to limited space, the generating data are not shown here.

From Table 7.2 we can see that four kinds of model GM(1,1) can all simulate the non-exponential increasing sequence to a certain degree. Generally speaking, the simulation error will increase with the increasing of the development index. In most cases, the simulation error of the difference, differential hybrid form of Even Model GM(1,1)(EGM), is smaller than that of the three discrete forms of Original Difference

**Table 7.2** The simulation errors of the non-exponential increasing sequence of four kinds of GM(1,1) (%)

| Code | $-, -a, +$ | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|---|
| $Y_1^{(0)}$ | 0.01 | 0.030994 | 0.030429 | 0.030432 | 0.03043 |
| $Y_2^{(0)}$ | 0.02 | 0.658978 | 0.659039 | 0.660095 | 0.659572 |
| $Y_3^{(0)}$ | 0.03 | 0.495833 | 0.495773 | 0.495768 | 0.49577 |
| $Y_4^{(0)}$ | 0.04 | 1.010474 | 1.010308 | 1.010329 | 1.010319 |
| $Y_5^{(0)}$ | 0.05 | 1.550886 | 1.550331 | 1.550468 | 1.550401 |
| $Y_6^{(0)}$ | 0.1 | 1.626294 | 1.70498 | 1.690324 | 1.697211 |
| $Y_7^{(0)}$ | 0.15 | 1.343565 | 1.4578 | 1.458993 | 1.458442 |
| $Y_8^{(0)}$ | 0.2 | 5.155856 | 5.100486 | 5.22948 | 5.171925 |
| $Y_9^{(0)}$ | 0.25 | 4.353253 | 4.893857 | 4.743792 | 4.808361 |
| $Y_{10}^{(0)}$ | 0.3 | 4.736323 | 5.345755 | 5.168529 | 5.244168 |
| $Y_{11}^{(0)}$ | 0.35 | 5.236438 | 5.377225 | 5.192273 | 5.269577 |
| $Y_{12}^{(0)}$ | 0.4 | 3.603875 | 4.166958 | 4.044567 | 4.096904 |
| $Y_{13}^{(0)}$ | 0.45 | 12.834336 | 15.36423 | 13.520184 | 14.246584 |
| $Y_{14}^{(0)}$ | 0.5 | 7.39677 | 8.276073 | 7.878017 | 8.044898 |
| $Y_{15}^{(0)}$ | 0.55 | 10.218727 | 10.084749 | 10.188912 | 10.143461 |
| $Y_{16}^{(0)}$ | 0.6 | 21.07307 | 23.709858 | 21.610863 | 22.440905 |
| $Y_{17}^{(0)}$ | 0.65 | 6.637022 | 7.906483 | 7.629068 | 7.731359 |
| $Y_{18}^{(0)}$ | 0.7 | 9.0889 | 11.000565 | 10.479505 | 10.677398 |
| $Y_{19}^{(0)}$ | 0.8 | 21.156265 | 30.606589 | 28.554915 | 29.245194 |
| $Y_{20}^{(0)}$ | 0.9 | 14.441947 | 20.378328 | 17.104 | 18.188008 |
| $Y_{21}^{(0)}$ | 1 | 11.685913 | 18.463203 | 17.357496 | 17.734931 |
| $Y_{22}^{(0)}$ | 1.1 | 13.011857 | 20.620317 | 19.396248 | 19.782271 |
| $Y_{23}^{(0)}$ | 1.2 | 17.176472 | 27.929743 | 26.16349 | 26.624283 |
| $Y_{24}^{(0)}$ | 1.5 | 26.327218 | 51.915584 | 50.006882 | 50.471089 |
| $Y_{25}^{(0)}$ | 1.8 | 62.460946 | 75.503705 | 73.434001 | 74.070128 |

Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM). As the non-exponential increasing sequence is closer to the homogeneous exponential sequence, the simulation accuracy of the three discrete models is higher than. When the non-exponential increasing sequence is close to the homogeneous exponential sequence to a certain extent, the simulation accuracy of the discrete models will be smaller than that of Even Model GM(1,1)(EGM). From the simulation results of the three discrete models GM(1,1), we can see that with the increasing of the development coefficient, the simulation accuracy of Original Difference Model GM(1,1)(ODGM), and Even Difference Model GM(1,1)(EDGM) is higher than that of Discrete Model GM(1,1)(DGM) in most cases. The statistics for sorting the simulation error of different models with the sequence $Y_1^{(0)}$, $Y_2^{(0)}$, …, $Y_{25}^{(0)}$ in ascending order are presented in Table 7.2. Table 7.3 shows the statistical results.

As can be seen from Table 7.3, among the four kinds of models, Even Model GM(1,1)(EGM) is the most suitable for modeling with a non-exponential increasing sequence, followed by the Original Differential Model GM (1,1) (ODGM) and Even Difference Model GM (1,1) (EDGM). The error is slightly larger when using the Discrete Model GM (1,1)(DGM) to simulate the non-exponential increasing sequence.

In theory, any simple model which describes a monotonous trend struggles to describe a change in the vibration sequence. Therefore, we add the limiting condition of the random number, then the research range is the vibration sequence $Z_1^{(0)}$, $Z_2^{(0)}$, …, $Z_{25}^{(0)}$. With that, when $k = 2, 3, \ldots, 5, z_i^{(0)}(k) < z_i^{(0)}(k-1), i = 1, 2, \ldots, 25$ will arise in the sequence data, but there is a growth trend as a whole. We can see from Table 7.4 that, for this specific vibration sequence, the simulation error of the four kinds of models is significantly higher than the non-exponential increasing sequence. Similar to the situation of the non-exponential increasing sequence, in most cases the simulation error of Even Model GM(1,1)(EGM) to the vibration sequence is smaller than that of the three discrete forms of Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM). For the vibration sequence being close to the homogeneous exponential sequence, the simulation error of the discrete model is smaller than one of the difference, differential hybrid form of Even Model GM(1,1)(EGM).

The statistics for sorting the simulation error of different models with the vibration sequence $Z_1^{(0)}$,$Z_2^{(0)}$, …, $Z_{25}^{(0)}$ in ascending order are presented in Table 7.4. Table 7.5 shows the statistical results.

As can be seen in Table 7.5, of the four kinds of models the Even Model GM(1,1)(EGM) is more suitable for modeling with vibration sequence than the other three discrete form models. The error using Discrete Model GM (1,1) (DGM) to simulate the vibration sequence is slightly larger than other two discrete form models.

**Table 7.3** Statistics for sorting the simulation error of the non-exponential increasing sequence of four kinds of model GM(1,1)

| Error sorting | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|
| 1 | 18 | 5 | 2 | 0 |
| 2 | 2 | 2 | 15 | 6 |
| 3 | 0 | 1 | 5 | 19 |
| 4 | 5 | 17 | 3 | 0 |

**Table 7.4** Simulation errors of the vibration sequence of four kinds of model GM(1,1)

| Code | $-, -a, +$ | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|---|
| $Z_1^{(0)}$ | 0.01 | 0.298392 | 0.299400 | 0.299118 | 0.299258 |
| $Z_2^{(0)}$ | 0.02 | 0.501223 | 0.505800 | 0.504877 | 0.505331 |
| $Z_3^{(0)}$ | 0.03 | 0.369630 | 0.378773 | 0.379089 | 0.378935 |
| $Z_4^{(0)}$ | 0.04 | 2.583662 | 2.586760 | 2.572109 | 2.579300 |
| $Z_5^{(0)}$ | 0.05 | 2.928035 | 2.953655 | 2.899369 | 2.925619 |
| $Z_6^{(0)}$ | 0.10 | 4.759929 | 4.791858 | 4.825226 | 4.807851 |
| $Z_7^{(0)}$ | 0.15 | 3.802630 | 3.770562 | 3.776330 | 3.773545 |
| $Z_8^{(0)}$ | 0.20 | 11.723459 | 11.946525 | 11.393483 | 11.642630 |
| $Z_9^{(0)}$ | 0.25 | 14.895391 | 14.979357 | 15.229595 | 15.130729 |
| $Z_{10}^{(0)}$ | 0.30 | 17.953543 | 17.992976 | 18.397577 | 18.241183 |
| $Z_{11}^{(0)}$ | 0.35 | 7.299184 | 8.980062 | 8.537865 | 8.708603 |
| $Z_{12}^{(0)}$ | 0.40 | 11.474779 | 11.519781 | 11.693309 | 11.619287 |
| $Z_{13}^{(0)}$ | 0.45 | 11.988111 | 12.321804 | 12.261075 | 12.286039 |
| $Z_{14}^{(0)}$ | 0.50 | 12.728220 | 11.753460 | 12.270432 | 12.038094 |
| $Z_{15}^{(0)}$ | 0.55 | 10.636507 | 10.285910 | 10.897796 | 10.623904 |
| $Z_{16}^{(0)}$ | 0.60 | 13.393234 | 13.515007 | 13.006751 | 13.227910 |
| $Z_{17}^{(0)}$ | 0.65 | 15.420377 | 15.457643 | 14.690315 | 15.004381 |
| $Z_{18}^{(0)}$ | 0.70 | 16.304197 | 16.365096 | 15.735103 | 15.998031 |
| $Z_{19}^{(0)}$ | 0.80 | 14.542100 | 14.579829 | 14.110548 | 14.310293 |
| $Z_{20}^{(0)}$ | 0.90 | 33.798587 | 33.160101 | 34.928437 | 34.293058 |
| $Z_{21}^{(0)}$ | 1.00 | 22.586380 | 22.384127 | 22.016157 | 22.145609 |
| $Z_{22}^{(0)}$ | 1.10 | 34.305920 | 34.481612 | 36.023522 | 35.484180 |
| $Z_{23}^{(0)}$ | 0.02 | 0.501223 | 0.505800 | 0.504877 | 0.505331 |
| $Z_{24}^{(0)}$ | 1.20 | 23.591927 | 24.133298 | 23.323921 | 21.511839 |
| $Z_{25}^{(0)}$ | 1.50 | 40.373380 | 40.475348 | 42.698005 | 41.917026 |

**Table 7.5** Statistics for sorting the simulation error of the vibration sequence of four kinds of model GM(1,1)

| Error sorting | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|
| 1 | 12 | 4 | 8 | 1 |
| 2 | 1 | 7 | 6 | 11 |
| 3 | 9 | 1 | 2 | 13 |
| 4 | 3 | 13 | 9 | 0 |

The authors once tried to use the original form (7.1) of Model GM(1,1) to estimate the parameter vector $\hat{a} = [a, b]^T$ and, in accordance with the solution of whitenization Eq. (7.6) along with the time response formula of Even Model GM(1,1)(EGM), modeled the original Model GM(1,1). After simulating the above data we found that, even in cases where the development index is very small, the simulation error was still comparatively large. Also, as the development index increases, the simulation error increases rapidly. Based on even transformation of the accumulation data to build the Even Model GM(1,1), the simulation accuracy improves greatly. Then a new method which can accurately simulate and predict the uncertain system involving small data and poor information comes into being.

Among the four basic forms of model GM(1,1) discussed in Sects. 7.3 and 7.4, three discrete models can all accurately simulate the homogeneous exponential sequence. In the real world, a mass of practical data are not the simple homogeneous exponential sequence or close to it. This is the fundamental reason that people prefer to choose Even Model GM(1,1)(EGM) in the modeling process of the uncertain system involving small data and poor information, and it can reflect a satisfactory result in most cases.

In Sects. 7.3 and 7.4, the definitions of four basic forms of model GM(1,1) are put forward, and the properties and characteristics of different models are studied in-depth. The suitable sequences of different models are studied by simulation and analysis with homogeneous exponential sequences, non-exponential increasing sequences, and vibration sequences. The main conclusions of the research are as follows:

(1) The four basic forms of model GM(1,1), namely Even Model GM(1,1)(EGM), Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM) are pairwise equivalent.
(2) Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) and Discrete Model GM(1,1)(DGM) can all simulate the homogeneous exponential sequence accurately.
(3) For the non-exponential increasing sequences and vibration sequences, we should first choose the difference, differential hybrid form of Even Model GM(1,1)(EGM).
(4) For the non-exponential increasing sequences and vibration sequences which are close to the homogeneous exponential sequences, we should first choose the discrete form of Original Difference Model GM(1,1)(ODGM), Even Difference Model GM(1,1)(EDGM) or Discrete Model GM(1,1)(DGM).

The conclusions above can be the reference and basis for choosing an appropriate model in the actual modeling process. There is a modeling software corresponding to the models. Interested readers can download it for free from the website of the Institute for Grey System Studies of Nanjing University of Aeronautics and Astronautics (http://igss.nuaa.edu.cn) or from the website of the Marie Curie International Incoming Fellowship project (FP7.People-IIF-GA-2013-629051) (http://preview.dmu.ac.uk/research/research-faculties-and-institutes/technology/cci/projects/).

**Example 6.2.1** Let sequences of $X_1^{(0)}, X_2^{(0)}$ and $X_3^{(0)}$ be as follows,

$$X_1^{(0)} = \left(x_1^{(0)}(1), x_1^{(0)}(2), x_1^{(0)}(3), x_1^{(0)}(4), x_1^{(0)}(5)\right) = (1.5,2.1, 3.0,4.5, 5.48)$$

$$X_2^{(0)} = \left(x_2^{(0)}(1), x_2^{(0)}(2), x_2^{(0)}(3), x_2^{(0)}(4), x_2^{(0)}(5), x_2^{(0)}(6)\right) = (1.5,1.3, 3.0,3.9, 7.2, 9.5)$$

$$X_3^{(0)} = \left(x_3^{(0)}(1), x_3^{(0)}(2), x_3^{(0)}(3), x_3^{(0)}(4), x_3^{(0)}(5)\right) = (2, 9, 32, 27, 55)$$

Try to build the Even Model GM(1,1)(EGM), Discrete Model GM(1,1)(DGM), Original Difference Model GM(1,1)(ODGM), and Even Difference Model GM(1,1)(EDGM) using sequences $X_1^{(0)}, X_2^{(0)}$ and $X_3^{(0)}$. Compare the simulation errors. Table 7.6

**Solution**:

(1) For $X_1^{(0)}$, we build Even Model GM(1,1)(EGM), Discrete Model GM(1,1)(DGM), Original Difference Model GM(1,1)(ODGM), and Even Difference Model GM(1,1)(EDGM) using 1.5, 2.1, 3.0, 4.5, 5.48. We then obtained the simulation results as follows.

Simulation results by EGM: $\hat{X}_1^{(0)}$= (1.5000, 2.2459, 3.0428, 4.1225, 5.5853)
Simulation results by DGM: $\hat{X}_1^{(0)}$= (1.5000, 2.2746, 3.0844, 4.1827, 5.6719).
Simulation results by ODGM: $\hat{X}_1^{(0)}$= (1.5000, 2.2600, 3.0726, 4.1772, 5.6789).
Simulation results by EDGM: $\hat{X}_1^{(0)}$= (1.5000, 2.2662, 3.0776, 4.1795, 5.6760).

(2) For $X_2^{(0)}$, we build EGM, DGM, ODGM, and EDGM using 1.5, 1.3, 3.0, 3.9, 7.2, 9.5. Then we obtained the simulation results as follows. Table 7.7
Simulation results by EGM: $\hat{X}_2^{(0)}$= (1.5000, 1.8632, 2.8290, 4.29556.5220, 9.9028)
Simulation results by DGM: $\hat{X}_2^{(0)}$= (1.5000, 1.9247, 2.9317, 4.4654, 6.8016, 10.3599).

**Table 7.6** Simulation errors of four different models with $X_1^{(0)}$

| Models | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|
| Mean relative errors (%) | 4.7363 | 5.3458 | 5.1685 | 5.2442 |

**Table 7.7** Simulation errors of four different models with $X_2^{(0)}$

| Models | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|
| Mean relative errors (%) | 11.9881 | 12.3218 | 12.2611 | 12.2860 |

**Table 7.8** Simulation errors of four different models with $X_3^{(0)}$

| Models | EGM | DGM | ODGM | EDGM |
|---|---|---|---|---|
| Mean relative errors (%) | 27.2510 | 25.9994 | 26.4180 | 26.1794 |

Simulation results by ODGM: $\hat{X}_2^{(0)}=$ (1.5000, 1.8793, 2.8771, 4.4047, 6.7433, 10.3236).

Simulation results by EDGM: $\hat{X}_2^{(0)}=$ (1.5000, 1.8973, 2.8988, 4.4290, 6.7669, 10.3388).

(3) For $X_3^{(0)}$, we build EGM, DGM, ODGM, and EDGM using 2, 9, 32, 27, 55. Then we obtained the simulation results as follows. Table 7.8

Simulation results by EGM: $\hat{X}_3^{(0)}$ =(2.0000, 13.9767, 21.6340, 33.4864, 51.8323)
Simulation results by DGM: $\hat{X}_3^{(0)}$ = (2.0000, 15.4516, 23.4647, 35.6332, 54.1122).
Simulation results by ODGM: $\hat{X}_3^{(0)}$=(2.0000, 13.4756, 21.3666, 33.8782, 53.7164).
Simulation results by EDGM: $\hat{X}_3^{(0)}=$ (2.0000,14.2602, 22.2313, 34.6581, 54.0311).

The simulation results with $X_1^{(0)}$, $X_2^{(0)}$ and $X_3^{(0)}$ confirmed the above conclusion once again.

### *7.3.2 Applicable Ranges of EGM*

**Proposition 7.3.1.** When $(n-1)\sum_{k=2}^{n}\left[z^{(1)}(k)\right]^2 \to \left[\sum_{k=2}^{n} z^{(1)}(k)\right]^2$, the EGM(1,1) becomes invalid.

***Proof*** By using the model parameters obtained by the least squared estimate, we have.

$$\hat{a} = \frac{\sum_{k=2}^{n} z^{(1)}(k)\sum_{k=2}^{n} x^{(0)}(k) - (n-1)\sum_{k=2}^{n} z^{(1)}(k)x^{(0)}(k)}{(n-1)\sum_{k=2}^{n}\left[z^{(1)}(k)\right]^2 - \left[\sum_{k=2}^{n} z^{(1)}(k)\right]^2}$$

$$\hat{b} = \frac{\sum_{k=2}^{n} x^{(0)}(k) \sum_{k=2}^{n} \left[z^{(1)}(k)\right]^2 - \sum_{k=2}^{n} z^{(1)}(k) \sum_{k=2}^{n} z^{(1)}(k) x^{(0)}(k)}{(n-1) \sum_{k=2}^{n} \left[z^{(1)}(k)\right]^2 - \left[\sum_{k=2}^{n} z^{(1)}(k)\right]^2}$$

When $(n-1)\sum_{k=2}^{n}\left[z^{(1)}(k)\right]^2 \to \left[\sum_{k=2}^{n} z^{(1)}(k)\right]^2$, $\hat{a} \to \infty, \hat{b} \to \infty$, so that the model parameters cannot be determined. Hence, the EGM(1,1) becomes invalid.

**Proposition 7.3.2.** When the development coefficient $a$ of the EGM(1,1) model satisfies $|a| \geq 2$, the GM(1,1) model becomes invalid.

***Proof*** From the following expression of the GM(1,1) model.

$$x^{(0)}(k) = \left(\frac{1-0.5a}{1+0.5a}\right)^{k-2} \left(\frac{b - ax^{(0)}(1)}{1+0.5a}\right); k = 2, 3, \ldots, n$$

it can be seen that when $a = -2, x^{(0)}(k) \to \infty$; when $a = 2, x^{(0)}(k) = 0$; and when $|a| > 2$, $\frac{b-ax^{(0)}(1)}{1+0.5a}$ becomes a constant, while the sign of $\left(\frac{1-0.5a}{1+0.5a}\right)^{k-2}$ changes with $k$ being even or odd. Thus, the sign of $x^{(0)}(k)$ flips with $k$ being even or odd.

The discussion above indicates that $(-\infty, -2] \cup [2, \infty)$ is the forbidden area for the development coefficient $(-a)$ of the GM(1,1) model. When $a \in (-\infty, -2] \cup [2, \infty)$, the GM(1,1) model loses its validity. In general, when $|a| < 2$, the GM(1,1) model is meaningful. However, for different values of $a$, the prediction effect of the model is different. For the case of $-2 < a < 0$, let us respectively take $-a = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1.5, 1.8 to conduct a simulation analysis. By taking $k =$ 0, 1, 2, 3, 4, 5, from $x_i^{(0)}(k+1) = e^{-ak}$, we obtain the following sequences:

$$\begin{aligned}
\text{If } -a = 0.1, X_1^{(0)} &= \left(x_1^{(0)}(1), x_1^{(0)}(2), x_1^{(0)}(3), x_1^{(0)}(4), x_1^{(0)}(5), x_1^{(0)}(6)\right) \\
&= (1, 1.1051, 1.2214, 1.3499, 1.4918, 1.6487). \\
\text{If } -a = 0.2, X_2^{(0)} &= (1, 1.2214, 1.4918, 1.8221, 2.2255, 2.7183). \\
\text{If } -a = 0.3, X_3^{(0)} &= (1, 1.3499, 1.8221, 2.4596, 3.3201, 4.4817). \\
\text{If } -a = 0.4, X_4^{(0)} &= (1, 1.4918, 2.225, 3.3201, 4.9530, 7.3890). \\
\text{If } -a = 0.5, X_5^{(0)} &= (1, 1.6487, 2.7183, 4.4817, 7.3890, 12.1825). \\
\text{If } -a = 0.6, X_6^{(0)} &= (1, 1.8821, 3.3201, 6.0496, 11.0232, 20.0855). \\
\text{If } -a = 0.8, X_7^{(0)} &= (1, 2.2255, 4.9530, 11.0232, 24.5325, 54.5982). \\
\text{If } -a = 1, X_8^{(0)} &= (1, 2.7183, 7.3890, 20.0855, 54.5982, 148.4132). \\
\text{If } -a = 1.5, X_9^{(0)} &= (1, 4.4817, 20.0855, 90.0171, 403.4288, 1808.0424). \\
\text{If } -a = 1.8, X_{10}^{(0)} &= (1, 6.0496, 36.5982, 221.4064, 1339.4308, 8103.0839).
\end{aligned}$$

Let us respectively apply $X_1^{(0)}$, $X_2^{(0)}$, …, and $X_9^{(0)}$ to establish a GM(1,1) model and obtain the following time response sequences:

$$\hat{x}_1^{(1)}(k+1) = 10.50754e^{0.09992182k} - 9.507541,$$
$$\hat{x}_2^{(1)}(k+1) = 5.516431e^{0.1993401k} - 4.516431,$$
$$\hat{x}_3^{(1)}(k+1) = 3.85832e^{0.297769k} - 2.858321,$$
$$\hat{x}_4^{(1)}(k+1) = 3.033199e^{0.394752k} - 2.033199,$$
$$\hat{x}_5^{(1)}(k+1) = 2.541474e^{0.4898382k} - 1.541474,$$
$$\hat{x}_6^{(1)}(k+1) = 2.216363e^{0.5826263k} - 1.216362,$$
$$\hat{x}_7^{(1)}(k+1) = 1.815972e^{0.7598991k} - 0.8159718,$$
$$\hat{x}_8^{(1)}(k+1) = 1.581973e^{0.9242348k} - 0.5819733,$$
$$\hat{x}_9^{(1)}(k+1) = 1.287182e^{1.270298k} - 0.2871823,$$
$$\hat{x}_{10}^{(1)}(k+1) = 0.198197e^{1.432596k} - 0.1981966.$$

From $\hat{x}_i^{(0)}(k+1) = \hat{x}_i^{(1)}(k+1) - \hat{x}_i^{(1)}(k)$, $i = 1, 2, \ldots, 10$, we obtain

$$\hat{x}_1^{(0)}(k+1) = 0.99918e^{0.09992182k},$$
$$\hat{x}_2^{(0)}(k+1) = 0.99698e^{0.1993401k},$$
$$\hat{x}_3^{(0)}(k+1) = 0.99362e^{0.297769k},$$
$$\hat{x}_4^{(0)}(k+1) = 0.989287e^{0.394752k},$$
$$\hat{x}_5^{(0)}(k+1) = 0.984248e^{0.4898382k},$$
$$\hat{x}_6^{(0)}(k+1) = 0.97868e^{0.5826263k},$$
$$\hat{x}_7^{(0)}(k+1) = 0.966617e^{0.7598991k},$$
$$\hat{x}_8^{(0)}(k+1) = 0.95419e^{0.9242348k},$$
$$\hat{x}_9^{(0)}(k+1) = 0.925808e^{1.270298k},$$
$$\hat{x}_{10}^{(0)}(k+1) = 0.91220e^{1.432596k}.$$

From the mean generation of $z^{(1)}(k) = \frac{1}{2}(x^{(1)}(k) + x^{(1)}(k-1))$ of GM(1,1) model $x^{(0)}(k) + az^{(1)}(k) = b$, it has the effect of weakening the growth for increasing sequences. For an exponential sequence, the established GM(1,1) has a small development coefficient.

Let us compare the errors between the original sequence $X_i^{(0)}$ and the simulation sequence $\hat{X}_i^{(0)}$, as seen in Table 7.9.

It can be seen that as the development coefficient increases, the simulation error grows drastically. When the development coefficient is smaller than or equal to 0.3, the simulation accuracy can reach above 98%. When the coefficient is smaller than or equal 0.5, the simulation accuracy can reach above 95%. When the coefficient is greater than 1, the simulation accuracy is lower than 70%. When the coefficient is greater than 1.5, the simulation accuracy is lower than 50%.

**Table 7.9** The simulation errors of different development coefficients $(-a)$

| Development coefficient $(-a)$ | $\frac{1}{5}\sum_{i=2}^{6}\left[\hat{x}^{(0)}(k)-x^{(0)}(k)\right]$ | Mean relative error $\frac{1}{5}\sum_{k=2}^{6}\Delta_k$ (%) |
|---|---|---|
| 0.1 | 0.004 | 0.104 |
| 0.2 | 0.010 | 0.499 |
| 0.3 | 0.038 | 1.300 |
| 0.4 | 0.116 | 2.613 |
| 0.5 | 0.307 | 4.520 |
| 0.6 | 0.741 | 7.074 |
| 0.8 | 3.603 | 14.156 |
| 1 | 14.807 | 23.544 |
| 1.5 | 317.867 | 51.033 |
| 1.8 | 1632.240 | 65.454 |

Let us now further focus on the first step, second step, fifth step, and 10th step prediction errors. See Table 7.10.

It can be seen that when the development coefficient is smaller than 0.3, the step 1 prediction accuracy is above 98%, with both steps 2 and 5 accuracies above 97%. When $0.3 < -a \leq 0.5$, the steps 1 and 2 prediction accuracies are all above 90%; and the step 10 prediction accuracy also above 80%. When the development coefficient is greater than 0.8, the step 1 prediction accuracy is below 70%. The horizontal bars in Table 4.5 represent that the relevant errors are greater than 100%.

From this analysis, we can draw the following conclusions: When $-a \leq 0.3$, GM(1,1) can be applied to make mid- to long-term predictions; when $0.3 < -a \leq 0.5$, GM(1,1) can be applied to make short- and mid-term predictions with caution; when $0.5 < -a \leq 0.8$ and GM(1,1) is used to make short-term predictions, one

**Table 7.10** Prediction errors

| $-a$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.8 | 1 | 1.5 | 1.8 |
|---|---|---|---|---|---|---|---|---|---|---|
| Step 1 error (%) | 0.129 | 0.701 | 1.998 | 4.317 | 7.988 | 13.405 | 31.595 | 65.117 | – | – |
| Step 2 error (%) | 0.137 | 0.768 | 2.226 | 4.865 | 9.091 | 15.392 | 36.979 | 78.113 | – | – |
| Step 5 error (%) | 0.160 | 0.967 | 2.912 | 6.529 | 12.468 | 21.566 | 54.491 | – | – | – |
| Step 5 error (%) | 0.855 | 1.301 | 4.067 | 9.362 | 18.330 | 32.599 | 88.790 | – | – | – |

needs to be very cautious about the prediction results; when $0.8 < -a \leq 1$, one should employ the remnant GM (1,1) model; and when $-a > 1$, GM(1,1) should not be applied.

## 7.4 Remnant GM(1,1) Model

When the accuracy of a GM(1,1) model does not meet the predetermined requirement, one can establish another GM(1,1) model using the error sequence to remedy the original model to improve the accuracy. We will use the remnant GM (1,1) of EGM(1) as an example.

**Definition 7.4.1** Assume that $X^{(0)}$ is a sequence of raw data, $X^{(1)}$ the accumulation generated sequence based on $X^{(0)}$, and the time response formula of the GM(1,1) model is

$$\hat{x}^{(1)}(k+1) = \left(x^{(0)}(1) - \frac{b}{a}\right) e^{-ak} + \frac{b}{a}$$

then

$$d\,\hat{x}^{(1)}(k+1) = (-a)\left(x^{(0)}(1) - \frac{b}{a}\right) e^{-ak} \tag{7.26}$$

is referred to as the restored value through derivatives.

Generally, $d\hat{x}^{(1)}(k+1) \neq \hat{x}^{(0)}(k+1)$, where $\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k)$ stands for the restored value through inverse accumulation. This very fact implies that the GM(1,1) is neither a differential equation nor a difference equation. However, when $|a|$ is sufficiently small, from $1 - e^{a} \approx -a$, it follows that $d\,\hat{x}^{(1)}(k+1) \approx \hat{x}^{(0)}(k+1)$, meaning that the results of differentiation and difference are quite close. Therefore, the GM(1,1) model in this case can be seen as both a differential equation and a difference equation.

Because the restored values through derivatives and through inverse accumulation are different, to reduce possible errors caused by reciprocating operators, the errors of $X^{(1)}$ are often used to improve the simulated values $\hat{x}^{(1)}(k+1)$ of $X^{(1)}$.

**Definition 7.4.2** Assume that $\varepsilon^{(0)} = \left(\varepsilon^{(0)}(1), \varepsilon^{(0)}(2), \ldots, \varepsilon^{(0)}(n)\right)$, where $\varepsilon^{(0)}(k) = x^{(1)}(k) - \hat{x}^{(1)}(k)$, is the error sequence of $X^{(1)}$. If there is a $k_0$ satisfying that $n - k_0 \geq 4$ and $\forall k \geq k_0$, the signs of $\varepsilon^{(0)}(k)$ stay the same, and $\left(\left|\varepsilon^{(0)}(k_0)\right|, \left|\varepsilon^{(0)}(k_0+1)\right|, , \ldots, \left|\varepsilon^{(0)}(n)\right|\right)$ is referred to as the error sequence of modelability, which is and still denoted $\varepsilon^{(0)} = \left(\varepsilon^{(0)}(k_0), \varepsilon^{(0)}(k_0+1), \ldots, \varepsilon^{(0)}(n)\right)$.

In this case, let the sequence $\varepsilon^{(1)} = \left(\varepsilon^{(1)}(k_0), \varepsilon^{(1)}(k_0+1), \ldots, \varepsilon^{(1)}(n)\right)$ be accumulation generated on $\varepsilon^{(0)}$ with the following GM(1,1) time response formula:

$$\hat{\varepsilon}^{(1)}(k+1) = \left(\varepsilon^{(0)}(k_0) - \frac{b_\varepsilon}{a_\varepsilon}\right)\exp[-a_\varepsilon(k-k_0)] + \frac{b_\varepsilon}{a_\varepsilon}, k \geq k_0$$

Then the simulation sequence of $\varepsilon^{(0)}$ is given by $\hat{\varepsilon}^{(0)} = (\hat{\varepsilon}^{(0)}(k_0), \hat{\varepsilon}^{(0)}(k_0+1), \ldots, \hat{\varepsilon}^{(0)}(n))$, where

$$\hat{\varepsilon}^{(0)}(k+1) = (-a_\varepsilon)\left(\varepsilon^{(0)}(k_0) - \frac{b_\varepsilon}{a_\varepsilon}\right)\exp[-a_\varepsilon(k-k_0)], k \geq k_0$$

**Definition 7.4.3** If $\hat{\varepsilon}^{(0)}$ is used to improve $\hat{X}^{(1)}$, the modified time response formula.

$$\hat{x}^{(1)}(k+1) = \begin{cases} \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak} + \frac{b}{a}, & k < k_0 \\ \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak} + \frac{b}{a} \pm a_\varepsilon(\varepsilon^{(0)}(k_0) - \frac{b_\varepsilon}{a_\varepsilon})\,e^{-a_\varepsilon(k-k_0)}, & k \geq k_0 \end{cases} \tag{7.27}$$

is referred to as the GM(1,1) model with error modification, or simply remnant GM (1,1) for short, where the sign of the error modification value

$$\hat{\varepsilon}^{(0)}(k+1) = a_\varepsilon \times \left(\varepsilon^{(0)}(k_0) - \frac{b_\varepsilon}{a_\varepsilon}\right)\exp[-a_\varepsilon(k-k_0)]$$

needs to stay the same as those in $\varepsilon^{(0)}$.

If a modeling of the error sequence $\varepsilon^{(0)} = \left(\varepsilon^{(0)}(k_0), \varepsilon^{(0)}(k_0+1), \ldots, \varepsilon^{(0)}(n)\right)$ of $X^{(0)}$ and $\hat{X}^{(0)}$ is used to modify the simulation value $\hat{X}^{(0)}$, then different methods of restoration from $\hat{X}^{(1)}$ to $\hat{X}^{(0)}$ can produce different time response sequences of error modification.

**Definition 7.4.4** Let

$$\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1) = \left(1-e^{a}\right)\left(x^{(0)}(1) - \frac{b}{a}\right)e^{-a(k-1)}$$

Then the corresponding time response sequence of error modification

$$\hat{x}^{(0)}(k+1) = \begin{cases} (1-e^{a})\left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak}, & k < k_0 \\ (1-e^{a})\left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak} \pm a_\varepsilon\left(\varepsilon^{(0)}(k_0) - \frac{b_\varepsilon}{a_\varepsilon}\right)e^{-a_\varepsilon(k-k_0)}, & k \geq k_0 \end{cases} \tag{7.28}$$

is called the error modification model of inverse accumulation restoration.

**Definition 7.4.5** Let

$$\hat{x}^{(0)}(k+1) = (-a)\left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak},$$

then the corresponding time response sequence of error modification

$$\hat{x}^{(0)}(k+1) = \begin{cases} (-a)\left(x^{(0)}(1) - \frac{b}{a}\right) e^{-ak}, & k < k_0 \\ (-a)\left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak} \pm a_\varepsilon\left(\varepsilon^{(0)}(k_0) - \frac{b_\varepsilon}{a_\varepsilon}\right)e^{-a_\varepsilon(k-k_0)}, & k \geq k_0 \end{cases} \tag{7.29}$$

is referred to as the error modification model of derivative restoration.

In the previous discussion, all the error simulation terms in remnant GM (1,1) have been taken as the derivative restoration. Of course, they can be taken as inverse accumulation restoration. That is, one can take

$$\hat{\varepsilon}^{(0)}(k+1) = \left(1 - e^{a_\varepsilon}\right)\left(\varepsilon^{(0)}(k_0) - \frac{b_\varepsilon}{a_\varepsilon}\right)e^{-a_\varepsilon(k-k_0)}, k \geq k_0$$

As long as $|a_\varepsilon|$ is sufficiently small, the effects of different error restoration methods on the modified $\hat{x}^{(0)}(k+1)$ are almost the same.

***Example 7.4.1*** Let.

$$\begin{aligned} \mathrm{X}^{(0)} &= \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(13)\right) \\ &= (6, 20, 40, 25, 40, 45, 35, 21, 14, 18, 15.5, 17, 15) \end{aligned}$$

be a sequence of raw data, and the creation of a EGM(1, 1) model produce the following time response sequence:

$$\hat{x}^{(1)}(k+1) = -567.999e^{-0.06486k} + 573.999$$

The application of inverse accumulating restoration gives:

$$\begin{aligned} \hat{X}^{(0)} = \{\hat{x}^{(0)}(k)\}_2^{13} = (&35.6704, 33.4303, 31.3308, 29.3682, 27.5192, 25.7900, \\ &24.1719, 22.6534, 21.2307, 19.8974, 18.6478, 17.4768) \end{aligned}$$

The errors and relative errors of the results can be seen in Table 7.11.

From Table 7.11, it can be seen that the simulation error is relatively large. Thus, it is necessary to apply a remnant model to remedy some of the errors.

Let $k_0 = 9$, we get the error sequence as follows

$$\begin{aligned} \varepsilon^{(0)} &= (\varepsilon^{(0)}(9), \varepsilon^{(0)}(10), \varepsilon^{(0)}(11), \varepsilon^{(0)}(12), \varepsilon^{(0)}(13)) \\ &= (-8.6534, -3.2307, -4.3974, -1.6478, -2.4768) \end{aligned}$$

which is an error sequence of modelability. Taking absolute value gives

$$\varepsilon^{(0)} = (8.6534, 3.2307, 4.3974, 1.6478, 2.4768)$$

**Table 7.11** The errors and relative errors of EGM(1,1)

| No | Real Data $x^{(0)}(k)$ | Simulated Values $\hat{x}^{(0)}(k)$ | Errors $\varepsilon(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)$ | Relative errors $\Delta_k = \frac{\lvert\varepsilon(k)\rvert}{x^{(0)}(k)}$ (%) |
|---|---|---|---|---|
| 2 | 20 | 35.6704 | − 15.6704 | 78.3540 |
| 3 | 40 | 33.4303 | 6.5697 | 16.4242 |
| 4 | 25 | 31.3308 | − 6.3308 | 25.3232 |
| 5 | 40 | 29.3682 | 10.6318 | 26.5795 |
| 6 | 45 | 27.5192 | 17.4808 | 38.8642 |
| 7 | 35 | 25.6901 | 9.2099 | 26.3140 |
| 8 | 21 | 24.1719 | − 3.1719 | 15.1043 |
| 9 | 14 | 22.6534 | − 8.6534 | 61.8100 |
| 10 | 18 | 21.2307 | − 3.2307 | 17.9483 |
| 11 | 15.5 | 19.8974 | − 4.3974 | 28.3703 |
| 12 | 17 | 18.6478 | − 1.6478 | 9.6926 |
| 13 | 15 | 17.4768 | − 2.4768 | 16.5120 |

In establishing a EGM(1, 1) for $\varepsilon^{(0)}$, we have the time response sequence of $\varepsilon^{(1)}\varepsilon^{(1)}$

$$\hat{\varepsilon}^{(1)}(k+1) = -24e^{-0.16855(k-9)} + 32.7$$

whose restored value of derivatives is

$$\hat{\varepsilon}^{(0)}(k+1) = (-0.16855)(-24)e^{-0.16855(k-9)} = 4.0452e^{-0.16855(k-9)}$$

From

$$\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k) = (1-e^{a})(x^{(0)}(1) - \frac{b}{a})e^{-ak} = 38.0614e^{-0.06486k}$$

We can obtain the remnant model of inverse accumulating restoration

$$\hat{x}^{(0)}(k+1) = \begin{cases} 38.0614e^{-0.06486k}, \ k < 9 \\ 38.0614e^{-0.06486k} - 4.0452e^{-0.16855(k-9)}, \ k \geq 9 \end{cases}$$

where the sign of $\hat{\varepsilon}^{(0)}(k+1)$ is the same as the original error sequence.

Based on this model, we can modify the four simulation values with k = 10, 11, 12, 13, with improved accuracy listed in Table 7.12.

From this table, we can compute the sum of squares of errors as follows,

$$s = \varepsilon^{T}\varepsilon = 3.1611$$

and the average relative error

**Table 7.12** Improved results

| No. | Real Data $x^{(0)}(k)$ | Simulated Values $\hat{x}^{(0)}(k)$ | Errors $\varepsilon(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)$ | Relative errors $\Delta_k = \frac{\lvert\varepsilon(k)\rvert}{x^{(0)}(k)}$ (%) |
|---|---|---|---|---|
| 10 | 18 | 17.1858 | 0.8142 | 4.52 |
| 11 | 15.5 | 16.4799 | – 0.9799 | 6.32 |
| 12 | 17 | 15.6604 | 1.2396 | 7.29 |
| 13 | 15 | 15.0372 | – 0.0372 | 0.25 |

$$\Delta = \frac{1}{12}\sum_{k=10}^{13}\Delta_k = 4.595\%$$

Here, the simulation accuracy of the remnant EGM(1, 1) has obviously increased. However, the current error sequence no longer satisfies the modeling requirement. Therefore, if the improved accuracy is still unsatisfactory, we will have to consider other models or some appropriate choice of data to the original sequence.

## 7.5 Group of GM(1,1) Models

In practice, one does not have to use all the available data in their modeling. Each subsequence of the original data can be employed to establish a model. Generally speaking, different subsequences lead to different models. Even though the same kind of GM(1,1) is applied, different subsequences lead to different $a, b$ values. These changes reflect the fact that varied circumstances and conditions have different effects on the system under consideration.

For example, for the grain production in China, if we use the data values collected since 1949 to establish a model GM(1, 1), the development coefficient (– a) will be on the small side. However, if only the values collected after 1978 are used, the corresponding development coefficient (– a) will obviously increase.

**Definition 7.5.1** For a given sequence $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$, if we take $x^{(0)}(n)$ as the origin of the time axis, then $t < n$ is seen as the past, $t = n$ the present, and $t > n$ the future.

**Definition 7.5.2** Assume that $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$ is a sequence of raw data, let.

$$\hat{x}^{(0)}(k+1) = \left(1 - e^{a}\right)\left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak}$$

be the restored values of inverse accumulation of the GM(1,1) time responses of $X^{(0)}$. Then:

(1) For $t \le n$, $\hat{x}^{(0)}(t)$ is referred to as the simulated value out of the model; and
(2) When $t > n$, $\hat{x}^{(0)}(t)$ is known as the prediction of the model.

The main purpose of modeling is to make predictions. To improve the prediction accuracy, one first needs to guarantee sufficiently high accuracy in his simulation, especially for the simulation of the time moment $t = n$. Therefore, in general, the data, including $x^{(0)}(n)$, used for modeling should be an equal-time-interval sequence.

**Definition 7.5.3** Assume that $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$ is a sequence of raw data, then:

(1) The GM(1,1) model established using the entire sequence $X^{(0)}$ is known as the all-data GM(1,1) );
(2) $\forall k_0 > 1$, the GM(1,1) model established on the tail sequence $X^{(0)} = \left(x^{(0)}(k_0), x^{(0)}(k_0+1), \ldots, x^{(0)}(n)\right)$ is known as a partial-data GM(1,1);
(3) If $x^{(0)}(n+1)$ stands for a piece of new information, then the GM(1,1) model established on the prolonged sequence $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n), x^{(0)}(n+1)\right) =$ is known as a new-information GM(1,1);
(4) The GM(1,1) model established on $X^{(0)} = \left(x^{(0)}(2), \ldots, x^{(0)}(n), x^{(0)}(n+1)\right)$ with the new information added and the oldest piece $x^{(0)}(1)$ of information removed is known as a metabolic GM(1,1).

***Example 7.5.1*** Let.

$$\mathrm{X}^{(0)} = (60.7, 73.8, 86.2, 100.4, 123.3)$$

and $x^{(0)}(6) = 149.5$ is a piece of new information. Try to establish a model with $\mathrm{X}^{(0)}$, a model of new information, and a metabolic EGM(1,1).

**Solution**:

(1) The model with $\mathrm{X}^{(0)}$. From

$$\mathrm{X}^{(0)} = (60.7, 73.8, 86.2, 100.4, 123.3)$$

We have

$$\hat{a} = (B^T B)^{-1} B^T Y = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.17241 \\ 55.889264 \end{bmatrix}$$

The time response sequence is as follows

$$\hat{x}^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-a(k-1)} + \frac{b}{a} = 384.865028e^{0.17241k} - 324.165028$$

Then we obtained the simulation sequence of $X^{(0)}$ as follows

$$\hat{X}^{(0)} = (60.7, 72.41804, 86.04456, 102.2351, 121.4721)$$

The corresponding error sequence is

$$\varepsilon = (0, 1.38196, 0.155434, -1.8351, 1.827829)$$

where $\varepsilon(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)$.

Therefore, we got the average relative error

$$\Delta = \frac{1}{4}\sum_{k=2}^{5}\Delta_k = 1.34\%$$

where $\Delta_k = \frac{|\varepsilon(k)|}{x^{(0)}(k)}$.

(2) The model of new information. In inserting a piece of new information $x^{(0)}(6) = 149.5$, the data sequence became

$$X^{(0)} = (60.7, 73.8, 86.2, 100.4, 123.3, 149.5)$$

We have

$$\hat{a} = (B^T B)^{-1} B^T Y = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.180888 \\ 54.254961 \end{bmatrix}$$

Its time response sequence is as follows:

$$\hat{x}^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-a(k-1)} + \frac{b}{a} = 360.63748e^{0.180888k} - 299.93748$$

The simulation sequence of the new information sequence $X^{(0}$, the corresponding error sequence ε,

and the average relative error △ are as follows:

$$\hat{X}^{(0)} = (60.7, 71.50736, 85.68587, 102.6757, 123.0342, 147.429)$$

$$\varepsilon = (0, 2.29264, 0.514129, -2.2757, 0.265712, 2.07041)$$

$$\Delta = \frac{1}{5}\sum_{k=2}^{6}\Delta_k = 1.51\%$$

(3) The metabolic EGM(1,1). In adding a piece of new information $x^{(0)}(6) = 149.5$, and deleting a piece of old information $x^{(0)}(1) = 60.7$, we have

$$X^{(0)} = (73.8, 86.2, 100.4, 123.3, 149.5)$$

and

$$\hat{a} = (B^T B)^{-1} B^T Y = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -0.187862 \\ 62.830896 \end{bmatrix}$$

The corresponding time response sequence is:

$$\hat{x}^{(1)}(k) = \left( x^{(0)}(1) - \frac{b}{a} \right) e^{-a(k-1)} + \frac{b}{a} = 408.251645 e^{0.187862k} - 334.451645$$

And the simulation sequence of the metabolic sequence $X^{(0}$, the corresponding error sequence ε, and the average relative error △ are as follows:

$$\hat{X}^{(0)} = (73.8, 84.37234, 101.8093, 122.85, 148.2391)$$

$$\varepsilon = (0, 1.827657, -1.4093, 0.45, 1.2609)$$

$$\Delta = \frac{1}{4} \sum_{k=3}^{6} \Delta_k = 1.18\%$$

Compared with these different results, it implies that the simulation accuracy can be improved by appropriately choosing the data to be used in the process of modeling. From the three different error sequences, we can see that for the simulation accuracy of value $x^{(0)}(5)$, both the new information model and the metabolic model are better than the model in (1). This implies that the new information EGM(1, 1) and the metabolic EGM(1, 1) have better prediction abilities than the old model. As a matter of fact, in the development process of a grey system, there always exists some stochastic interferences or driving forces entering the system as time goes on, so that the consequent development of the system is accordingly affected.

Therefore, when using the EGM(1, 1) model to do predictions, high accuracy can be achieved only for the first or the second data values after the last origin value $x^{(0)}(n)$. In general, the farther away into the future, and the farther away from the last origin value, the weaker the prediction ability of EGM(1, 1) becomes. In practical applications, one needs to constantly consider those interferences and driving factors entering the system as time goes on and promptly add new pieces of information to the original sequence $X^{(0)}$ and establish consequent new information EGM(1, 1) models.

From the simulation accuracy of value $x^{(0)}(6)$, it can be seen that the metabolic model is better than the new information model. From the angle of prediction, it can be seen that the metabolic model is the best prediction model. As the system develops further, the significance of the older data reduces so that, when new data are added, the older data are deleted promptly, and the constantly renewing modeling sequence can better reflect the current characteristics of the system. Specifically, as the accumulation of quantitative changes increases, a jump or sudden change in the system will occur. At this very moment, compared with the older system, the current system is completely different. Hence, the practice of deleting old data is very reasonable. Indeed, the ongoing replacement of old data can avoid computation difficulties in modeling due to the fact that increased information can increase computer storage space requirements tremendously.

## 7.6 The Fractional Grey Model

**Definition 7.6.1** Assume that $X_1^{(0)} = \left(x_1^{(0)}(1), x_1^{(0)}(2), \ldots, x_1^{(0)}(n)\right)$ is a non-negative sequence, then.

$$x^{\left(\frac{p}{q}\right)}(k) = \sum_{i=1}^{k} C_{k-i+\frac{p}{q}-1}^{k-i} x^{(0)}(i)$$

is called a $\frac{p}{q}$ order accumulation operator. Let $C_{\frac{p}{q}-1}^{0} = 1, C_k^{k+1} = 0, k = 0, 1, \ldots, n-1$,

$$C_{k-i+\frac{p}{q}-1}^{k-i} = \frac{\left(k-i+\frac{p}{q}-1\right)\left(k-i+\frac{p}{q}-2\right)\ldots\left(\frac{p}{q}+1\right)\frac{p}{q}}{(k-i)!}$$

Then $X^{\left(\frac{p}{q}\right)} = \left(x^{\left(\frac{p}{q}\right)}(1), x^{\left(\frac{p}{q}\right)}(2), \ldots, x^{\left(\frac{p}{q}\right)}(n)\right)$ is called a $\frac{p}{q}$ order accumulation sequence (Wu et al., 2013).

**Definition 7.6.2** Assume that $X^{(0)}$ is a non-negative sequence, then.

$$\alpha^{(1)} x^{\left(1-\frac{p}{q}\right)}(k) = x^{\left(1-\frac{p}{q}\right)}(k) - x^{\left(1-\frac{p}{q}\right)}(k-1)$$

is called a $\frac{p}{q}\left(0 < \frac{p}{q} < 1\right)$ order inverse accumulation operator. And

$$\alpha^{\left(\frac{p}{q}\right)} X^{(0)} = \alpha^{(1)} X^{\left(1-\frac{p}{q}\right)} = \left(\alpha^{(1)} x^{\left(1-\frac{p}{q}\right)}(1), \alpha^{(1)} x^{\left(1-\frac{p}{q}\right)}(2), \ldots, \alpha^{(1)} x^{\left(1-\frac{p}{q}\right)}(n)\right)$$

is called a $\frac{p}{q}\left(0 < \frac{p}{q} < 1\right)$ order inverse accumulation sequence.

**Definition 7.6.3** Assume that $X^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)\right)$ is a non-negative sequence, and.

$x^{\left(\frac{p}{q}\right)} = \left(x^{\left(\frac{p}{q}\right)}(1), x^{\left(\frac{p}{q}\right)}(2), \ldots, x^{\left(\frac{p}{q}\right)}(n)\right)$ is the $\frac{p}{q}$ order accumulation sequence of $X^{(0)}$, then the following

$$x^{\left(\frac{p}{q}\right)}(k+1) = \beta_1 x^{\left(\frac{p}{q}\right)}(k) + \beta_2 (k = 1, 2, \ldots n-1) \tag{7.30}$$

is called a $\frac{p}{q}$ order accumulation discrete grey model (Wu et al., 2013).

**Theorem 7.6.1** Assume that $x^{\left(\frac{p}{q}\right)}(k+1) = \beta_1 x^{\left(\frac{p}{q}\right)}(k) + \beta_2$ is called a $\frac{p}{q}$ order accumulation discrete grey model, then.

$$\begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} = \left(B^T B\right)^{-1} B^T Y$$

where

$$B = \begin{bmatrix} 1 & x^{\left(\frac{p}{q}\right)}(1) \\ 1 & x^{\left(\frac{p}{q}\right)}(2) \\ \vdots & \vdots \\ 1 & x^{\left(\frac{p}{q}\right)}(n-1) \end{bmatrix}, Y = \begin{bmatrix} x^{\left(\frac{p}{q}\right)}(2) \\ x^{\left(\frac{p}{q}\right)}(3) \\ \vdots \\ x^{\left(\frac{p}{q}\right)}(n) \end{bmatrix}$$

**Definition 7.6.4** Assume that $X^{(0)} = \left(x^{(0)}(1),\ x^{(0)}(2),\ \ldots, x^{(0)}(n)\right)$ is a non-negative sequence, $p(0 < p < 1)$, then.

$$\alpha^{(1)} x^{(1-p)}(k) + a z^{(0)}(k) = b \tag{7.31}$$

is called a grey model of GM($p$,1).

where $\alpha^{(1)} x^{(1-p)}(k)$ is the p order difference of $x^{(0)}(k)$. We can calculated the $1-p$ order accumulation of $x^{(0)}(k)$ at first, then acted by the first order inverse accumulation operator on $x^{(1-p)}(k)$ $\alpha^{(1)} x^{(1-p)}(k) = x^{(1-p)}(k) - x^{(1-p)}(k-1)$, let

$$\begin{bmatrix} a \\ b \end{bmatrix} = (B^T B)^{-1} B^T Y$$

where

$$B = \begin{bmatrix} -z^{(0)}(2) & 1 \\ -z^{(0)}(3) & 1 \\ \vdots & \vdots \\ -z^{(0)}(n) & 1 \end{bmatrix}, Y = \begin{bmatrix} \alpha^{(1)}x^{(1-p)}(2) \\ \alpha^{(1)}x^{(1-p)}(3) \\ \vdots \\ \alpha^{(1)}x^{(1-p)}(n) \end{bmatrix}$$

and $z^{(0)}(k) = \frac{x^{(0)}(k)+x^{(0)}(k+1)}{2}$.

The whitening equation of the model GM($p$,1) as follows

$$\frac{d^p x^{(0)}(t)}{dt^p} + ax^{(0)}(t) = b \tag{7.32}$$

Let $\hat{x}^{(0)}(1) = x^{(0)}(1)$, we obtained the time response sequence of (7.32) by fractional Laplace transform

$$x^{(0)}(k) = (x^{(0)}(1) - \frac{b}{a}) \sum_{i=0}^{\infty} \frac{(-at^p)^i}{\Gamma(pi+1)} + \frac{b}{a} \tag{7.33}$$

where $\Gamma(pi+1)$ is Gamma function.

***Example 7.6.1*** Let.

$$\mathrm{X}^{(0)} = (247.839,\ 273.021,\ 289.014,\ 285.208,\ 288.818,\ 297.078)$$

Please try to build a 0.1 order accumulation discrete grey model.

**Solution**: The 0.1 order accumulation sequence of $X^{(0)}$ as follows.

$\mathrm{X}^{(0.1)} = (247.839,\ 297.805,\ 329.947,\ 338.667,\ 351.141,\ 366.983)$,

From $\begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} = \left(B^T B\right)^{-1} B^T Y$, we have

$$\hat{x}^{(0.1)}(k+1) = -126.356 \times 0.6101^{k-1} + 374.195$$

The simulated sequence is.

$\hat{x}^{(0.1)}(k) = (247.839,\ 297.105,\ 327.163,\ 345.501,\ 356.689,\ 363.515)$, its 0.9 order accumulation sequence is

$$\hat{x}^{(1)}(k) = (247.839,\ 520.160,\ 806.460,\ 1098.811,\ 1392.639,\ 1685.479)$$

Acted by a first order inverse accumulation operator, we have

$$\hat{x}^{(0)}(k) = (247.839,\ 272.321,\ 286.299,\ 292.351,\ 293.828,\ 292.841)$$

## 7.7 The Models of GM(*r*,*h*)

### *7.7.1 The Model of GM(0,N)*

**Definition 7.7.1** Assume that $X_1^{(0)} = \left(x_1^{(0)}(1), x_1^{(0)}(2), \ldots, x_1^{(0)}(n)\right)$ is a data sequence of a system's characteristic variable,

$$\begin{aligned} X_2^{(0)} &= \left(x_2^{(0)}(1), x_2^{(0)}(2), \ldots, x_2^{(0)}(n)\right) \\ X_3^{(0)} &= \left(x_3^{(0)}(1), x_3^{(0)}(2), \ldots, x_3^{(0)}(n)\right) \\ &\ldots\ldots\ldots\ldots\ldots \\ X_N^{(0)} &= \left(x_N^{(0)}(1), x_N^{(0)}(2), \ldots, x_N^{(0)}(n)\right) \end{aligned}$$

the data sequences of relevant factors, and $X_i^{(1)}$ the accumulation generated sequence of $X_i^{(0)}$, $i = 2, 3, \ldots, N$. Then

$$x_1^{(1)}(k) = a + b_2 x_2^{(1)}(k) + b_3 x_3^{(1)}(k) + \ldots + b_N x_N^{(1)}(k) \tag{7.34}$$

is called the model of GM(0,*N*). Because this model does not contain any derivative, it is a static model. Although its form looks like a multivariate linear regression model, it is essentially different from any of the statistical models. In particular, the general multivariate linear regression model is established on the basis of the original data sequences, while the model of GM(0,*N*) is constructed on the accumulation generation of the original data.

**Theorem 7.7.1** Assume $X_i^{(0)}$ and $X_i^{(1)}$ $(i = 1, 2, \ldots, N)$ as given in Definition 7.6.1, let.

$$B = \begin{bmatrix} 1 & x_2^{(1)}(2) & x_3^{(1)}(2) & \ldots & x_N^{(1)}(2) \\ 1 & x_2^{(1)}(3) & x_3^{(1)}(3) & \ldots & x_N^{(1)}(3) \\ \ldots & \ldots & \ldots & & \ldots\ldots \\ 1 & x_2^{(1)}(n) & x_3^{(1)}(n) & \ldots & x_N^{(1)}(n) \end{bmatrix}, Y = \begin{bmatrix} x_1^{(1)}(2) \\ x_1^{(1)}(3) \\ \vdots \\ x_1^{(1)}(n) \end{bmatrix}$$

then the least squares estimate of the parametric sequence $\hat{a} = [a, b_1, b_2, \ldots, b_N]^T$ is given by

$$\hat{a} = (B^T B)^{-1} B^T Y$$

***Example 7.7.1*** Let.

$$X_1^{(0)} = (2.874, 3.278, 3.307, 3.39, 3.679) = \{x_1^{(0)}(k)\}_1^5$$

be a data sequence of a system's characteristic variable, and

$$X_2^{(0)} = (7.04, 7.645, 8.075, 8.53, 8.774) = \{x_2^{(0)}(k)\}_1^5$$

the data sequences of a relevant factor. Try to establish the model of GM(0,2).

**Solution**: Assume the model of GM(0,2) as follows:

$$X_1^{(1)} = bX_2^{(1)} + a$$

From

$$B = \begin{bmatrix} x_2^{(1)}(2) & 1 \\ x_2^{(1)}(3) & 1 \\ x_2^{(1)}(4) & 1 \\ x_2^{(1)}(5) & 1 \end{bmatrix} = \begin{bmatrix} 14.685 & 1 \\ 22.76 & 1 \\ 31.29 & 1 \\ 40.064 & 1 \end{bmatrix}, Y = \begin{bmatrix} x_1^{(1)}(2) \\ x_1^{(1)}(3) \\ x_1^{(1)}(4) \\ x_1^{(1)}(5) \end{bmatrix} = \begin{bmatrix} 6.152 \\ 9.459 \\ 12.849 \\ 16.528 \end{bmatrix}$$

We have

$$\hat{b} = \begin{bmatrix} b \\ a \end{bmatrix} = (B^T B)^{-1} B^T Y = \begin{bmatrix} 0.412435 \\ -0.482515 \end{bmatrix}$$

It follows that

$$\hat{x}_1^{(1)}(k) = 0.412435 x_2^{(1)}(k) - 0.482515$$

Therefore, the simulation results are as shown in Table 7.13.

The average relative error is

$$\overline{\Delta} = \frac{1}{4}\sum_{k=2}^{5}\Delta_k = \frac{1}{4}\sum_{k=2}^{5}\frac{|\varepsilon(k)|}{x^{(0)}(k)} = 2.475\%$$

**Table 7.13** Simulation results with errors

| Ordinality | Real Data $x^{(0)}(k)$ | Simulated Values $\hat{x}^{(0)}(k)$ | Errors $\varepsilon(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)$ | Relative errors $\Delta_k = \frac{\lvert\varepsilon(k)\rvert}{x^{(0)}(k)}$ (%) |
|---|---|---|---|---|
| 2 | 3.278 | 3.153 | 0.125 | 3.8 |
| 3 | 3.307 | 3.331 | − 0.024 | 0.7 |
| 4 | 3.390 | 3.518 | − 0.128 | 3.8 |
| 5 | 3.679 | 3.619 | 0.06 | 1.6 |

### 7.7.2 *The Model of GM(1,* N*)*

**Definition 7.7.2** Assume that $X_i^{(0)}$ and $X_i^{(1)}$ $(i = 1, 2, \ldots, N)$ as given in definition 7.6.1. Let $X_i^{(1)}$ be the accumulated sequences of $X_i^{(0)}$, $i = 1, 2, ..., N$, and $Z_1^{(1)}$ the adjacent neighbor average sequence of $X_1^{(1)}$. Then,

$$x_1^{(0)}(k) + az_1^{(1)}(k) = \sum_{i=2}^{N} b_i x_i^{(1)}(k) \tag{7.35}$$

is called the model of GM(1,$N$).

The constant $(-a)$ is known as the system's development coefficient, $b_i x_i^{(1)}(k)$ the driving term, $b_i$ the driving coefficient, and $\hat{a} = [a, b_1, b_2, \ldots, b_N]^T$ the sequence of parameters.

**Theorem 7.7.2** For the previously defined terms $X_i^{(0)}$, $X_i^{(1)}$, and $Z_1^{(1)}$, $i = 1, 2, ..., N$, let.

$$B = \begin{bmatrix} -z_1^{(1)}(2) & x_2^{(1)}(2) & \cdots & x_N^{(1)}(2) \\ -z_1^{(1)}(3) & x_2^{(1)}(3) & \cdots & x_N^{(1)}(3) \\ \cdots & \cdots & \cdots & \cdots \\ -z_1^{(1)}(n) & x_2^{(1)}(n) & \cdots & x_N^{(1)}(n) \end{bmatrix}, Y = \begin{bmatrix} x_1^{(0)}(2) \\ x_1^{(0)}(3) \\ \vdots \\ x_1^{(0)}(n) \end{bmatrix}$$

Then the least squares estimate of the sequence $\hat{a} = [a, b_1, b_2, \ldots, b_N]^T$ of parameters satisfies

$$\hat{a} = (B^T B)^{-1} B^T Y.$$

***Example 7.7.2*** Let.

$$X_1^{(0)} = (2.874, 3.278, 3.307, 3.39, 3.679) = \{x_1^{(0)}(k)\}_1^5$$

is a data sequence of a system's characteristic variable, and

$$X_2^{(0)} = (7.04, 7.645, 8.075, 8.53, 8.774) = \{x_2^{(0)}(k)\}_1^5$$

the data sequences of a relevant factor. Try to establish the model of GM(1,2).

**Solution**: Assume that the model of GM(1,2) is as follows:

$$x_1^{(0)}(k) + az_1^{(1)}(k) = bx_2^{(1)}(k)$$

From

$$X_1^{(1)} = \left[x_1^{(1)}(1), x_1^{(1)}(2), x_1^{(1)}(3), x_1^{(1)}(4), x_1^{(1)}(5)\right]$$
$$= (2.874, 6.152, 9.459, 12.849, 16.528)$$

$$X_2^{(1)} = \left[x_2^{(1)}(1), x_2^{(1)}(2), x_2^{(1)}(3), x_2^{(1)}(4), x_2^{(1)}(5)\right]$$
$$= (7.04, 14.685, 22.76, 31.29, 40.064)$$

We have

$$Z_1^{(1)} = \left[z_1^{(1)}(2), z_1^{(1)}(3), z_1^{(1)}(4), z_1^{(1)}(5)\right]$$
$$= (4.513, 7.8055, 11.154, 14.6885)$$

It follows that

$$B = \begin{bmatrix} -z_1^{(1)}(2) & x_2^{(1)}(2) \\ -z_1^{(1)}(3) & x_2^{(1)}(3) \\ -z_1^{(1)}(4) & x_2^{(1)}(4) \\ -z_1^{(1)}(5) & x_2^{(1)}(5) \end{bmatrix} = \begin{bmatrix} -4.513 & 14.685 \\ -7.8055 & 22.76 \\ -11.154 & 31.29 \\ -14.6885 & 40.064 \end{bmatrix}, Y = \begin{bmatrix} x_1^{(0)}(2) \\ x_1^{(0)}(3) \\ x_1^{(0)}(4) \\ x_1^{(0)}(5) \end{bmatrix}$$
$$= \begin{bmatrix} 3.278 \\ 3.307 \\ 3.390 \\ 3.679 \end{bmatrix}, Y = \begin{bmatrix} x_1^{(0)}(2) \\ x_1^{(0)}(3) \\ x_1^{(0)}(4) \\ x_1^{(0)}(5) \end{bmatrix} = \begin{bmatrix} 3.278 \\ 3.307 \\ 3.390 \\ 3.679 \end{bmatrix}.$$

Therefore, we have

$$\hat{a} = \begin{bmatrix} a \\ b \end{bmatrix} = (B^T B)^{-1} B^T Y = \begin{bmatrix} 2.2273 \\ 0.9068 \end{bmatrix}$$

and

$$x_1^{(0)}(k) + 2.2273 z_1^{(1)}(k) = 0.9068 x_2^{(1)}$$

That is,

$$\hat{x}_1^{(0)}(k) = -2.2273 z_1^{(1)}(k) + 0.9068 x_2^{(1)}$$

The simulation results are as shown in Table 7.14.

**Table 7.14** Simulation results with errors

| Ordinality | Real data $x^{(0)}(k)$ | Simulated values $\hat{x}^{(0)}(k)$ | Errors $\varepsilon(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)$ | Relative errors $\Delta_k = \frac{\lvert\varepsilon(k)\rvert}{x^{(0)}(k)}$ (%) |
|---|---|---|---|---|
| 2 | 3.278 | 3.265 | 0.013 | 0.4 |
| 3 | 3.307 | 3.254 | 0.053 | 1.6 |
| 4 | 3.390 | 3.530 | – 0.140 | 4.1 |
| 5 | 3.679 | 3.614 | 0.065 | 1.8 |

The average relative error is

$$\overline{\Delta} = \frac{1}{4}\sum_{k=2}^{5}\Delta_k = \frac{1}{4}\sum_{k=2}^{5}\frac{|\varepsilon(k)|}{x^{(0)}(k)} = 1.975\%$$

### *7.7.3 The Grey Verhulst Model*

The GM(1,1) model is suitable for sequences that show an obvious exponential pattern and can be used to describe monotonic changes. As for non-monotonic wavelike development sequences, or saturated sigmoid sequences, one can consider establishing a grey Verhulst model.

**Definition 7.7.3** Assume that $X^{(0)}$ is a sequence of raw data, $X^{(1)}$ the accumulation sequence of $X^{(0)}$, and $Z^{(1)}$ the adjacent neighbor average sequence of $X^{(1)}$. Then,

$$x^{(0)}(k) + az^{(1)}(k) = b\left[z^{(1)}(k)\right]^{\alpha} \tag{7.36}$$

is known as the power model of GM(1,1). Also,

$$dx^{(1)}/dt + ax^{(1)} = b(x^{(1)})^{\alpha} \tag{7.37}$$

is known as the shadow equation of the power model of GM(1,1) (Deng, 1985).

**Theorem 7.7.3** The solution of the whitenization equation of the power model of GM(1,1) is.

$$x^{(1)}(t) = \left\{e^{-(1-a)at}[(1-a)\int be^{(1-a)at}dt + c]\right\}^{\frac{1}{1-a}} \tag{7.38}$$

**Theorem 7.7.4** Let $X^{(0)}$, $X^{(1)}$, and $Z^{(1)}$ be defined as above. Let.

$$B = \begin{bmatrix} -z^{(1)}(2) & \left[z^{(1)}(2)\right]^\alpha \\ -z^{(1)}(3) & \left[z^{(1)}(3)\right]^\alpha \\ \vdots & \vdots \\ -z^{(1)}(n) & \left[z^{(1)}(n)\right]^\alpha \end{bmatrix}, Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}$$

Then the least squares estimate of the parametric sequence $\hat{a} = [a, b]^T$ of the power model of GM(1,1) is

$$\hat{a} = (B^T B)^{-1} B^T Y.$$

**Definition 7.7.4** When the power $\alpha = 2$ in the power model of GM(1,1), the resultant model.

$$x^{(0)}(k) + az^{(1)}(k) = b(z^{(1)}(k))^2 \tag{7.39}$$

is known as the grey Verhulst model; and

$$dx^{(1)}/dt + ax^{(1)} = b(x^{(1)})^2 \tag{7.40}$$

is known as the whitenization equation of the grey Verhulst model (Deng, 1985).

**Theorem 7.7.5**

(1) The solution of the Verhulst whitenization equation is

$$x^{(1)}(t) = \frac{1}{e^{at}[\frac{1}{x^{(1)}(0)} - \frac{b}{a}(1 - e^{-at})} = \frac{ax^{(1)}(0)}{e^{at}[a - bx^{(1)}(0)(1 - e^{-at})}$$

That is

$$x^{(1)}(t) = \frac{ax^{(1)}(0)}{bx^{(1)}(0) + \left[a - bx^{(1)}(0)\right]e^{-at}} \tag{7.41}$$

(2) The time response sequence of the grey Verhulst model is

$$\hat{x}^{(1)}(k+1) = \frac{ax^{(1)}(0)}{bx^{(1)}(0) + \left[a - bx^{(1)}(0)\right]e^{-ak}} \tag{7.42}$$

The Verhulst model is mainly used to describe and study processes with saturated states (or sigmoid processes). For instance, this model is often used in the prediction of human populations, biological growth, reproduction, and economic life span of consumable products. From the solution of the Verhulst equation, it can be seen that when $t \to \infty$, if $a > 0$, then $x^{(1)}(t) \to 0$; if $a < 0$, then $x^{(1)}(t) \to \frac{a}{b}$. That is, there is a sufficiently large $t$ such that for any $k > t$, both $x^{(1)}(k+1)$ and $x^{(1)}(k)$ will be sufficiently close to each other. In this case, $x^{(0)}(k+1) = x^{(1)}(k+1) - x^{(1)}(k) \approx 0$, which means that the system approaches distinction.

In practice, one often faces sigmoid processes in the original data sequences. When such an instance appears, we can simply take the original sequence as $X^{(1)}$ with its accumulation generation as $X^{(0)}$ to establish a grey Verhulst model to directly simulate $X^{(1)}$.

***Example 7.7.3*** Assume that the expenditures on the research of a certain kind of torpedo are given in Table 7.15. Try to employ the grey Verhulst model to simulate the data and make predictions (Liang et al., 2005).

The accumulated expenditures are given in Table 7.16.

From Theorem 7.7.5, we compute the parameters as follows:

$$\hat{a} = [a, b]^T = \begin{bmatrix} -0.98079 \\ -0.00021576 \end{bmatrix}$$

so that the whitenization equation is

$$x^{(1)}/dt - 0.98079x^{(1)} = -0.00021576(x^{(1)})^2.$$

By taking $x^{(1)}(0) = x^{(0)}(1) = 496$, we obtain the time response sequence

$$\hat{x}^{(1)}(k+1) = \frac{ax^{(1)}(0)}{bx^{(1)}(0) + [a - bx^{(1)}(0)]e^{-ak}} = \frac{-486.47}{-0.10702 - 0.87378e^{-0.98079k}}.$$

On the basis of this formula, we produce the simulated values $\hat{x}^{(1)}(k)$ as shown in Table 7.17

From Table 7.17, we can obtain the average relative error

$$\Delta = \frac{1}{9}\sum_{k=2}^{10} \Delta_k = 4.3354\%$$

and predict the research expenditure for the year of 2005 on the special kind of torpedo as

**Table 7.15** Expenditures on the research of a certain kind of torpedo (in million Yuan)

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 |
|---|---|---|---|---|---|---|---|---|---|---|
| Expenditure | 496 | 779 | 1187 | 1025 | 488 | 255 | 157 | 110 | 87 | 79 |

**Table 7.16** Accumulated expenditures (in ten thousand Yuan)

| Year | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 |
|---|---|---|---|---|---|---|---|---|---|---|
| Expenditure | 496 | 1275 | 2462 | 3487 | 3975 | 4230 | 4387 | 4497 | 4584 | 4663 |

**Table 7.17** The simulation results with errors

| Ordinality | Actual data $x^{(0)}(k)$ | Simulated data $\hat{x}^{(0)}(k)$ | Error $\varepsilon(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)$ | Relative error $\Delta_k = \frac{\lvert\varepsilon(k)\rvert}{x^{(0)}(k)}$ |
|---|---|---|---|---|
| 2 | 1275 | 1119.1 | 155.9 | 0.12226 |
| 3 | 2462 | 2116 | 346 | 0.14053 |
| 4 | 3487 | 3177.5 | 309.5 | 0.08876 |
| 5 | 3975 | 3913.7 | 61.3 | 0.01541 |
| 6 | 4230 | 4286.2 | – 56.2 | 0.01328 |
| 7 | 4387 | 4444.8 | -57.8 | 0.01318 |
| 8 | 4497 | 4507.4 | – 10.4 | 0.0023 |
| 9 | 4584 | 4531.3 | 52.7 | 0.0115 |
| 10 | 4663 | 4540.3 | 122.7 | 0.02631 |

$$\hat{x}_1^{(0)}(11) = \hat{x}_1^{(1)}(11) - \hat{x}_1^{(1)}(10) = 9.0342.$$

This value indicates that the research work on the torpedo is nearing its conclusion.

### *7.7.4 The Self-memory Grey Model*

For unimodal series or nonlinear saturated growth series, self memory GM (1,1) power model can also be established to describe its evolution law (Guo et al. 2014a).

**Definition 7.7.5** Assume that.

$$F(x, t) = -ax^{(1)} + b(x^{(1)})^{\gamma} \tag{7.43}$$

where $x$ is variable, $t$ is time, then formula (7.43) is called a self-memory dynamic equation.

**Definition 7.7.6** Assume that $\beta(t)$, $|\beta(t)| \le 1$ is a memory function,

the variable $x$, memory function $\beta(t)$ and self-memory dynamic equation $F(x, t)$ all meet the conditions of continuity, differentiability, integrability, then following

$$\beta_t x_t - \beta_{-p} x_{-p} - \sum_{i=-p}^{0} x_i^m (\beta_{i+1} - \beta_i) - \int_{t_{-p}}^{t} \beta(\tau) F(x, \tau) \mathrm{d}\tau = 0 \tag{7.44}$$

is called a self-memory prediction model.

Where $T = \{t_{-p}, t_{-p+1}, \ldots, t_{-1}, t_0, t\}$ is the time set.

Let $x_{-p-1}^m \equiv x_{-p}$, $\beta_{-p-1} \equiv 0$, we can obtained the following

$$x_t = \frac{1}{\beta_t}\sum_{i=-p-1}^{0} x_i^m(\beta_{i+1}-\beta_i) + \frac{1}{\beta_t}\int_{t_{-p}}^{t}\beta(\tau)F(x,\tau)\mathrm{d}\tau = S_1 + S_2 \tag{7.45}$$

where $S_1$ is the self-memory item which represents the influence of historical statistical data on the predicted value $x_t$, $S_2$ is other effective item which represents the influence of the dynamic equation $F(x,t) = -ax^{(1)} + b(x^{(1)})^{\gamma}$ on the predicted value $x_t$ within the backtracking period $[t_{-p}, t_0]$.

In (7.45), we use addition to approximately replace integration and difference to approximately replace differentiation and let $x_i^m = \frac{1}{2}(x_{i+1}+x_i) \equiv y_i, \Delta t_i = t_{i+1}-t_i = 1$ further, then we obtained the discrete self-memory prediction model as follows

$$x_t = \sum_{i=-p-1}^{-1}\alpha_i y_i + \sum_{i=-p}^{0}\theta_i F(x,i) \tag{7.46}$$

where $\alpha_i = (\beta_{i+1}-\beta_i)\big/\beta_t$, $\theta_i = \beta_i\big/\beta_t$, and $F(x,t) = -ax^{(1)} + b(x^{(1)})^{\gamma}$.

**Theorem 7.7.6** Assume that.

$$\underset{L\times 1}{X_t} = \begin{bmatrix} x_{t1} \\ x_{t2} \\ \vdots \\ x_{tL} \end{bmatrix}, \underset{L\times(p+1)}{Y} = \begin{bmatrix} y_{-p-1,1} & y_{-p,1} & \cdots & y_{-1,1} \\ y_{-p-1,2} & y_{-p,2} & \cdots & y_{-1,2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{-p-1,L} & y_{-p,L} & \cdots & y_{-1,L} \end{bmatrix}, \underset{(p+1)\times 1}{A} = \begin{bmatrix} \alpha_{-p-1} \\ \alpha_{-p} \\ \vdots \\ \alpha_{-1} \end{bmatrix}$$

$$\Gamma_{L\times(p+1)} = \begin{bmatrix} F(x,-p)_1 & F(x,-p+1)_1 & \cdots & F(x,0)_1 \\ F(x,-p)_2 & F(x,-p+1)_2 & \cdots & F(x,0)_2 \\ \vdots & \vdots & \ddots & \vdots \\ F(x,-p)_L & F(x,-p+1)_L & \cdots & F(x,0)_L \end{bmatrix}, \underset{(p+1)\times 1}{\Theta} = \begin{bmatrix} \theta_{-p} \\ \theta_{-p+1} \\ \vdots \\ \theta_0 \end{bmatrix}$$

Let $Z = [Y, \Gamma]$, $W = \begin{bmatrix} A \\ \Theta \end{bmatrix} W = \begin{bmatrix} A \\ \Theta \end{bmatrix}$

then the least squares estimate of the parametric vector $W = \begin{bmatrix} A \\ \Theta \end{bmatrix}$ satisfies

$$W = (Z^{\mathrm{T}}Z)^{-1}Z^{\mathrm{T}}X_t \tag{7.47}$$

### 7.7.5 The Models of GM(r,h)

In this subsection, we focus on the investigation of the structure of the models of GM(r,h), and its relationships with models GM(1,1), GM(1,$N$), GM(0,$N$), and the grey Verhulst model.

**Definition 7.7.7** Assume that $X_i^{(0)} = \left(x_i^{(0)}(1), x_i^{(0)}(2), \ldots, x_i^{(0)}(n)\right)$, $i = 1, 2, ..., h$, where $X_1^{(0)}$ stands for a data sequence of a system's characteristic, and $X_i^{(0)}$, $i = 2, 3, \ldots, h$ data sequences of relevant factors. Let.

$$\alpha^{(1)}\hat{x}_1^{(1)}(k) = \hat{x}_1^{(1)}(k) - \hat{x}_1^{(1)}(k-1) = \hat{x}_1^{(0)}(k)$$

$$\alpha^{(2)}\hat{x}_1^{(1)}(k) = \alpha^{(1)}\hat{x}_1^{(1)}(k) - \alpha^{(1)}\hat{x}_1^{(1)}(k-1) = \hat{x}_1^{(0)}(k) - \hat{x}_1^{(0)}(k-1)$$

$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$

$$\alpha^{(r)}\hat{x}_1^{(1)}(k) = \alpha^{(r-1)}\hat{x}_1^{(1)}(k) - \alpha^{(r-1)}\hat{x}_1^{(1)}(k-1) = \alpha^{(r-2)}\hat{x}_1^{(0)}(k) - \alpha^{(r-2)}\hat{x}_1^{(0)}(k-1)$$

and $z^{(1)}(k) = \frac{1}{2}\left(x^{(1)}(k) + x^{(1)}(k-1)\right)$, then

$$\alpha^{(r)}\hat{x}_1^{(1)}(k) + \sum_{i=1}^{r-1} a_i \alpha^{(r-i)} x_1^{(1)}(k) + a_r z_1^{(1)}(k) = \sum_{j=1}^{h-1} b_j x_{j+1}^{(1)}(k) + b_h \tag{7.48}$$

is referred to as the model of GM($r$,$h$). The GM($r$,$h$) model is a $r$th order grey model in $h$ variables.

**Definition 7.7.8** In the model of GM($r$,$h$), $-\hat{a} = [-a_1, -a_2, \ldots, -a_r]^T$ is referred to as the development coefficient vector, $\sum_{j=1}^{h-1} b_j x_{j+1}^{(1)}(k)$ the driving term, and $\hat{b} = [b_1, b_2, \ldots, b_h]^T$ the vector of driving coefficients.

**Theorem 7.7.7** Let $X_1^{(0)}$ be a data sequence of a system's characteristic, $X_i^{(0)}$, $i = 2, 3, \ldots, h$, the data sequences of relevant factors, $X_i^{(1)}$ the accumulation sequence of $X_i^{(0)}$, $Z_1^{(1)}$ the adjacent neighbor average sequence from $X_1^{(1)}$, and $\alpha^{(r-i)}X_1^{(1)}$ the $(r-i)$th order inverse accumulation sequence of $X_1^{(1)}$. Define.

$$B = \begin{bmatrix} -\alpha^{(r-1)}x_1^{(1)}(2) & -\alpha^{(r-2)}x_1^{(1)}(2) & \cdots & -\alpha^{(1)}x_1^{(1)}(2) & -z_1^{(1)}(2) & x_2^{(1)}(2) & \cdots & x_h^{(1)}(2) & 1 \\ -\alpha^{(r-1)}x_1^{(1)}(3) & -\alpha^{(r-2)}x_1^{(1)}(3) & \cdots & -\alpha^{(1)}x_1^{(1)}(3) & -z_1^{(1)}(3) & x_2^{(1)}(3) & \cdots & x_h^{(1)}(3) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\alpha^{(r-1)}x_1^{(1)}(n) & -\alpha^{(r-2)}x_1^{(1)}(n) & \cdots & -\alpha^{(1)}x_1^{(1)}(n) & -z_1^{(1)}(n) & x_2^{(1)}(n) & \cdots & x_h^{(1)}(n) & 1 \end{bmatrix},$$

$$Y = \begin{bmatrix} \alpha^{(r)} x_1^{(1)}(2) \\ \alpha^{(r)} x_1^{(1)}(3) \\ \vdots \\ \alpha^{(r)} x_1^{(1)}(n) \end{bmatrix}$$

then the parametric sequence $\hat{c} = [-\hat{a}, \hat{b}]^T = [-a_1, -a_2, \ldots, -a_r; b_1, b_2, \ldots, b_h]^T$ of the least squares estimate satisfies

$$\hat{a} = (B^T B)^{-1} B^T Y.$$

The model of GM($r$,$h$) is the general form of grey systems models. In particular,

(1) When $r = 1$ and $h = 1$, the previous (7.48) reduces to:

$dx_1^{(1)}/dt + a_1 x_1^{(1)} = b_1$ and $\alpha^{(1)} x_1^{(1)}(k) + a_1 z_1^{(1)}(k) = b_1$.
which is the model of GM(1,1).

(2) When $r = 1$ and $h = N$, the previous (7.48) takes the form of

$$x_1^{(0)}(k) + a_1 z_1^{(1)}(k) = \sum_{i=2}^{N} b_i x_i^{(1)}(k)$$

which is the GM(1,$N$) model.

(3) When $r = 0$ and $h =$ N, the previous model (7.48) is

$$x_1^{(1)}(k) = b_1 x_2^{(1)}(k) + b_2 x_3^{(1)}(k) + \cdots + b_{N-1} x_N^{(1)}(k) + b_N$$

which is the GM(0,$N$) model.

(4) When $r = 1$ and $h = 1$, and $b_1$ in the model of GM(1,1) is changed to $b(z^{(1)}(k))^2$, then we have the following grey Verhulst model:

$$x^{(0)}(k) + az^{(1)}(k) = b(z^{(1)}(k))^2$$

Based on this discussion, it can be seen that models GM(1,1), GM(1,N), GM(0,N), etc., are all special cases of model GM($r$,$h$). So, it is very important to further the study of model GM($r$,$h$).

## 7.8 Practical Applications

***Example 7.8.1*** (Liu, 1991) Let us look at the revenue predictions of private enterprises at Changge County, Henan Province, The People's Republic of China, which we mentioned in Example 4.3.1. For the years from 1983 to 1986, the overall business revenue of private enterprises in Changge county was recorded as.

$$X = (10155, 12588, 23480, 35388)$$

We obtained the following second-order buffered sequence

$$XD^2 = (27260, 29547, 32411, 35388)$$

in Example 4.3.1 by a second-order average weakening buffer operator (AWBO) as follows:

$$x(k)d = \frac{1}{n-k+1}[x(k) + x(k+1) + \cdots + x(n)], k = 1, 2, \cdots, n$$

We denote the $XD^2$ as $X^{(0)}$, that is, let

$$X^{(0)} = (27260, 29547, 32411, 35388).$$

Then the 1-AGO sequence $X^{(1)}$ of $X^{(0)}$ is as follows.

$$X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), x^{(1)}(3), x^{(1)}(4)\right) = (27260, 56807, 89218, 124606)$$

Assume that

$$x^{(0)}(k) + az^{(1)}(k) = b$$

Based on the least squares method, we obtain the estimated values for a and b as follows:

$$\hat{a} = -0.089995, \hat{b} = 25790.28$$

Thus, the resultant whitenization equation of EGM(1, 1) is given by

$$\frac{dx^{(1)}}{dt} - 0.089995x^{(1)} = 25790.28$$

and its time response sequence is

$$\begin{cases} \hat{x}^{(1)}(k+1) = 313834e^{0.089995k} - 286574 \\ \hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k) \end{cases}$$

From these results, we obtain the simulated sequence

$$\hat{X} = (\hat{x}(1), \hat{x}(2), \hat{x}(3), \hat{x}(4)) = (27260, 29553, 32337, 35381)$$

with the sequence of errors

$$\varepsilon^{(0)} = \left(\varepsilon^{(0)}(1,)\varepsilon^{(0)}(2), \varepsilon^{(0)}(3), \varepsilon^{(0)}(4)\right) = (0, -6, 74, 7)$$

The sequence of relative errors

$$\Delta = \left[\left|\frac{\varepsilon^{(0)}(1)}{x^{(0)}(1)}\right|, \left|\frac{\varepsilon^{(0)}(2)}{x^{(0)}(2)}\right|, \left|\frac{\varepsilon^{(0)}(3)}{x^{(0)}(3)}\right|, \left|\frac{\varepsilon^{(0)}(4)}{x^{(0)}(4)}\right|\right]$$
$$= (0, 0.0002, 0.00228, 0.0002)$$

And the average relative error

$$\overline{\Delta} = \frac{1}{4}\sum_{k=1}^{n4} \Delta_k = 0.00067 = 0.067\% < 0.01$$

$$\Delta_4 = 0.0002 = 0.02\% < 0.01$$

Therefore, the accuracy of our simulation is in level one.

Now, we can compute the absolute degree $\varepsilon$ of grey incidences of $X$ and $\hat{X}$.

$$|s| = \left|\sum_{k=2}^{3}[x(k) - x(1)] + \frac{1}{2}[x(4) - x(1)]\right| = 11502$$

$$|\hat{s}| = \left|\sum_{k=2}^{3}[\hat{x}(k) - \hat{x}(1)] + \frac{1}{2}[\hat{x}(4) - \hat{x}(1)]\right| = 11430.5$$

$$|\hat{s} - s| = \left|\sum_{k=2}^{3}[x(k) - x(1) - (\hat{x}(k) - \hat{x}(1))] + \frac{1}{2}[x(4) - x(1) - (\hat{x}(4) - \hat{x}(1))]\right|$$
$$= 71.5$$

Thus,

$$\varepsilon = \frac{1 + |s| + |\hat{s}|}{1 + |s| + |\hat{s}| + |\hat{s} - s|} = \frac{1 + 11502 + 11430.5}{1 + 11502 + 11430.5 + 71.5} = 0.997 > 0.90$$

That is, the degree of incidence is in level one.

Compute the ratio of mean square deviations $C$:

$$\bar{x} = \frac{1}{4}\sum_{k=1}^{4} x(k) = 31151.5,\ S_1^2 = \frac{1}{4}\sum_{k=1}^{4}(x(k)-\bar{x})^2 = 37252465,\ S_1 = 6103.48$$

$$\bar{\varepsilon} = \frac{1}{4}\sum_{k=1}^{4} \varepsilon(k) = 18.75,\ S_2^2 = \frac{1}{4}\sum_{k=1}^{4}(\varepsilon(k)-\bar{\varepsilon})^2 = 4154.75,\ S_2 = 64.46$$

It follows that

$$C = \frac{S_2}{S_1} = \frac{64.46}{6103.48} = 0.01 < 0.35$$

which is in level one.

Compute the small error probability. From

$$0.6745 S_1 = 4116.80$$

$$|\varepsilon(1)-\bar{\varepsilon}| = 18.75,\ |\varepsilon(2)-\bar{\varepsilon}| = 24.75,\ |\varepsilon(3)-\bar{\varepsilon}| = 55.25,\ |\varepsilon(4)-\bar{\varepsilon}| = 11.75$$

Therefore

$$p = P(|\varepsilon(k)-\bar{\varepsilon}| < 0.6745 S_1) = 1 > 0.95$$

With our accuracy checks in place, we can apply the grey model

$$\begin{cases} \hat{x}^{(1)}(k+1) = 313834e^{0.089995k} - 286574 \\ \hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k) \end{cases}$$

to make predictions. Here, we list five predicted values as follows:

$$\begin{aligned} \hat{X}^{(0)} &= \left[\hat{x}^{(0)}(5), \hat{x}^{(0)}(6), \hat{x}^{(0)}(7), \hat{x}^{(0)}(8), \hat{x}^{(0)}(9)\right] \\ &= (38714, 42359, 46348, 50712, 55488) \end{aligned}$$

These predictions indicated an average 9.4% annual growth. When we look back today, this predicted rate of growth agreed very well with the recorded values over the time span of our predictions.

***Example 7.8.2*** Subgrade settlement prediction (Guo et al. 2015).

Subgrade settlement is one important indicator affecting road safety because the major hidden danger could result in road traffic accidents. So subgrade settlement prediction is one of the major research topics in the field of geotechnical engineering. Three monitoring points (Points A, B and C) at certain roadbed sections of Beijing-Harbin freeway (G102 line) were arranged, the method of single point extensometer was employed to monitor its subgrade settlement.

The three groups of accumulated subgrade settlement data at different monitoring points are listed in Table 7.18 (Liu et al., 2013).

Step 1: Analyze the coupling relationship between the data of different monitoring points.

Analyze the coupling relationship between the data of different monitoring points to determine whether the data of monitoring points A, B and C are relevant. Let

$$X_1 = (13.42, 15.38, 22.18, 23.30, 24.55, 25.41, 26.91, 28.02, 28.44, 28.64)$$
$$X_2 = (9.89, 12.20, 16.27, 17.66, 19.07, 20.85, 21.91, 23.40, 23.77, 24.12)$$
$$X_3 = (12.03, 15.60, 19.57, 20.80, 22.03, 23.38, 24.60, 25.79, 26.36, 27.16)$$

Calculate the grey absolute relational degree between $X_1$, $X_2$, and $X_3$ respectively, we have

$$\epsilon_{12} = 0.9923, \epsilon_{13} = 0.9721, \epsilon_{23} = 0.9648.$$

The results shows that there is coupling relationship and certain relationship exists of the data at monitoring point A, B and C.

Step 2: Determining the self-memory dynamic equation.

$$\begin{cases} \frac{\mathrm{d}x_1^{(1)}}{\mathrm{d}t} + 4.0920x_1^{(1)} + 2.8789x_2^{(1)} - 7.0870x_3^{(1)} = 7.2671 \\ \frac{\mathrm{d}x_2^{(1)}}{\mathrm{d}t} + 1.7787x_1^{(1)} + 1.5361x_2^{(1)} - 3.3707x_3^{(1)} = 7.4767 \\ \frac{\mathrm{d}x_3^{(1)}}{\mathrm{d}t} + 1.9224x_1^{(1)} + 1.5285x_2^{(1)} - 3.5145x_3^{(1)} = 10.9312 \end{cases} \tag{7.49}$$

**Table 7.18** The accumulated subgrade settlement data of monitoring points A, B and C (unit: mm)

| Period | Number of days | Accumulated subgrade settlement value | | |
|---|---|---|---|---|
| | | Point A | Point B | Point C |
| 1 | 35 | 13.42 | 9.89 | 12.03 |
| 2 | 50 | 15.38 | 12.20 | 15.60 |
| 3 | 65 | 22.18 | 16.27 | 19.57 |
| 4 | 80 | 23.30 | 17.66 | 20.80 |
| 5 | 95 | 24.55 | 19.07 | 22.03 |
| 6 | 110 | 25.41 | 20.85 | 23.38 |
| 7 | 125 | 26.91 | 21.91 | 24.60 |
| 8 | 140 | 28.02 | 23.40 | 25.79 |
| 9 | 155 | 28.44 | 23.77 | 26.36 |
| 10 | 170 | 28.64 | 24.12 | 27.16 |

The matrix form $\mathrm{d}X^{(1)}\big/\mathrm{d}t = -AX^{(1)} + B$ of Eq. (7.49) was taken as the dynamic kernel $F(X, t)$ of the self-memory equation of the SMGM(1,3) model.

Step 3: Deducing the self-memory prediction equation system.

The value of retrospective order is determined as $p = 1$ by trial calculation method under the principle of minimum error of fitting root-mean-square. Then the self-memorization equation system of the SMGM(1,3) model can be established for subgrade settlement forecasting as follows.

$$\begin{cases} x_{1t} = \sum_{i=-2}^{-1} \alpha_{1i} y_{1i} + \sum_{i=-1}^{0} \theta_{1i} F_1(x, i) \\ x_{2t} = \sum_{i=-2}^{-1} \alpha_{2i} y_{2i} + \sum_{i=-1}^{0} \theta_{2i} F_2(x, i) \\ x_{3t} = \sum_{i=-2}^{-1} \alpha_{3i} y_{3i} + \sum_{i=-1}^{0} \theta_{3i} F_3(x, i) \end{cases} \tag{7.50}$$

Step 4: Estimate the memory coefficients matrix by the least square method.

$$W = [W_1,\ W_2,\ W_3] = \begin{bmatrix} \alpha_{1,-2} & \alpha_{2,-2} & \alpha_{3,-2} \\ \alpha_{1,-1} & \alpha_{2,-1} & \alpha_{3,-1} \\ \theta_{1,-1} & \theta_{2,-1} & \theta_{3,-1} \\ \theta_{1,0} & \theta_{2,0} & \theta_{3,0} \end{bmatrix} = \begin{bmatrix} -0.05200.08900.0260 \\ 1.04680.90670.9712 \\ 0.21410.36160.2931 \\ 1.27611.27691.2702 \end{bmatrix}$$

Step 5: Simulation.

Substituting the memory coefficient matrix into Eq. (7.50), The simulation values of original subgrade settlement data matrix $X^{(0)}$ can be obtained.

The simulated values and their corresponding APE of three compared models, SMGM(1,3), GM(1,1) and MGM(1,3) are presented in Tables 7.19, 7.20, and 7.21 respectively.

Step 6: Simulation accuracy comparison.

The values of accuracy criteria (MSE, AME and MAPE) of different subgrade settlement prediction models are shown in Table 7.22.

From the viewpoint of error analysis, the multi-variable models of MGM(1,3) and SMGM(1,3) always show lower error values than the uni-variable model GM(1,1). It is shown that the multi-point prediction models can take the relationship among variables into account, and are able to adequately reflect the integral evolution laws of subgrade settlement system. The self-memory technique helped model SMGM(1,3) to further reduce the modeling errors compared with the traditional model MGM(1,3). Meanwhile, the model SMGM(1,3) has passed the modeling simulation and prediction accuracy test, and the single-step and two-step rolling prediction precisions are also generally superior than that of the other two grey models. In summary, the model SMGM (1,3) markedly promoted the predictive performance compared with other grey prediction models.

**Table 7.19** The simulated values and APE of the SMGM(1,3), GM(1,1) and MGM(1,3) at point A (unit: mm)

| No | Actual value | GM(1,1) | | MGM(1,3) | | SMGM(1,3) | |
|---|---|---|---|---|---|---|---|
| | | Simulated value | APE/% | Simulated value | APE/% | Simulated value | APE/% |
| 1 | 13.42 | – | – | – | – | – | – |
| 2 | 15.38 | 18.931 | 23.088 | 16.283 | 5.871 | – | – |
| 3 | 22.18 | 20.333 | 8.327 | 21.182 | 4.500 | 22.168 | 0.055 |
| 4 | 23.30 | 21.839 | 6.270 | 23.276 | 0.103 | 23.331 | 0.132 |
| 5 | 24.55 | 23.456 | 4.456 | 24.601 | 0.208 | 24.625 | 0.307 |
| 6 | 25.41 | 25.193 | 0.854 | 25.715 | 1.200 | 25.105 | 1.201 |
| 7 | 26.91 | 27.059 | 0.554 | 26.775 | 0.502 | 27.229 | 1.184 |
| 8 | 28.02 | 29.063 | 3.722 | 27.824 | 0.700 | 27.913 | 0.381 |
| 9 | 28.64 | 31.216 | 8.994 | 29.057 | 1.456 | 28.594 | 0.161 |
| 10 | 28.44 | 30.298 | 6.533 | 30.340 | 6.681 | 29.746 | 4.591 |

**Table 7.20** The simulated values and APE of the SMGM(1,3), GM(1,1) and MGM(1,3) at point B (unit: mm)

| No | Actual value | GM(1,1) | | MGM(1,3) | | SMGM(1,3) | |
|---|---|---|---|---|---|---|---|
| | | Simulated value | APE/% | Simulated value | APE/% | Simulated value | APE/% |
| 1 | 9.89 | – | – | – | – | – | – |
| 2 | 12.20 | 14.171 | 16.156 | 12.625 | 3.484 | – | – |
| 3 | 16.27 | 15.484 | 4.831 | 15.789 | 2.956 | 16.258 | 0.075 |
| 4 | 17.66 | 16.920 | 4.190 | 17.713 | 0.300 | 17.737 | 0.434 |
| 5 | 19.07 | 18.488 | 3.052 | 19.256 | 0.975 | 19.165 | 0.496 |
| 6 | 20.85 | 20.202 | 3.108 | 20.662 | 0.902 | 20.373 | 2.288 |
| 7 | 21.91 | 22.075 | 0.753 | 22.001 | 0.415 | 22.222 | 1.423 |
| 8 | 23.40 | 24.121 | 3.081 | 23.300 | 0.427 | 23.520 | 0.512 |
| 9 | 23.77 | 26.357 | 10.883 | 24.654 | 3.719 | 24.527 | 3.186 |
| 10 | 24.12 | 26.624 | 10.381 | 25.131 | 4.192 | 24.842 | 2.993 |

**Table 7.21** The simulated values and APE of the SMGM(1,3), GM(1,1) and MGM(1,3) at point C (unit: mm)

| No | Actual value | GM(1,1) | | MGM(1,3) | | SMGM(1,3) | |
|---|---|---|---|---|---|---|---|
| | | Simulated value | APE/% | Simulated value | APE/% | Simulated value | APE/% |
| 1 | 12.03 | – | – | – | – | – | – |
| 2 | 15.60 | 17.443 | 11.814 | 16.044 | 2.846 | – | – |
| 3 | 19.57 | 18.696 | 4.466 | 19.085 | 2.478 | 19.564 | 0.032 |
| 4 | 20.80 | 20.039 | 3.659 | 20.810 | 0.048 | 20.836 | 0.171 |
| 5 | 22.03 | 21.479 | 2.501 | 22.153 | 0.558 | 22.107 | 0.351 |
| 6 | 23.38 | 23.022 | 1.531 | 23.378 | 0.009 | 23.084 | 1.266 |
| 7 | 24.60 | 24.676 | 0.309 | 24.558 | 0.171 | 24.771 | 0.694 |
| 8 | 25.79 | 26.449 | 2.555 | 25.719 | 0.275 | 25.886 | 0.374 |
| 9 | 26.36 | 28.349 | 7.546 | 26.954 | 2.253 | 26.813 | 1.720 |
| 10 | 27.16 | 29.087 | 7.095 | 27.445 | 1.049 | 27.412 | 0.927 |

**Table 7.22** Simulation error of different subgrade settlement prediction models

| Monitoring point | Model | MSE | AME | MAPE (%) |
|---|---|---|---|---|
| Point A | MGM(1,3) | 3.924 | 1.595 | 10.174 |
| | OMGM(1,3) | 3.088 | 1.511 | 9.277 |
| | SMGM(1,3) | 0.170 | 0.316 | 1.572 |
| Point B | MGM(1,3) | 4.538 | 1.665 | 10.096 |
| | OMGM(1,3) | 3.336 | 1.542 | 9.017 |
| | SMGM(1,3) | 0.232 | 0.370 | 1.741 |
| Point C | MGM(1,3) | 5.200 | 1.735 | 10.051 |
| | OMGM(1,3) | 3.700 | 1.607 | 8.961 |
| | SMGM(1,3) | 0.343 | 0.451 | 2.026 |
| Subgrade system | MGM(1,3) | 4.554 | 1.665 | 10.107 |
| | OMGM(1,3) | 3.375 | 1.553 | 9.085 |
| | SMGM(1,3) | 0.248 | 0.379 | 1.780 |

# References

Benítez, R. B. C., Paredes, R. B. C., et al. (2013). Damp trend Grey Model forecasting method for airline industry. *Expert Systems with Applications, 40*(12), 4915–4921.

Bo, X. B., Wei, J., Qian, Y. F., et al. (2012). Study on Ground Settlement of High-speed Rail at the Transition Section between Bridge and Plain Track and Predicating Method. *Railway Construction Technology, 8*, 88–93.

Chen, X.D; Xia, J., & Xu, Q. (2009). Self-memory prediction mode with grey differentiamodel. *China Science E: Technological Science*, *39*(2), 341–350.

Chirwa, E. C., & Mao, M. (2006). Application of grey model GM (1, 1) to vehicle fatality risk estimation. *Technological Forecasting and Social Change*, *73*(5), 588–605.

Comert, G., Khan, Z., Rahman, M., & Chowdhury, M. (2021). Grey models for short-term queue length predictions for adaptive traffic signal control. *Expert Systems with Applications*, *185*, 115618.

Cempel, C. (2008). Decomposition of the symptom observation matrix and grey forecasting in vibration condition monitoring of machines. *International Journal of Applied Mathematics and Computer Science*, *18*(4), 569–579.

Dang, Y. G., & Liu, S. F. (2004). The GM models that x(n) be taken as initial value, kybernetes. *The International Journal of Systems & Cybernetics, 33*(2), 247–254.

Deng, J. L. (1982). Control problems of grey systems. *Systems & Control Letters, 1*(5), 288–294.

Deng, J. L. (1985). *Grey control systems*. Press of Huazhong University of Science and Technology.

Duman, G. M., Kongar, E., & Gupta, S. M. (2019). Estimation of electronic waste using optimized multivariate grey models. *WASTE MANAGEMENT, 95*, 241–249.

Evans, M. (2014). An alternative approach to estimating the parameters of a generalised Grey Verhulst model: an application to steel intensity of use in the UK. *Expert Systems with Applications*, *41*(4), 1236–1244.

Guo, X. J., Liu, S. F., Wu, L. F., et al. (2014a). Grey GM(1,1) power pharmacokinetics model coupling self-memory principle of dynamic system. *The Journal of Grey System*, *26*, (4), 122–138.

Guo, X. J., Liu, S. F., & Fang, Z. G. (2014b). Self-memory prediction model of interval grey number based on grey degree of compound grey number. *Systems Engineering and Electronics*, *36*(6), 1124–1129.

Guo, X. J., Liu, S. F., Wu, L. F., et al. (2015). A multi-variable grey model with a self-memory component and its application on engineering prediction. *Engineering Applications of Artificial Intelligence, 42*, 82–93.

Gurden, S. P., Westerhuis, J. A., Bijlsma, S., et al. (2001). Modelling of spectroscopic batch process data using grey models to incorporate external information. *Journal of chemometrics*, *15*(2), 101–121.

Hao, Y., Cao, B., Chen, X., Yin, J., Sun, R., & Yeh, T. C. J. (2013). A piecewise grey system model for study the effects of anthropogenic activities on karst hydrological processes. *Water Resources Management*, *27*(5), 1207–1220.

Hao, Y., Zhao, J., Li, H., Cao, B., Li, Z., & Yeh, T. C. J. (2012). Karst Hydrological Processes and Grey System Model 1. *JAWRA Journal of the American Water Resources Association*, *48*(4), 656–666.

Hsu, Y. T., & Yeh, J. (2000). A novel image compression using grey models on a dynamic window. *International Journal of Systems Science*, *31*(9), 1125–1141.

Ji, P. R., Huang, W. S., & Hu, X. Y. (2001). On characteristics of grey prediction models. *Systems Engineering: Theory and Practice, 21*(9), 105–108.

Kose, E., & Tasci, L. (2015). Prediction of the Vertical Displacement on the Crest of Keban Dam. *The Journal of Grey System*, *27*(1), 12–20.

Li, P., Yang, H., Sun, L., & Deng, Z. (2011). Application of gray prediction and time series model in spacecraft prognostic. *Computer Measurement & Control*, *19*(1), 111–113.

Li, T., Ren, M. F., & Chen, H. R. (2010). Calculation method for fatigue crack propagation rate based on grey theory. *Journal of Mechanical Strength, 32*(3), 472–475.

Li, X., Yuan, Z., Zhang, G., & Cheng, S. H. (2014). Some properties of grey differential equation GM (1, 1, β). *Systems Engineering–Theory and Practice*, *34*(5), 1249–1255.

Liang, Q. W., Song, B. W., & Jia, Y. (2005). The grey Verhulst model of the costs on the research of torpedoes. *Journal of System Simulation, 17*(2), 257–258.

Lin, Y., Wang, T., Wang, L., & Chen, H. C. (2005). Stability analysis of high excavated slope in Three Gorges Project. *Journal of Tianjing University*, *38*(10), 936–940. (In Chinese).

Liu, H. B., Xiang, Y. M., & Ruan, Y. X. (2013). A multivariable grey model based on background value optimization and its application to subgrade settlement prediction. *Rock Soil Mechanics, 34*(1), 173–181.

Liu, J. F., Liu, S. F., & Fang, Z. G. (2015). Fractional-order reverse accumulation generation GM(1,1) model and its applications. *The Journal of Grey System, 27*(4), 52–62.

Liu, S. F., Zeng, B., Liu, J. F., et al. (2015). Four basic models of GM(1, 1) and their suitable sequences. *Grey Systems: Theory and Application. 5*(2), 141–156.

Liu, S. F. (1991). The three axioms of buffer operator and their application. *The Journal of Grey System, 3*(1), 39–48.

Liu, S. F., & Guo, T. B. (1991). *Grey system theory and its applications*. Henan University Press. In Chinese.

Liu, S. F. (2024). *Grey system theory and its application* (10th ed.). Science Press. (In Chinese).

Liu, S. F., & Deng, J. L. (2000). The range suitable for GM(1,1). *Systems Engineering Theory & Practice, 20*(5), 121–124.

Liu, B., Liu, S. F., Cui, Z. J., et al. (2003). Optimization of time response functions of the GM(1,1) model. *Management Science of China*, *11*(4), 54–57.

Liu, S. F., Sheng, K. Q., & Forrest, J. (2012). On uncertain systems and uncertain models. *Kybernetes, 41*(5–6), 548–558.

Liu, S. F., Zeng, B., Liu, J. F., et al. (2015). Four basic models of GM(1, 1) and their suitable sequences. *Grey Systems: Theory and Application, 5*(2), 141–156.

Liu, S. F., Yang, Y. J., & Forrest, J. (2017). *Grey data analysis*. Springer-Verlag.

Manickam, A., Indrakala, S., & Kumar, P. (2023). A novel mathematical study on the predictions of volatile price of gold using grey models. *Contemporary Mathematics, 4*(2), 182–378.

Mao, S. H., Gao, M. Y., Xiao, X. P. (2015). Fractional order accumulation time-lag GM(1,N,t) model and its application. *Systems Engineering - Theory & Practice*, *35*(2): 430–436.

Mao, S., Gao, M., Xiao, X., & Zhu, M. (2016). A novel fractional grey system model and its application. *Applied Mathematical Modelling*, *40*(7–8), 5063–5076.

Özdemir, A., & Özdagoglu, G. (2017). Predicting product demand from small-sized data: grey models. *Grey Systems: Theory and Application*, *7*(1), 80–96.

Öztürk, Z., Bilgil, H., & Erdinç, Ü. (2022). An optimized continuous fractional grey model for forecasting of the time dependent real world cases. *Hacettepe Journal of Mathematics and Statistics*, *51*(1), 308–326.

Wuyong, Q., Yaoguo, D., & Sifeng, L. (2012). Grey GM (1, 1, t) model with time power and application. *Systems Engineering Theory and Practice*, *32*(10), 2247–2252.

Salmeron, J. L. (2010). Modelling grey uncertainty with fuzzy grey cognitive maps. *Expert Systems with Applications., 37*, 7581–7588.

Saxena, A. (2023). Optimized fractional overhead power term polynomial grey model (OFOPGM) for market clearing price prediction. *Electric Power Systems Research*, *214*, 108800.

Song, Z. M., Xiao, X. P., & Deng, J. L. (2002). The character of opposite direction AGO and class ratio. *The Journal of Grey System, 4*(1), 9–14.

Tabaszewski, M., & Cempel, C. (2015). Using a set of GM(1,1) models to predict values of diagnostic symptoms. *Mechanical Systems and SignaL Processing*, *52–53*, 416–425.

Tan, G. J. (2005). The structure method and application of background value in grey system GM(1,1) model. *Systems Engineering Theory & Practice, 25*(1), 98–103. In Chinese.

Tang, W. M. (2006). New forecasting model based on grey support vector machine. *Journal of System Engineering, 21*(4), 410–413.

Tien, T. L. (2003). A research on the deterministic grey dynamic model with multiple inputs DGDMMI(1,1,1)[J]. *Applied Mathematics & Computation*, *139*(2), 401–416.

Tong, X. J., Chen, M. Y. & Zhou, L. (2002). On AGO Effect of the Grey Model.Systems Engineering-Theory and Prectace. *22*(11), 121–125.

Wang, Y. N., et al. (2001). GM(1,1) Modelling based onGrey derivative value optimization. *Systems Engineering: Theory and Practice*, *21*(5), 124–128.

Wang, X., & Nie, H. (2008). On a method of fatigue life prediction based on grey system model GM (1, 1). *Transaction of Nanjing University of Aeronautics and Astronautics*, *40*(6), 845–848.
Wang, Y. N. (2003). An extended step by step optimum direct modeling method of GM(1, 1). *Systems Engineering Theory & Practice, 23*(2), 120–124. (In Chinese).
Wang, Y. M., Dang, Y. G., & Wang, Z. X. (2008). Optimization of the background values of non-equal-distant GM(1,1) model. *Management Science of China, 16*(4), 159–162. (In Chinese).
Wu, L. F., Liu, S. F., Yao, L. G., et al. (2015). Using fractional order accumulation to reduce errors from inverse accumulated generating operator of grey model. *Soft Computing*, *19*(2), 483–488.
Yanyang, W., & Yihua, C. (2010). Gray neural network model of aviation safety risk. *Journal of Aerospace Power*, *25*(5), 1036–1042.
Wang, Z. X. (2013). On derivative model of power model GM(1,1). *Systems Engineering Theory and Practice, 33*(11), 2894–2902.
Wu, L. F., Liu, S. F., Yao, L. G., et al. (2013). Grey system model with the fractional order accumulation. *Communications in Nonlinear Science and Numerical Simulation, 18*(7), 1775–1785.
Wu, L. Y., Wu, Z. P., & Li, M. (2013). Quadratic time-varying parameters discrete grey model. *Systems Engineering-Theory & Practice*, *33*(11), 2887–2892.
Wu, W. Z., Zeng, L., Liu, C., Xie, W., & Goh, M. (2022). A time power-based grey model with conformable fractional derivative and its applications. *Chaos, Solitons & Fractals*, *155*, 111657.
Xia, T.B., Jin, X.N., Xi, L.F., et al. (2015). Operating load based real-time rolling grey forecasting for machine health prognosis in dynamic maintenance schedule. *Journal of intelligent manufacturing*, *26*(2), 269-228.
Xiao, X. P. (2000). On parameters in grey models. *The Journal of Grey System, 11*(4), 73–78.
Xiao, X. P., & Wang, H. H. (2014). Change of GM(1,1,a) model background value on the influences of relative error. *System Engineering Theory and Practice*, *34*(2), 408–415.
Xinping, X., Jun, L., & Huan, G. (2013). Properties and optimization of generalized accumulation grey model. *Systems Engineering Theory and Practice*, *33*(1), 1–9. (In Chinese).
Xie, N. M., & Liu, S. F. (2005). Discrete GM(1,1) model and modeling mechanism of grey prediction models. *Systems Engineering: Theory and Practice, 28*(4), 93–99. (In Chinese).
Xie, N. M., & Liu, S. F. (2009). Discrete grey forecasting model and its optimization. *Applied Mathematical Modelling, 33*(1), 1173–1186.
Xie, N., Liu, S., Yuan, C., & Yang, Y. (2014). Grey number sequence forecasting approach for interval analysis: a case of China's gross domestic product prediction. *The Journal of Grey System*, *26*(1), 45–58.
Xie, X., Liu, X., & Blanco, C. (2023). Evaluating and forecasting the niche fitness of regional innovation ecosystems: A comparative evaluation of different optimized grey models. *Technological Forecasting and Social Change*, *191*, 122473.
Yang, J., & Wong, W. G. (2014). Improved unbiased grey model for prediction of gas supplies. *Journal of Tsinghua University (Sci. & Technol), 54*(2), 145–148.
Tianshe, Y., Ping, Y., & Xiaoshe, D. (2008). Method for predicting fault status of satellite based on gray system theory. *Computer Measurement & Control*, *9*, 1284–1285, 1307
Yao, T. X., Liu, S. F., & Xie, N. M. (2010). Study on the properties of new information discrete GM (1, 1) model. *Journal of Systems Engineering*, *25*(2), 164–170.
Zadeh, L. A. (1994). Soft computing and fuzzy logic. *IEEE Software, 11*(6), 48–56.
Zeng, B., Li, H., Mao, C. W., & Wu, Y. (2023). Modeling, prediction and analysis of new energy vehicle sales in China using a variable-structure grey model. *Expert Systems with Applications, 213*, 118879.
Zeng, B., & Liu, S. F. (2012). Prediction model of stochastic oscillation sequence based on amplitude compression. *Systems Engineering Theory and Practice, 34*(8), 2084–2091 (In Chinese).
Zeng, B., Yin, F., Yang, Y., Wu, Y., & Mao, C. (2023). Application of the novel-structured multivariable grey model with various orders to forecast the bending strength of concrete. *Chaos, Solitons & Fractals*, *168*, 113200.

Zhang, K. (2014). Multi-variables discrete grey model based on driver control. *Systems Engineering–Theory and Practice*, *34*(8), 2084–2091.

Zhang, Q. S. (2007). Improving the precision of GM(1, 1) model using particle swarm optimization. *Chinese Journal of Management Science, 15*(5), 126–129. (In Chinese).

Zhang, X. L., Rao, C. J., Xiao, X. P., et al. (2024). Prediction of demand for staple food and feed grain by a novel hybrid fractional discrete multivariate grey model. *Applied Mathematical Modelling, 125*, 85–107.

Zhang, X., Wang, Z., & Masatake, N. (2006). Research on affective interaction models of robot. *Jisuanji Gongcheng/ Computer Engineering*, *32*(24), 6–12.

# Chapter 8
# Combined Grey Models

## 8.1 Introduction

Along with the disciplinary development of systems science and systems engineering, methods and modeling techniques established for systems evaluation, prediction, decision-making, and optimization are enriched constantly. Generally, each method and every model have their strengths and weaknesses, so in practical applications several different methods and modeling techniques are combined to form hybrid methods or techniques in order to successfully deal with the problems at hand. Such combinations and mixtures are used to capitalize upon the strengths and advantages of different methods so that they complement each other and at the same time improve the weaknesses of individual methods and modeling techniques. This explains why combined or mixed systems are superior to individual component methods. Additionally, the availability of many different methods and modeling techniques also provides us with different ways to deal with information and systems. Therefore, how to combine and mix different methods and techniques has become a research direction with wide-ranging applicability in areas of data mining and knowledge discovery.

The grey system model has the effect of weakening the randomness of the sequence and mining the evolution law of the system, and has strong fusion and penetration to the general model. Integrating the grey system model with the traditional model to achieve functional complementarity can often get more satisfactory results.

In the process of modeling, the grey system theory advocates respecting the original data but not rigidly adhering to the original data, allowing the necessary screening and processing of the experimental, observational and statistical data of the research object based on scientific qualitative analysis. Using the idea and method of grey system theory to process the original observation data will greatly improve the statistical characteristics of the model.



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_8

Any model is just an image of one (or several) side of the research object. At the same time, because the development and evolution process of the system is often the result of the interaction of many known factors and unknown factors, deterministic factors and uncertain factors, it is difficult to fully reveal the development and change law of the research object with a single model. Among the numerous models, different models have their own characteristics, and have different advantages in revealing the change law of one side of the research object. Therefore, the organic combination of the grey system model and other models may deepen the understanding of the system evolution law.

In 1996, based on the grey relational analysis method, the explanatory variables of the model are selected, and the GM (1,1) model of explanatory variables is established to predict the values of explanatory variables. Then, the GM (1,1) model simulation values of all variables are used as the basic data to estimate the model parameters, Liu and Zhu proposed a grey-econometrics combined model (Liu & Zhu, 1996).

In 2004, the author of this book combines the possibility degree function of grey system with the capital asset pricing model (CAPM), and proposes a new comprehensive utility evaluation model for venture capital investment. This method overcomes the shortcomings of the expectation-variance method and Sharpe's index method, and avoids the difficulty of constructing utility function to a certain extent (Liu et al., 2004a). Combined the thought and method of grey system theory with Cobb–Douglas production function, a grey Cobb–Douglas (G-C-D) production function model is proposed (Liu et al., 2004a).

In 2008, Li and Liu studied the grey matrix and grey input–output model in depth. Based on these, they put forward an analytical model for enterprise grey input–output in 2012 (Li & Liu, 2008; Li et al., 2012) the grey physical input–output analysis model. Jian et al. (2011) developed a series of the grey rough set hybrid models. Wang and Liu (2009) conducted research on the grey DEA model, which has made significance research results.

Lin et al. (2001) proposed a Markov-Fourier grey prediction model. They compared the performance of the new model with different prediction schemes, such as back propagation neural networks and fuzzy models. The simulation results show that the new approach can predict the future more accurately and also use less computation time than other methods. Kose and Forrest (2015) combined the grey system theory with the classic N-person game theory and sets up the N-person grey game with grey payoff functions. Chen and Chang (2000) proposed a new approach of grey fuzzy dynamic modeling for the prediction of solid waste generation in the urban area based on a set of limited samples. Luo et al. (2001) proposed a hierarchical grey fuzzy motion decision-making algorithm, which is capable of integrating multiple sequential data for decision making and for the design of the control kernel of the target tracking system. Bahrami et al. (2014) proposed a new model based on the combination of the wavelet transform and GM for short-term electric load forecasting. Samet and Mojallal (2014) proposed a rolling GM and a Grey-Markov method to predict the actual reactive power of Mobarakeh Steel Company in Iran. Verma et al. (2014) used GRA coupled with fuzzy logic to model the stator winding fault and to predict the optimal setting for running the induction motor within its parameters range. The results indicate that the proposed novel approach is very effective in predicting the stator winding fault (Aydemir et al., 2015). Oztaysi (2014) proposed a AHP integrated grey-TOPSIS method, and applied in a Turkish foreign trade company. Zhang and Chen (2002) proposed a new genetic algorithm method based on random simulation to solve the general grey nonlinear programming problem.

Liu et al. (2010) optimized the railway digital mobile communication system scheme under the condition of limit frequency planning based on grey cluster and rough set models. Guo et al. (2013) combined grey prediction and Markov chain to improve the prediction accuracy of pollutants. Yuan et al. (2014) forecasted fire accidents based on portfolio optimization model of grey neural network. Meng et al. (2012) predicted gun tube life using the grey linear regression combination model and enhanced the prediction accuracy. Mi et al. (2014) constructed an optimal portfolio model based on the diagnosis results of three kinds of diagnosis methods such as fuzzy fault diagnosis method, genetic algorithm and grey system theory and made the fault diagnosis on 25 Hz phase sensitive track circuit. Xu et al. (2010) used grey econometrics model to predict the traffic volume of highway. Jiang (2012) diagnose the fault of wind turbine drive chain based on the grey rough set theory. Yin et al. (2012) evaluated groundwater quality in Taonan City based on the grey cluster method and matter element extension method. Tang et al. (2012) studied the main influential factors of gas well productivity of Permian Shan 2 gas reservoir in Zizhou Gas Field using grey system method, the method of principal component analysis and R cluster analysis method comprehensively.

The Chinese academician Wu et al. (2012) with The National Key Laboratory on Hydrology Water Resources and Hydraulic Engineering at Hehai University applied grey system theory and a variety of scientific methods to research the slope stability and dam safety service status, which made a series of vital achievements (Zheng et al., 2005).

Wang and Huang applied GM-BP combined model to deformation forecasting of foundation pit (Wang & Huang, 2016). Chen, et al. used three-point-method-gray GM(1,1) combination model in settlement prediction for a railway line (Chen et al., 2015). Khuman, et al. studied the problem of quantification of subjectivity by the R-fuzzy grey analysis framework (Khuman et al., 2019). Tian analyzed the radial velocity of projectile based on combined model of ARIMA, GM(1,1) and linear regression model (Tian, 2022). He, et al. Solved the problem of segmented calibration of transducer (He et al., 2015), Huang and Su's research on failure prediction based on combined model of grey neural network (Huang & Su, 2020). Karimi and Hojati designed a medical rule model system by using rough-grey modelling (Karimi & Hojati, 2020). Nain, et al. studied the cutting speed, wire wear ratio, and dimensional deviation of wire electric discharge machining of super alloy Udimet-L605 using support vector machine and grey relational analysis model (Nain et al., 2018). Vyavahare, et al. investigated FDM manufactured auxetic structures bymachine learning techniques and GRA model (Vyavahare et al., 2023). Muthukumar, et al. analyzed the localization and classification of gender focus in epilepsy patients based on Rough Set and GRA model (Muthukumar et al., 2023). Rajesh predited sustainability performance of supply chains by grey and rough set theoretical approaches (Rajesh, 2022).

Grey systems theory and methods strongly complement many of the traditional technologies and soft computing techniques. In this chapter, we explore various combinations, mixtures and applications of grey systems models and models developed in econometrics, production functions, artificial neural networks, linear regression, Markov models, and rough sets.

## 8.2 Grey Econometrics Models

### 8.2.1 Select Explanatory Variables Using Grey Relational Analysis Method

In analyzing systems, due to the complications of mutually crossing influences of the endogenous variables, at the very start of modeling, the first problem that needs to be addressed is how to select the variables that will be part of the eventual model. To revolve this problem, the researcher needs not only rely on his qualitative analysis of the system, but also have sufficiently adequate tools for conducting quantitative analysis. Grey relational analysis model provide an effective method for this class of problems.

Let $y$ be an endogenous variable of the system of our concern (for systems with many endogenous variables, these variables can be studied individually), and $x_1, x_2, \ldots, x_n$ be pre-images of influencing factors that are correlated either positively or negatively to $y$. Calculate the grey relational degree $\varepsilon_i$ between $y$ and $x_i$, $i = 1, 2, \ldots, n$, at first. For a chosen lower threshold value $\varepsilon_0$, when $\varepsilon_i < \varepsilon_0$, remove the variable $x_i$ out of consideration. By doing so, some of the system's endogenous variables with weak grey relational degrees with $y$ can be removed from further consideration. Assume that the remaining illustrative variables of $y$ are $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$. Next, consider the grey relational degrees $\varepsilon_{i_j i_k} (i_j, i_k = i_1, i_2, \ldots, i_m)$ between these remaining variables. For a chosen threshold value $\varepsilon_0\prime$, when $\varepsilon_{i_j i_k} \geq \varepsilon_0\prime$, the variables $x_{i_j}$ and $x_{i_k}$ are seen as the same kind so that the remaining variables are divided into several subsets. Now, choose one representative from each of these subsets to enter into the eventual model. By going through this possess, the resultant econometrics model can be greatly simplified without losing the needed power of explanation. At the same time, to a certain degree the difficult problem of collinearity of the variables can be avoided.

### 8.2.2 Grey Econometrics Model

In econometrics, there are many different kinds of models, such as linear regression models in one or multiple variables, nonlinear models, systems of equations, among others. When estimating the parameters of these models, one often faces phenomena that are difficult to explain. For instance, the coefficients of the major illustrative variables are nearly zero; the signs of some estimated values of the parameters do not agree with reality or contradict theoretical economic analysis; small vibrations in a few individual observations cause drastic changes in many other estimated parametric values. Among the main reasons underlying these difficulties are:

(1) During the time period the observations are done, the internal structure of the system goes through major changes;
(2) There is a problem of collinearity between the illustrative variables; and
(3) There are randomness and noise in the observed data.

For the first two scenarios, there is a need to repeat the investigation of the model structure or a need to recheck the illustrative variables. For the third scenario, one can consider establishing models using the GM(1,1) simulated values of the original observations to eliminate the effect of the randomness or noise existing in the available data. The combined grey econometrics model, obtained this way, can more accurately reflect the relationship between the system's variables. At the same time, the prediction results made on the endogenous variables of the grey econometrics model system, which is based on the GM(1,1) predicted values of the illustrative variables, possess more solid scientific foundation than qualitative estimate values of the illustrative variables. Besides, by comparing the results of grey predictions of the endogenous variables with those obtained out of econometrics models, one can further improve the reliability of the predictions.

The steps for establishing and applying grey econometrics models are as follows:

Step 1: Design the theoretical model. Study the economic activity of interest closely. According to the purpose of the investigation, select the variables that will potentially enter the model. Discover the relationships between these variables based on theories of economic behavior and experience and/or analyze the sampled data. Develop the mathematical expressions, which are the theoretical model, that describe the relationships between these variables. This stage is the most important and difficult phase of the entire modeling process, and the following work need to be done:

(1) Study relevant theories of economics

Theoretical models summarize the fundamental characteristics and laws of development of the objective matters. They are abstract pictures of reality. Therefore, in the stage of model design, one first needs to conduct a qualitative analysis using economic theories. With different theories, various models can be established. For instance, according to the theory of equilibrium of labor markets, the rate $y$ of wage increase is related to the unemployment rate $x_1$ and inflation rate $x_2$, that is, $y = f(x_1, x_2)$.

The greater the unemployment rate increases, the smaller the rate of wage increase due to the fact that the supply of labor is clearly greater than the demand. This is the well-established Alban W. Phillips curve, which has been widely accepted and applied in the economic models of Western countries. However, this model may not necessarily hold true in the socialist market economy of China. As a second example, according to Keynes's theory of consumption, it is believed that, on average, when income grows, people tend to increase their consumption. However, the degree of increase in consumption is not as high as that of income. Assume that $y$ stands for consumption, and $x$ for income. Then, a mathematical expression for the relationship between these variables is

$$y = f(x) = b_0 + b_1 x + \varepsilon$$

where the parameter $b_1 = dy/dx$ stands for the marginal consumption tendency, and $\varepsilon$ a random noise, representing the inherent randomness of consumption. According to Keynes, $0 < b_1 < 1$. However, Simon Kuznets does not agree with Keynes's opinion of a declining marginal consumption tendency. His work indicates that there is a stable proportion of increase between consumption and income. That is, the previous model is only a product of Keynes's theory.

(2) Variables and the form of the eventual model

The established model should reflect the objective economic activity. However, it is impossible for such a reflection to include all details. This is why we need reasonable assumptions. Employing the method of this section to select the major variables to be included in the model using grey relational analysis will help to eliminate minor relationships and factors. It focuses on the dominant connections while simplifying the eventual model, making it convenient to handle and apply.

The specific works of this stage of model design include: (i) Determine which variables to include, which ones are dependent variables, and which ones are independent. Here, each independent variable is also known as illustrative variable. (ii) Determine the number of parameters to be included in the model and their (positive or negative) signs. (iii) Determine the mathematical form of the model expression. Is it linear or nonlinear?

(3) Collection and organization of statistical data

After having decided on which variables to consider, one needs to collect all the relevant data. That is the foundation of establishing models. Generally speaking, all the collected raw data need to be statistically categorized and organized so that they become the empirical evidence of the characteristics of the problem of concern and are systematically usable for the purpose of modeling. The basic types of statistical data, as discussed in Chap. 3, include behavioral sequences, time series, index sequences, horizontal sequences, among others.

Step 2: Establish the GM(1,1) model and obtain its simulated values. In order to eliminate the random effect or error noise existing in the observational values of individual variables of the model, establish the GM(1,1) models for the individually observed sequences and then apply the simulated values of these GM(1,1) models as the base sequences on which to construct the eventual model.

Step 3: Estimate the parameters. After having designed the econometrics model, the next task is to estimate the parameters, which are the constant coefficients of the quantitative relationship between the chosen variables of the model. They connect the individual variables within the model. More specifically, these parameters explain how independent variables affect the dependent variable. Before using observed data to make estimations, these parameters are unknown. After the form of the model is established on the basis of the GM(1,1) model, simulated sequences solve the estimated values of the parameters using an appropriate method, such as that of least squares estimate. As soon as the parameters are clearly specified, the relationships between model variables become known and the model can be determined.

The estimated values of the parameters provide realistic and empirical contents and verification for the theories of economics. For instance, in the previously mentioned consumption model, if the estimated value of parameter $b_1$ is $\hat{b}_1 = 0.8$, it not only classifies the realistic content of the marginal consumption tendency, but also provides a piece of evidence for the assumption of Keynes's theory of consumption that this parameter is between 0 and 1.

Step 4: Test the model. After the parameters are estimated, the abstract model becomes specific and determined. However, to determine whether or not the model agrees with objective reality, and whether or not it can explain realistic economic processes, it still has to go through tests. The tests consist of two aspects, the test of economic meanings and statistical tests. The test of economic meanings checks whether or not the individual estimated values of the parameters agree with economic theories and relevant experiences. Statistical tests check the reliability of the estimate, the effectiveness of the data sequence simulation, the correctness of various econometrics assumptions, as well as the overall structure of the model and its prediction ability using the principles of statistical reasoning. It is only after the model passes through these tests that it can be applied in practice. If the model does not pass the tests, then the model needs to be modified and improved.

Step 5: Apply the established model. Grey econometrics models have been mainly employed to analyze economic structures, evaluate policies and decisions, simulate economic systems, and predict economic development. Each application process is also a process of verifying the model and its underlying theory. If the prediction contains small errors, it means that the model is of high accuracy and quality, with a strong ability to explain reality and an underlying theory that agrees with reality. Otherwise, the model and the economic theory on which the model was initially developed need to be modified.

Combined grey econometrics models can be employed not only to situations of known system structures, but also to situations of system structures that need further study and exploration. Combined grey econometrics models have produced satisfactory results in practical applications. To this end, please consult Liu and Lin (2006, p. 247–254) to see how applications are carried out.

## 8.3 Combined Grey Linear Regression Models

Combined grey linear regression models can improve the weakness of original linear regression models where no exponential growth is considered. They can also improve the weakness of GM(1,1) models that do not involve enough linear factors. Thus, such combined models are suited for studying sequences with both linear tendencies and exponential growth tendencies. For such a sequence, the modeling process can be described as follows.

**Definition 8.2.1** Assume that $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\}$ is a sequence of raw data. Its first order accumulation sequence is $X^{(1)} = \{x^{(1)}(1), x^{(1)}(2), ..., x^{(1)}(n)\}$.

$$\hat{x}^{(1)}(k) = \mathrm{C}_1 e^{-vk} + \mathrm{C}_2 k + C_3 \tag{8.1}$$

is called a combined grey linear regression model, where $v$, $C_1$, $C_2$, $C_3$ are parameters that need to be estimated.

In fact, combined grey linear regression model (8.1) is a simulation model of $X^{(1)}$, which can be seen as the sum of a linear regression model of $y = ak + b$ and an exponential model of $y = \mathrm{C}_1 e^{-ak} + \mathrm{C}_2$.

From the model GM(1,1), we can obtain

$$\hat{x}^{(1)}(k+1) = \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-ak} + \frac{b}{a} \tag{8.2}$$

Let $C_1 = (x^{(0)}(1) - \frac{b}{a})$, $C_3 = \frac{b}{a}$, which can be written as shown below:

$$\hat{x}^{(1)}(k+1) = C_1 e^{-ak} + C_3 \tag{8.3}$$

By adding a linear term $\mathrm{C}_2 k$ to Formula (8.3), we can obtain the same formula as (8.1).

**Lemma 8.2.1** *Assume that $X^{(0)}$ and $X^{(1)}$ are the same as in Definition 8.2.1, then the parameter $v$ in Formula* (8.1) *can be estimated by the following Formula (8.4)*:

$$\hat{V} = \frac{\sum_{m=1}^{n-3} \sum_{k=1}^{n-2-m} \tilde{V}_{\mathrm{m}}(k)}{(n-2)(n-3)/2} \tag{8.4}$$

where $\tilde{V}_m(k) = \ln[y_m(k+1)/y_m(k)]$, $y_m(k) = x^{(1)}(k+m+1) - x^{(1)}(k+m) - x^{(1)}(k+1) + x^{(1)}(k)$, $k, m = 1, 2, \ldots, n-3$

**Theorem 8.2.1** *Assume that $X^{(0)}$ and $X^{(1)}$ are the same as in Definition 8.2.1. Let*

$$X^{(1)} = \begin{bmatrix} x^{(1)}(1) \\ x^{(2)}(2) \\ \vdots \\ x^{(1)}(n) \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}, A = \begin{bmatrix} e^{v} & 1 & 1 \\ e^{2v} & 2 & 1 \\ \vdots & \vdots & \vdots \\ e^{nv} & n & 1 \end{bmatrix},$$

*then we have the matrix form (8.5) of (8.1):*

$$X^{(1)} = AC \tag{8.5}$$

Therefore, we have

$$C = (A^T A)^{-1} A^T X^{(1)} \tag{8.6}$$

With the estimated values of parameters $v$, $C_1$, $C_2$, $C_3$, the following Formula (8.7) can be used as a simulating or forecasting model:

$$\hat{x}^{(1)}(k) = \mathrm{C}_1 e^{-\hat{V}k} + \mathrm{C}_2 k + C_3 \tag{8.7}$$

From Eq. (8.7), it can be seen that if $C_1 = 0$, then the first order accumulation sequence stands for a linear regression model. If $C_2 = 0$, then the accumulation sequence stands for a GM(1,1) model. This new model improves the weaknesses of the original linear regression model with no exponential growth and that of the GM(1,1) model where no linear factors are considered.

By applying the inverse accumulation generation operator to Eq. (8.7), we can obtain the simulated and predicted values $\hat{X}^{(0)}$ of the original sequence.

***Example 8.2.1*** At a certain observation station of ore and rock movement, the sequence of recorded subsides of a specific location from February 1995 to April 1996 is given in Table 8.1. Try to make predictions for the sinking dynamics of this specific location.

**Table 8.1** The original sequence of recorded subsides

| Time | 9502 | 9504 | 9506 | 9508 | 9510 | 9512 | 9602 | 9604 |
|---|---|---|---|---|---|---|---|---|
| Amount of subside | 12 | 22 | 31 | 43 | 51 | 57 | 75 | 83 |

**Table 8.2** Simulated and predicted values and their errors

| Time | 9502 | 9504 | 9506 | 9508 | 9510 | 9512 | 9602 | 9604 | 9606 | 9608 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x^{(0)}(k)$ | 12 | 22 | 31 | 43 | 51 | 57 | 75 | 83 | | |
| $\hat{x}^{(0)}(k)$ | 12.34 | 21.75 | 31.35 | 41.15 | 51.15 | 61.36 | 71.79 | 82.43 | 93.29 | 104.38 |
| Error (%) | − 2.85 | 1.15 | − 1.12 | 4.31 | − 0.30 | − 7.66 | 4.28 | 0.69 | | |

**Solution** Due to the small amount of available data, grey systems models are the most appropriate models for this prediction task. However, grey systems models employ exponential functions to simulate accumulation generated sequences. They are generally only suitable for modeling situations of exponential development, as it is difficult for such models to describe linear tendencies of change. Therefore, in this case study we will apply a grey linear exponential regression model to predict the subsides of the specified location.

The original sequence of data is

$$X^{(0)} = (12, 22, 31, 43, 51, 57, 75, 83).$$

Its first order accumulation sequence is

$$X^{(1)} = (12, 34, 65, 108, 159, 216, 291, 374).$$

For different $m$ values, from Eqs. (8.6) and (8.7) we obtain the estimated value $\hat{V} = 0.02058096$ for $v$. Also, from Eq. (6.10), we obtain the estimated value of $C$:

$$C = (A^T A)^{-1} A^T X^{(1)} = (21750.995, -439.9523, -21751.078).$$

Thus, the combined model of the first order accumulation generation sequence is

$$\hat{x}^{(1)}(k) = 21750.995e^{0.02058096k} - 439.9523k - 21751.078.$$

Out of this model, we obtain the simulated and predicted values for each of the time moments as listed in Table 8.2

## 8.4 Grey Cobb–Douglas Model

In this section, we study the Cobb–Douglas or production function model. Let $K$ be the capital input, $L$ the labor input, and $Y$ the production output. Then,

$$Y = A_0 e^{\gamma t} K^{\alpha} L^{\beta}$$

is known as the C-D production function model, where $\alpha$ stands for capital elasticity, βlabor elasticity, and $\gamma$ the parameter for the progress of technology. The log-linear form of this production function model is given below:

$$\ln Y = \ln A_0 + \gamma t + \alpha \ln K + \beta \ln L$$

For given time series data of the production output $Y$, capital input $K$, and labor input $L$,

$$Y = (y(1), y(2), \ldots, y(n)), K = (k(1), k(2), \ldots, k(n))\text{and} L = (l(1), l(2), \ldots, l(n)),$$

one can employ the method of multivariate least squares estimate to approximate the parameters $\ln A_0$, $\gamma$, $\alpha$, and $\beta$.

When $Y$, $K$, and $L$ represent the time series of a specific department, district, or business, it is often the case that, due to severe fluctuations existing in the data, the estimated parameters contain errors leading to incorrect results. For instance, the estimated coefficient $\gamma$ for progress of technology is too small or becomes a negative number; the estimated values $\alpha$ and $\beta$ for elasticity go beyond their reasonable ranges. Under such circumstances, if one considers using the GM(1,1) simulated data of $Y$, $K$, and $L$ as the original data for their least squares estimates, then to a certain degree they can eliminate some of the random fluctuations, produce more reasonable estimated parameter values, and obtain a model that can more accurately reflect the relationship between the production output and labor, and capital inputs and the progress of technology.

**Definition 8.3.1** Assume that

$$\hat{Y} = (\hat{y}(1), \hat{y}(2), \ldots, \hat{y}(n)),$$

$$\hat{K} = (\hat{k}(1), \hat{k}(2), \ldots, \hat{k}(n)),$$

and

$$\hat{L} = (\hat{l}(1), \hat{l}(2), \ldots, \hat{l}(n)).$$

are respectively the GM(1,1) simulated sequences of $Y$, $K$, and $L$. Then $\hat{Y} = A_0 e^{\gamma t} \hat{K}^{\alpha} \hat{L}^{\beta}$ is known as the grey model of production function.

In the grey production function model, although no grey parameters appear explicitly, it stands for an expression that combines the idea of grey systems modeling into the C-D production function model. That is, this model possesses a very deep intension of the greyness. It embodies the non-uniqueness principle of solutions and the absoluteness principle of greyness. This is why, in practical applications, this model has produced satisfactory results. To this end, please consult Liu and Lin (2006, pp. 256–258) to see how applications are carried out.

## 8.5 Grey Artificial Neural Network Models

### 8.5.1 BP Artificial Neural Model and Computational Schemes

Each artificial neural network is made up of a large amount of elementary information processors, known as neurons or nodes. The model with multi-layered nodes, or the scheme known as error back propagation, represents the currently well developed and widely employed artificial neural network system and computational method. It translates the input–output problem of an available sample into a nonlinear optimization problem. It is a powerful tool that can be employed to uncover the laws and patterns hidden in large amounts of data. The use of artificial neural networks to simulate data sequences has several latent advantages. First, it has the ability to model multiple kinds of functions, including nonlinear functions, piecewise defined functions, among others. Secondly, artificial neural networks are unlike the traditional methods of distinguishing data sequences, which, to work properly, must have presumed types of functional relationships between data sequences. This means that artificial neural networks can establish the needed functional relationship by using the attributes and intension naturally existing in the provided data variables, without presuming the kinds of distributions the parameters satisfy. Thirdly, this method possesses the advantage of making use of available information very efficiently, while avoiding the problem of losing the real meanings and pictures of the data due to various combinations, such as additions of positive and negative values of data mining methods. That is, the artificial neural networks method is especially useful for improving the GM(1,1) model.

Figure 8.1 shows a back propagation network with three layers. The network consists of an input layer, an implicit (or latent) layer, and an output layer. An entire process of learning consists of forward and backward propagation. The particular scheme of learning is given below:

(1) Apply random numbers to initialize $W_{ij}$ (the connection weight between nodes $i$ and $j$ of different layers) and $\theta_j$ (the threshold value of node $j$);

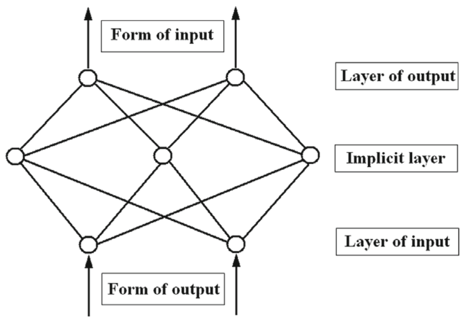



**Fig 8.1** A back propagation neural network

(2) Feed in the preprocessed training samples $\{X_{PL}\}$ and $\{Y_{PK}\}$;
(3) Compute the output of the nodes of each layer, $O_{pj} = f \sum_i (W_{ij} I_{pi} - \theta_j)$ for the $p$th sample point, where $I_{pi}$ stands for the output of node $i$ and the input of node $j$;
(4) Compute the information error of each layer. For the input layer, $\delta_{pk} = O_{pk}(y_{pk} - O_{pk})(1 - O_{pk})$; for the latent layer, $O_{pi} = O_{pi}(1 - O_{pi}) \sum_i \delta_{pi} W_{ij}$;
(5) For the backward propagation, the modifiers of the weights are $W_{ij}(t+1) = \alpha \delta_{pi} O_{pi} + W_{ij}(t)$, and the modifiers of the thresholds $\theta_j(t+1) = \theta_j(t) + \beta \delta_{pi}$, where $\alpha$ stands for the learning factor and $\beta$ the momentum factor for accelerated convergence; and
(6) Calculate the error $E_p = (\sum_p \sum_k)(O_{pk} - Y_{pk})^2/2$.

### 8.5.2 Steps in Grey BP Neural Network Modeling

The steps to establish a grey BP neural network model are as follows:

Step 1: Assume that a time series $\left\{x^{(0)}(i)\right\}, i = 1, 2, \ldots, n$, is given. We then obtain the restored values $\hat{x}^{(0)}(t), i = 1, 2, \ldots, n$, using the outputs of the GM(1,1) model.

Step 2: Establish the back propagation network model for the error sequence $\left\{e^{(0)}(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)\right\}, k = 1, 2, \ldots, n.$

If the order of prediction is $S$, it means that we use the information of $e^{(0)}(i-1)$, $e^{(0)}(i-2)$, ..., $e^{(0)}(i-S)$ to predict the value at the $i$th moment; we will treat $e^{(0)}(i-1), e^{(0)}(i-2), \ldots, e^{(0)}(i-S)$ as the input sample points of the back propagation network training, while using the value of $e^{(0)}(i)$ as the expected prediction of the back propagation network training. By using the back propagation computational scheme outlined earlier, train this network through enough amount of cases of error sequences so that output values (along with empirical test values) are produced in ways that correspond to different input vectors. The resultant weights and thresholds represent the correct internal representations through the self-learning and adaptation of the network. A well trained back propagation network model can be an effective tool for error sequence prediction.

Step 3: Determine the simulation values of $\left\{e^{(0)}(k) = x^{(0)}(k) - \hat{x}^{(0)}(k)\right\}, k = 1, 2, \ldots, n$. Assume that the simulation sequence is $\left\{\hat{e}^{(0)}(k)\right\}, k = 1, 2, \ldots, n$, which is obtained by the BP neural network.

Step 4: Based on $\left\{\hat{x}^{(0)}(i)\right\}$ and $\left\{\hat{e}^{(0)}(k)\right\}, i, k = 1, 2, \ldots, n$, we have the following result

$$\hat{x}^{(0)}(i, k) = \hat{x}^{(0)}(i) + \hat{e}^{(0)}(k) \tag{8.8}$$

which is the predicted sequence of the grey artificial neural network model.

**Table 8.3** The GM(1,1) simulations and errors

| Year | Investment $x^{(0)}(i)$ | GM(1,1) simulation $\hat{x}^{(0)}(i)$ | Errors $e^{(0)}(k)$ |
|---|---|---|---|
| 1985 | 110.20 | 110.20 | 0 |
| 1986 | 146.34 | 164.39 | − 19.05 |
| 1987 | 185.36 | 187.65 | − 2.29 |
| 1988 | 221.14 | 214.22 | 6.92 |
| 1989 | 255.16 | 244.54 | 10.52 |
| 1990 | 289.18 | 279.17 | 9.01 |
| 1991 | 320.54 | 319.69 | 1.85 |
| 1992 | 352.79 | 363.81 | − 11.02 |

**Table 8.4** Simulation results of the grey artificial neural network model

| Year | Actual value $x^{(0)}(i)$ | Simulated value $\hat{x}(i, k)$ | Relative errors (%) |
|---|---|---|---|
| 1988 | 221.14 | 221.12 | 0.01 |
| 1989 | 255.16 | 255.29 | 0.05 |
| 1990 | 289.18 | 289.11 | 0.02 |
| 1991 | 320.54 | 320.79 | 0.08 |
| 1992 | 352.79 | 352.70 | 0.03 |

***Example 8.4.1*** Given the actual yearly investments in environmental protection over a period of time of a certain location, and the GM(1,1) simulations and relevant errors in Table 8.3, establish an artificial neural network model for the error sequence.

**Solution** Based on and using the GM(1,1) error sequence data given in Table 8.3, we apply the previously outlined method to establish a back propagation network model. Our projected back propagation network will have three characteristic parameters, one latent layer, within which there are 6 nodes, and one input layer within which there is one node. Let the learning rate be 0.6, the convergence rate 0.001, and the variance limited within the range of 0.01. Let us conduct the training and testing of the network on a computer. Then, Table 8.4 lists the simulation results of the combined back propagation network model.

## 8.6 Grey Markov Model

### *8.6.1 Grey Moving Probability Markov Model*

**Definition 8.5.1** Assume that $\{X_n, n \in T\}$ is a stochastic process. If for any whole number $n \in T$ and any states $i_0, i_1, \ldots, i_{n+1} \in I$, the following conditional probability satisfies

$$P(X_{n+1} = i_{n+1}|X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_{n+1} = i_{n+1}|X_n = i_n) \quad (8.9)$$

then $\{X_n, n \in T\}$ is known as a Markov chain. Equation (8.9) is seen without any post-effect. It means that the future state of the system at $t = n + 1$ is only related to the current state at $t = n$, without any influence from any other earlier state $t \leq n-1$.

For any $n \in T$ and states $i, j \in I$, the following

$$p_{ij}(n) = P(X_{n+1} = j|X_n = i) \quad (8.10)$$

is known as the transition probability of the Markov chain. If the transition probability $p_{ij}(n)$ in this equation does not have anything to do with the index $n$, then $\{X_n, n \in T\}$ is known as a homogeneous Markov chain. For such a Markov chain, the transition probability $p_{ij}(n)$ is often denoted as $p_{ij}$. Because our discussion will be mainly on homogeneous Markov chains, the word "homogeneous" will be omitted. When all the transition probabilities $p_{ij}(n)$ are placed in a matrix, such as $P = [p_{ij}]$, this matrix is referred to as the transition probability matrix of the system's state.

**Proposition 8.6.1** *The entries of the transition probability matrix $P$ satisfy*

(1) $p_{ij} \geq 0$, $i, j \in I$; and
(2) $\sum_{j \in I} p_{ij} = 1$, $i \in I$.

The probability $p_{ij}^{(n)} = P(X_{m+n} = j|X_m = i)$, $i, j \in I$, $n \geq 1$ is known as the $n$th step transition probability of the given Markov chain, and $P^{(n)} = [p_{ij}^{(n)}]$ the $n$th step transition probability matrix.

**Proposition 8.6.2** *The nth step transition probability matrix $P^{(n)}$ satisfies*

(1) $p_{ij}^{(n)} \geq 0$, $i, j \in I$;
(2) $\sum_{j \in I} p_{ij}^{(n)} = 1$, $i \in I$; and
(3) $P^{(n)} = P^n$.

Any Markov chain with grey transition probabilities is known as a grey Markov chain. When studying practical problems, due to a lack of sufficient information, it is often difficult to determine the exact values of the transition probabilities. In such cases, it might be possible to determine the grey ranges $p_{ij}(\otimes)$ of these uncertain probabilities based on available information. When the transition probability matrix is grey, the entries of its whitenization $\tilde{P}(\otimes) = [\tilde{P}_{ij}(\otimes)]$ are generally required to satisfy the following properties:

(1) $\tilde{P}_{ij}(\otimes) \geq 0$, $i, j \in I$; and
(2) $\sum_{j \in I} \tilde{P}_{ij}(\otimes) = 1$, $i \in I$.

**Proposition 8.6.3** *Assume that the initial distribution of a finite-state grey Markov chain is* $P^T(0) = (p_1, p_2, \ldots, p_n)$ *and the transition probability matrix* $P(\otimes) = [P_{ij}(\otimes)]$. *Then, the system's distribution of the next sth state is*

$$P^T(s) = P^T(0)P^s(\otimes) \quad (8.11)$$

That is, when the system's initial distribution and the transition probability matrix are known, one can predict the system's distribution for any future state.

### *8.6.2 Grey State Markov Model*

Assume that a stationary process $X^{(0)}$ satisfies the condition of Markov chains. If we divide it into $n$ states and each of the states $\otimes_i$ is expressed by

$$(i = 1, 2, \ldots, s)(i = 1, 2, \ldots, s)$$

where $a_i, b_i$ are constants and determined according to the states. The steps to establish a grey state Markov model are outlined next.

Step 1: Determine the states for a stationary process $X^{(0)}$ which satisfies the condition of Markov chains

$$(i = 1, 2, \ldots, s) \quad (i = 1, 2, \ldots, s).$$

Step 2: Compute the initial probability distribution. Assume that there are s different states $\otimes_1, \otimes_2, \ldots, \otimes_s$. If state $\otimes_i (i = 1, 2, \ldots, s)$ occurs $M_i$ times in total in M experimentations, then the frequency of $M_i$ can be calculated by

$$f_i = \frac{M_i}{M}(i = 1, 2, \ldots, s).$$

We can use $f_i(i = 1, 2, \ldots, s)$ as an approximation of the initial probability $p_i(i = 1, 2, \ldots, s)$, that is, let $f_i \approx p_i(i = 1, 2, \ldots, s)$.

Step 3: Compute the transition probability. Just like computing the initial probability, we take the frequency as an approximation of the transition probability.

Firstly, we calculate the one step transition frequency of $\otimes_i \rightarrow \otimes_j$ (from state $\otimes_i$ transfer to state $\otimes_j$ through one step) by

$$f_{ij} = f(\otimes_j | \otimes_i)$$

If state $\otimes_i (i = 1, 2, \ldots, s)$ occurs $M_i$ times in total in M experimentations, let $M_{ij}$ be the number of transfers to the state $\otimes_j$ from $M_i$ state $\otimes_i$. Then we have

$$f_{ij} = \frac{M_{ij}}{M_i}$$

Then, if $f_{ij} \approx p_{ij}$, we have the transition probability matrix $P = (p_{ij})_{s \times s}$. Similarly, we can calculate the approximation of m steps transition probability as follows (8.12):

$$p_{ij}(m) = \frac{M_{ij}(m)}{M_i}, (i = 1, 2, \ldots, s), \tag{8.12}$$

where $M_{ij}(m)$ is the number of transfers to the state $\otimes_j$ from $M_i$ state $\otimes_i$ through m steps.

Step 4: Prediction using the transition probability. Assume that the object of prediction is located at state $\otimes_k$, then consider the $k$th row of $P$. If

$$\max_j p_{kj} = p_{kl}$$

then it can be inferred that, at the next time moment, the system will most likely transform from state $\otimes_k$ to state $\otimes_l$. If there are two or more entries in the $k$th row of $P$ that are equal or roughly equal, then the direction of change in the system's state is difficult to determine. In this case, one needs to look at the two-step or $n$-step transition probability matrix $P^{(2)}$ or $P^{(n)}$, where $n \geq 3$.

## 8.7 Combined Grey-Rough Model

Grey systems theory and rough set theory are two mathematical tools developed to address uncertain and incomplete information. To a certain degree they complement each other. They both apply the idea of lowering the preciseness of expression of the available data to gain the extra generality of the expression. In particular, grey systems theory employs the method of grey sequence generations to reduce the accuracy of data expressions, while rough set theory makes use of the idea of data scattering to uncover patterns hidden in the data by ignoring unnecessary details. Neither grey systems theory nor rough set theory requires any prior knowledge, such as probability distribution or degree of membership. On one hand, rough set theory investigates rough, non-intersecting classes and concepts of roughness, with emphasis placed on the indistinguishability of objects. On the other hand, grey systems theory focuses on grey sets with clear extension and unclear intension, with emphasis placed on uncertainties caused by insufficient information. Thus, if rough set theory and grey systems methodology are mixed, their individual weaknesses both in theory and application can be improved so that greater theoretical strength and practical applicability can be achieved (Jian & Liu, 2005).

### *8.7.1 Rough Membership, Grey Membership and Grey Numbers*

Rough set theory can be seen as an expansion of the classic set theory. It makes use of rough membership functions to define rough sets, where each membership function is explained and understood as those of conditional probabilities.

The concepts of rough approximation sets and rough membership functions of the rough set theory are closely related to those of greyness of grey numbers. When either $\mu_X(x) = 0$ or $\mu_X(x) = 1$, the object is assured either to belong or not to belong to set $X$. In such cases, the classification is definite and clear; the involved greyness is the smallest. If $0 < \mu_X(x) < 1$, then object $x$ belongs to set $X$ with the degree of confidence $\mu_X(x)$. In this case, object $x$ projects a kind of grey state of transition between definitely being in set $X$ and definitely not being in $X$. When $\mu_X(x) = 0.5$, the probability of object $x$ to either belong to set $X$ or not to belong to $X$ is 50%. For this situation, the degree of uncertainty is the highest. That is, the degree of greyness is the highest. When the rough membership function $\mu_X(x)$ is near 1 or 0, the uncertainty for object $x$ to belong or not to belong to set $X$ is decreased, and the corresponding degree of greyness should also decrease. The closer to 0.5 the rough membership is, the greater the uncertainty for object $x$ to belong or not to belong to set $X$; the corresponding degree of greyness is also greater in such cases. We categorize all rough membership functions into two groups: upper and lower rough membership functions, where a rough membership function is upper if its values come from the interval [0.5, 1], denoted $\overline{\mu}_X(x)$; the corresponding grey membership function is also referred to as upper and denoted by $\overline{g}_X(x)$. A lower rough membership function is one that takes values from the interval [0, 0.5], denoted $\underline{\mu}_X(x)$. The corresponding grey membership function is referred to as a lower grey membership function, denoted $\underline{g}_X(x)$.

Evidently, upper, lower and general rough membership functions satisfy the following properties:

(1) $\overline{\mu}_X(x) = 1 - \underline{\mu}_X(x)$;
(2) $\mu_{X\cup Y}(x) = \mu_X(x) + \mu_Y(x) - \mu_{X\cap Y}(x)$; and
(3) $\max(0, \mu_X(x) + \mu_Y(x) - 1) \le \mu_{X\cap Y}(x) \le \min(1, \mu_X(x) + \mu_Y(x))$.

Based on the discussion above, we introduce the following definition of grey membership functions using the concept of rough membership functions.

**Definition 8.6.1** Assume that $x$ is an object with its field of discourse $U$. That is, $x \in U$. Let $X$ be a subset of $U$. Then mappings from $U$ to the closed interval [0, 1]:

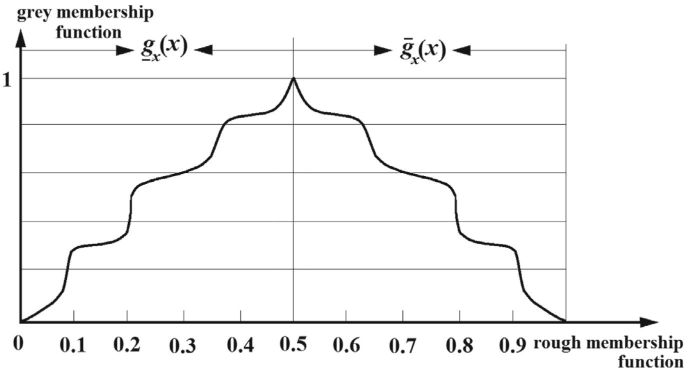



**Fig. 8.2** A conceptual depiction of grey membership functions

$$\overline{\mu}_X : U \to [0.5, 1], \overline{\mu|} \to \overline{g}_X(x) \in [0, 1].$$

and

$$\underline{\mu}_X : U \to [0, 0.5], \mu| \to \underline{g}_X(x) \in [0, 1]$$

are respectively referred to as upper and lower grey membership functions of $X$, where $\overline{\mu}_X \geq \underline{\mu}_X$; $\overline{g}_X(x)$ and $\underline{g}_X(x)$ are respectively referred to as upper and lower grey membership functions of object $x$ with respect to $X$.

The defined concept of grey membership functions based on rough membership functions is depicted in Fig. 8.2

**Definition 8.6.2** Assume that $x \in U$, $X \subseteq U$, the grey number scale of the uncertainty for $x$ to belong to $X$ is $g_c$, the grey number scale of the upper grey membership function $\overline{g}_X(x)$ is $\overline{g}_c$, and the grey number scale of the lower grey membership function $\underline{g}_X(x)$ is $\underline{g}_c$. Then the greyness scales $\overline{g}_c$ and $\underline{g}_c$ of the upper grey number and the lower grey number of the greyness scales $g_c$ of different grey numbers are respectively given as outlined below.

The greyness of white numbers ($g_c = 0$): if $\mu_X(x) = 0$, then $\underline{g}_c = 0$; if $\mu_X(x) = 1$, then $\overline{g}_c = 0$.

For first class grey numbers ($g_c = 1$): if $\mu_X(x) \in (0,0.1]$, then $\underline{g}_c = 1$; if $\mu_X(x) \in [0.9,1)$, then $\overline{g}_c = 1$.

For second class grey numbers ($g_c = 2$): if $\mu_X(x) \in (0.1,0.2]$, then $\underline{g}_c = 2$; if $\mu_X(x) \in [0.8,0.9)$, then $\overline{g}_c = 2$.

For third class grey numbers ($g_c = 3$): if $\mu_X(x) \in (0.2,0.3]$, then $\underline{g}_c = 3$; if $\mu_X(x) \in [0.7,0.8)$, then $\overline{g}_c = 3$.

For fourth class grey numbers ($g_c = 4$): if $\mu_X(x) \in (0.3,0.4]$, then $\underline{g}_c = 4$; if $\mu_X(x) \in [0.6,0.7)$, then $\overline{g}_c = 4$.

For fifth class grey numbers ($g_c = 5$): if $\mu_X(x) \in (0.4,0.5)$, then $\underline{g}_c = 5$; if $\mu_X(x) \in (0.5,0.6)$, then $\overline{g}_c = 5$.

The greyness of black numbers ($g_c > 5$): if $\mu_X(x) = 0.5$, then $\underline{g}_c = \overline{g}_c > 5$.

When $\mu_X(x) \in [0,1]$, $\underline{g}_X(x) = 0$ and $\overline{g}_X(x) = 1$. In this case, there is no uncertain information, so it is referred to as the greyness of white numbers. That is, $g_c = \underline{g}_c = \overline{g}_c = 0$. When $\mu_X(x) = 0.5$, $\underline{g}_X(x) = \overline{g}_X(x) = 1$, the degree of uncertainty for object $x$ to belong or not to belong to set $X$ reaches its maximum, which is referred to as the greyness of black numbers $g_c > 5$. From Definition 8.6.1, it follows that the higher the greyness of a grey number, the less clear the information is; the lower the greyness of a grey number, the clearer the information is.

From Definition 8.6.1, it can be readily obtained that $\overline{\mu}_X(x) = 1 - \underline{\mu}_X(x)$. If we use the greyness of the upper grey number to represent the degree of uncertainty for object $x$ to belong to set $X$, and the greyness of the lower grey number to illustrate the degree of uncertainty for object $x$ not to belong to set $X$, then these two degrees of uncertainty are supplementary.

According to Definition 8.6.2, the scale of the greyness of a grey number is determined by the grey interval to which the maximum rough membership value of the information granularity could belong. Thus, the whitenizations of grey numbers of different degrees of greyness are defined as the maximum possible rough membership value of the grey numbers of corresponding scales. For example, if the possible maximum rough membership value of a certain conditional subset computed out of the available decision-making table is $\mu_X(x) = 0.75$, because $0.75 \in [0.7,0.8)$, then $\mu_X(x) = 0.75$ stands for the white value of such a grey number whose upper greyness is $\overline{g}_c = 3$.

### 8.7.2 *Grey Rough Approximation*

**Definition 8.6.3** Assume that $S = (U, A, V, f)$, $A = C \cup D$, $X \subseteq U$, $P \subseteq C$, and the greyness scale $g_c \leq 5$ of a grey number. Then

$$\underline{apr}_P^{g_c}(X) = \cup\left\{\frac{|I_P(x) \cap X|}{|I_P(x)|} \leq \overline{g}_c\right\} \tag{8.13}$$

and

$$\overline{apr}_P^{g_c}(X) = \cup\left\{\frac{|I_P(x) \cap X|}{|I_P(x)|} > \underline{g}_c\right\} \tag{8.14}$$

are respectively referred to as the $g_c$-lower approximation and $g_c$-upper approximation of $X$ with respect to $I_P$, where the upper rough membership function corresponding to the upper scale $\overline{g}_c$ of grey-number greyness satisfies $\overline{\mu}_X(x) \in (0.5, 1]$, and the lower rough membership function corresponding to the lower scale $\underline{g}_c$ of grey-number greyness satisfies $\underline{\mu}_X(x) \in [0, 0.5)$.

The $g_c$-lower approximation of the set $X \subseteq U$ under the grey-number greyness scale $g_c$ equals the union of all the equivalence classes of $U$ that belong to $X$, with grey-number greyness scales less than or equal to the upper grey-number greyness scale $\overline{g}_c$. The $g_c$-upper approximation is equal to the intersection of all the equivalence classes of $U$ that belong to $X$, with grey-number greyness scales greater than the lower grey-number greyness scale $\underline{g}_c$.

**Definition 8.6.4** The quality of $g_c$-classification is

$$\gamma_P^{g_c}(P, D) = \frac{\left| \cup \left\{ \frac{|X \cap I_P(x)|}{|I_P(x)|} \leq \overline{g}_c \right\} \right|}{|U|} \tag{8.15}$$

The classification quality $\gamma_P^{g_c}(P, D)$ measures the percentage of the knowledge in the field of discourse that can be clearly classified for a given grey-number greyness scale $g_c \leq 5$, in the totality of current knowledge.

For a given grey-number greyness scale $g_c \leq 5$, let approximate reduction $red_P^{g_c}(C, D)$ stand for the set of attributes with the minimum condition that still produces clear classification without containing any extra attributes.

In rough set theory, the classification of the elements located along the boundary regions is not clear. Whether or not an element in such a region can be clearly classified is determined most commonly by the pre-fixed greyness scale. The concept of grey rough approximation so defined is analogous to that of variable precision rough approximation. When the interval grey numbers in which the upper greyness scale $\overline{g}_c$ and the lower greyness scale $\underline{g}_c$ of the grey-number greyness $g_c$ of the grey rough approximation respectively belong to their corresponding white values, the grey rough approximation is consequently transformed into rough approximation under the meaning of variable precision rough sets. Evidently, variable precision rough approximation can be seen as a special case of grey rough approximation. When compared to models of variable precision rough sets of the sets of variable precision, whether or not elements in a relatively rough set $X$ can be correctly classified is mostly determined by the pre-fixed maximum critical confidence threshold parameter $\beta$. This is where classification can be done if smaller than or equal to the upper bound of $\beta$, and indistinguishability appears when this upper bound is surpassed. However, the parameter of the maximum critical confidence threshold $\beta$ in general is difficult to determine beforehand, especially for large data sets. In other words, the parameter of maximum critical confidence threshold $\beta$ generally stands for a grey number. Thus, the concept of interval grey numbers provides a practical quantitative tool which appoints upper and lower endpoints. For cases where we cannot obtain much information about the degree of accuracy of the actual data, this method of representation becomes extremely useful.

**Proposition 8.6.1** *Given the greyness scale $g_c \leq 5$, the following hold true*:

(1) $\overline{apr}_P^{g_c}(X \cup Y) \supseteq \overline{apr}_P^{g_c}(X) \cup \overline{apr}_P^{g_c}(Y)$;
(2) $\underline{apr}_P^{g_c}(X \cap Y) \subseteq \underline{apr}_P^{g_c}(X) \cap \underline{apr}_P^{g_c}(Y)$;
(3) $\underline{apr}_P^{g_c}(X \cup Y) \supseteq \underline{apr}_P^{g_c}(X) \cup \underline{apr}_P^{g_c}(Y)$; and

(4) $\overline{apr}_P^{g_c}(X \cap Y) \subseteq \overline{apr}_P^{g_c}(X) \cap \overline{apr}_P^{g_c}(Y)$.

***Proof***

(1) For any $X \subseteq U$ and $Y \subseteq U$, and given the greyness scale $g_c$, we have

$$\frac{|I_P(x) \cap (X \cup Y)|}{|I_P(x)|} \geq \frac{|I_P(x) \cap X|}{|I_P(x)|}$$

and

$$\frac{|I_P(x) \cap (X \cup Y)|}{|I_P(x)|} \geq \frac{|I_P(x) \cap Y|}{|I_P(x)|}.$$

Therefore, $\overline{apr}_P^{g_c}(X \cup Y) \supseteq \overline{apr}_P^{g_c}(X) \cup \overline{apr}_P^{g_c}(Y)$.

(2) For any $X, Y \subseteq U$, and given the greyness scale $g_c \leq 5$, we have

$$\frac{|I_P(x) \cap (X \cap Y)|}{|I_P(x)|} \leq \frac{|I_P(x) \cap X|}{|I_P(x)|}$$

and

$$\frac{|I_P(x) \cap (X \cap Y)|}{|I_P(x)|} \leq \frac{|I_P(x) \cap Y|}{|I_P(x)|}.$$

Therefore, $\underline{apr}_P^{g_c}(X \cap Y) \subseteq \underline{apr}_P^{g_c}(X) \cap \underline{apr}_P^{g_c}(Y)$. Similarly, we can prove (3) and (4). QED.

**Proposition 8.6.2** $\underline{apr}_P^{g_c}(X) \subseteq \overline{apr}_P^{g_c}(X)$.

***Proof*** Let $x \in \underline{apr}_P^{g_c}(X)$. Because the equivalence relation $I_P$ is reflective, we have $x \in I_P(x)$. From Definition 8.6.2, it follows that $g_c \leq 5$ and that the interval grey number to which the rough membership value corresponding to the upper grey-number greyness scale belongs is greater than the interval grey number to which the rough membership value corresponding to the lower grey-number greyness scale belongs. Hence, we have $x \in \overline{apr}_P^{g_c}(X)$ and consequently $\underline{apr}_P^{g_c}(X) \subseteq \overline{apr}_P^{g_c}(X)$. QED.

### *8.7.3 The Combined Grey Clustering and Rough Set Model*

When employing the expansion dominant rough set model to probabilistic decision-making, one needs to have a multi-criteria decision-making table. However, in many practical applications involving uncertain multi-criteria decision-making, the researcher has to rely on existing data sets to generate his multi-criteria information table instead of being able to obtain their own multi-criteria decision making table. For instance, we can easily collect the financial data of a publically-traded company, such as income per share, net asset per share, net profit, reliability, operating profit, and so on. Based on the collected financial data, we can establish a multi-criteria information table. Given that such a company's style of decision-making is unknown ahead of time, it is difficult to classify it according to whether it presents a high risk, moderate risk, or low risk decision-making style. Thus, it is also difficult, if not impossible, to generate a relevant multi-criteria decision-making table. Therefore, dominant rough set models and expanded dominant rough set models cannot be directly employed to conduct decision-making analysis of these problems. However, the method of grey clustering of grey systems theory generally groups objects into different preference categories by considering attribute preference information and decision-makers' preference behaviors. In particular, the method of grey fixed weight clustering provides an effective way to transform a multi-criteria information table, which is made of preferred attributes of various dimensions, into a multi-criteria decision-making table. For instance, based on the collected financial data of companies, the distributions of the preferred attributes' values of the criteria, and the preferred behaviors of the decision-makers, we can establish possibility functions. On this basis, we can group the companies into different risk classes, such as high risk, moderate risk, and low risk class.

When considering the strengths of the methods of dominant rough sets and grey fixed weight clustering, we can construct a hybrid method combining grey fixed weight clustering and dominant rough sets, where grey fixed weight clustering can be seen as a processing tool used before the method of dominant rough sets is employed. The purpose of doing so is to generalize the dominant rough sets to a method that can be employed to conduct decision-making analysis based on multi-criteria information tables, and to extract the most precise expression of knowledge from the multi-criteria information table.

By following the steps below, one can establish the needed model combining grey fixed weight clustering and dominant rough sets:

(1) Develop a system of knowledge expressions using the values of preferred conditional attributes (criteria);
(2) Determine the ordered decision-making evaluation grey classes $g$ according to the specific circumstances;
(3) Establish the possibility function for the field of each criterion. Let the possibility function of the $k$th subclass of the $j$th criteria be $f_i^k(\cdot)$ $(j = 1, 2, \ldots, m; k = 1, 2, \ldots, g)$;
(4) Determine the clustering weight $\eta_j, j = 1, 2, \ldots, m$, for each criterion;
(5) Based on the observed value $x_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, of object $i$ with respect to criterion $j$, compute the coefficients $\sigma_i^k = \sum_{j=1}^m f_j^k(x_{ij})\eta_j$ of the grey fixed weight clustering $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, g$;
(6) Obtain the clustering coefficient vector

$$\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^g) = \left( \sum_{j=1}^m f_j^1(x_{ij})\eta_j, \sum_{j=1}^m f_j^2(x_{ij})\eta_j, \ldots \sum_{j=1}^m f_j^g(x_{ij})\eta_j \right);$$

(7) Generate the clustering coefficient matrix

$$\sum = (\sigma_i^k) = \begin{bmatrix} \sigma_1^1 & \sigma_1^2 & \ldots & \sigma_1^g \\ \sigma_2^1 & \sigma_2^2 & \ldots & \sigma_2^g \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_n^1 & \sigma_n^2 & \ldots & \sigma_n^g \end{bmatrix};$$

(8) Based on the clustering coefficient matrix Σ, determine the classes to which individual objects belong. If $\max_{1 \le k \le g}\{\sigma_i^k\} = \sigma_i^{k^*}$, then object $i$ belongs to grey class $k^*$;
(9) Establish the decision-making table using preferred conditional attributes and preferred decision-making grey classes; and
(10) Employ the method of dominant rough sets to conduct decision-making analysis.

## 8.8 Practical Applications

***Example 8.7.1*** Let us look at how to choose regional key technologies using a hybrid model combining the methods of grey fixed weight clustering and dominant rough sets. For a specific geographic area, the evaluation criteria system and relevant evaluation values for its key technologies are given in Table 8.5 (Liu & Jian, 2009).

**Table 8.5** The criteria system for evaluating key regional technologies

| Code | Meaning of criterion | Criterion weight | Evaluation values |
|---|---|---|---|
| $a_1$ | Time lag of technology | 0.1 | A: > 10 years; B: 5–10 years; C: 3–5 years; D: < 3 years |
| $a_2$ | Time length technological bottleneck existed | 0.09 | A: > 10 years; B: 5–10 years; C: 3–5 years; D: < 3 years |
| $a_3$ | Ability to create own knowledge right | 0.14 | A: complete own right; B: partial right; C: no right at all |
| $a_4$ | Coverage of technology | 0.09 | A: widely applicable; B: applied in profession; C: special technique |
| $a_5$ | Promotion and lead of technological fields | 0.11 | A: strong; B: relatively strong; C: general; D: weak |
| $a_6$ | Time needed for technology transfer | 0.07 | A: within 1 year; B: 1–3 years; C: 4–5 years; D: > 5 years |
| $a_7$ | Input/output ratio | 0.13 | A: high; B: relatively high; C: normal; D: low |
| $a_8$ | Effect on environmental protection | 0.12 | A: strong; B: relatively strong; C: normal; D: weak |

Based on the evaluations of relevant experts on 11 key technologies candidates, we generate the knowledge expression system as shown in Table 8.6.

In the following graph we present a decision-making analysis for this region's key technologies candidates.

For the evaluation criteria of the region's key technologies, the preference orders are the same as A > B > C > D. Quantify the set of criteria evaluations by letting the set be $V = (A, B, C, D) = (7, 5, 3, 1)$. According to practical needs, we divide each criterion into three grey classes of decision-making: the class of weak need for key technologies (coded with 1), the class of general need for key technologies (coded with 2), and the class of strong need for key technologies (coded with 3). Let us take the possibility function of the class of weak need as the measurement of the low bound, that of the class of general need as the moderate measurement, and that of the class of strong need as the measurement of the upper bound. For details, see Fig. 8.3. Based on decision-making goals and specific distributions of experts' evaluation values, we introduce the possibility functions for each grey class as shown in Table 8.7.

**Table 8.6** The knowledge system on key regional technologies

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ |
|---|---|---|---|---|---|---|---|---|
| $n_1$ | B | B | C | B | D | C | B | A |
| $n_2$ | D | D | B | B | C | B | C | D |
| $n_3$ | D | D | B | B | C | A | D | A |
| $n_4$ | B | C | C | B | B | B | A | C |
| $n_5$ | B | B | B | B | C | B | B | C |
| $n_6$ | D | D | B | B | B | B | B | C |
| $n_7$ | D | D | B | C | D | A | C | C |
| $n_8$ | C | B | B | C | C | C | B | C |
| $n_9$ | B | B | B | B | A | B | A | B |
| $n_{10}$ | C | B | B | B | B | B | B | B |
| $n_{11}$ | B | B | B | B | C | B | C | B |

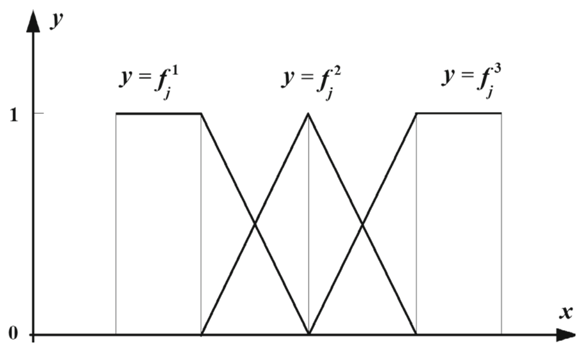



**Fig. 8.3** Possibility functions of the three grey classes

**Table 8.7** Possibility functions for key regional technologies

| Criterion name | Weak need class (1) | Moderate need class (2) | Strong need class (3) |
|---|---|---|---|
| Time lag of technology | $f(x) = \begin{cases} 1 & 1 \le x < 2 \\ 3 - x & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} x - 2 & 2 \le x < 3 \\ 4 - x & 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 1.5 & 3 \le x < 5 \\ 1 & 5 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |
| Time length technological bottleneck existed | $f(x) = \begin{cases} 1 & 1 \le x < 2 \\ 3 - x & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} x - 2 & 2 \le x < 3 \\ 4 - x & 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 1.5 & 3 \le x < 5 \\ 1 & 5 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |
| Ability to create own knowledge right | $f(x) = \begin{cases} 1 & 3 \le x < 4 \\ 5 - x & 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} x - 4 & 4 \le x < 5 \\ 6 - x & 5 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} x - 5 & 5 \le x < 6 \\ 1 & 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |
| Coverage of technology | $f(x) = \begin{cases} 1 & 3 \le x < 4 \\ 5 - x & 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} x - 4 & 4 \le x < 5 \\ 6 - x & 5 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} x - 5 & 5 \le x < 6 \\ 1 & 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |
| Promotion and lead of technological fields | $f(x) = \begin{cases} 1 & 1 \le x < 2 \\ 2 - 0.5x & 2 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 1 & 2 \le x < 4 \\ 3 - 0.5x & 4 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 2 & 4 \le x < 6 \\ 1 & 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |
| Time needed for technology transfer | $f(x) = \begin{cases} 1 & 1 \le x < 2 \\ 2 - 0.5x & 2 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 1 & 2 \le x < 4 \\ 3 - 0.5x & 4 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 2 & 4 \le x < 6 \\ 1 & 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |
| Input/output ratio | $f(x) = \begin{cases} 1 & 1 \le x < 3 \\ 4 - x & 3 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} x - 3 & 3 \le x < 4 \\ 3 - 0.5x & 4 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 2 & 4 \le x < 6 \\ 1 & 6 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |
| Effect on environmental protection | $f(x) = \begin{cases} 1 & 1 \le x < 3 \\ 2.5 - 0.5x & 3 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 1.5 & 3 \le x < 5 \\ 3.5 - 0.5x & 5 \le x < 7 \\ 0 & \text{otherwise} \end{cases}$ | $f(x) = \begin{cases} 0.5x - 25 & 5 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$ |

**Table 8.8** Evaluation decision-making table for key regional technologies

| $U$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $Cl$ |
|---|---|---|---|---|---|---|---|---|---|
| $n_1$ | B | B | C | B | D | C | B | A | 3 |
| $n_2$ | D | D | B | B | C | B | C | D | 1 |
| $n_3$ | D | D | B | B | C | A | D | A | 1 |
| $n_4$ | B | C | C | B | B | B | A | C | 3 |
| $n_5$ | B | B | B | B | C | B | B | C | 2 |
| $n_6$ | D | D | B | B | B | B | B | C | 1 |
| $n_7$ | D | D | B | C | D | A | C | C | 1 |
| $n_8$ | C | B | B | C | C | C | B | C | 2 |
| $n_9$ | B | B | B | B | A | B | A | B | 3 |
| $n_{10}$ | C | B | B | B | B | B | B | B | 2 |
| $n_{11}$ | B | B | B | B | C | B | C | B | 2 |

From formula $\sigma_i^k = \sum_{j=1}^{m} f_j^k(x_{ij}) \cdot \eta_j$, we can compute the clustering coefficient for each grey class of each key technology. Based on such coefficients we can establish the evaluation decision-making (Table 8.8) for choosing key technologies for the region.

Because the values of all the conditional attributes have the preference order A > B > C > D, these attributes contain preference information. Based on the decision-making attributes, the comprehensive evaluation can be divided into three preference ordered classes: $Cl_1 = \{1\}$, $Cl_2 = \{2\}$, $Cl_3 = \{3\}$. Based on this result, we divide the field of discourse and obtain the following unions of the decision-making classes:

$Cl_1^{\leq} = Cl_1$, with the comprehensive evaluation 1 (the need for key technologies is weak);

$Cl_2^{\leq} = Cl_1 \cup Cl_2$, with the comprehensive evaluation $\leq 2$ (the need for key technologies is at most moderate);

$Cl_2^{\geq} = Cl_2 \cup Cl_3$, with the comprehensive evaluation $\geq 2$ (the need for key technologies is at least moderate);

$Cl_3^{\leq} = Cl_1 \cup Cl_2 \cup Cl_3$, with the comprehensive evaluation $\leq 3$ (the need for key technologies is at most strong); and

$Cl_3^{\geq} = Cl_3$, with the comprehensive evaluation 3 (the need for key technologies is strong).

A reduction found by using the method of dominant rough sets is $\{a_2, a_7\}$. The sets $D_{\geq}$ and $D_{\leq}$ of the least amounts of preference rules generated from this reduction are respectively given in Tables 8.9 and 8.10

Based on the set $D_{\geq}$ of preference decision-making rules generated by employing our hybrid model that combines grey fixed weight clustering and dominant rough sets, all the 11 key technologies considered are correctly classified. That is, the quality of classification is 100%. Based on the set $D_{\leq}$ of preference decision-making rules, a total of 7 key technologies are classified correctly so that the quality of classification is 63.6%.

**Table 8.9** Set $D_{\geq}$ of preference rules

| Rule | Confidence (%) | Support number |
|---|---|---|
| If the length of time for technology bottleneck to exist ≥ C and input/output ratio = A, then the urgency for needing key technologies = 3 (strong) | 100 | 2 |
| If the length of time for technology bottleneck to exist ≥ B and input/output ratio ≥ C, then the urgency for needing key technologies ≥ 2 (moderate) | 100 | 5 |
| If the length of time for technology bottleneck to exist = D, then the urgency for needing key technologies = 1 (weak) | 100 | 4 |

**Table 8.10** Set $D_{\leq}$ of preference rules

| Rule | Confidence (%) | Support number |
|---|---|---|
| If the length of time for technology bottleneck to exist ≤ C and input/output ratio = A, then the need for key technologies = 3 (strong) | 100 | 2 |
| If the length of time for technology bottleneck to exist ≤ B and input/output ratio ≤ C, then the need for key technologies ≤ 2 (moderate) | 100 | 1 |
| If the length of time for technology bottleneck to exist = D, then the need for key technologies = 1 (weak) | 100 | 4 |

## References


Aydemir, E., Bedir, F., & Ozdemir, G. (2015). Degree of greyness approach for an EPQ model with imperfect items in copper wire industry. *The Journal of Grey System, 27*(2), 13–26.

Bahrami, S., Hooshmand, R. A., & Parastegari, M. (2014). Short term electric load forecasting by wavelet transform and grey model improved by PSO (particle swarm optimization) algorithm. *Energy, 72*, 434–442.

Chen, H. W., & Chang, N. B. (2000). Prediction analysis of solid waste generation based on grey fuzzy dynamic modelling. *Resources Conservation and Recycling, 29*(1–2), 1–18.

Chen, W., Tang, D.Y., & Zhang, J. (2015). Application of three-point-method-gray GM(1,1) combination model in settlement prediction for a railway line. *Science of Surveying and Mapping, 40*(7), 37–40 (In Chinese).

Guo, X., Liu, S. F., & Wu, L. (2013). Modeling and algorithm of prediction about pollutants' emission reduction bycombining gray theory and Markov chain. *Research on Computer Application, 30*(12), 3670–3673.

Han, X. D., & He, Z. L. (1997). Combination model of GM(1,1) and linear regression and its application to deformation prediction. *Journal of Huainan Mining Institute, 17*(4), 51–54.

He, W. M., Shui, H. W., Song, X. Q., et al. (2015). Segmented calibration of transducer based on grey neural network. *China Mechanical Engineering, 26*(15), 2056–2061. In Chinese.

Huang, K., & Su, C. (2020). Failure prediction based on combined model of grey neural network. *Systems Engineering & Electronics, 42*(1), 238–244 (In Chinese).

Jian, L. R., & Liu, S. F. (2005). Extended rough set method for preference multi-attribute decision analysis. *Journal of Nanjing University of Aeronautics and Astronautics, 37*(4), 270–275.

Jian, L. R., Liu, S. F., & Lin, Y. (2011). *Hybrid rough sets and applications in uncertain decision-making*. Taylor & Francis Group.

Jiang, W. (2012). An intelligent diagnosis method based on grey rough set theory for wind turbine driving chain. *Power System and Clean Energy, 28*(12), 79–83. In Chinese.

Karimi, T., & Hojati, A. (2020). Designing a medical rule model system by using rough-grey modeling. *Grey Systems: Theory and Application, 10*(4), 513–527.

Khuman, AS; Yang, YJ and John, R.(2019). The quantification of subjectivity: The R-fuzzy grey analysis framework. *Expert Systems with Applications, 136*, 201–216.

Kose, E., & Forrest, J.Y.-L. (2015). N-person grey game. *Kybernetes, 44*(2), 271–282.

Li, Q-X., & Liu, S.-F. (2008). The foundation of the grey matrix and the grey input–output analysis. *Applied Mathematical Modelling, 32*, 267–291.

Li, Q. X., Liu, S. F., & Lin, Y. (2012). Grey enterprise input-output analysis. *Journal of Computational and Applied Mathematics, 236*(7), 1862–1875.

Lin, C. B., Su, S. F., & Hsu, Y. T. (2001). High-precision forecast using grey models. *International Journal of Systems Science, 32*(5), 609–619.

Liu, Q. Y., Zhong, Z. D., & Ai, B. (2010). Research on frequency planning of GSM-R system based on grey cluster theory with rough set. *Journal of the China Railway Society, 32*(5), 53–58

Liu, S. F., & Jian, L. R. (2009). Evaluation and selection of regional key technologies for international cooperation. *Science & Technology Progress and Policy., 26*(10), 126–129. In Chinese.

Liu, S. F., & Zhu, Y. D. (1996). Grey-econometrics combined model. *Journal of Grey System, 8*(1), 103–110.

Liu, S. F., Dang, Y. G., & Lin, Y. (2004a). A new method for Venturous capital pricing. *Kybernetes, 33*(2), 288–294.

Liu, S. F., Li, B. J., & Dang, Y. G. (2004b). The G-C.D model and technical advance. *Kybernetes, 33*(2), 303–309.

Liu,S. F., & Lin, Y. (2006). Grey Information-Theory and Practical Applications. Springer-Verlag. London LtD.

Liu, S. F., Lin, Y., et al. (2006). On measures of information content of grey numbers. *Kybernetes, 35*(5), 899–904.

Luo, R. C., Chen, T. M., & Su, K. L. (2001). Target tracking using hierarchical grey-fuzzy motion decision-making method. *IEEE Transactions on Systems Man and Cybernetics Part A-Systems and Humans, 31*(3), 179–186.

Meng, X. F., Wang, C. M., & He, B. X. (2012). On life prediction of gun tube based on combined model of grey linear regression. *Journal of Nanjing University of Science and Technology, 36*(4), 635–638.

Mi, G. S., Yang, R. X., & Liang, L. (2014). Research on the method of diagnosis of complex fault in track circuit based on combined model. *Journal of the China Railway Society, 36*(10), 65–69.

Muthukumar, B., Murugan, S., & Bharathi, B. (2023). Localization and classification of gender focus in epilepsy patients based on Rough Set. *Malaysian Journal of Computer Science, 36*(1), 40–52.

Nain, S. S., Garg, D., & Kumar, S. (2018). Evaluation and analysis of cutting speed, wire wear ratio, and dimensional deviation of wire electric discharge machining of super alloy Udimet-L605 using support vector machine and grey relational analysis. *Advances in Manufacturing, 6*(2), 225–246.

Oztaysi, B. (2014). A decision model for information technology selection using AHP integrated TOPSIS-Grey: the case of content management systems. *Knowledge-Based Systems, 70*(SI), 44–54.

Rajesh, R. (2022). Sustainability performance predictions in supply chains: Grey and rough set theoretical approaches. *Annals of Operations Research, 310*(1), 171–200

Samet, H., & Mojallal, A. (2014). Enhancement of electric arc furnace reactive power compensation using Grey-Markov prediction method. *IET Generation Transmission & Distribution, 8*(9), 1626–1636.

Tang, K., Zhou, N. N., & Fan, X. Y. (2012). Analysis on the factors influencing the gas well productivity of S2 gas pool in Permian of Zizhou gas field. *Computing Techniques for Geophysical and Geochemical Exploration, 34*(6), 723–728.

Tian, K. (2022). A Method for High Precision Prediction of Radial Velocity of Projectile Based on Combined Model. *Journal of Ballistics, 34*(3), 49–57. In Chinese.
Verma, A., Sarangi, S., & Kolekar, M. H. (2014). Stator winding fault prediction of induction motors using multiscale entropy and grey fuzzy optimization methods. *Computers & Electrical Engineering, 40*(7), 2246–2258.
Vyavahare, S., Teraiya, S., & Kumar, S. (2023). FDM manufactured auxetic structures: An investigation of mechanical properties using machine learning techniques. *International Journal of Solids and Structures, 265–266*, 112126.
Wang, J. F., & Liu, S. F. (2009). On measuring and sorting for the efficiency index of interval DEA based on interval position and grey incidence model. *Systems Engineering and Electronics.* 31(6), 2146–2150. In Chinese.
Wang, X. P., & Huang, S. X. (2016). Application of GM-BP combined model to deformation forecasting of foundation pit. *Engineering of Surveying and Mapping, 25*(3), 61–63, 67.
Wu, Z. R., Xu, B., Gu, C. S., et al. (2012). On comprehensive evaluation methods of the service condition of the dam. *China Science: Technological Sciences, 42*(11), 1243–1254. In Chinese.
Xu, C., Sun, X. Y., & Wang, H. L. (2010). Application grey-economitrics model in traffic volume prediction. *Highway Engineering, 35*(5), 34–38. In Chinese.
Yin, J. J., Liang, X. J., Xiao, C. L., et al. (2012). Application of matter-element extension method based on grey clustering theory in the ground water quality evaluation—Example with Taonan City. *Water Saving Irrigation, 6*, 52–55. In Chinese.
Yuan, P. W., Song, S. X., & Dong, X. Q. (2014). Study on fire prediction based on combined model of optimum grey neural network. *China Safety Science and Technology, 10*(3), 119–124. In Chinese.
Zhang, S. H., & Chen, M. Y. (2002). The problem of grey nonlinear programming and its genetic algorithm method. *Systems Engineering—Theory and Practice, 22*(7), 128–130.
Zheng, D. J., Gu, C. S., & Wu, Z. R. (2005). On time varying prediction model of the deformation of slope with multi-factors. *Journal of Rock Mechanics and Engineering, 24*(17), 3180–3184. In Chinese.

# Chapter 9
# Techniques for Grey Systems Forecasting

## 9.1 Introduction

No matter what needs to be done, one should always get familiar with the situation, think through the details, make educated predictions, and lay out a detailed plan before he could potentially arrive at his desired successful conclusions. For matters as great as international affairs, national events and citizens' lives, the development of regional or business entities, and for matters as small as daily work or living arrangements, scientifically sound predictions are needed everywhere (Deng, 1990; Liu & Guo, 1991).

Prediction is about foretelling the possible course of development of societal events, political matters, economic ups and downs, and so on, using scientific methods and techniques based on attainable historical and present data so that appropriate actions can be planned and carried out. In short, prediction is about making scientific inferences regarding the evolution of materials and events ahead of time. General prediction includes not only static inference about unknown matters based on what is known within a specific time frame, but also dynamic inference about the future based on history and the present state of affairs of a certain matter. A specific prediction is a dynamic forecast within which a scientific inference about the future evolution of a certain event is given.

In 2010, Zeng, et al. proposed a prediction model of interval grey number based on DGM (1, 1) ) (Zeng et al., 2010). In 2014, Xie, et al. proposed a grey number sequence forecasting model of interval analysis (Xie et al., 2014). Then, they studied interval grey number sequence prediction by non-homogenous exponential discrete grey forecasting model (Xie et al., 2015). Luo et al. proposed a grey interval forecasting method in 2016. Firstly, two non-equidistance GM(1,1) models are built for upper and lower sequences respectively, and the development boundary of the system are described by the upper and lower envelope curves. Then, the computing method for the interval and basic forecasting values of the original sequence are proposed, and the algorithm is constructed. Finally, the grey exponent law and timeliness of interval



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_9

forecasting model are studied. The numerical experiment shows that the value of the development coefficient is not the only factor influencing the timeliness. The application example shows that the forecasting accuracy can be effectively enhanced (Luo et al., 2016). In the same year, Liu et al. proposed an improved interval forecasting model based on fuzzy multi-objective programming combined with discrete grey model theory (Liu et al., 2016). Chen et al. proposed a time series interval forecast method using GM(1,1) and NGBM(1, 1) models in 2019 (Chen et al., 2019). In 2020, Xiong et al. predicted the fog and haze pollution using amulti-variable grey model based on interval number sequences (Xiong et al., 2020). Zeng et al. proposed a multi-variable grey model based on dynamic background algorithm for forecasting the interval sequence (Zeng et al., 2020). Zeng et al. proposed a novel grey interval forecasting model to predict sulfur dioxide concentration in Beijing. They extended the original modeling data to the area sequence and coordinate sequence with equal information, which is used to deduce the boundary equation of the original sequence (Zeng et al., 2021). Hu applied grey prediction models and neural networks to develop interval models for tourism demand forecasting (Hu, 2021). Please refer to Wang's review article for details if the readers interested in interval forecasting (Wang et al., 2024).

## 9.2 Criteria for Model Verification

Grey prediction makes scientific, quantitative forecasts about the future states of systems based on understandings of unascertained characteristics of such systems. It makes use of sequence operators on the original data sequences in order to generate, treat, and excavate the hidden laws of systems evolution, so that grey systems models can be established to predict future outcomes. All the methods of the grey systems theory studied so far can be employed to make predictions. For a given problem, the appropriate prediction model is chosen by making use of the conclusions of a sufficiently and carefully done qualitative analysis. Also, the choice of models should vary along with changing conditions. Each model chosen has to be tested through many different methods in order to decide its appropriateness and effectiveness. Only the models that pass various tests can be meaningfully employed to make predictions (Liu, et al., 2020; Liu, 2024; Liu, et al., 2022).

**Definition 9.2.1** Let $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$ be a sequence of raw data, $\hat{X}^{(0)} = (\hat{x}^{(0)}(1), \hat{x}^{(0)}(2), \ldots, \hat{x}^{(0)}(n))$ the simulated data out of a chosen prediction model, $\varepsilon^{(0)} = (\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n)) = \left(x^{(0)}(1) - \hat{x}^{(0)}(1), x^{(0)}(2) - \hat{x}^{(0)}(2), \ldots, x^{(0)}(n) - \hat{x}^{(0)}(n)\right)$ the error sequence, and

$$\Delta = \left(\left|\frac{\varepsilon(1)}{x^{(0)}(1)}\right|, \left|\frac{\varepsilon(2)}{x^{(0)}(2)}\right|, \ldots, \left|\frac{\varepsilon(n)}{x^{(0)}(n)}\right|\right) = \{\Delta_k\}_1^n$$

the relative error sequence. Then:

(1) For $k \le n$, $\Delta_k = \left|\frac{\varepsilon(k)}{x^{(0)}(k)}\right|$ is known as relative error of the simulation at point $k$, and $\overline{\Delta} = \frac{1}{n}\sum_{k=1}^{n}\Delta_k$ the average relative error;
(2) $1 - \overline{\Delta}$ is known as the average relative accuracy, and $1 - \Delta_k$ the simulation accuracy at point $k$, $k = 1, 2, \ldots, n$; and
(3) For a given $\alpha$, when $\overline{\Delta} < \alpha$ and $\Delta_n < \alpha$ hold true, the prediction model is said to be error-satisfactory.

**Definition 9.2.2** Let $\varepsilon$ stand for the absolute grey relational degree between the raw data $X^{(0)}$ and the simulated values $\hat{X}^{(0)}$. If for a given $\varepsilon_0 > 0$ the absolute grey relational degree $\varepsilon$ satisfies $\varepsilon > \varepsilon_0$, then the simulation model is said to be grey relational satisfactory.

**Definition 9.2.3** Assume that the sequences $X^{(0)}$, $\hat{X}^{(0)}$, and $\varepsilon^{(0)}$ are the same as above, and consider the relevant means and variances.

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n} x^{(0)}(k),\ \ S_1^2 = \frac{1}{n}\sum_{k=1}^{n}(x^{(0)}(k) - \bar{x})^2$$

and

$$\bar{\varepsilon} = \frac{1}{n}\sum_{k=1}^{n}\varepsilon(k),\ \ S_2^2 = \frac{1}{n}\sum_{k=1}^{n}(\varepsilon(k) - \bar{\varepsilon})^2.$$

(1) If for a given $C_0 > 0$, the ratio of root-mean-square deviation (RMSD) is $C = \frac{S_2}{S_1}$ < $C_0$, then the model is said to be RMSD ratio satisfactory.
(2) If $p = P(|\,\varepsilon(k) - \bar{\varepsilon})\,| < 0.6745 S_1)$ is seen as a small error probability and for a given $p_0 > 0$, when $p > p_0$, then the model is said to be small-error probability satisfactory.

The discussion above shows three different ways to test a chosen model. Each of them is based on observations of the error to determine the accuracy of the model. For both the mean relative error $\overline{\Delta}$ and the simulation error, the smaller they are, the better. With regards to the grey relational degree $\varepsilon$, the greater it is the better. As for the RMSD ratio $C$, the smaller the value is, the better. This is because a small $C$ indicates that $S_2$ is relatively small, while $S_1$ is relatively large. This means that the error variance is small while the variance of the original data is large, so that the errors are relatively more concentrated with little fluctuation compared to the original data. Therefore, for better simulation results, the smaller $S_2$ is when compared to $S_1$, the better. With regards to small error probability $p$, as soon as a set of $\alpha$, $\varepsilon_0$, $C_0$, and $p_0$ values are chosen, a scale of accuracy for testing models is determined. The most commonly used scales of accuracy for testing models are listed in Table 9.1.

In most applications published so far in the area of grey systems, the most commonly used is the criterion of relative errors.

**Table 9.1** Commonly used scales of accuracy for model testing

| Accuracy scale | Threshold | | | |
|---|---|---|---|---|
| | Relative error $\alpha$ | Grey relational degree $\varepsilon_0$ | RMSD $C_0$ | Small error probability $p_0$ |
| 1st level | 0.01 | 0.90 | 0.35 | 0.95 |
| 2nd level | 0.05 | 0.80 | 0.50 | 0.80 |
| 3rd level | 0.10 | 0.70 | 0.65 | 0.70 |
| 4th level | 0.20 | 0.60 | 0.80 | 0.60 |

## 9.3 Interval Forecasting

If a given sequence of raw data is chaotic and it is difficult for any model to pass the accuracy test, the researcher will then have trouble producing accurate quantitative predictions. In this case, one can consider providing a range for future values to fall within.

**Definition 9.3.1** Let $X(t)$ be a zigzagged line. If there are smooth and continuous curves $f_u(t)$ and $f_s(t)$, satisfying that for any $t$, $f_u(t) < X(t) < f_s(t)$, then $f_u(t)$ is known as the lower bound function of $X(t)$, $f_s(t)$ the upper bound function, and $S = \{(t, X(t)) | X(t) \in [f_u(t), f_s(t)]\}$ the value domain of $X(t)$. If the upper and lower bound of $X(t)$ are the same kind of function, then $S$ is known as a uniform domain. When $S$ is a uniform band with exponential functions as its upper and lower bounds $f_u(t)$ and $f_s(t)$, then S is known as a uniform exponential domain. If a uniform band $S$ has linear upper and lower bound functions $f_u(t)$ and $f_s(t)$, then S is known as a uniform linear domain or a straight domain for short. If for $t_1 < t_2, f_s(t_1) - f_u(t_1) < f_s(t_2) - f_u(t_2)$ always holds true, then S is known as a trumpet-like domain.

***Example 9.3.1*** Let $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$ be a sequence of raw data, and its accumulation generation be $X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \ldots, x^{(1)}(n))$. Define

$$\sigma_{\max} = \max_{1 \le k \le n} \{x^{(0)}(k)\}, \sigma_{\min} = \min_{1 \le k \le n} \{x^{(0)}(k)\}$$

and respectively take the upper and lower bound functions $f_u(n+t)$ and $f_s(n+t)$ of $X^{(1)}$ as follows:

$$f_u(n+t) = x^{(1)}(n) + t\sigma_{\min},\ f_s(n+t) = x^{(1)}(n) + t\sigma_{\max}.$$

That is, both the upper and lower bound functions of a proportional band are increasing straight lines of time with slopes $\sigma_{\min}$ and $\sigma_{\max}$, respectively.

Then $S = \{(t, X(t)) | t > n, X(t) \in [f_u(t), f_s(t)]\}$ is known as the proportional domain (see Fig. 9.1).

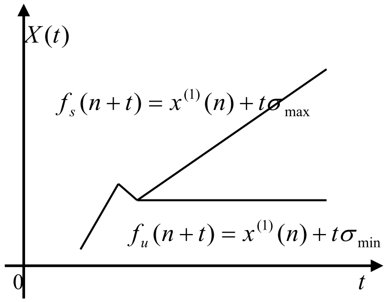



**Fig. 9.1** A trumpet-like domain

***Example 9.3.2*** For a sequence $X^{(0)}$ of raw data, let $X_u^{(0)}$ be the sequence corresponding to the curve that connects all the low points of $X^{(0)}$, and $X_s^{(0)}$ the sequence corresponding to the curve of all the upper points of $X^{(0)}$. Assume that

$$\hat{x}_u^{(1)}(k+1) = \left(x_u^{(0)}(1) - \frac{b_u}{a_u}\right)\exp(-a_u k) + \frac{b_u}{a_u}$$

and

$$\hat{x}_s^{(1)}(k+1) = \left(x_s^{(0)}(1) - \frac{b_s}{a_s}\right)\exp(-a_s k) + \frac{b_s}{a_s}$$

are respectively the GM(1,1) time response sequences of $X_u^{(0)}$ and $X_s^{(0)}$. Then

$$S = \{(t, X(t)) \big| X(t) \in [\hat{X}_u^{(1)}(t), \hat{X}_s^{(1)}(t)]\}$$

is known as a wrapping domain (see Fig. 9.2).

***Example 9.3.3*** For a given sequence $X^{(0)}$ of raw data, let us take $m$ different subsequences to establish $m$ GM(1, 1) models with the corresponding parameters $\hat{a}_i = [a_i, b_i]^T$; $i = 1, 2, \ldots, m$. Let

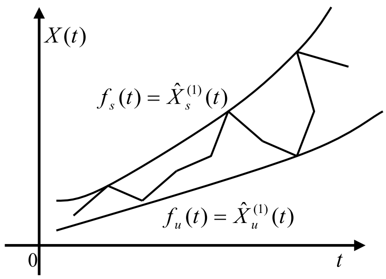



**Fig. 9.2** A wrapping domain

$-a_{\max} = \max_{1 \le i \le m}\{-a_i\}$, $-a_{\min} = \min_{1 \le i \le m}\{-a_i\}$, and

$$\hat{x}_u^{(1)}(k+1) = \left(x_u^{(0)}(1) - \frac{b_{\min}}{a_{\min}}\right)\exp(-a_{\min}k) + \frac{b_{\min}}{a_{\min}}$$

$$\hat{x}_s^{(1)}(k+1) = \left(x_s^{(0)}(1) - \frac{b_{\max}}{a_{\max}}\right)\exp(-a_{\max}k) + \frac{b_{\max}}{a_{\max}}$$

Then $S = \{(t, X(t)) \big| X(t) \in [\hat{X}_u^{(1)}(t), \hat{X}_s^{(1)}(t)]\}$ is known as a development domain. The wrapping domain and development domain are exponential domains.

**Definition 9.3.2** For a sequence $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$ of raw data, let $f_u(t)$ and $f_s(t)$ be a upper and a lower bound function of the accumulation sequence $X^{(1)}$ of $X^{(0)}$. For any $k > 0$,

$$\hat{x}^{(0)}(n+k) = \frac{1}{2}[f_u(n+k) + f_s(n+k)]$$

is known as basic prediction value, and $\hat{x}_u^{(0)}(n+k) = f_u(n+k)$ and $\hat{x}_s^{(0)}(n+k) = f_s(n+k)$, respectively, the lowest and highest predicted values.

***Example 9.3.4*** The data (in tens of thousands) for electric car sales in a certain city are given as follows:

$$X^{(0)} = (x^{(0)}(1), x^{(0)}(2), x^{(0)}(3), x^{(0)}(4), x^{(0)}(5), x^{(0)}(6))$$
$$= (5.0810, 4.6110, 5.1177, 9.3775, 11.0574, 11.3524)$$

where $x^{(0)}(1) = 5.0810$ is the annual sales for the year of 2018, …, and $x^{(0)}(6) = 11.3524$ for the year of 2023. Try to make a prediction using development domain.

**Solution**: Take the following sub-sequences

$$X_1^{(0)} = (x^{(0)}(1), x^{(0)}(2), x^{(0)}(3), x^{(0)}(4), x^{(0)}(5), x^{(0)}(6))$$
$$= (5.0810, 4.6110, 5.1177, 9.3775, 11.0574, 11.3524)$$
$$X_2^{(0)} = (x^{(0)}(1), x^{(0)}(2), x^{(0)}(3), x^{(0)}(4), x^{(0)}(5))$$
$$= (5.0810, 4.6110, 5.1177, 9.3775, 11.0574)$$
$$X_3^{(0)} = (x^{(0)}(2), x^{(0)}(3), x^{(0)}(4), x^{(0)}(5), x^{(0)}(6))$$
$$= (4.6110, 5.1177, 9.3775, 11.0574, 11.3524)$$
$$X_4^{(0)} = (x^{(0)}(3), x^{(0)}(4), x^{(0)}(5), x^{(0)}(6))$$
$$= (5.1177, 9.3775, 11.0574, 11.3524)$$

Based on each of these sub-sequences, let us establish the corresponding the models of EGM(1, 1):

$$\frac{dx^{(1)}}{dt} + a_i x^{(1)} = b_i,\ i = 1, 2, 3, 4$$

Their individual parameters $\hat{a}_i = [a_i, b_i]^T, i = 1, 2, 3, 4$, are given below:

$$\hat{a}_1 = [a_1, b_1]^T = [-0.2202, 3.4689]^T,\ \hat{a}_2 = [a_2, b_2]^T = [-0.3147, 2.1237]^T.$$

$$\hat{a}_3 = [a_3, b_3]^T = [-0.2013, 5.0961]^T,\ \hat{a}_4 = [a_4, b_4]^T = [-0.0911, 8.7410]^T.$$

Because

$$-a_{\min} = \min_{1 \le i \le 4}\{-a_i\} = \min\{0.2202, 0.3147, 0.2013, 0.0911\} = 0.0911 = -a_4$$

$$-a_{\max} = \max_{1 \le i \le 4}\{-a_i\} = \max\{0.2202, 0.3147, 0.2013, 0.0911\} = 0.3147 = -a_2$$

the upper bound time response sequence of the development domain is

$$\begin{cases} \hat{x}_s^{(1)}(k+1) = \left(x^{(0)}(1) - \dfrac{b_2}{a_2}\right)e^{-a_2 k} + \dfrac{b_2}{a_2} = 11.8293e^{0.3147k} - 6.7483 \\ \hat{x}_s^{(0)}(k+1) = \hat{x}_s^{(1)}(k+1) - \hat{x}_s^{(1)}(k) \end{cases}$$

That is, $\hat{x}_s^{(0)}(k+1) = 11.8293e^{0.3147k} - 11.8293e^{0.3147k-0.3147} = 3.1938e^{0.3147k}$. Thus, the highest predicted values are $\hat{x}_s^{(0)}(7) = 21.1029$, $\hat{x}_s^{(0)}(8) = 28.9078$, and $\hat{x}_s^{(0)}(9) = 39.5993$. Because the starting value of $X_4^{(0)}$ is $x^{(0)}(3)$, the lower bound time response sequence of the development domain is

$$\begin{cases} \hat{x}_u^{(1)}(k+3) = \left(x^{(0)}(3) - \dfrac{b_4}{a_4}\right)e^{-a_4 k} + \dfrac{b_4}{a_4} = 101.0672e^{0.0911k} - 95.9495 \\ \hat{x}_u^{(0)}(k+3) = \hat{x}_u^{(1)}(k+3) - \hat{x}_u^{(0)}(k+2) \end{cases}$$

That is, $\hat{x}_u^{(0)}(k+3) = 101.0672e^{0.0911k} - 101.0672e^{0.0911k-0.0911} = 8.8003e^{0.0911k}$. Therefore, we obtain the lowest predicted values: $\hat{x}_u^{(0)}(7) = 12.6694$, $\hat{x}_u^{(0)}(8) = 13.8777$, and $\hat{x}_u^{(0)}(9) = 15.2014$. From the highest and lowest predicted values, we obtain the basic prediction values:

$$\hat{x}^{(0)}(7) = \frac{1}{2}[\hat{x}_s^{(0)}(7) + \hat{x}_u^{(0)}(7)] = 16.8862$$

$$\hat{x}^{(0)}(8) = \frac{1}{2}[\hat{x}_s^{(0)}(8) + \hat{x}_u^{(0)}(8)] = 21.3928$$

$$\hat{x}^{(0)}(9) = \frac{1}{2}[\hat{x}_s^{(0)}(9) + \hat{x}_u^{(0)}(9)] = 27.4004$$

Based on the qualitative analysis of the estimated amount of electric car ownership in the given city and the improvement in public transportation systems, we conclude that the basic predicted values are the most reliable.

## 9.4 Grey Distortion Forecasting

The basic idea of grey distortion prediction is essentially the prediction of abnormal values. The kinds of values that are considered abnormal are commonly determined based on individuals' experiences. The objective of grey distortion predictions is to provide the time moments of the forthcoming abnormal values so that relevant parties can prepare for the worst ahead of time.

**Definition 9.4.1** Let $X = (x(1), x(2), \ldots, x(n))$ be a sequence of raw data. Then

(1) For a given upper abnormal value $\xi$, the sub-sequence of $X$

$$X_\xi = (x[q(1)], x[q(2)], \ldots, x[q(m)]) = \{x[q(i)] | x[q(i)] \geq \xi; i = 1, 2, \ldots, m\}$$

is known as the upper distortion sequence.

(2) For a given lower abnormal value $\zeta$, the sub-sequence

$$X_\zeta = (x[q(1)], x[q(2)], \ldots, x[q(l)]) = \{x[q(i)] | x[q(i)] \geq \zeta; i = 1, 2, \ldots, l\}$$

is known as the lower distortion sequence. Together, these upper and lower distortion sequences are referred to as distortion sequences. Because the idea behind the discussion of distortion sequences is the same, in the following discussion we will not distinguish between upper and lower distortion sequences.

**Definition 9.4.2** Assume that $X = (x(1), x(2), \ldots, x(n))$ is a sequence of raw data. The following sub-sequence of $X$

$$X_\xi = (x[q(1)], x[q(2)], \ldots, x[q(m)]) \subset X$$

is a distortion sequence. Then,

$$Q^{(0)} = (q(1), q(2), \ldots, q(m))$$

will be referred to as the distortion date sequence. Distortion prediction is about finding patterns, if any, through the study of distortion date sequences to predict future dates of occurrences of distortion. In grey system theory, each distortion prediction is realized through establishing GM(1,1) models for relevant distortion date sequences.

**Definition 9.4.3** If $Q^{(0)} = (q(1), q(2), \ldots, q(m))$ is a distortion date sequence, the following

$$Q^{(1)} = (q(1)^{(1)}, q(2)^{(1)}, \ldots, q(m)^{(1)})$$

is the 1-AGO sequence of the distortion date sequence $Q^{(0)}$,$Z^{(1)}$ is the adjacent neighbor mean generated sequence of $Q^{(1)}$, and

$$q(k) + az^{(1)}(k) = b$$

is referred to as a distortion model of GM(1, 1). For the available sequence $X = (x(1), x(2), \ldots, x(n))$ of raw data, if $n$ stands for the present and the last entry $q(m)(\leq n)$ in the corresponding distortion date sequence $Q^{(0)}$ represents when the last abnormal value occurred, then the predicted value $\hat{q}(m+1)$ represents the next forthcoming abnormal value and for any $k > 0, \hat{q}(m+k)$ stands for the predicted date for the $k$th abnormal value to occur in the future.

***Example 9.4.1*** The following sequence gives the annual average precipitations (in mm) of a certain region for 17 years, where x(1), x(2), …, x(17) are respectively the data for the years of 2006,2007, …, 2023:

$$\begin{aligned} X &= (x(1), x(2), x(3), x(4), x(5), x(6), x(7), x(8), x(9), x(10) \\ &\quad x(11), x(12), x(13, x(14), x(15), x(16), x(17)) \\ &= (390.6, 412.0, 320.0, 559.2, 380.8, 542.4, 553.0, 310.0, 561.0, 300.0 \\ &\quad 632.0, 540.0, 406.2, 313.8, 576.0, \ 586.6, 318.5) \end{aligned}$$

Take $\xi = 320$ mm as a lower abnormal (drought) value. Carry out a drought prediction for this specific region.

**Solution:** If $\xi = 320$, we obtain the following lower distortion sequence

$$X_\xi = (x(3), x(8), x(10), x(14), x(17)) = (320.0, 310.0, 300.0, 313.8, 318.5)$$

with the corresponding distortion date sequence

$$Q^{(0)} = (q(1), q(2), q(3), q(4), q(5)) = (3, 8, 10, 14, 17)$$

and it's 1-AGO sequence

$$Q^{(1)} = (3, 11, 21, 35, 52)$$

The mean sequence based on consecutive neighbors of $Q^{(1)}$ is given by

$$Z^{(1)} = (7, 16, 28, 43.5)$$

Let $q(k) + az^{(1)}(k) = b$. From

$$B = \begin{bmatrix} -7 & 1 \\ -16 & 1 \\ -28 & 1 \\ -43.5 & 1 \end{bmatrix}, \ Y = \begin{bmatrix} 8 \\ 10 \\ 14 \\ 17 \end{bmatrix}$$

it follows that

$$\hat{a} = \begin{bmatrix} a \\ b \end{bmatrix} = (B^T B)^{-1} B^T Y = \begin{bmatrix} -0.25361 \\ 6.258339 \end{bmatrix}$$

Therefore, the GM(1, 1) ordinality response sequence of the distortion date sequence is

$$\hat{q}^{(1)}(k+1) = 27.667e^{0.25361k} - 24.667$$

$$\hat{q}(k+1) = \hat{q}^{(1)}(k+1) - \hat{q}^{(1)}(k)$$

That is,

$$\hat{q}(k+1) = 27.667e^{0.25361k} - 24.667e^{0.25361(k-1)} = 6.1998e^{0.25361k}$$

Thus, we can obtain a simulated sequence for $Q^{(0)}$ as follows:

$$\begin{aligned} \hat{Q}^{(0)} &= (\hat{q}(1), \hat{q}(2), \hat{q}(3), \hat{q}(4), \hat{q}(5)) \\ &= (6.1998, 7.989, 10.296, 13.268, 17.098) \end{aligned}$$

From

$$\varepsilon(k) = q(k) - \hat{q}(k), k = 1, 2, 3, 4, 5$$

we obtain the error sequence as follows:

$$\begin{aligned} \varepsilon^{(0)} &= (\varepsilon(1), \varepsilon(2), \varepsilon(3), \varepsilon(4), \varepsilon(5)) \\ &= (-3.1998, 0.011, -0.296, 0.732, -0.098) \end{aligned}$$

And from

$$\Delta_k = \left| \frac{\varepsilon(k)}{q(k)} \right|; \ k = 1, 2, 3, 4, 5$$

it follows that the sequence of relative errors is

$$\Delta = (\Delta_2, \Delta_3, \Delta_4, \Delta_5) = (0.1\%, 2.96\%, 5.1\%, 0.6\%)$$

From this sequence, we calculate the average relative error

$$\overline{\Delta} = \frac{1}{4}\sum_{k=2}^{5} \Delta_k = 2.19\%$$

With $1 - \overline{\Delta} = 97.81\%$ as the average relative accuracy, and $1 - \Delta_5 = 99.4\%$. Therefore, we can use

$$\hat{q}(k+1) = 6.1998e^{0.25361k}$$

to carry out our predictions. Because

$$\hat{q}(5+1) = \hat{q}(6) \approx 22,\ \hat{q}(6) - \hat{q}(5) \approx 22 - 17 = 5$$

we predict that in five years, counting from the time of the last drought in 2023, there might be a drought. In order to improve the accuracy of our prediction, we can take several different abnormal values to build various models to make predictions.

## 9.5 Wave Form Forecasting

When the available data sequence vibrates widely with large magnitudes, it is often difficult, if not impossible, to find an appropriate simulation model. In this case, one can consider making use of the pattern of fluctuation of the data to predict the future development of the wavy movement. This kind of prediction is known as a wave form forecasting.

**Definition 9.5.1** Let $X = (x(1), x(2), \ldots, x(n))$ be the sequence of raw data, then

$$x_k = x(k) + (t-k)[x(k+1) - x(k)]$$

is known as a k-piece zigzagged line of the sequence $X$, and

$$\{x_k = x(k) + (t-k)[x(k+1) - x(k)] | k = 1, 2, \ldots, n-1\}$$

the zigzagged line, still denoted by using X.

**Definition 9.5.2** Assume that $X$ is a zigzagged line, let

$\sigma_{\max} = \max_{1\le k\le n}\{x(k)\}$ and $\sigma_{\min} = \min_{1\le k\le n}\{x(k)\}$.

Then

(1) For any $\forall \xi \in [\sigma_{\min}, \sigma_{\max}]$, $X = \xi$ is known as the $\xi$-contour (line); and
(2) The solutions $(t_i, x(t_i))(i = 1, 2, \ldots)$ of system of equations

$$\begin{cases} X = \{x(k) + (t-k)[x(k+1) - x(k)] | k = 1, 2, \ldots, n-1\} \\ X = \xi \end{cases}$$

is called the $\xi$-contour points. The $\xi$-contour point is the intersection of the zigzagged line $X$ and the $\xi$-contour line.

**Proposition 9.5.1** If on the $i$th segment of $X$ there is a $\xi$-contour point, then the coordinates of this point are given by $\left(i + \frac{\xi - x(i)}{x(i+1) - x(i)}, \xi\right)$.

***Proof*** The equation of i-piece zigzagged line of the sequence $X$ is as follows:

$$X = x(i) + (t_i - i)[x(i+1) - x(i)]$$

From

$$\begin{cases} X = x(i) + (t_i - i)[x(i+1) - x(i)] \\ X = \xi \end{cases}$$

We have

$$t_i = i + \frac{\xi - x(i)}{x(i+1) - x(i)}$$

**Definition 9.5.3** Let $X_\xi = (P_1, P_2, \ldots, P_m)$ be the sequence of $\xi$-contour points of $X$ such that point $P_i$ is located on the $i$th segment. Let

$$q(i) = t_i + \frac{\xi - x(t_i)}{x(t_i + 1) - x(t_i)}, i = 1, 2, \ldots, m$$

Then $Q^{(0)} = (q(1), q(2), \ldots, q(m))$ is known as the $\xi$-contour time moment sequence. By establishing a GM(1, 1) model using this $\xi$-contour moment sequence, one can produce the predicted values for future $\xi$-contour time moments:

$$\hat{q}(m+1), \hat{q}(m+2), \ldots, \hat{q}(m+k).$$

**Definition 9.5.4** The lines $X = \xi_i (i = 0, 1, 2, \ldots, s)$, where $\xi_0 = \sigma_{\min}$, $\xi_1 = \frac{1}{s}(\sigma_{\max} - \sigma_{\min}) + \sigma_{\min}, \ldots \xi_i = \frac{i}{s}(\sigma_{\max} - \sigma_{\min}) + \sigma_{\min}, \ldots, \xi_{s-1} = \frac{s-1}{s}(\sigma_{\max} - \sigma_{\min}) +$

$\sigma_{\min}$, $\xi_s = \sigma_{\max}$ are known as equal time distanced contours. When taking contour lines, one needs to make sure that the corresponding contour moments satisfy the conditions for establishing valid GM(1,1) models.

**Definition 9.5.5** Let $X = \xi_i (i = 1, 2, \ldots, s)$ be $s$ different contours,

$$Q_i^{(0)} = (q_i(1), q_i(2), \ldots, q_i(m_1), \; i = 1, 2, \ldots, s,$$

stand for the sequence of $\xi_i$-contour time moments, and

$$\hat{q}_i(m_i + 1), \hat{q}_i(m_i + 2), \ldots, \hat{q}_i(m_i + k_i), \; i = 1, 2, \ldots, s,$$

the GM(1,1) predicted $\xi_i$-contour time moments. If there are $i \neq j$ such that

$$\hat{q}_i(m_i + l_i) = \hat{q}_j(m_j + l_j),$$

then these values are known as a pair of invalid moments.

**Proposition 9.5.2** Let $\hat{q}_i(m_i + j)$,$j = 1, 2, \ldots, k_i$,$i = 1, 2, \ldots, s$, be the GM(1,1) predicted $\xi_i$-contour time moments. After deleting all invalid predictions, order the rest in terms of their magnitudes as follows:

$$\hat{q}(1) < \hat{q}(2) < \cdots < \hat{q}(n_s),$$

where $n_s \leq k_1 + k_2 + \cdots + k_s$. If $X = \xi_{\hat{q}(k)}$ is the contour line corresponding to $\hat{q}(k)$. Then the predicted wavy curve of $X^{(0)}$ is given below:

$$X = \hat{X}^{(0)} = \{\xi_{\hat{q}(k)} + [t - \hat{q}(k)][\xi_{\hat{q}(k+1)} - \xi_{\hat{q}(k)}] | k = 1, 2, \ldots, n_s\}.$$

## 9.6 System Forecasting

### 9.6.1 The Five-Step Modeling Process

Generally, when studying a system one should first establish a mathematical model through which the overall functionality of the system, abilities of coordination, incidence relations, causal relations, and dynamic relationships between different parts can be quantitatively investigated. This kind of study has to be guided by an early qualitative analysis, and there must be close connection between the quantitative and qualitative studies. As for the development of the system's model, one generally goes through the following five steps: development of thoughts, analysis of relevant

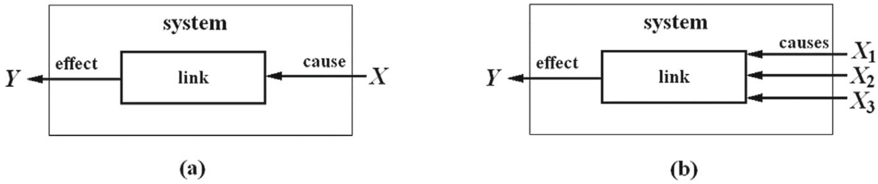



**Fig. 9.3** Depicted causal relationships

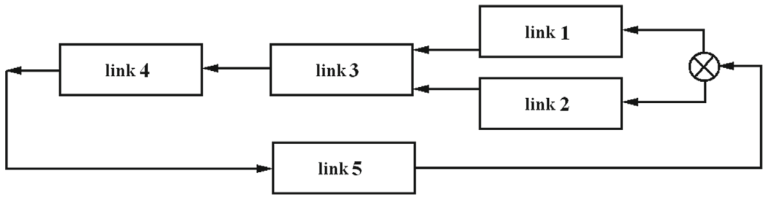



**Fig. 9.4** Line drawing of an abstract system

factors, quantification, dynamics, and optimization. This is the so-called five-step modeling (Deng, 1985).

**Step 1**: Develop thoughts and form concepts. Through an initial qualitative analysis, one clarifies his goal, possible paths and specific procedures, and then verbally and precisely describes the desired outcomes. This is the initial language model of the problem (see Fig. 9.3).

**Step 2:** Examine all the factors involved in the language model and their mutual relationships in order to pinpoint the causes and conclusions. Then, construct a line-drawing to depict the causal relationships (Fig. 9.3). Each pair (or a group) of causes and effect form a link. A system might be made up of many of such links. At the same time, a quantity can be a cause of a link and also a consequence of another link. When several of these links are connected, one obtains a line drawing of many links that organically form the system of our concern (Fig. 9.4).

**Step 3**: Quantitatively study each causality link and obtain an approximate quantitative relationship, which is a quantified model.

**Step 4**: For each link, collect additional input–output data, on which dynamic GM models are established. Such dynamic models are higher level quantitative models. They can further reveal the relationships between input and output, and their laws of transformation. They are the foundation of systems analysis and optimization.

**Step 5**: Systematically investigate the established dynamic models by adjusting their structures, mechanisms, and parameters, in order to arrive at the purpose of optimizing the outcome and realizing the desired conclusions. Models obtained in this way are known as optimal models.

The procedure of five-step modeling is such a holistic process that at five different stages five different kinds of models are established: language models, network models, quantified models, dynamic models, and optimized models. In the entire process of modeling, the conclusions of the next level should be repeatedly fed back so that the modeling exercise itself becomes a feedback system making the model system as perfect as possible.

### *9.6.2 System Models for Prediction*

For a system with many mutually related factors and many autonomous controlling variables, no single model can reflect adequately the development and change of the system. To effectively study such a system and to predict its future behaviors, one should consider establishing a system of models.

**Definition 9.6.1** Assume that

$$X_i^{(0)} = \left(x_i^{(0)}(1), x_i^{(0)}(2), \ldots, x_i^{(0)}(n)\right), i = 1, 2, \ldots m,$$

are sequences of raw data for the state variables of a system, and

$$U_i^{(0)} = \left(u_i^{(0)}(1), u_i^{(0)}(2), \ldots, u_i^{(0)}(n)\right), j = 1, 2, \ldots s,$$

are sequences of data of the control variables. Then the following

$$\begin{aligned}
x_1^{(0)} &= a_{11}z_1^{(1)} + a_{12}x_2^{(1)} + \cdots + a_{1m}x_m^{(1)} + b_{11}u_1^{(1)} + b_{12}u_2^{(1)} + \cdots + b_{1s}u_s^{(1)} \\
x_2^{(0)} &= a_{21}x_1^{(1)} + a_{22}z_2^{(1)} + \cdots + a_{2m}x_m^{(1)} + b_{21}u_1^{(1)} + b_{22}u_2^{(1)} + \cdots + b_{2s}u_s^{(1)} \\
&\ldots \\
x_m^{(0)} &= a_{m1}x_1^{(1)} + a_{m2}x_2^{(1)} + \cdots + a_{mm}z_m^{(1)} + b_{m1}u_1^{(1)} + b_{m2}u_2^{(1)} + \cdots + b_{ms}u_s^{(1)}
\end{aligned}$$

$$\frac{du_1^{(1)}}{dt} = c_1u_1^{(1)} + d_1, \frac{du_2^{(1)}}{dt} = c_2u_2^{(1)} + d_2, \ldots, \frac{du_s^{(1)}}{dt} = c_su_s^{(1)} + d_s$$

are known as system models for prediction. As a matter of fact, each system model for prediction consists of $m$ DGM$(1, m+s)$ and $s$ EGM$(1, 1)$ models. If we write the previous system models for prediction using the terminology of matrices, we have

$$\begin{cases} X^{(0)} = AX^{(1)} + BU^{(1)} \\ U^{(0)} = CU^{(1)} + D \end{cases}$$

where $X^{(1)} = \left(x_1^{(1)}, x_1^{(1)}, \ldots, x_m^{(1)}\right)^T$, $U^{(1)} = \left(u_1^{(1)}, u_1^{(1)}, \ldots, u_s^{(1)}\right)^T$, $A = [a_{kl}]_{m\times m}$,$B = \left[b_{pq}\right]_{m\times s}$, $C = diag\left[c_j\right]_{s\times s}$, and $D = \left[d_j\right]_{s\times 1}$.

$X$ is known as the state vector, $U$ the control vector, $A$ the state matrix, $B$ the control matrix, $C$ the development matrix, and $D$ the grey effect vector.

**Proposition 9.6.1** For the previous system models for prediction, the time response sequences are given as follows:

$$\hat{x}_i^{(0)}(k) = a_{i1}x_1^{(1)}(k) + a_{i2}x_2^{(1)}(k) + \cdots + a_{im}x_m^{(1)}(k)$$
$$+ b_{i1}u_1^{(1)}(k) + b_{i2}u_2^{(1)}(k) + \cdots + b_{is}u_s^{(1)}(k) \ i = 1, 2, \ldots, m$$

$$\hat{u}_j^{(0)}(k) = (1 - e^{cj})\left(u_j^{(0)}(1) - \frac{d_j}{c_j}\right)e^{-cj(k-1)},\ j = 1, 2, \ldots, s$$

## 9.7 Practical Applications

***Example 9.7.1*** Let us look at a wavy curve prediction for the (synthetic) stock index of Shanghai stock exchange. Using the stock index data of the stock index weekly closes of Shanghai stock exchange, the time series plot from February 21, 1997, through to October 31, 1998, is shown in Fig. 9.5 (Dang & Liu, 2009).

Let us take
$\xi_1 = 1140$, $\xi_2 = 1170$, $\xi_3 = 1200$, $\xi_4 = 1230$, $\xi_5 = 1260$,
$\xi_6 = 1290$, $\xi_7 = 1320$, $\xi_8 = 1350$, $\xi_9 = 1380$.
Then the corresponding $\xi_i$-contour time moment sequences are given below:

(1) For $\xi_1 = 1140$,

$$Q_1^{(0)} = \{q_1(k)\}_1^7 = (4.4, 31.7, 34.2, 41, 42.4, 76.8, 78.3)$$

(2) For $\xi_2 = 1170$,

$$Q_2^{(0)} = \{q_2(k)\}_2^{12} = (5.2, 19.8, 23, 25.6, 26.9, 31.2, 34.8, 39.5, 44.6, 76, 76.2, 79.2)$$

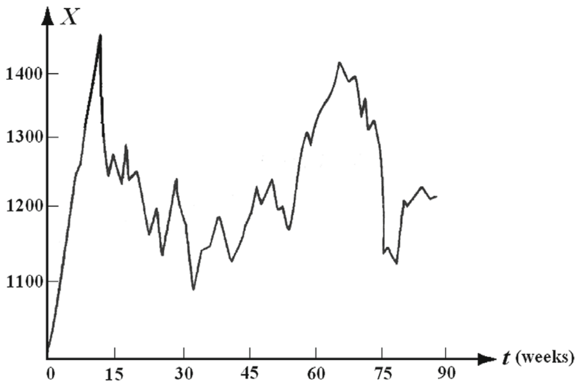



**Fig. 9.5** Shanghai stock exchange index (Feb. 21, 1997, to Oct. 31, 1998)

(3) For $\xi_3 = 1200$,

$$Q_3^{(0)} = \{q_3(k)\}_3^{11} = (5.9,19.5, 24.8,25.2, 26.5,30.3, 46.2,53.4, 55.4,75.5, 79.7)$$

(4) For $\xi_4 = 1230$,

$$Q_4^{(0)} = \{q_4(k)\}_4^{10} = (6.5,19.2, 28.3,29.5, 49.7,50.8, 56.2,76.4, 82.9,85)$$

(5) For $\xi_5 = 1260$,

$$Q_5^{(0)} = \{q_5(k)\}_5^{7} = (7, 14.2,16.5, 16.4,18.8, 56.7,75.2)$$

(6) For $\xi_6 = 1290$,

$$Q_6^{(0)} = \{q_6(k)\}_6^{5} = (8.3,13.4, 16.9,56.2, 74.6)$$

(7) For $\xi_7 = 1320$,

$$Q_7^{(0)} = \{q_7(k)\}_7^{6} = (8.8,12.8, 60.2,71.8, 72.7,73.6)$$

(8) For $\xi_8 = 1350$,

$$Q_8^{(0)} = \{q_8(k)\}_8^{6} = (9.6,12.5, 61.8,69.8, 70.9,71.8)$$

(9) For $\xi_9 = 1380$,

$$Q_9^{(0)} = \{q_9(k)\}_9^{4} = (10.8,12.4, 64.1,69)$$

Applying the 1-AGO on $Q_i^{(0)}(i = 1, 2, \ldots, 9)$ produces $Q_i^{(1)}(i = 1, 2, \ldots, 9)$, whose EGM(1,1) response sequences are respectively given by:

$$\hat{q}_1^{(1)}(k+1) = 113.91e^{0.215k} - 109.51, \hat{q}_2^{(1)}(k+1) = 98.58e^{0.159k} - 93.83,$$

$$\hat{q}_3^{(1)}(k+1) = 102.08e^{0.166k} - 96.18, \hat{q}_4^{(1)}(k+1) = 151.66e^{0.160k} - 145.16,$$

$$\hat{q}_5^{(1)}(k+1) = 13e^{0.435k} - 6, \hat{q}_6^{(1)}(k+1) = 21.94e^{0.539k} - 13.64,$$

$$\hat{q}_7^{(1)}(k+1) = 185.08e^{0.192k} - 176.28, \hat{q}_8^{(1)}(k+1) = 193.19e^{0.186k} - 182.57,$$

$$\hat{q}_9^{(1)}(k+1) = 45.22e^{0.490k} - 35.39.$$

By letting $\hat{q}_i(k+1) = \hat{q}_i^{(1)}(k+1) - \hat{q}_i^{(1)}(k)$, we obtain the following $\xi_i$-contour prediction sequences, $i = 1, 2, \ldots, 9$,

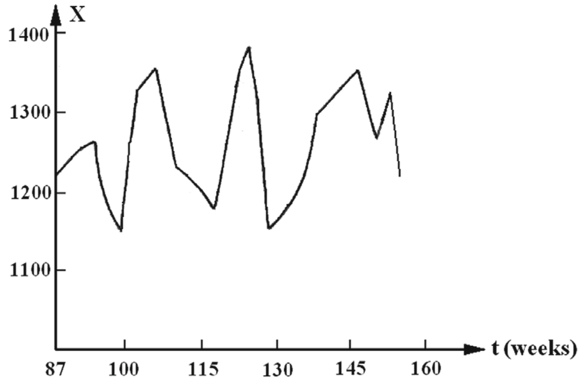



**Fig. 9.6** The predicted wavy curve of Shanghai stock exchange index (Nov. 1998 to March 2000)

$$\hat{Q}_1^{(0)} = (\hat{q}_1(12), \hat{q}_1(13)) = (99.8, 127.7)$$

$$\hat{Q}_2^{(0)} = (\hat{q}_2(13), \hat{q}_2(14), \hat{q}_2(15)) = (96.8, 116.7, 131.4)$$

$$\hat{Q}_3^{(0)} = (\hat{q}_3(12), \hat{q}_3(13), \hat{q}_3(14)) = (95.7, 114.2, 133.8)$$

$$\hat{Q}_4^{(0)} = (\hat{q}_4(11), \hat{q}_4(12), \hat{q}_4(13)) = (110.9, 134.2, 152.8)$$

$$\hat{Q}_5^{(0)} = (\hat{q}_5(8), \hat{q}_5(9)) = (94.2, 148.8)$$

$$\hat{Q}_6^{(0)} = (\hat{q}_6(6)) = (135.5)$$

$$\hat{Q}_7^{(0)} = (\hat{q}_7(7), \hat{q}_7(8), \hat{q}_7(9)) = (101.9, 123.4, 149.5)$$

$$\hat{Q}_8^{(0)} = (\hat{q}_8(7), \hat{q}_8(8), \hat{q}_8(9)) = (105, 119.8, 144.6)$$

$$\hat{Q}_9^{(0)} = (\hat{q}_9(5)) = (122.3)$$

Based on these predictions, we construct the predicted wavy curve for the Shanghai stock exchange index for the time period from November 1998 to the end of 1999 (see Fig. 9.6).

## References

Chen, Y. Y., Liu, H. T., & Hsieh, H. L. (2019). Time series interval forecast using GM(1, 1) and NGBM(1, 1) models. *Soft Computing, 23*(5), 1541–1555.

Dang, Y. G., & Liu, S. F. (2009). *On grey prediction and decision-making models.* Science Press.

Deng, J. L. (1985). *Grey control system.* Press of Huazhong University of Science and Technology.

Deng, J. L. (1990). *Grey system tutorial*. Press of Huazhong University of Science and Technology.

Hu, Y. C. (2021). Forecasting the demand for tourism using combinations of forecasts by neural network-based interval grey prediction models. *Asia Pacific Journal of Tourism Research, 26*(12), 1350–1363.

Liu, S. F. (2024). *Grey systems theory and its applications* (10th ed.). Science Press.

Liu, S. F., & Guo, T. B. (1991). *Grey systems theory and its application.* Henan University Press.

Liu, S. F., Yang, Y. J., & Forrest, J. (2022). *Grey systems analysis.* Springer.

Liu, S. F., Jian, L. R., & Mi, C. M. (2020). *Management forecasting and decision-making Methods* (4th ed.). Science Press.

Liu, X. J., Deng, Z., & Bai, E. J. (2016). Fuzzy multi-objective interval forecasting based on a discrete grey model. *Fuzzy Systems and Mathematics, 30*(2), 97–102.

Luo, D., Wei, B. L., Li, H. T. et al. (2016). Grey interval forecasting model and its properties. *Control and Decision, 31*(12), 2293–2298 (In Chinese).

Wang, P., Gurmani, S. H., Tao, Z. F., Liu, J., & Chen, H. (2024). Interval time series forecasting: A systematic literature review. *Journal of Forecasting, 43*(2), 249–285.

Xie, N. M., Liu, S. F., Yuan, C., & Yang, Y. (2014). Grey number sequence forecasting approach for interval analysis: A case of China's gross domestic product prediction. *The Journal of Grey System, 26*, 45–58.

Xie, N. M., & Liu, S. F. (2015). Interval grey number sequence prediction by using non-homogenous exponential discrete grey forecasting model. *Journal of Systems Engineering and Electronics, 26*(1), 96–102.

Xiong, P. P., Shen, H., Mao, P. A., Peng, M., & Wu, X. (2020). Examination and prediction of fog and haze pollution using a multi-variable grey model based on interval number sequences. *Applied Mathematical Modelling, 77*, 1531–1544.

Zeng, B., Liu, S. F., & Xie, N. M. (2010). Prediction model of interval grey number based on DGM (1, 1). *Journal of Systems Engineering and Electronics, 21*(4).

Zeng, B., Zhou, W. H., & Zhou, M. (2021). Forecasting the concentration of sulfur dioxide in Beijing using a novel grey interval model with oscillation sequence. *Journal of Cleaner Production, 311*, 127500.

Zeng, X. Y., Yan, S. L., He, F. L., & Shi, Y. (2020). Multi-variable grey model based on dynamic background algorithm for forecasting the interval sequence. *Applied Mathematical Modelling, 8*, 99–114.

# Chapter 10
# Grey Models for Decision-Making

## 10.1 Introduction

The so-called decision-making is to determine the action to be taken according to the actual situation and the predetermined goal. The essential meaning of decision-making is "making a decision" or "deciding countermeasures". Decision making activities are not only an important part of various management activities, but also run through everyone's work, study and life process. The understanding of decision-making can be divided into broad sense and narrow sense. In a broad sense, decision-making refers to the whole process of raising questions, collecting information, determining objectives, formulating options, evaluating and selecting options, and implementing, feeding back and revising a series of activities; In a narrow sense, decision-making only refers to the process of selecting schemes in the whole decision-making process, which is traditionally called "clappers". Some people only understand decision-making as choosing a scheme under uncertain conditions, that is, making a choice, which largely depends on the decision-maker's personal experience, attitude and determination, and has to bear certain risks. Grey decision-making is the case that the decision-making model contains grey elements or when the general decision-making model is combined with the grey model. It focuses on the problem of scheme selection (Liu et al., 2020; Liu, 2024).

Professor Deng proposed the concept of grey target decision-making at first (Deng, 1990; Deng, 2002). A grey target is defined as a satisfying region, which a decision maker wants to reach. The grey targets can be divided into rectangular grey targets and spherical grey targets. As the name suggests, grey target decision-making mainly focuses on "hitting the target" or "off target" to help people make judgments or choices in uncertain situations.

In 2010, Liu et al. proposed a multi-attribute weighted intelligent grey target decision model (Liu et al., 2010). This model converts the target effect values of different types into additive consistent effect measurement values by determining the critical value of the target effect and using consistent effect measurement functions,



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_10

including effect measurement function for benefit type objective, effect measurement function for cost type objective, cost based target effect measurement functions, lower effect measurement function for moderate objective, and upper effect measurement function for moderate objective. Based on this, comparable comprehensive effect measurement values can be calculated (Liu et al., 2013). According to the results obtained from the multi-attribute weighted intelligent grey target decision model, it is not only possible to determine whether the decision is "hitting the target" or "off target", but also to compare the advantages and disadvantages of different decision-making options.

The general decision model usually uses the "maximum value criterion" or "maximum expected value criterion" as the basis for decision-making. For example, grey clustering evaluation decision is to determine the grey class to which the decision object belongs based on the maximum component of the clustering coefficient vector. Dang et al. researched the problem of decision object attribution when there is no significant difference in the components of the grey clustering coefficient vector (Dang et al., 2005). Liu et al. (2014) proposed a two-stage decision model to address the situation where the maximum component value of the grey comprehensive clustering coefficient vector has low discrimination from other components, and the decision made according to the "maximum value criterion" conflicts with the conclusion obtained from the overall evaluation of the decision coefficient vector, known as the "maximum value criterion" decision paradox. In 2015, Liu and Yang defined a general synthetic weight vectors for decision making and the decision coefficient vectors with grey synthetic measures (Liu & Yang, 2015). Then, the new concepts of weight vector group of kernel clustering and weighted coefficient vector of kernel clustering for decision-making were put forward to solve the problem of decision paradox or the dilemma in supplier selection (Liu et al., 2018, 2022).

Wu and Chang (2004) studied optimization problems of the company production plan under variable environmental costs using the grey compromise programming model. Li et al. (2007) constructed the grey planning model of missile nuclear optimal allocation, which provided a theoretical basis for the order, storage, position allocation and operational application of the missile nuclear weapons. Yu et al. (2009) applied the grey cluster decision method to aircraft large parts automatic docking assembly system, which enhanced the stability of the system, reduced the risk of equipment failure and reduced repair costs as well. Cui et al. (2012b) researched the selection problem of ground and air missile weapon system applying the multi-objective grey decision model.

Cui et al. (2012a) gave the weighting formula evaluation value on each stage detection based on the new information priority principle, which provided a new thinking to solve multiple stages grey decision problems. Golmohammadi and Mellat-Parast (2012) gave the grey decision model about supplier selection. Liang et al. (2012) came up with a case reasoning method based on the grey system theory and logistic regression model and applied it to safety assessment for thermal power plant. Bai et al. researched the regional leading industry selection of "Kashgar urban agglomerations" by multi-attribute weighted intelligent grey target decision-making evaluation model (Bai et al., 2021). Rogulj et al. studied historic bridges reconstruction using

Intuitionistic fuzzy decision support based on EDAS and grey relational analysis model (Rogulj et al., 2022).

Dang Luo studied grey decision models of different types and obtained a series of achievements (Luo & Wang, 2012a, 2012b). Guo et al. (2014) researched the grey double layers and multi-objective linear programming and solving problems. Yan et al. (2014) proposed a new method to determine the weights of decision makers and attributes for group decision making with interval grey numbers. In 2019, Yazdani et al. proposed a combined compromise solution (CoCoSo) method for multi-criteria decision-making problems (Yazdani et al., 2019). Jafarian et al. put forward a novel multi-objective co-evolutionary approach for supply chain gap analysis with consideration of uncertainties in 2020 (Jafarian et al., 2020). In 2021, Dahooie proposed a novel dynamic credit risk evaluation method using data envelopment analysis with common weights and combination of multi-attribute decision-making methods (Dahooie et al., 2021). Aslani et al. researched the problem of integrated information fusion and grey multi-criteria decision-making framework for sustainable supplier selection (Aslani et al., 2021). Chatterjee et al. solved the optimization problem of green machining processes using grey-based multi-criteria decision making methods (Chatterjee et al., 2024). Li and Li proposed a grey decision model based on generalized greyness of interval grey number (Li & Li, 2024).

## 10.2 Event and Decision Scheme

In this chapter, we define an event as the problem waiting to be resolved, the event needing to be handled, and the current state of a system's behavior. Events are where we begin our investigation.

**Definition 10.2.1** Events, countermeasures, objectives, and effects are known as the four key elements of decision-making.

**Definition 10.2.2** The totality of all events within the range of a research is known as the set of events of the study, denoted $A = \{a_1, a_2, ...a_n\}$, where $a_i, i = 1, 2, 3, \ldots, n$, stands for the $i$th event. The totality of all possible countermeasures is known as the set of countermeasures, denoted $B = \{b_1, b_2, \ldots, b_m\}$ with $b_j, j = 1, 2, \ldots, m$, be the $j$th countermeasure.

**Definition 10.2.3** The Cartesian product $A \times B = \left\{(a_i, b_j)|a_i \in A, b_j \in B\right\}$ of the event set $A$ and the countermeasure set $B$ is known as the set of decision schemes, written as $S = A \times B$, where each ordered pair $s_{ij} = \left(a_i, b_j\right)$, for any $a_i \in A, b_j \in B$, is known as a decision scheme.

For example, in the decision-making on what to plant in agriculture, weather conditions can be used as the set of events, with a normal year denoted as $a_1$, a drought year as $a_2$, and a flood year as $a_3$. Then, the set of events is

$$A = \{a_1, a_2, a_3\}$$

Different strains of crops can be seen as countermeasures, with corn denoted as $b_1$, Chinese sorghum as $b_2$, soybeans as $b_3$, sesame $b_4$, potatoes and yams as $b_5, \ldots$; then the countermeasure set is given as

$$B = \{b_1, b, b_3, b_4, b_5, \ldots\}$$

Therefore, the set of decision scheme is

$$S = A \times B = \{s_{11}, s_{12}, \ldots, s_{15}, \ldots, s_{21}, \ldots, s_{25}, \ldots, \ldots, s_{31}, \ldots, s_{35}, \ldots\}$$

where $s_{ij} = (a_i, b_j)$.

Here, events and countermeasures are simple. Therefore, the constructed decision schemes are relatively simple, too. In practical decision-making, events are often complicated, consisting of many kinds of simple events, so the countermeasures are complicated, too. Hence, the resultant decision schemes can be extremely complicated.

Let us continue to use the previous agricultural decision-making example. The set of events is the organic body consisting of weather, soil, irrigation, fertilizer, agricultural chemicals, work force, and technology. The countermeasures are not simply the individual strains of crops, but various proportional combinations of many different strains of crops. Let us define $a_1$ as an event characterized by a normal year, loam, 50% effective irrigation area, sufficient fertilizer and agricultural chemicals, sufficient work force, and medium level of technology. Additionally, let us define $a_2$ as an event characterized by a drought year, black earth, 50% effective irrigation area, sufficient fertilizer and work force, lack of agricultural chemicals, and medium level of technology. Then, we have the set of events:

$$A = \{a_1, a_2, \ldots\}$$

Let us write $b_1$ as the countermeasure including 30% corn + 10% Chinese sorghum + 20% soybeans + 15% sesame + 15% potatoes and yams + 10% others. Also, let us write $b_2$ as the countermeasure including 10% corn + 20% Chinese sorghum + 30% soybeans + 30% sesame + 10% others. Then, we have the countermeasure set:

$$B = \{b_1, b_2, \ldots\}$$

Now, the decision scheme $s_{11} = (a_1, b_1)$ is that, under the conditions of a normal year, loam, 50% effective irrigation area, with sufficient fertilizer and agricultural chemicals, sufficient workforce, and medium level of technology, we should plant 30% corn, 10% Chinese sorghum + 20% soybeans + 15% sesame + 15% potatoes and yams + 10% others.

Let us look at the example of teaching scheduling. The collection of all course offerings of a fixed semester at a certain school can be seen as the set of events; all teaching faculty of this school, and various teaching methods, such as laboratory, interns, and multimedia, are seen as the set of countermeasures. Based on the circumstances, one teacher can teach several courses, or several teachers teach one course together. The work load could be 100% teaching, or 60% teaching, 20% laboratory, 10% interns, and 10% multimedia and others.

For a given decision scheme $s_{ij} \in S$, evaluating the effects under a set of predetermined objectives and deciding on what to take and what to let go based on evaluation is the decision-making we discuss in this chapter. In the following sections, we will study several different kinds of grey decision-making methods.

## 10.3 Grey Target Decisions

**Definition 10.3.1** Let $S = \{s_{ij} = (a_i, b_j) \big| a_i \in A, b_j \in B\}$ be a set of decision schemes, $u_{ij}^{(k)}$ the effect value of decision scheme $s_{ij}$ with respect to objective $k$, and $R$ the set of all real numbers. Then $u_{ij}^{(k)}: S \mapsto R$, defined by $s_{ij} \mapsto u_{ij}^{(k)}$, is known as the effect mapping of S with respect to object $k$.

**Definition 10.3.2** If $u_{ij}^{(k)} = u_{ih}^{(k)}$, then we say that the countermeasures $b_j$ and $b_h$ of event $a_i$ are equivalent with respect to objective $k$, written as $b_j \cong b_h$; and the set

$$B_i^{(k)} = \{b | b \in B, b \cong b_h\}$$

is known as the effect equivalence class of countermeasure $b_h$ of event $a_i$ with respect to objective $k$.

**Definition 10.3.3** If $k$ is such an objective that the greater the effect value is the better, and $u_{ij}^{(k)} > u_{ih}^{(k)}$, then we say that the countermeasure $b_j$ is superior to $b_h$ in terms of event $a_i$ with respect to objective $k$, written as $b_j \succ b_h$. The set $B_{ih}^{(k)} = \{b | b \in B, b \succ b_h\}$ is known as the superior set of countermeasure $b_h$ of event $a_i$ with respect to objective $k$.

Similarly, we can define the concept of superior classes of countermeasures for situations where the closer to a fixed moderate value the effect value is the better, and where the smaller the effect value is the better.

**Definition 10.3.4** If $u_{ij}^{(k)} = u_{ih}^{(k)}$, then events $a_i$ and $a_j$ are said to be equivalent in terms of the countermeasure $b_h$ with respect to objective $k$, written $a_i \cong a_j$. The set

$$A_{jh}^{(k)} = \{a | a \in A, a \cong a_i\}$$

is known as the effect equivalence class of events of the countermeasure $b_h$ with respect to objective $k$.

**Definition 10.3.5** If $k$ is such an objective that the greater the effect value is the better, and $u_{ih}^{(k)} > u_{jh}^{(k)}$, then we say that event $a_i$ is superior to event $a_j$ in terms of countermeasure $b_h$ with respect to objective $k$, denoted $a \succ a_j$. The set

$$A_{jh}^{(k)} = \left\{ a \middle| a \in A, a \succ a_j \right\}$$

is known as the superior class of event $a_j$ in terms of countermeasure $b_h$ with respect to objective $k$.

Similarly, the concept of superior classes can be defined for situations where the closer to a fixed moderate value the effect value is the better, and where the smaller the effect value is the better.

**Definition 10.3.6** If $u_{ij}^{(k)} = u_{hl}^{(k)}$, then scheme $s_{ij}$ is equivalent to scheme $s_{hl}$ under objective $k$, denoted $s_{ij} \cong s_{hl}$. The set

$$S^{(k)} = \{s | s \in S, s \cong s_{hl}\}$$

is known as the effect equivalence class of scheme $s_{hl}$ under objective $k$.

**Definition 10.3.7** If $k$ is such an objective that the greater the effect value is the better, and $u_{ij}^{(k)} > u_{hl}^{(k)}$, then scheme $s_{ij}$ is said to be superior to scheme $s_{hl}$ under objective $k$, denoted $s_{ij} \succ s_{hl}$. The set

$$S_{hl}^{(k)} = \{s | s \in S, s \succ s_{hl}\}$$

is known as the effect superior class of scheme $s_{hl}$ under objective $k$.

Similarly, the concept of superior classes for scheme effects can be defined for scenarios where the closer to a fixed moderate value the effect value of a scheme is the better, and where the smaller the effect value of the scheme is the better.

**Proposition 10.3.1** *Assume that* $S = \{s_{ij} = (a_i, b_j) | a_i \in A, b_j \in B\} \neq \emptyset$ *and* $U^{(k)} = \left\{ u_{ij}^{(k)} \middle| a_i \in A, b_j \in B \right\}$ *is the set of effects under objective k, and* $\left\{S^{(k)}\right\}$ *the set of effect equivalence classes of schemes under objective k. Then the mapping* $u^{(k)} : \left\{S^{(k)}\right\} \to U^{(k)}$*, defined by* $S^{(k)} \mapsto u_{ij}^{(k)}$*, is bijective.*

**Definition 10.3.8** Let $d_1^{(k)}$ and $d_2^{(k)}$ be the upper and lower threshold values of the decision effects of $s_{ij}$ under objective $k$. Then $S^1 = \left\{ r \middle| d_1^{(k)} \leq r \leq d_2^{(k)} \right\}$ is known as the one-dimensional grey target of objective $k$, $u_{ij}^{(k)} \in [d_1^{(k)}, d_2^{(k)}]$ a satisfactory effect under objective $k$, the corresponding $s_{ij}$ a desirable scheme with respect to objective $k$, and $b_j$ a desirable countermeasure of event $a_i$ with respect to objective $k$.

**Proposition 10.3.2** *Assume that* $u_{ij}^{(k)}$ *stands for the effect value of scheme* $s_{ij}$ *with respect objective k. If* $u_{ij}^{(k)} \in S^1$*, that is,* $s_{ij}$ *is a desirable scheme with respect to objective k. Then for any* $s \in S_{ij}^{(k)}$*, s is also a desirable scheme. That is, when* $s_{ij}$ *is desirable, all schemes in its effect superior class are desirable.*

*The discussion above applies to cases involving a single objective. Nevertheless, grey targets of decision-making with multi-objectives can also be addressed.*

**Definition 10.3.9** Assume that $d_1^{(1)}$ and $d_2^{(1)}$ are the threshold values of decision effects of objective 1, $d_1^{(2)}$ and $d_2^{(2)}$ the threshold values of decision effects of objective 2. Then

$$S^2 = \left\{ (r^{(1)}, r^{(2)}) \middle| d_1^{(1)} \le r^{(1)} \le d_2^{(1)}, d_1^{(2)} \le r^{(2)} \le d_2^{(2)} \right\}$$

is known as a grey target of two-dimensional decision-making. If the effect vector of scheme $s_{ij}$ satisfies $u_{ij} = \left\{ u_{ij}^{(1)}, u_{ij}^{(2)} \right\} \in S^2$, then $s_{ij}$ is seen as a desirable scheme with respect to objectives 1 and 2, and $b_j$ a desirable countermeasure for event $a_i$ with respect to objectives 1 and 2.

**Definition 10.3.10** Assume that $d_1^{(1)}, d_2^{(1)}; d_1^{(2)}, d_2^{(2)}; \cdots ; d_1^{(s)}, d_2^{(s)};$ are respectively the threshold values of decision effects under objectives $1, 2, \ldots, s$. Then the following region of the $s$–dimensional Euclidean space

$$S^s = \left\{ (r^{(1)}, r^{(2)}, \ldots, r^{(s)}) \middle| d_1^{(1)} \le r^{(1)} \le d_2^{(1)}, d_1^{(2)} \le r^{(2)} \le d_2^{(2)}, \ldots, d_1^{(s)} \le r^{(s)} \le d_2^{(s)} \right\}$$

is known as a grey target of an $s$–dimensional decision-making. If the effect vector of scheme $s_{ij}$ satisfies

$$u_{ij} = (u_{ij}^{(1)}, u_{ij}^{(2)}, \ldots, u_{ij}^{(s)}) \in S^s$$

where $u_{ij}^{(k)}$ stands for the effect value of the scheme $s_{ij}$ with respect to objective $k$, $k = 1, 2, \ldots, s$,, then $s_{ij}$ is known as a desirable scheme with respect to objectives $1, 2, \ldots, s$, and $b_j$ a desirable countermeasure of event $a_i$ with respect to objectives $1, 2, \ldots, s$.

Intuitively, the grey targets of a decision-making essentially represent the location of satisfactory effects in terms of relative optimization. In many practical circumstances, it is impossible to obtain the absolute optimization so that people are happy if they can achieve a satisfactory outcome. Of course, based on the need, one can gradually shrink the grey targets of his decision-making to a single point in order to obtain the ultimate optimal effect, where the corresponding scheme is the most desirable, and the corresponding countermeasure the optimal countermeasure.

**Definition 10.3.11** The following equation

$$R^s = \left\{ (r^{(1)}, r^{(2)}, \ldots, r^{(s)}) \middle| (r^{(1)} - r_0^{(1)})^2 + (r^{(2)} - r_0^{(2)})^2 + \cdots + (r^{(s)} - r_0^{(s)})^2 \le R^2 \right\}$$

is known as an $s$-dimensional spherical grey target centered at $r_0 = (r_0^{(1)}, r_0^{(2)}, \ldots, r_0^{(s)})$ with radius $R$. The vector $ss$ is seen as the optimum effect vector.

For $r_1 = (r_1^{(1)}, r_1^{(2)}, \ldots, r_1^{(s)}) \in R$,

$$|r_1 - r_0| = \left[(r_1^{(1)} - r_0^{(1)})^2 + (r_1^{(2)} - r_0^{(2)})^2 + \cdots + (r_1^{(s)} - r_0^{(s)})^2\right]^{1/2}$$

is known as the bull's-eye distance of vector $r_1$. The values of this distance reflect the superiority of the corresponding decision effect vectors.

**Definition 10.3.12** Let $s_{ij}$ and $s_{hl}$ be two different schemes, and $u_{ij} = (u_{ij}^{(1)}, u_{ij}^{(2)}, \ldots, u_{ij}^{(s)})$ and $u_{hl} = (u_{hl}^{(1)}, u_{hl}^{(2)}, \ldots, u_{hl}^{(s)})$ their effect vectors, respectively. If

$$\left|u_{ij} - r_0\right| \geq |u_{hl} - r_0| \tag{10.1}$$

then scheme $s_{hl}$ is said to be superior to $s_{ij}$, denoted $s_{hl} \succ s_{ij}$. When the equal sign in Eq. (10.1) holds true, schemes $s_{ij}$ and $s_{hl}$ are said to be equivalent, written $s_{hl} \cong s_{ij}$.

If for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m,$, $u_{ij} \neq r_0$ always holds true, then the optimum scheme does not exist, and the event does not have any optimum countermeasure. If the optimum scheme does not exist, however, there are $h$ and $l$ such that for any $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m,$, $|u_{hl} - r_0| \leq \left|u_{ij} - r_0\right|$ holds true, that is, for any $s_{ij} \in S$, $s_{hl} \succ s_{ij}$ holds, then $s_{hl}$ is known as a quasi-optimum scheme, $a_h$ a quasi-optimum event, and $\mathrm{b_l}$ a quasi-optimum countermeasure.

**Theorem 10.3.1** *Let $S = \{s_{ij} = (a_i, b_j)\big|a_i \in A, b_j \in B\}$ be a set of schemes, and*

$$R^s = \left\{(r^{(1)}, r^{(2)}, \ldots, r^{(s)})\middle|(r^{(1)} - r_0^{(1)})^2 + (r^{(2)} - r_0^{(2)})^2 + \ldots + (r^{(s)} - r_0^{(s)})^2 \leq R^2\right\}$$

*an s-dimensional spherical grey target. The S becomes an ordered set with "superiority" as its order relation $\prec$.*

**Theorem 10.3.2** *There must be quasi-optimum scheme in the set of decision schemes of $(S, \succ)$.*

***Proof*** This is a restatement of Zorn's Lemma in set theory.

***Example 10.3.1*** Consider event $a_1$ of reconstructing an old building. There are three possibilities: $b_1$ = renovate the building completely; $b_2$ = tear down the building and reconstruct another; and $b_3$ = simply maintain what the building is by fixing up minor problems. Let us make a grey target decision using three objectives: cost, functionality, and construction speed.

**Solution** Let us denote the cost as objective 1, the functionality as objective 2, and the construction speed as objective 3. Then, we have the following three decision schemes:

$$s_{11} = (a_1, b_1) = (\textit{reconstruction}, \ \textit{renovation}),$$
$$s_{12} = (a_1, b_2) = (\textit{reconstruction}, \ \textit{new building}), \ \textit{and}.$$
$$s_{13} = (a_1, b_3) = (\textit{reconstruction}, \ \textit{maintenance}).$$

Evidently, different decision schemes with respect to different objectives have different effects; and the standards for measuring the effects are also accordingly different. For instance, regarding cost, the lesser the better; for functionality, the higher the better; and for speed, the faster the better. Let us divide the effects of the decision schemes into three classes: good, okay, and poor.

The effect vectors of the decision schemes are respectively defined as follows:

$$u_{11} = \left(u_{11}^{(1)}, u_{11}^{(1)}, u_{11}^{(3)}\right) = (2, 2, 2),$$
$$u_{12} = \left(u_{12}^{(1)}, u_{12}^{(2)}, u_{12}^{(3)}\right) = (3, 1, 3), \; and$$
$$u_{13} = \left(u_{13}^{(1)}, u_{13}^{(2)}, u_{13}^{(3)}\right) = (1, 3, 1)\cdot$$

Let the bull's eye be located at $r_0 = (1, 1, 1)$ and compute the bull's-eye distances

$$|u_{11} - r_0| = \left[\left(u_{11}^{(1)} - r_0^{(1)}\right)^2 + \left(u_{11}^{(2)} - r_0^{(2)}\right)^2 + \left(u_{11}^{(3)} - r_0^{(3)}\right)^2\right]^{1/2}$$
$$= \left[(2-1)^2 + (2-1)^2 + (2-1)^2\right]^{1/2} = 1.73$$
$$|u_{12} - r_0| = \left[\left(u_{12}^{(1)} - r_0^{(1)}\right)^2 + \left(u_{12}^{(2)} - r_0^{(2)}\right)^2 + \left(u_{12}^{(2)} - r_0^{(3)}\right)^2\right]^{1/2}$$
$$= \left[(3-1)^2 + (1-1)^2 + (3-1)^2\right]^{1/2} = 2.83$$
$$|u_{13} - r_0| = \left[\left(u_{13}^{(1)} - r_0^{(1)}\right)^2 + \left(u_{13}^{(2)} - r_0^{(2)}\right)^2 + \left(u_{13}^{(3)} - r_0^{(3)}\right)^2\right]^{1/2}$$
$$= \left[(1-1)^2 + (3-1)^2 + (1-1)^2\right]^{1/2} = 2$$

where $|u_{11} - r_0|$ is the smallest. So, the effect vector $u_{11} = (2, 2, 2)$ of the decision scheme $s_{11}$ enters the grey target. Hence, renovation is a satisfactory decision.

## 10.4 Other Approaches for Grey Decision

### *10.4.1 Grey Relational Decision*

The bull's-eye distance between a decision effect vector and the center of the target measures the superiority of the scheme in comparison with other schemes. At the same time, the grey relational degree between the effect vector of a decision scheme and the optimum effect vector can be seen as another way to evaluate the superiority of a decision scheme.

**Definition 10.4.1** Let $S = \{s_{ij} = (a_i, b_j) | a_i \in A, b_j \in B\}$ be a set of decision schemes, and $u_{i_0j_0} = \{u^{(1)}_{i_0j_0}, u^{(2)}_{i_0j_0}, \ldots, u^{(s)}_{i_0j_0}\}$ the optimum effect vector. If the decision scheme corresponding to $u_{i_0j_0}$ satisfies $u_{i_0j_0} \notin S$, then $u_{i_0j_0}$ is known as an imagined optimum effect vector, and $s_{i_0j_0}$ the imagined optimum scheme.

**Proposition 10.4.1** *Let S be the same as above and the effect vector of scheme* $s_{ij}$ *is* $u_{ij} = \{u^{(1)}_{ij}, u^{(2)}_{ij}, \ldots, u^{(s)}_{ij}\}, = \{u^{(1)}_{ij}, u^{(2)}_{ij}, \ldots, u^{(s)}_{ij}\}$, *for* $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

(1) When $k$ is an objective such that the greater its effect value is the better, let $u^{(k)}_{i_0j_0} = \max_{1\le i\le n, 1\le j\le m}\left\{u^{(k)}_{ij}\right\}$;
(2) When $k$ is an objective such that the closer to a fixed moderate value $u_0$ its effect value is the better, let $u^{(k)}_{i_0j_0} = u_0$; and
(3) When $k$ is an objective such that the smaller its effect value is the better, let $u^{(k)}_{i_0j_0} = \min_{1\le i\le n, 1\le j\le m}\left\{u^{(k)}_{ij}\right\}$,

then $u_{i_0j_0} = \left\{u^{(1)}_{i_0j_0}, u^{(2)}_{i_0j_0}, \ldots, u^{(s)}_{i_0j_0}\right\}$ is the imagined optimum effect vector.

**Proposition 10.4.2** *Assume the same as in Proposition 10.3.1 and let* $u_{i_0j_0} = \left\{u^{(1)}_{i_0j_0}, u^{(2)}_{i_0j_0}, \ldots, u^{(s)}_{i_0j_0}\right\}$ *be the imagined optimum effect vector,* $\varepsilon_{ij}$ *the grey absolute relational degree between* $u_{ij}$ *and* $u_{i_0j_0}$, *for* $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$. *If for any* $i \in \{1, 2, \ldots, n\}$ *and* $j \in \{1, 2, \ldots, m\}$ *satisfying* $i \ne i_1$ *and* $j \ne j_1$, $\varepsilon_{i_1j_1} \ge \varepsilon_{ij}$ *always holds true, then* $u_{i_1j_1}$ *is a quasi-optimum effect vector and* $s_{i_1j_1}$ *a quasi-optimum decision scheme.*

Grey relational decisions can be made by following the following steps:

Step 1: Determine the set of events $A = \{a_1, a_2, \ldots, a_n\}$ and the set of countermeasures $B = \{b_1, b_2, \ldots, b_m\}$. And then construct the set of decision schemes $S = \{s_{ij} = (a_i, b_j) | a_i \in A, b_j \in B\}$.
Step 2: Choose the objectives 1, 2, …, $s$, for the decision-making.
Step 3: Compute the effect values $u^{(k)}_{ij}$ of the individual decision scheme $s_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, with respect to objective $k$, obtained in the decision effect sequence $u^{(k)}$

$$u^{(k)} = \left(u^{(k)}_{11}, u^{(k)}_{12}, \ldots u^{(k)}_{1m}; u^{(k)}_{21}, u^{(k)}_{22}, \ldots u^{(k)}_{2m}; \ldots; u^{(k)}_{n1}, u^{(k)}_{n2}, \ldots u^{(k)}_{nm}\right); k = 1, 2, \ldots, s\cdot$$

Step 4: Compute the average image of the decision effect sequence $u^{(k)}$ with respect to objective $k$, which is still written the same as

$$u^{(k)} = \left(u^{(k)}_{11}, u^{(k)}_{12}, \ldots u^{(k)}_{1m}; u^{(k)}_{21}, u^{(k)}_{22}, \ldots u^{(k)}_{2m}; \ldots; u^{(k)}_{n1}, u^{(k)}_{n2}, \ldots u^{(k)}_{nm}\right); k = 1, 2, \ldots, s$$

Step 5: Based on the results of Step 4, write out the effect vector $u_{ij} = \left\{u^{(1)}_{ij}, u^{(2)}_{ij}, \ldots, u^{(s)}_{ij}\right\}$ of decision scheme $s_{ij}$, for $i = 1, 2, \ldots, n$,$j = 1, 2, m$.

Step 6: Compute the imagined optimum effect vector $u_{i_0j_0} = \left\{ u_{i_0j_0}^{(1)}, u_{i_0j_0}^{(2)}, \ldots, u_{i_0j_0}^{(s)} \right\}$.

Step 7: Calculate the grey absolute relational degree $\varepsilon_{ij}$ between $u_{ij}$ and $u_{i_0j_0}$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$.

Step 8: From $\max_{1 \le i \le n, 1 \le j \le m}\left\{\varepsilon_{ij}\right\} = \varepsilon_{i_1j_1}$, the quasi-optimum effect vector $u_{i_1j_1}$ and the quasi-optimum decision scheme $s_{i_1j_1}$ are obtained.

**Theorem 10.4.1** *Let us look at grey relational decision-making regarding the evaluation of looms.*

**Solution** Let us denote the event of evaluating loom models by $a_1$. Then the event set is $A = \{a_1\}$. There are three loom models under consideration: Model 1: purchase projectile loom, which is treated as countermeasure $b_1$; Model 2: select air jet loom, which is treated as countermeasure $b_2$; Model 3: choose rapier loom, which is treated as countermeasure $b_3$. Thus, the set of countermeasure is $B = \{b_1, b_2, b_3\}$, and the set of decision schemes is $S = \left\{s_{ij} = \left(a_i, b_j\right)\middle| a_i \in A, b_j \in B\right\} = \{s_{11}, s_{12}, s_{13}\}$.

Now, let us determine the objectives. According to the functionality of looms, eleven objectives are chosen. The weft-insertion rate (m/min) of the looms is objective 1. The efficiency of the looms is objective 2. The total investment (in ten thousand US$) on the looms is objective 3. The total energy cost (W/a) is objective 4. The total area ($m^2$) of the land to be occupied by the looms is objective 5. The total manpower (person) is objective 6. The quantity of weft yarn waste (cm/weft) is objective 7. The cost of replacement parts (ten thousand Yuan/a) is objective 8. Noise (dB) is objective 9. The quality of the produced fabric is objective 10. And, the adaptability of the type of loom is objective 11.

Under the assumptions that the above-mentioned three loom models produce the same kind of grey fabric meeting the same set of requirements, and that these looms will produce the same amount of annual output, let us conduct the associated computations for the said loom models. Our quantitative calculations lead to relevant values for the objectives, some of which determined from the literature and field investigations (see Table 10.1).

In the following equations, we compute decision effect sequences $U^k (k = 1, 2, \ldots, 11)$ with respect to the objectives.

For objective 1, we have $U^{(1)} = \left(u_{11}^{(1)}, u_{12}^{(1)}, u_{13}^{(1)}\right) = (1000, 1200, 800)$.

For objective 2, we have $U^{(2)} = \left(u_{11}^{(2)}, u_{12}^{(2)}, u_{13}^{(2)} = (92, 90, 92)\right)$.

For objective 3, we have $U^{(3)} = (u_{11}^{(3)}, u_{12}^{(3)}, u_{13}^{(3)}) = (880, 336, 612)$.

For objective 4, we have $U^{(4)} = (u_{11}^{(4)}, u_{12}^{(4)}, u_{13}^{(4)}) = (374, 924, 816)$.

For objective 5, we have $U^{(5)} = \left(u_{11}^{(5)}, u_{12}^{(5)}, u_{13}^{(5)}\right) = (1760, 1092, 2124)$.

For objective 6, we have $U^{(6)} = \left(u_{11}^{(6)}, u_{12}^{(6)}, u_{13}^{(6)}\right) = (18, 22, 24)$.

For objective 7, we have $U^{(7)} = \left(u_{11}^{(7)}, u_{12}^{(7)}, u_{13}^{(7)}\right) = (5, 6, 10)$.

For objective 8, we have $U^{(8)} = \left(u_{11}^{(8)}, u_{12}^{(8)}, u_{13}^{(8)}\right) = (37, 35, 75)$.

**Table 10.1** Objective values for the looms

| Model | Projectile loom | Air jet loom | Rapier loom |
|---|---|---|---|
| Weft-insertion rate (m/min) | 1000 | 1200 | 800 |
| Efficiency (%) | 92 | 90 | 92 |
| Total investment (10 K US$) | 880 | 336 | 612 |
| Total energy consumption (W/a) | 374 | 924 | 816 |
| Total land needed ($m^2$) | 1760 | 1092 | 2124 |
| Total manpower (person) | 18 | 22 | 24 |
| Quantity of weft yarn waste (cm/weft) | 5 | 6 | 10 |
| Cost of parts (10K ¥/a) | 37 | 35 | 75 |
| Noise (dB) | 85 | 91 | 91 |
| Quality | Best | Good | Fine |
| Adaptability | Good | Better | Best |

For objective 9, we have $U^{(9)} = \left(u_{11}^{(9)}, u_{12}^{(9)}, u_{13}^{(9)}\right) = (85, 91, 91)$.

For objective 10, we have $U^{(10)} = \left(u_{11}^{(10)}, u_{12}^{(10)}, u_{13}^{(10)}\right) = (best, good, fine)$.

For objective 11, we have $U^{(11)} = \left(u_{11}^{(11)}, u_{12}^{(11)}, u_{13}^{(11)}\right) = (good, better, best)$.

Quantify the last two qualitative objectives as follows:

$$U^{(10)} = \left(u_{11}^{(10)}, u_{12}^{(10)}, u_{13}^{(10)}\right) = (9, 8, 7)$$

$$U^{(11)} = \left(u_{11}^{(11)}, u_{12}^{(11)}, u_{13}^{(11)}\right) = (8, 7, 9)$$

We now compute the average images of the decision effect sequences for each of the objectives:

$U^{(1)} = (1, 1.2, 0.8)$; $U^{(2)} = (1.01, 0.98, 1.01)$; $U^{(3)} = (1.44, 0.55, 1.01)$

$U^{(4)} = (0.53, 1.31, 1.16)$; $U^{(5)} = (1.06, 0.66, 1.28)$; $U^{(6)} = (0.84, 1.03, 1.13)$

$U^{(7)} = (0.71, 0.86, 1.43)$; $U^{(8)} = (0.76, 0.71, 1.53)$; $U^{(9)} = (0.96, 1.02, 1.02)$

$U^{(10)} = (1.13, 1, 0.87)$; *and* $U^{(11)} = (1, 0.87, 1.13)$.

We also compute the effect vectors $U_{ij}$ of decision schemes $s_{ij}$, $i = 1, j = 1, 2, 3$:

$$U_{11} = \left(u_{11}^{(1)}, u_{11}^{(2)}, \ldots, u_{11}^{(11)}\right) = (1, 1.01, 1.44, 0.53, 1.06, 0.84, 0.71, 0.76, 0.96, 1.13, 1)$$

$$U_{12} = \left(u_{12}^{(1)}, u_{12}^{(2)}, \ldots, u_{12}^{(11)}\right) = (1.2, 0.98, 0.55, 1.31, 0.66, 1.03, 0.86, 0.71, 1.02, 1, 0.87)$$

$$U_{13} = \left(u_{13}^{(1)}, u_{13}^{(2)}, \ldots, u_{13}^{(11)}\right) = (0.8, 1.01, 1.01, 1.16, 1.28, 1.13, 1.43, 1.53, 1.02, 0.87, 1.13), \textit{ and}$$

According to the principle of constituting optimum reference sequences, from the average images of the decision effect sequences of the objectives, it follows that:

For objective 1, the greater the effect value is the better, so $U_{i_0j_0}^{(1)} = \max\left\{u_{ij}^{(1)}\right\} = u_{12}^{(1)} = 1.2$;

For objective 1, the higher the effect value is the better, so $U_{i_0j_0}^{(2)} = \max\left\{u_{ij}^{(2)}\right\} = u_{11}^{(2)} = 1.01$;

For objective 3, the smaller effect value is the better, so $U_{i_0j_0}^{(3)} = \min\left\{u_{ij}^{(3)}\right\} = u_{12}^{(3)} = 0.55$;

For objective 4, the smaller effect value is the better, so $U_{i_0j_0}^{(4)} = \min\left\{u_{ij}^{(4)}\right\} = u_{11}^{(4)} = 0.53$;

For objective 5, the smaller effect value is the better, so $U_{i_0j_0}^{(5)} = \min\left\{u_{ij}^{(5)}\right\} = u_{11}^{(5)} = 0.66$;

For objective 6, the smaller effect value is the better, so $U_{i_0j_0}^{(6)} = \min\left\{u_{ij}^{(6)}\right\} = u_{11}^{(6)} = 0.84$;

For objective 7, the smaller effect value is the better, so $U_{i_0j_0}^{(7)} = \min\left\{u_{ij}^{(7)}\right\} = u_{11}^{(7)} = 0.71$;;

For objective 8, the smaller effect value is the better, so $U_{i_0j_0}^{(8)} = \min\left\{u_{ij}^{(8)}\right\} = u_{12}^{(8)} = 0.71$;;

For objective 9, the smaller effect value is the better, so $U_{i_0j_0}^{(9)} = \min\left\{u_{ij}^{(9)}\right\} = u_{11}^{(9)} = 0.96$;

For objective 10, the higher effect value is the better, so $U_{i_0j_0}^{(10)} = \max\left\{u_{ij}^{(10)}\right\} = u_{11}^{(10)} = 1.13$; *and*.

For objective 11, the higher effect value is the better, so $U_{i_0j_0}^{(11)} = \max\left\{u_{ij}^{(11)}\right\} = u_{13}^{(11)} = 1.13$;.

That is, we obtain the following optimum reference sequence:

$$\begin{aligned} U_{i_0j_0} &= \left(u_{i_0j_0}^{(1)}, u_{i_0j_0}^{(2)}, \ldots, u_{i_0j_0}^{(11)}\right) \\ &= (1.2, 1.01, 0.55, 0.53, 0.66, 0.84, 0.71, 0.71, 0.96, 1.13, 1.13) \end{aligned}$$

From $u_{ij}$ and $u_{i_0j_0}$, we compute the absolute grey relational degrees:

$$\varepsilon_{11} = 0.628, \varepsilon_{12} = 0.891, \varepsilon_{13} = 0.532$$

From the definition of grey relational decision-making, it follows that because $\max\left\{\varepsilon_{ij}\right\} = \varepsilon_{12} = 0.891$, $U_{12}$ is the quasi-optimum vector and $s_{12}$ the quasi-optimum decision scheme. That is to say, in terms of producing general grey fabric, the air jet loom is the best choice among the available loom models.

### 10.4.2 Grey Development Decision

Grey development decision-making is done based on the development tendency or the future behaviors of the decision scheme of concern. It does not necessarily place specific emphasis on the current effect of the scheme. Instead it focuses more on the change of the decision effect over time. This method of decision-making can be and has been employed for long-term planning as well as the decision-making of large scale engineering projects and urban planning. It looks at problems from the angle of development while attempting to make feasible arrangements and avoiding repetitious constructions so that great savings of capital and manpower can be achieved. What we have discussed earlier are static decision schemes with a fixed time moment. Because we now involve the concept of time, as time moves, constantly changing decision effects are considered.

**Definition 10.4.2** Assume that $A = \{a_1, a_2, \ldots, a_n\}$ is a set of events, $B = \{b_1, b_2, \ldots, b_m\}$ a set of countermeasures, and $S = \{s_{ij} = (a_i, b_j) | a_i \in A, b_j \in B\}$ the set of decision schemes. Then,

$$u_{ij}^{(k)} = \left(u_{ij}^{(k)}(1), u_{ij}^{(k)}(2), \ldots, u_{ij}^{(k)}(h)\right)$$

is known as the decision effect time series of scheme $s_{ij}$ with respect to objective $k$.

**Definition 10.4.3** Let the decision effect time series of the scheme $s_{ij}$ with respect to objective $k$ be

$$u_{ij}^{(k)} = \left(u_{ij}^{(k)}(1), u_{ij}^{(k)}(2), \ldots, u_{ij}^{(k)}(h)\right)$$

$\hat{a}_{ij}^{(k)} = \left[a_{ij}^{(k)}, b_{ij}^{(k)}\right]^T$ the least squares estimate of the parameters of the EGM(1,1) model of $u_{ij}^{(k)}$. Then the inverse accumulation restoration of the EGM(1,1) time response of $u_{ij}^{(k)}$ is given by

$$\hat{u}_{ij}^{(k)}(l+1) = \left[1 - \exp\left(a_{ij}^{(k)}\right)\right] \cdot \left[u_{ij}^{(k)}(1) - \frac{b_{ij}^{(k)}}{a_{ij}^{(k)}}\right] \exp\left(-a_{ij}^{(k)} \cdot l\right)$$

Assume that the restored sequence through inverse accumulation of the EGM(1,1) time response of the decision effect time series of the scheme $s_{ij}$ with respect to objective $k$ is

$$\hat{u}_{ij}^{(k)}(l+1) = \left[1 - \exp\left(a_{ij}^{(k)}\right)\right] \cdot \left[u_{ij}^{(k)}(1) - \frac{b_{ij}^{(k)}}{a_{ij}^{(k)}}\right] \exp\left(-a_{ij}^{(k)} \cdot l\right)$$

When objective $k$ satisfies that the greater the effect value is the better, if

(1) $\max_{1\le i\le n,1\le j\le m}\left\{-a_{ij}^{(k)}\right\} = -a_{i_0j_0}^{(k)}$, then $s_{i_0j_0}$ is known as the optimum scheme of development coefficients with respect to objective $k$;

(2) $\max_{1\le i\le n,1\le j\le m}\left\{\hat{u}_{ij}^{(k)}(h+l)\right\} = \hat{u}_{i_0j_0}^{(k)}(h+l)$, then $s_{i_0j_0}$ is known as the optimum scheme of predictions with respect to objective $k$.

Similarly, the concepts of optimum schemes of development coefficients and predictions can be defined for cases of objectives satisfying that the smaller the effect value is the better, and that the closer to a moderate value the effect value is the better, respectively. In particular, for objectives satisfying that the smaller the effect value is the better, one only needs to replace "max" in the items (1) and (2) above by "min"; if $k$ is an objective satisfying that the closer to a fixed moderate value the effect value is the better, one can determine the moderate value of the development coefficients or predicted values at first; then define the optimum scheme based on the distances of the development coefficients or predicted values to the moderate value.

In practical applications, one may face the scenarios that either both the optimum scheme of development coefficients and predictions are the same, or that they are different. Even so, the following theorem tells us that eventually these optimum schemes would converge into one.

**Theorem 10.4.1** *Assume that k is such an objective that the greater its effect value is the better,* $s_{i_0j_0}$ *is the optimum scheme of development coefficients, that is,* $-a_{i_0j_0}^{(k)} = \max_{1\le i\le n,1\le j\le m}\left\{-a_{ij}^{(k)}\right\}$, and $\hat{u}_{i_0j_0}^{(k)}(h+l+1)$ *is the predicted value for the decision effect of* $s_{i_0j_0}$. *Then there must be* $l_0 > 0$ *such that*

$$\hat{u}_{i_0j_0}^{(k)}(h+l_0+1) = \max_{1\le i\le n,1\le j\le m}\left\{\hat{u}_{ij}^{(k)}(h+l_0+1)\right\}$$

*That is, in a sufficiently distant future,* $s_{i_0j_0}$ *will also be the optimum scheme of predictions.*

***Proof*** See Liu and Lin (2006, p. 340–341) for details.

Similar results hold true for those objectives satisfying either that the smaller the effect value is the better or that the closer to a fixed moderate value the effect value is the better.

At this junction, careful readers might have noticed that Theorem 10.4.1 does not state the case that there are some increasing and decreasing sequences among decision effect time series at the same time. As a matter of fact, for objectives satisfying that the greater the effect value is the better, there is no need to consider decreasing decision effect time series. For objectives satisfying that the smaller the effect value is the better, all increasing decision effect time series are deleted in advance in all discussions. As for objectives satisfying that the closer to a moderate value the effect value is the better, one can consider only either increasing or decreasing decision effect time series depending on the circumstances involved.

### *10.4.3 Grey Clustering Decision*

Grey cluster decision is useful for synthetic evaluations of objects with respect to several different criteria so that decisions can be made about whether or not an object meets the given standards for inclusion in or exclusion from a set. This method has often been employed for classification decision-making regarding objects or people. For instance, school students can be classified based on their individual capabilities to receive information, to comprehend what is provided, and to grow so that different teaching methods can be applied and different students can be enrolled in different programs. As a second example, based on different sets of criteria, comprehensive evaluations can be done for general employees, technicians, and administrators respectively so that decisions can be made regarding who is qualified for his/her job, who is ready for a promotion, and so on.

**Definition 10.4.4** Assume that there are $n$ objects to make decisions on, $m$ criteria, $s$ different grey classes, the quantified evaluation value of object $i$ with respect to criterion $j$ is $x_{ij}, f_j^k(*)$ are the possibility functions of the $k$th grey class with respect to the $j$th criterion, and $w_j$ is the synthetic decision-making weight of criterion $j$ such that $\sum_{j=1}^{m} w_j = 1, i = 1, 2, \ldots, n, j = 1,\ 2, \ldots, m, k = 1, 2, \ldots, s$. Then

$$\sigma_i^k = \sum_{j=1}^{m} f_j^k\left(x_{ij}\right)w_j$$

is known as the decision coefficient for the object $i$ to belong to grey class $k$; $\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^s)$ is known as the decision coefficient vector of object $i$, $i = 1, 2, \ldots, n$; and $\sum = \left(\sigma_i^k\right)_{n\times s}$ the decision coefficient matrix. If $\max_{1\le k\le s}\left\{\sigma_i^k\right\} = \sigma_i^{k^*}$, then the decision is that the object $i$ belongs to grey class $k^*$.

In practical applications, it is quite often the case that many objects belong to the same decision grey class at the same time, while there is a constraint on how many objects are allowed in the grey class. When this occurs, we can further determine individual objects' precedence in grey class $k^*$ on the basis of the size of integrate clustering coefficients.

## 10.5 Multi-attribute Weighted Intelligent Grey Target Decision Model

In this section, we will study a new decision model, which is constructed on the basis of four new functions of uniform effect measures. This new decision model sufficiently considers the two different scenarios of whether or not the effect values of the objectives actually hit the targets with very clear physics significance. First, a grey target is defined as a satisfying region, which a decision maker wants to reach, with an inside ideal point across multiple objectives. To facilitate the uniform distance measure of a decision strategy to the pre-defined grey target, four kinds of measure procedures are designed including the effect measures for benefit-type objectives and cost-type objectives, the lower effect measure for moderate-type objectives, and the upper effect measure for moderate-type according to three types of decision objective including benefit objective, cost objective, and non-monotonic objective with a most preferred middle value. Then, a matrix of synthetic effect measures can be easily obtained based on the uniform distance measure of a decision strategy to the grey target over different objectives. Based upon the obtained matrix information, different decision strategies can be evaluated easily and comprehensively. The proposed method has a clear physical meaning as missing target, hitting target as well as hitting performance.

### *10.5.1 The Uniform Effect Measure*

**Definition 10.5.1**

(1) Let $k$ be a benefit type objective, that is, for $k$ the larger the effect value is the better, and the decision grey target of objective $k$ is $u_{ij}^{(k)} \in \left[u_{i_0j_0}^{(k)}, \max_i \max_j\left\{u_{ij}^{(k)}\right\}\right]$, that is, $u_{i_0j_0}^{(k)}$ stands for the threshold effect value of objective $k$. Then

$$r_{ij}^{(k)} = \frac{u_{ij}^{(k)} - u_{i_0j_0}^{(k)}}{\max_i \max_j\left\{u_{ij}^{(k)}\right\} - u_{i_0j_0}^{(k)}}. \tag{10.2}$$

is referred to as the effect measure of a benefit-type objective.

(2) Let $k$ be a cost-type objective, that is, for $k$ the smaller the effect value is the better, and the decision grey target of objective $k$ is $u_{ij}^{(k)} \in \left[\min_i \min_j\left\{u_{ij}^{(k)}\right\}, u_{i_0j_0}^{(k)}\right]$, that is, $u_{i_0j_0}^{(k)}$ stands for the threshold effect value of objective $k$. Then

$$r_{ij}^{(k)} = \frac{u_{i_0j_0}^{(k)} - u_{ij}^{(k)}}{u_{i_0j_0}^{(k)} - \min_i \min_j\left\{u_{ij}^{(k)}\right\}} \tag{10.3}$$

is referred to as the effect measure of cost-type objective.

(3) Let $k$ be a moderate-value type objective, that is, for $\eta_k$ the closer to a moderate value $A$ the effect value is the better, and the decision grey target of objective $\eta_k$ is $u_{ij}^{(k)} \in \left[A - u_{i_0j_0}^{(k)}, A + u_{i_0j_0}^{(k)}\right]$, that is, both $A - u_{i_0j_0}^{(k)}$ and $A + u_{i_0j_0}^{(k)}$ are respectively the lower and upper threshold effect values of objective $k$. Then,

(i) When $u_{ij}^{(k)} \in \left[A - u_{i_0j_0}^{(k)}, A\right]$,

$$r_{ij}^{(k)} = \frac{u_{ij}^{(k)} - A + u_{i_0j_0}^{(k)}}{u_{i_0j_0}^{(k)}} \tag{10.4}$$

is referred to as the lower effect measure of moderate-value type objective.

(ii) When $u_{ij}^{(k)} \in \left[A, A + u_{i_0j_0}^{(k)}\right]$,,

$$r_{ij}^{(k)} = \frac{A + u_{i_0j_0}^{(k)} - u_{ij}^{(k)}}{u_{i_0j_0}^{(k)}} \tag{10.5}$$

is referred to as the upper effect measure of moderate-value type objective.

The effect measures of benefit-type objectives reflect the degrees of both how close the effect sample values are to the maximum sample values and how far away they are from the threshold effect values of the objectives. Similarly, the effect measures of cost-type objectives represent how close the effect sample values are to the minimum effect sample values and how far away the effect sample values are from the threshold effect values of the objectives; the lower effect measures of moderate-value type objectives indicate how far away the effect sample values that are smaller than the moderate value $A$ are from the lower threshold effect value, and the upper effect measures indicate how far away the effect sample values that are greater than the moderate value $A$ are from the upper threshold effect values of the objectives.

For situations of missing targets, there are the following four different possibilities:

(1) The effect value of a benefit-type objective is smaller than the threshold value $u_{i_0j_0}^{(k)}$, that is,$u_{ij}^{(k)} < u_{i_0j_0}^{(k)} l$;
(2) The effect value of a cost-type objective is greater than the threshold value $u_{i_0j_0}^{(k)}$, that is,$u_{ij}^{(k)} > u_{i_0j_0}^{(k)}$;
(3) The effect value of a moderate-value type objective is smaller than the lower threshold effect value $A - u_{i_0j_0}^{(k)}$, that is, $u_{ij}^{(k)} < A - u_{i_0j_0}^{(k)}$; and
(4) The effect value of a moderate-value type objective is greater than the upper threshold effect value $A + u_{i_0j_0}^{(k)}$, that is,$u_{ij}^{(k)} > A + u_{i_0j_0}^{(k)}$.

In order for the effect measures of each type of objective to satisfy the condition of normality, that is, $r_{ij}^{(k)} \in [-1, 1]$, without loss of generality, we can assume that:

For a benefit-type objective, $u_{ij}^{(k)} \geq -\max_i \max_j \left\{u_{ij}^{(k)}\right\} + 2u_{i_0j_0}^{(k)}$;

For a benefit-type objective, $u_{ij}^{(k)} \leq -\min_i \min_j \left\{u_{ij}^{(k)}\right\} + 2u_{i_0j_0}^{(k)}$;

For cases where the effect value of a moderate-value type objective is smaller than the lower threshold effect value $A - u_{i_0j_0}^{(k)}$, $u_{ij}^{(k)} \geq A - 2u_{i_0j_0}^{(k)}$, ; and

For cases where the effect value of a moderate-value type objective is greater than the upper threshold effect value $A + u_{i_0j_0}^{(k)}$, $u_{ij}^{(k)} \leq A + 2u_{i_0j_0}^{(k)}$.

With these assumptions, we have the proposition below.

**Proposition 10.5.1** *The effect measures* $r_{ij}^{(k)}(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, s)$,*, as defined in Definition 4.1, satisfy the following properties*:

(1) $r_{ij}^{(k)}$ is non-dimensional; (2) the more ideal the effect, the larger $r_{ij}^{(k)}$ is; and (3) $r_{ij}^{(k)} \in [-1, 1]$.

**Definition 10.5.2** $r_{ij}^{(k)}(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, s)$, as defined in Definition 4.1, is called uniform effect measure of decision scheme $s_{ij}$.For decision scheme $s_{ij}$ of hitting the target, $r_{ij}^{(k)} \in [0, 1]$; and for decision scheme $s_{ij}$ of missing the target, $r_{ij}^{(k)} \in [-1, 0]$.

**Definition 10.5.3** For a given set $S$, define $R^{(k)} = \left(r_{ij}^{(k)}\right)_{n\times m}$ as the matrix of uniform effect measure of $S$ with respect to objective $k$. For $s_{ij} \in S$, $r_{ij} = \left(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(s)}\right)$ is known as the vector of uniform effect measure of the decision scheme $s_{ij}$.

## 10.5.2 The Weighted Synthetic Effect Measure

**Definition 10.5.4** Assume that $\eta_k$ stands for the decision weight of objective $k$, $k = 1, 2, \ldots, s$, satisfying $\sum_{k=1}^{s} \eta_k = 1$, then $\sum_{k=1}^{s} \eta_k \cdot r_{ij}^{(k)}$ is called a weighted synthetic effect measure of the decision scheme $s_{ij}$, which is still denoted as $r_{ij} = \sum_{k=1}^{s} \eta_k \cdot r_{ij}^{(k)}$; and $R = \left(r_{ij}\right)_{n\times m}$ is known as the matrix of weighted synthetic effect measures.

In the case of weighted synthetic effect measures, $r_{ij} \in [-1, 0]$ belongs to the decision scheme $s_{ij}$ of missing the target, while $r_{ij} \in [0, 1]$ belongs to the decision scheme $s_{ij}$ of hitting the target. For the decision scheme of hitting the target, we can further compare the superiority of events $a_i$, countermeasures $b_j$, and decision schemes $s_{ij}$ respectively by using the magnitudes of the weighted synthetic effect measures, $i = 1, 2, ..., n, j = 1, 2, ..., m$.

**Definition 10.5.5** (1) If $\max_{1\le j\le m}\left\{r_{ij}\right\} = r_{ij_0}$, then $b_{j_0}$ is known as the optimum countermeasure of event $a_i$; (2) If $\max_{1\le i\le n}\left\{r_{ij}\right\} = r_{i_0 j}$, then $a_{i_0}$ is known as the optimum event corresponding to countermeasure $b_j$; (3) If The weighted multi-attribute grey target decision can be made by following the below:

Step 1: Based on the set $A = \{a_1, a_2, \ldots, a_n\}$ of events and the set $B = \{b_1, b_2, \ldots, b_m\}$ of countermeasures, construct the set of decision schemes $S = \left\{s_{ij} = \left(a_i, b_j\right) \middle| a_i \in A, b_j \in B\right\}$;

Step 2: Determine the decision objectives $k = 1, 2, \ldots, s$;

Step 3: Determine the decision weights $\eta_1, \eta_2, \ldots, \eta_s$ of the objectives;

Step 4: For each objective $k = 1, 2, \ldots, s$, compute the corresponding observed effect matrix $U^{(k)} = \left(u_{ij}^{(k)}\right)_{n\times m}$;

Step 5: Determine the threshold effect value of objective $k = 1, 2, \ldots, s$;

Step 6: Calculate the matrix $R^{(k)} = \left(r_{ij}^{(k)}\right)_{n\times m}$ of uniform effect measures of objective $k = 1, 2, \ldots, s$;

Step 7: From $r_{ij} = \sum_{k=1}^{s} \eta_k \cdot r_{ij}^{(k)}$, compute the matrix of synthetic effect measures $R = \left(r_{ij}\right)_{n\times m}$; and

Step 8: Determine the optimum decision scheme $s_{i_0 j_0}$.

$\max_{1\le i\le n, 1\le j\le m}\left\{r_{ij}\right\} = r_{i_0 j_0}$, then $s_{i_0 j_0}$ is known as the optimum decision scheme.

The proposed model here has a unique feature of clear physical meaning presented as missing target, hitting target and hitting performance of different decision strategies with a pre-defined grey target. The distance of a strategy to the grey target over different objectives is calculated through effect measure functions as follows: the concept of upper effect measure reflects the distance of the observed effect value from the maximum observed effect value; the concept of lower effect measure indicates the distance between the observed effect value from the minimum observed effect value; and the concept of moderate effect measure tells the distance of the observed effect value from the pre-defined most preferred effect value in the middle.

To aggregate the performance of a strategy over different objectives, one can make use of the concept of upper effect measure for benefit objectives where the larger or the more the effect sample values are the better; for cost objectives where the smaller or the fewer the effect sample values are the better, one can utilize the concept of lower effect measure. As for non-monotonic objectives that require "neither too large nor too small" and/or "neither too many nor too few," one can apply the concept of moderate effect measure. The effect measure for benefit and cost type objectives, the lower effect measure for moderate type, and the upper effect measure for moderate type can be further integrated as uniform effect measures by incorporating weight information over different objectives. The value of uniform effect measures is located in the interval of [-1,1] and has a crystal physical meaning: if a strategy hits the target, the value will be positive and the larger the closer to the ideal point in the grey target; if a strategy misses the target, the value will be negative. The new model has been applied to the selection of the supplier of a key component used in the production of large commercial aircrafts and this application confirmed its feasibility.

***Example 10.5.1*** Let us look at the selection of the supplier of a key component used in the production of large commercial aircrafts.

In China, the production of large commercial aircrafts is managed using the model of main manufacturers—suppliers, where a great amount of key components comes from international suppliers. So, the scientific approach to decision-making regarding the selection of relevant suppliers is a key determinant of the success or failure of the operation. As a typical decision-making problem involved in the production process of sophisticated products, the selection of suppliers is generally accomplished through public bidding. Usually the main manufacturer first lists his demands, then each potential supplier puts together their proposal to outline how they meet the needs of the manufacturer. After collecting the proposals, the manufacturer comprehensively evaluates all the suppliers' submissions to select the optimum proposal and sign the purchase agreement. As for what factors actually affect the manufacturer's decision, it is an extremely complicated matter. In order to arrive at educated and scientifically sound decisions, there is a need to analyze all the involved factors closely and holistically.

During the selection of international suppliers for a specific key component of the production of large commercial aircrafts, there were there suppliers accepted into the second round of the tender. To decide on the eventual supplier, let us go through the following steps.

Step 1: Establish the sets of events, countermeasures, and situations. Let us define event $a_1$ as the selection of a supplier for the said component for the production of large commercial aircrafts. So, the set of events is $A = \{a_1\}$. Define the selection of supplier 1, supplier 2, or supplier 3 to be our countermeasures $b_1$, $b_2$, and $b_3$, respectively, so that the set of countermeasures is $B = \{b_1, b_2, b_3\}\cdot$. Therefore, our set of situations in this case is $S = -\left\{s_{ij} = \left(a_i, b_j\right)\middle| a_i \in A, b_j \in B, i = 1; j = 1, 2, 3\right\} = \{s_{11}, s_{12}, s_{13}\}$.

Step 2: Determine the decision objectives. Through three rounds of surveys with relevant experts, the following 5 objectives are considered: quality, price, time of delivery, design proposal, and competitiveness.

Among these objectives, competitiveness, quality, and design proposal are qualitative. They are scored by relevant experts' evaluations, and the higher the evaluation scores the better. That is, they are benefit-type objectives. Let us take the threshold value $u_{i_0 j_0}^{(k)} = 9, k = 1, 4, 5\cdot$ For the objective of cost, the lower the cost the better. So, it is a cost-type objective. Let us take the threshold value $u_{i_0 J_0}^{(2)} = 15.$ The objective of time of delivery is one of moderate-value type. The main manufacturer desires the delivery at the end of the 16th month with 2 months' deviation allowed. That is $u_{i_0 j_0}^{(3)} = 2$, the lower threshold effect value is $16 - 2 = 14$, and the upper threshold effect value is $16 + 2 = 18$.

Step 3: Determine the decision weights of the objectives. To this end, we apply the Analytic Hierarchy Process(AHP) method (see Table 10.2 for details).

Step 4: Determine the effect sample vectors of each of the objectives:

$$U^{(1)} = (9.5, 9.4, 9), U^{(2)} = (14.2, 15.1, 13.9), U^{(3)} = (15.5, 17.5, 19),$$
$$U^{(4)} = (9.6, 9.3, 9.4), U^{(5)} = (9.5, 9.7, 9.2).$$

**Table 10.2** The weights of objectives

| Objective | Quality | Price | Delivery | Design | Competitiveness |
|---|---|---|---|---|---|
| Unit | Qualitative | Million US$ | Month | Qualitative | Qualitative |
| Order # | 1 | 2 | 3 | 4 | 5 |
| Weight | 0.25 | 0.22 | 0.18 | 0.18 | 0.17 |

Step 5: Assign the threshold effect values for the objectives. Because competitiveness, quality, and design proposal are all benefit-type ss us take the threshold value $u_{i_0 j_0}^{(2)} = 15$. Because time of delivery is a moderate value-type objective and the main manufacturer desires the delivery at the end of the 16th month with a tolerance of $\pm$ 2 months, we set $u_{i_0 j_0}^{(3)} = 2$, the lower threshold effect value $16 - 2 = 14$, and the upper threshold effect value $16 + 2 = 18$.

Step 6: Calculate the vectors of uniform effect measures. For the three qualitative objectives, competitiveness, quality, and design proposal, we employ the effect measures of benefit-type. For the objective of price, we utilize the effect measures of cost-type. For the objective of time of delivery, we apply the lower and upper effect measures. Thus, we obtain the following vectors of uniform effect measures:

$$R^{(1)} = [1, 0.8, 0], R^{(2)} = [0.73, -0.09, 1], R^{(3)} = [0.75, 0.25, -0.5]$$
$$R^{(4)} = [1, 0.5, 0.67], andR^{(5)} = [0.71, 1, 0.29]\cdot$$

Step 7: From, $r_{ij} = \sum_{k=1}^{5} \eta_k \cdot r_{ij}^{(k)}$, we compute the following vector of synthetic effect measures:

$$\left\{\delta_j^k\right\} = 0.40 = \delta_j^3$$

Step 8: Make the final decision. Because $\left\{\delta_j^k\right\} = 0.40 = \delta_j^3$, it means that all these three suppliers have hit the target. This result implies that it is reasonable for these suppliers to enter the second round of the tender. However, based on $\max_{1 \le j \le 3}\left\{r_{1j}\right\} = r_{11} = 0.08463$,, it follows that the main manufacturer should sign the agreement with supplier 1.

## 10.6 On Paradox of Rule of Maximum Value and Its Solution

In cases where more than one object belongs to a class or cluster, people may be confronted with a decision paradox. For example, assume that $\delta_1 = (0.4, 0.35, 0.25)$ and $\delta_2 = (0.41, 0.2, 0.39)$ are the clustering coefficient vectors of objects 1 and 2, respectively. It is demonstrably the case that objects 1 and 2 both belong to class 1, according to the principles of maximum value of clustering coefficient. Also, object 2 is better than object 1 given that $0.41 > 0.4$. However, if we were to consider the values of all the components of $\delta_1, \delta_2$ in an integrated manner, object 1 could be perceived as being superior to object 2. This is a paradox.

In this section, we try and find a solution for the decision paradox by using weight vector group with kernel, weighted coefficient vector of kernel clustering for decision-making and a two-stages decision model.

### 10.6.1 The Weight Vector Group with Kernel

Clustering coefficient vectors cannot be compared with each other because usually they are not unit vectors. Therefore, firstly all clustering coefficient vectors need to be unitized.

**Definition 10.6.1** Assume that $\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^s)$, $i = 1, 2, \ldots, n$ are $n$ clustering coefficient vectors, $\delta_i^k = \frac{\sigma_i^k}{\sum_{k=1}^s \sigma_i^k}$, $\delta_i^k$, $\delta_i^k$ is called unitized clustering coefficient of decision-making object $i$ belonging to class $k$. Clearly, $\delta_i^k (k = 1, 2, \ldots s)$ satisfy $\sum_{i=1}^s \delta_i^k = 1$.

**Definition 10.6.2** $\delta_i = \left(\delta_i^1, \delta_i^2, \ldots, \delta_i^s); (i = 1, 2, \ldots, n\right)$ is called unitized clustering coefficient vectors of decision-making object $i$. The following conclusion about unitized clustering coefficient vector $\delta_i$ is also suitable for non-unitized clustering coefficient vector $\sigma_i$. Therefore, the "unitized" can be omitted.

Sort the components of $\delta_i$ according to their values, that is, $\delta_i^{k_1} \geq \delta_i^{k_2} \geq \cdots \geq \delta_i^{k_l} \geq \cdots \geq \delta_i^{k_s}$.

**Definition 10.6.3** Assume that $\max_{1 \leq k \leq s}\left\{\delta_i^k\right\} = \delta_i^{k^*}$, then $\delta_i^{k^*}$ is called the maximum component of clustering coefficient vector $\delta_i$.

Given that all the corresponding coefficients of two decision coefficient vectors $\delta_i$,$\delta_j$ are equal, then there is no difference between $\delta_i$,$\delta_j$. When two objects $i, j$ belong to a class $k^*$ and the maximum component $\delta_i^{k^*} > \delta_j^{k^*}$, it means that $\delta_i$ is better than $\delta_j$ by the rule of maximum value; but it is possible to think that $\delta_j$ is better than $\delta_i$ if we consider the values of all the components of $\delta_1$, $\delta_2$ in an integrated manner. This is a decision paradox of rule of maximum value.

To solve the decision paradox of rule of maximum value, firstly the weight vector group of kernel clustering is defined. The basic step to solve the paradox is to cluster the information which is included in other components around $\eta_k$, and supporting objects $i$ come under class k into component k. Then it is necessary to obtain a new decision coefficient vector which contains factors included in other components around $\eta_k$.

**Definition 10.6.4** Assume that there are s classes of decision-making, and real numbers $w_k \geq 0, k = 1, 2, \ldots, s$, let

$$\eta_1 = \frac{1}{\sum_{k=1}^s w_k}(w_s, w_{s-1}, w_{s-2}, \ldots, w_1),$$

$$\eta_2 = \frac{1}{w_{s-1} + \sum_{k=2}^s w_k}(w_{s-1}, w_s, w_{s-1}, w_{s-2}, \ldots, w_2),$$

$$\eta_3 = \frac{1}{w_{s-1} + w_{s-2} + \sum_{k=3}^s w_k}(w_{s-2}, w_{s-1}, w_s, w_{s-1}, \ldots, w_3),$$

$$\eta_k = \frac{1}{\sum_{i=s-k+1}^{s-1} w_i + \sum_{i=k}^s w_i}(w_{s-k+1}, w_{s-k+2}, \ldots, w_{s-1}, w_s, w_{s-1}, \ldots, w_k),$$

$$\eta_{s-1} = \frac{1}{w_{s-1} + \sum_{k=2}^{s} w_k}(w_2, w_3, \ldots w_{s-1}, w_s, w_{s-1}),$$
$$\eta_s = \frac{1}{\sum_{k=1}^{s} w_k}(w_1, w_2, w_3, \ldots, w_{s-1}, w_s),$$

then $\eta_k (k = 1, 2, \ldots, s)$ is called a weight vector group with kernel.

Note: s-dimensional vector $\eta_k = \left(\eta_k^1, \eta_k^2, \ldots, \eta_k^s\right)(k = 1, 2, \ldots, s)$ is the multiplication of scalar $a_k = \frac{1}{\sum_{i=s-k+1}^{s-1} w_i + \sum_{i=k}^{s} w_i}$ with vector $\zeta_k$, where the function of scalar factor $a_\text{k}$ is to ensure $\eta_k (k = 1, 2, \ldots, s)$ is a normalized vector. Also, the k-th component of vector factor $\zeta_k (k = 1, 2, \ldots, s)$ is $w_s$, which is the maximum component of $\zeta_{k.}$ Then the k-th component $w_s$ can be taken as a center, and the other components on both sides of the k-th component $w_s$ descend step by step. The k-th component with the largest contribution for the decision-making object belongs to grey class k, so the k-th component of $\zeta_k$ should take the maximum weight $w_s$. The values of other components are set by the principle which states that "the component which is closest to the k-th component has the largest contribution for object *I* belonging to class k, so it is given the largest weight; the component which is farthest from the k-th component has the smallest contribution for object *I* belonging to class k, so it is given the smallest weight".

## *10.6.2 The Weighted Coefficient Vector of Kernel Clustering for Decision-Making*

**Definition 10.6.5** Assume there are $n$ decision objects and $s$ different grey classes, then $\omega_i^k = \eta_k \cdot \delta_i^T$ is called the weighted coefficient of kernel clustering for decision-making of object $_i$ about grey class $k$. And.

$$\omega_i = \left(\omega_i^1, \omega_i^2, \ldots, \omega_i^s\right); i = 1, 2, \ldots, n$$

is called the weighted coefficient vector of kernel clustering for decision-making of object $i$.

**Definition 10.6.6** Let $\max_{1 \le k \le s}\left\{\omega_{i_1}^k\right\} = \omega_{i_1}^{k^*}$, $\max_{1 \le k \le s}\left\{\omega_{i_2}^k\right\} = \omega_{i_2}^{k^*}$, when $\omega_{i_1} > \omega_{i_2}$, then decision object $i_1$ is better than decision object $i_2$ in grey class $k^*$.

**Definition 10.6.7** Let $\max_{1 \le k \le s}\left\{\omega_{i_1}^k\right\} = \omega_{i_1}^{k^*}, \max_{1 \le k \le s}\left\{\omega_{i_2}^k\right\} = \omega_{i_2}^{k^*}, \max_{1 \le k \le s}\left\{\omega_{i_l}^k\right\} = \omega_{i_l}^{k^*}$, in other words, objects $i_1, i_2, \ldots, i_l$ all belong to grey class $k^*$. Also, $\omega_{i_1} > \omega_{i_2} > \cdots > \omega_{i_l}$, and if the number of objects contained in the decision grey class $k^*$ is $l_1$, then objects $i_1, i_2, \ldots, i_{l_1}$ are called the taken object of grey class $k^*$, and the rest of the objects are called the candidates of grey class $k^*$.

The two stages decision model to solve the decision paradox by the weight vector group with kernel and the weighted coefficient vector of kernel clustering for decision-making can be constructed step by step as outlined below.

**Stage 1**

Step 1: Compute normalized clustering coefficient vector $\delta_i$

$$\delta_i = (\delta_i^1, \delta_i^2, \ldots, \delta_i^s); \ (i = 1, 2, \ldots, n)$$

Step 2: Estimate the distinguishability of the clustering coefficient vectors of objects belonging to class $k^*$. If the order of priority of the objects i belonging to class $k^*$ is easy to identify, turn to step 6**;** in cases where the order of priority of the objects belonging to class $k^*$ is difficult to identify, turn to step 3;

**Stage 2**

Step 3: Set the weight vector group with kernel $(\eta_1, \eta_2, \ldots, \eta_s)$;

Step 4: Calculate the weighted coefficient vector of kernel clustering for decision-making of object $i$.
$\omega_i = (\omega_i^1, \omega_i^2, \ldots, \omega_i^s); \ i = 1, 2, \ldots, n;$

Step 5: Determine object $i$ belonging to grey class $k^*$ by $\max_{1 \le k \le s}\left\{\omega_i^k\right\} = \omega_i^{k^*}$;

Step 6: Sort the decision objects which belong to class $k^*$ according to the values of $\delta_{i_1}^{k^*}, \delta_{i_2}^{k^*}, \ldots, \delta_{i_l}^{k^*}$ for case where there are $l$ objects belonging to class $k^*$.

### *10.6.3 Several Functional Weight Vector Groups with Kernel*

**Proposition 10.6.1** *Assume that*

$$\eta_1 = \frac{2}{s(s+1)}(s, s-1, s-2, \ldots, 1)$$

$$\eta_2 = \left(\frac{1}{\frac{s(s+1)}{2} + (s-2)}\right)(s-1, s, s-1, s-2, \ldots, 2)$$

$$\eta_3 = \left(\frac{1}{\frac{s(s+1)}{n} + (2s-6)}\right)(s-2, s-1, s, s-1, \ldots, 3)$$

$$\eta_k = \left\{\frac{1}{\frac{s(s+1)}{2} + \left[(k-1)s - \frac{k(k-1)}{2}\right]}\right\}(s-k+1, s-k+2, \ldots, s$$
$$-1, s, s-1, \ldots, k),$$

$$\eta_{s-1} = \frac{2}{\frac{s(s+1)}{2} + (s-2)}(2, 3, \ldots s-1, s, s-1)$$

$$\eta_s = \frac{2}{s(s+1)}(1, 2, 3, \ldots s-1, s)$$

*Then* $\eta_k (k = 1, 2, \ldots, s)$ *is a weight vector group with kernel.*

**Proposition 10.6.2** *Assume that*

$$\eta_1 = \frac{1}{\sum_{k=1}^{s}\frac{1}{2^k}}\left(\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\dots\frac{1}{2^{s-1}},\frac{1}{2^s}\right)$$

$$\eta_2 = \left(\frac{1}{\frac{1}{2^2}+\sum_{k=1}^{s-1}\frac{1}{2^k}}\right)\left(\frac{1}{2^2},\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\dots,\frac{1}{2^{s-1}}\right)$$

$$\eta_3 = \left(\frac{1}{\frac{1}{2^3}+\frac{1}{2^2}+\sum_{k=1}^{s-2}\frac{1}{2^k}}\right)\left(\frac{1}{2^3},\frac{1}{2^2},\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3}\dots,\frac{1}{2^{s-2}}\right),$$

$$\eta_k = \left\{\frac{1}{\sum_{i=2}^{k}\frac{1}{2^i}+\sum_{i=1}^{s-k+1}\frac{1}{2^i}}\right\}\left(\frac{1}{2^k},\frac{1}{2^{k-1}},\dots,\frac{1}{2^2},\frac{1}{2},\frac{1}{2^2},\dots,\frac{1}{2^{s-k+1}}\right),$$

$$\eta_{s-1} = \frac{1}{\frac{1}{2^2}+\sum_{k=1}^{s-1}\frac{1}{2^k}}\left(\frac{1}{2^{s-1}},\frac{1}{2^{s-2}},\dots\frac{1}{2^2},\frac{1}{2},\frac{1}{2^2}\right)$$

$$\eta_s = \frac{1}{\sum_{k=1}^{s}\frac{1}{2^k}}\left(\frac{1}{2^s},\frac{1}{2^{s-1}},\dots\frac{1}{2^3},\frac{1}{2^2},\frac{1}{2}\right)$$

*Then* $\eta_k(k = 1, 2, \dots, s)$ *is a weight vector group with kernel.*

**Proposition 10.6.3** *For case* $s = 10$*, assume that*

$$\eta_1 = \frac{1}{5.5}(1, 0.9, 0.8, 0.7\dots, 0.1)$$

$$\eta_2 = \frac{1}{6.3}(0.9, 1, 0.9, 0.8, \dots, 0.2)$$

$$\eta_3 = \frac{1}{6.9}(0.8, 0.9, 1, 0.9, \dots, 0.3),$$

$$\eta_k = \frac{1}{1+\sum_{i=1}^{k}0\cdot(10-i)+\sum_{i=k}^{9}0.i}(0.(10-k), 0.8, 0.9, 1, 0, 9\dots, 0.k),$$

$$\eta_9 = \frac{1}{6.3}(0.2, \dots 0.8, 0.9, 1, 0.9)$$

$$\eta_{10} = \frac{1}{5.5}(0.1, \dots, 0.7, 0.8, 0.9, 1)$$

*Then* $\eta_k$ $(k = 1, 2, \dots, s)$ *is a weight vector group with kernel.*

## 10.7 Practical Applications

***Example 10.7.1*** Strategic supplier selection for the C919 cooperative development. C919 is the first large commercial aircraft developed by Commercial Aircraft Corporation of China Ltd. (COMAC). Many domestic and overseas suppliers joined the development program. Suppliers A and B took part in the development task of the C919 program for a specific key component. One of the two suppliers, either A or B, should be chosen and confirmed as a strategic supplier according to COMAC's criteria. A dilemma for strategic supplier selection will be presented to demonstrate the feasibility of the two-stage decision model based on the weight vector group with kernel and the weighted coefficient vector of kernel clustering for decision-making to solve the selection dilemma.

The consulting group collected all data according the evaluation index system, which was determined in advance. Then the clustering coefficient vectors of A and B are defined as follows:

$$\delta_A = (\delta_A^1, \delta_A^2, \delta_A^3, \delta_A^4, \delta_A^5) = (0.246, 0.338, 0.292, 0.124, 0)$$
$$\delta_B = (\delta_B^1, \delta_B^2, \delta_B^3, \delta_B^4, \delta_B^5) = (0.089, 0.352, 0.312, 0.197, 0)$$

Here, classes 1, 2, 3, 4, 5 correspond to 'especially excellent', 'excellent', 'good', 'moderate', and 'poor', respectively.

From $\max_{1 \le k \le 5}\{\delta_A^k\} = 0.338 = \delta_A^2$, $\max_{1 \le k \le 5}\{\delta_B^k\} = 0.352 = \delta_B^2$, it is known that the two suppliers A and B both belong to class 'excellent'. It seems that B should be selected and confirmed as the strategic supplier if we compare the clustering coefficients $\delta_A^2$ of A belonging to class excellent with $\delta_B^2$ of B belonging to class excellent, because $\delta_A^2 = 0.338 < \delta_B^2 = 0.352$. But we found that the clustering coefficients $\delta_A^1 = 0.246$ of A belonging to class 'especially excellent' is greater than the clustering coefficients $\delta_B^1 = 0.089$ of B belonging to class 'especially excellent' if we compare $\delta_A$ and $\delta_B$ in an integrated way. Therefore, the values of each component of the clustering coefficient vectors $\delta_A$ and $\delta_B$ should be integrated by a weight vector group with ke rnel.

The weight vector group with kernel presented in Proposition 2 is used to integrate the values of each component of the clustering coefficient vectors $\delta_A$ and $\delta_B$. Notice that $s = 5$. We obtain:

$$\eta_1 = \frac{32}{31}\left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}\right), \eta_2 = \frac{16}{19}\left(\frac{1}{2^2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}\right),$$
$$\eta_3 = \frac{4}{5}\left(\frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}\right), \eta_4 = \frac{16}{19}\left(\frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}, \frac{1}{2^2}\right),$$
$$\eta_5 = \frac{32}{31}\left(\frac{1}{2^5}, \frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2}\right)$$

Then, from $\omega_j^k = \eta_k \cdot \delta_j^T, j = A, B$, we haves

$$\omega_A^1 = \eta_1 \cdot \delta_A^T = \frac{32}{31}\left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}\right) \cdot (0.2466_20.338, 0.292.0.124.0)^T = 0.26$$
$$\omega_A^2 = \eta_2 \cdot \delta_A^T = \frac{16}{19}\left(\frac{1}{2^2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}\right) \cdot (0.2460.338, 0.292.0.124.0)^T = 0.27$$
$$\omega_A^3 = \eta_3 \cdot \delta_A^T = 0.23, \omega_A^4 = \eta_4 \cdot \delta_A^T = 0.16, \omega_A^5 = \eta_5 \cdot \delta_A^T = 0.10$$

$$\omega_A = \left(\omega_A^1, \omega_A^2, \omega_A^3, \omega_A^4, \omega_A^5\right) = (0.26, 0.27, 0.23, 0.16, 0.10)$$
$$\omega_B^1 = \eta_1 \cdot \delta_B^T = 0.19, \omega_B^2 = \eta_2 \cdot \delta_B^T = 0.25, \omega_B^3 = \eta_3 \cdot \delta_B^T = 0.24,$$
$$\omega_B^4 = \eta_4 \cdot \delta_B^T = 0.19, \omega_B^5 = \eta_5 \cdot \delta_B^T = 0.12$$
$$\omega_B = \left(\omega_B^1, \omega_B^2, \omega_B^3, \omega_B^4, \omega_B^5\right) = (0.19, 0.25, 0.24, 0.19, 0.12)$$

When comparing the weighted coefficient vector of kernel clustering for decision-making of $\omega_A$ and $\omega_B$, we found that $\omega_A^1 = 0.26 > \omega_B^1 = 0.19, \omega_A^2 = 0.27 > \omega_B^2 = 0.25$;at the same time, $\omega_A^4 = 0.16 < \omega_B^4 = 0.19$, $\omega_A^5 = 0.10 < \omega_B^5 = 0.12$.

So, it can be judged that the supplier A is better than vendor B. Supplier A should be selected and confirmed as the strategic supplier. The outcome can provide a basis for COMAC's strategic supplier selection.

We can obtain the same conclusion if the weight vector group with kernel presented in Proposition 1 or proposition 3 is used to integrate the values of each component of the clustering coefficient vectors $\delta_A$ and $\delta_B$.

It is directed against the decision paradox that the conclusion we arrive at by comparing the maximum components $\delta_i^k$ and $\delta_j^k$ of $\delta_i$ and $\delta_j$ is in conflict with the conclusion we arrive at by comparing $\delta_i$ and $\delta_j$, in an integrated way. The decision paradox that the value of the maximum component $\delta_i^k$ of $\delta_i$ is close to the maximum component $\delta_j^k$ of $\delta_j$ is solved effectively

## References

Aslani, B., Rabiee, M., & Tavana, M. (2021). An integrated information fusion and grey multi-criteria decision-making framework for sustainable supplier selection. *International Journal of Systems Science: Operations & Logistics, 8*(4), 348–370.

Bai, X., Xu, Y., & Liu, S. F. (2021). Research on the regional leading industry selection of "Kashgar urban agglomerations" based on multi-attribute weighted intelligent grey target decision-making evaluation model. *Grey Systems: Theory and Application,11*(3), 418–433.

Chatterjee, S., Das, P. P., & Chakraborty, S. (2024). Optimization of green machining processes using grey-based multi-criteria decision making methods: A comparative analysis. *International Journal on Interactive Design and Manufacturing, 18*(1), 33–53.

Cui, J., Liu, S. F., & Xie, N. M. (2012a). Novel grey decision making model and its numerical simulation. *Transactions of Nanjing University of Aeronautics & Astronautics, 29*(2), 112–117.

Cui, J. P., Xin, Y. P., & Liu, X. J. (2012b). Research on surface to air missile type selection based on multi-criteria gray decision-making. *Tactical Missile Technology, 1*(2), 7–10. In Chinese.

Dahooie, J. H., Hajiagha, S. H. R., Farazmehr, S., Zavadskas, E. K., & Antucheviciene, J. (2021). A novel dynamic credit risk evaluation method using data envelopment analysis with common

weights and combination of multi-attribute decision-making methods. *Computers & Operations Research, 129*, 105223.
Dang, Y. G., Liu, S. F., Liu, B., & Zhai, Z. (2005). Research on grey comprehensive clustering method with no significant difference in clustering coefficient. *Chinese Management Science, 13*(4), 69–73 (In Chinese).
Deng, J. L. (1990). *A course on grey system theory*. Press of Huazhong University of Science and Technology.
Deng, J. L. (2002). *Grey forecasting and grey decision*. Press of Huazhong University of Science and Technology.
Golmohammadi, D., & Mellat-Parast, M. (2012). Developing a grey-based decision-making model for supplier selection. *International Journal of Production Economics, 137*(2), 191–200.
Guo, H., Xiao, X. P., & Forest, J. (2014). Problem of grey bilevel multi-objective linear programming and its algorithm. *Control and Decision, 29*(7), 1193–1198.
Jafarian, A., Rabiee, M., & Tavana, M. (2020). A novel multi-objective co-evolutionary approach for supply chain gap analysis with consideration of uncertainties. *International Journal of Production Economics, 228*, 107852.
Li, L., & Li, X. C. (2024). The grey decision model and its application based on generalized greyness of interval grey number. *Grey Systems: Theory and Application*, published online https://doi.org/10.1108/GS-01-2024-0003
Li, X. Q., Tan, S. L., & Tang, B. G. (2007). The optimal collocation model of missile nuke based on the gray decision theory. *Fire Control and Command Control, 32*(2), 42–47.
Liang, C. Y., Gu, D. X., & Bichindaritz, I. (2012). Integrating gray system theory and logistic regression into case-based reasoning for safety assessment of thermal power plants. *Expert Systems with Applications, 39*(5), 5154–5167.
Liu, S. F. & Lin, Y. (2006). Grey Information: Theory and Practical Applications. Springer-Verlag. London.
Liu, S. F., Yuan, W. F., & Sheng, K. Q. (2010). Multi-attribute intelligent grey target decision model. *Control and Decision, 25*(8), 1159–1163. In Chinese.
Liu, S. F., Xu, B., Forrest, J., et al. (2013). On uniform effect measure functions and a weighted multi-attribute grey target decision model. *The Journal of Grey System, 25*(1), 1–11.
Liu, S. F., Fang, Z. G., & Yang, Y. J. (2014). On the two stages decision model with grey synthetic measure and a betterment of triangular whitenization weight function. *Control and Decision, 29*(7), 1232–1238.
Liu, S. F., & Yang, Y. (2015). A new decision-making model to solve the clustering dilemma. In A keynote speech at Leverhulme Trust Workshop on Grey System, Nanjing, June 18–20.
Liu, S. F., Zhang, H. Y., & Yang, Y. J. (2018). The decision paradox of "maximum value criterion" and its solution model. *Systems Engineering-Theory & Practice, 38*(7), 1830–1835 (In Chinese).
Liu, S. F., Jian, L. R., & Mi, C. M. (2020). Forecasting and Decision-making Methods, Science Press, Beijing In Chinese.
Liu, S. F., Liu, T., Yuan, W. F., & Yang, Y. J. (2022). Solving the dilemma in supplier selection by the group of weight vector with kernel. *Grey Systems: Theory and Application, 12*(3), 624–634.
Liu, S. F. (2024). Grey System Theory and Its Applications, 10th edition, Science Press, Beijing In Chinese.
Luo, D., & Wang, J. F. (2012a). *Theory and methods of grey decision-making*. Science Press. In Chinese.
Luo, D., & Wang, X. (2012b). The multi-attribute grey target decision method for attribute value within three-parameter interval grey number. *Applied Mathematical Modelling, 36*(5), 1957–1963.
Rogulj, K., Pamukovic, J. K., Antucheviciene, J., & Zavadskas, E. K. (2022). Intuitionistic fuzzy decision support based on EDAS and grey relational degree for historic bridges reconstruction priority. *Soft Computing, 26*(18), 9419–9444.

Wu, C. C., & Chang, N. B. (2004). Corporate optimal production planning with varying environmental costs: A grey compromise programming approach. *European Journal of Operational Research, 155*(1), 68–95.

Yan, S. L., Liu, S. F., Fang, Z. G., et al. (2014). Method of determining weights of decision makers and attributes for group decision making with interval grey numbers. *Systems Engineering - Theory & Practice*, *34*(9), 2372–2378.

Yazdani, M., Zarate, P., Zavadskas, E. K., & Turskis, Z. (2019). A combined compromise solution (CoCoSo) method for multi-criteria decision-making problems. *Management Decision, 57*(9), 2501–2519.

Yu, F. J., Ke, Y. L., & Ying, Z. (2009). Decision of fault repair in aircraft assembly system of automated docking. *Computer Integrated Manufacturing System, 15*(9), 1823–1830.

# Chapter 11
# Grey Control Systems

## 11.1 Introduction

Grey system theory originated from control system. The first paper on grey system theory is "Control problems of grey systems" (Deng, 1982). Since then, Professor Deng and his students have published several papers on the research of grey control system (Deng & Zhou, 1986; Zhou & Deng, 1989). In 1985, the book of Grey Control System published by Press of Huazhong University of Science and Technology (Deng, 1985).

As a scientific concept, the so-called control stands for a special effect a controlling device exerts on controlled equipment. It is a purposeful, selective and dynamic activity. A control system contains at least three parts, including a controlling device, controlled equipment, and an information path. A control system made up of these three parts is known as an open loop control system, as shown in Fig. 11.1. Each open loop control system is quite elementary in that the input directly controls the output, with no resistance against disturbances.

A control system with a feedback return is known as a closed loop control system, as shown in Fig. 11.2. The closed loop control system materializes its control through the combined effect of the input and the feedback of the output. One of the outstanding characteristics of closed loop systems is their strong ability to assist disturbances, with their outputs constantly vibrating around pre-determined objectives. Therefore, closed loop control systems possess a degree of stability.

A grey control system stands for such a system whose control information is only partially known, and is known as a grey system for short. The control of grey systems is different to that of general white systems, mainly due to the existence of grey elements in such systems. Under such conditions, one first needs to understand the possible connection between the systems' behaviors and the parametric matrices of the grey elements, how the systems' dynamics differ from one moment to the next and, in particular, how to obtain a white control function to alter the characteristics of the systems and to materialize control of the process of change of the systems. Grey



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_11

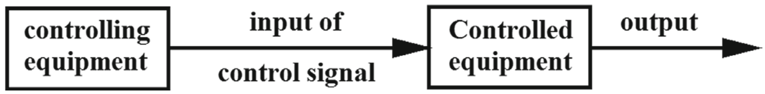



**Fig. 11.1** Open loop control system

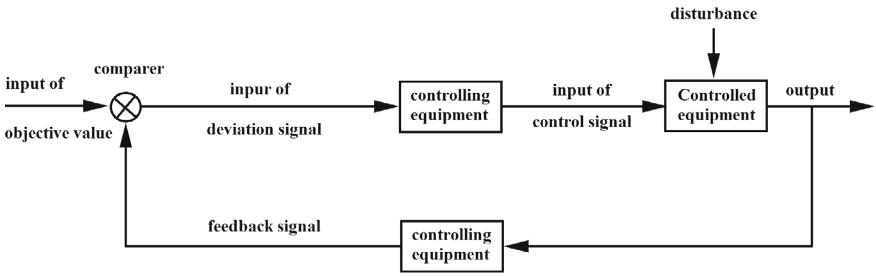



**Fig. 11.2** Closed loop control system

control contains not only the general situation of systems involving grey parameters, but also the construction of controls based on grey systems analysis, modeling, prediction, and decision-making. Grey control thinking can reveal the essence of the problems at hand and help materialize the purpose of control.

In the past 40 years, the research and application of grey control have achieved a lot of valuable results.

In 1982, Chen applied the grey dynamic control method to the dynamic control of boring machine spindle system and feed system, and achieved satisfactory control effect (Chen, 1982). In 1994, Ni et al. developed a grey predictive controller for EDM servo system (Ni et al., 1994). Wang and Qiu studied the problem of grey prediction guidance and simulation of air defense missile (Wang & Qiu, 1999). Liu et al. researched the problem of rotor vibration applying grey relational control method and Grey Verhuslt predictive optimal control model (Liu et al., 2004a, 2004b).

Su and Liu (2008, 2009) used several methods such as the Lyapuonv function, Lyapunov–Krasovskii function and model transformation and combined formula, matrix inequality, Holder inequality, Schur complement and other mathematical tools and decomposition technique of continuous matrix cover of grey matrix, and studied the robust stability problem of grey stochastic time-delay systems in depth, especially the distribution type, neutral type and neutral-distribution type exponential robust stability problem of grey stochastic time-delay systems. They investigated in details, gave the effective criterion, and obtained several useful achievements (Su, 2012).

Li et al. (2015) proposed an improved GM to acquire high-control system performance. Liem et al. (2015) set up a new method for estimating the load torque of a DC motor shaft by using a novel modeling method based on an adaptive control technique, named as online tuning grey fuzzy PID (OTGFPID). Huang and Huang (2000) proposed a grey prediction model combined with a proportional plus derivative controller to balance an inverted pendulum. Luo and Chen (2000) developed an autonomous mobile target tracking system based on grey fuzzy control algorithm. Chou et al. (2000) designed an optimal grey fuzzy controller of a constant turning force system by Taguchi-genetic method. Li et al. (2001) used the "grey system" analysis methodology for automated boiler water chemistry control in electric power plants. Lee and Liao (2003) proposed a selftuning fuzzy control system which adopted the grey predictor to compensate the timedelayed R-ab caused by the low pass filter data processing.

Gao et al. (2012) built up high-speed train speed controller model based on the model of grey genetic algorithm according to the fitness grey number of train operation target design. Lu and Wang (2013) studied the problem on modeling and simulation of the automatic train operation speed controller. Tian and Lu (2007) set up the grey forecast model of billet heating furnace temperature and put forward billet temperature control method. In the view of the flue temperature control problems with strong nonlinearity, large time delay, multi disturbance characteristics, Wang et al. (2010) brought forth an improved fuzzy expert control method based on the combination of grey prediction model. Zhang et al. (2004) designed self-adjustable grey prediction controller combining the traditional feedback control methods and grey prediction controlling. The simulation results showed that the new controller with more excellent dynamic performance and robustness. Salmeron et al. studied the problem of. Fuzzy grey cognitive maps and nonlinear Hebbian learning in process control (Salmeron & Palos-Sanchez, 2019; Salmeron & Papageorgiou, 2014). Abdulshahed et al. applied ANFIS prediction models for thermal error compensation on CNC machine tools (Abdulshahed et al., 2015). Dinh et al. researched the robust predictive tracking control of nonlinear systems (Dinh et al., 2017, 2018).

Chinese academician Yexiang Liu and his research group with the National Key Laboratory of Powder Metallurgy at Central South University made a number of achievements using grey system methods and models on control problems of aluminum electrolysis process (Lai et al., 2004; Liu et al., 2004a, 2004b, 2004c). In the light of the characteristics such as stochastic, nonlinear, time-varying and difficult to establish precise mathematical model of deep sea walking mechanism at the bottom of the complex operation environment of deep sea, Qiao et al. (2009) put forward the grey prediction and fuzzy PID control method, and realized the effective control of the deep sea walking mechanism. Wang et al. proposed a SINS/GPS integrated navigation system based on improved grey prediction model (Wang et al., 2015). Nie et al. researched grey prediction PID control method for pressure control system of pressure regulator (Nie et al., 2016). Luo designed an intelligent control system for biomass gasification furnace (Luo, 2016). Pang et al. designed a mine sweeping pear control system for a weapon based on variable step grey prediction fuzzy PID (Pang et al., 2017). Liang, Wei, Wang, Huang and Gao studied grey or fuzzy neuron PID control methods (Liang et al., 2018; Wei et al., 2019; Wang et al., 2021; Gao et al., 2022; Huang et al., 2024).

Li et al. applied grey prediction on dual hydraulic cylinder synchronous control (Li et al., 2020). Wang et al. Used grey prediction PI control on direct drive permanent magnet synchronous wind turbine (Wang et al., 2021). Chen et al. Researched the problem of spray burner swing angle reheat steam temperature control based on improved multivariate grey prediction model (Chen et al., 2023). Guo, Liu and Hu applied grey control technology to different scenes, such as control for multi autonomous underwater vehicles (Guo, 2023), control for micro cracking degree of cement stabilized crushed stone based on CT technology (Liu et al., 2024) and control on the noise of automotive cooling fans (Hu et al., 2024). Xu and Liao researched the problem of global Control of solar power generation stability based on active fault tolerant synovial prediction (Xu & Liao, 2024).

## 11.2 Controllability and Observability of Grey System

The concepts of controllability and observability are two fundamental structural characteristics of systems seen from the angle of control and observation. This section focuses on the problems of controllability and observability of grey linear systems.

**Definition 11.2.1** Assume that $U = [u_1, u_2, \ldots, u_s]^T$ is a control vector, $X = [x_1, x_2, \ldots, x_n]^T$ a state vector, and $Y = [y_1, y_2, \ldots, y_m]^T$ the output vector. Then

$$\begin{cases} \dot{X} = A(\otimes)X + B(\otimes)U \\ Y = C(\otimes)X \end{cases} \tag{11.1}$$

is known as the mathematical model of a grey linear control system, where $A(\otimes) \in G^{n\times n},\ B(\otimes) \in G^{n\times s},\ C(\otimes) \in G^{m\times n}$. Correspondingly, $A(\otimes)$ is known as the grey state matrix, $B(\otimes)$ the grey control matrix, and $C(\otimes)$ the grey output matrix.

In some studies, to emphasize the fact that *U, X,* and *Y* change the dynamic characteristics of the system over time, we also respectively write the control vector, state vector, and the output vector as $U(t),\ X(t)$, and $Y(t)$.

The first group of equations

$$\dot{X}(t) = A(\otimes)X(t) + B(\otimes)U(t) \tag{11.2}$$

in the mathematical model of grey linear control systems in Eq. (11.1) is known as the state equation, while the second group of equations

$$Y(t) = C(\otimes)X(t) \tag{11.3}$$

is known as the output equation.

**Definition 11.2.2** For a given precision and an objective vector $J = \left[j_1, j_2, \ldots, j_m\right]^T$, with a controlling device and a control vector $U(t)$ such that the output of the system can reach objective $J$ while satisfying the required precision through controlling the input, then the system is said to be controllable.

**Definition 11.2.3** For a given time moment $t_0$ and a pre-determined precision, if there is $t_1 \in (t_0, \infty)$ such that based on the system's output $Y(t), t \in [t_0, t_1]$, one can measure the system's state $X(t)$ within the required precision, then the system is said to be observable within the time interval $[t_0, t_1]$. If for any $t_0, t_1$, the system is observable within the interval $[t_0, t_1]$, then the system is said to be observable.

According to control theory, it follows that whether or not a grey system is controllable or observable is determined by whether or not the controllability matrix and the observability matrix, made up of $A(\otimes)$, $B(\otimes)$, are of full rank. That is, the following result holds true.

**Theorem 11.2.1** *For the system in Eq. (11.1), define*

$$L(\otimes) = \left[B(\otimes) \quad A(\otimes)B(\otimes) \quad A^2(\otimes)B(\otimes) \quad \cdots \quad A^{n-1}(\otimes)B(\otimes)\right]^T$$

$$D(\otimes) = \left[C(\otimes) \quad C(\otimes)A(\otimes) \quad C(\otimes)A^2(\otimes) \quad \cdots \quad C(\otimes)A^{n-1}(\otimes)\right]^T$$

Then the following hold true (Su & Liu, 2008):

(1) When $rank(L(\otimes)) = n$, the system is controllable; and
(2) When $rank(D(\otimes)) = n$, the system is observable.

Based on this result, the following four theorems can be established.

**Theorem 11.2.2** *For the system in Eq. (11.1), if the grey control matrix* $B(\otimes) \in G^{n\times n}$ *satisfies* $B(\otimes) \underset{\rightharpoonup}{=} \text{diag}[\otimes_{11}, \otimes_{22}, \ldots, \otimes_{nn}]$, *where each grey entry along the diagonal is non-zero, then the system is controllable.*

**Theorem 11.2.3** *For the system in Eq. (11.1), if the grey output matrix* $C(\otimes) \in G^{n\times n}$ *satisfies* $C(\otimes) \underset{\rightharpoonup}{=} \text{diag}[\otimes_{11}, \otimes_{22}, \ldots, \otimes_{nn}]$, *where each grey entry along the diagonal is non-zero, then the system is observable.*

**Theorem 11.2.4** *For the system in Eq. (11.1), if the control matrix* $B(\otimes) \in G^{n\times n}$ *satisfies* $B(\otimes) \underset{\rightharpoonup}{=} \text{diag}[\otimes_{11}, \otimes_{22}, \ldots, \otimes_{nn}, 0.., 0]$ *with* $\text{rank}B(\otimes) = m < n$, *and the grey state matrix* $A(\otimes)_{n\times n} = \text{diag}\left[0, \ldots 0 \otimes_{m+1,1}, \otimes_{m+2,2}, \ldots \otimes_{n,n-m}\right]$ *with* $\text{rank}A(\otimes) = n - m < n$, *then the system is controllable.*

**Theorem 11.2.5** *For the system in Eq. (11.1), if the grey output matrix* $C(\otimes) \in G^{m\times n}$ *satisfies* $C(\otimes) = \text{diag}[\otimes_{11}, \otimes_{22}, \cdots \otimes_{mm}]$, *with* $\text{rank}C(\otimes) = m < n$ *and the grey state matrix.*

$$A(\otimes) = \begin{pmatrix} 0 \cdots 0 & \otimes_{1,m+1} & 0 & \cdots & 0 \\ 0 \cdots 0 & 0 & \otimes_{2,m+2} & \cdots & 0 \\ \vdots \cdots \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 \cdots 0 & 0 & 0 & \cdots & \otimes_{n-m,n} \\ 0 \cdots 0 & 0 & 0 & \cdots & 0 \\ \vdots \cdots \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 \cdots 0 & 0 & 0 & \cdots & 0 \end{pmatrix}; \ \mathrm{rank}A(\otimes) = n - m < \mathrm{n}$$

*then the system is observable.*

## 11.3 Transfer Functions of Grey System

The concept of transfer functions stands for a fundamental relationship between the input and output of time invariant, linear grey control systems. Its rich connection with the expressions of the systems' state spaces can be described by using the concepts of controllability and observability.

### *11.3.1 Grey Transfer Function*

**Definition 11.3.1** Assume that the mathematical model of an $n$th order linear system with grey parameters is given as follows:

$$\otimes_n \frac{d^n x}{dt^n} + \otimes_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + \otimes_0 x = \otimes \cdot u(t) \tag{11.4}$$

After applying Laplace transform to both sides of this equation, we obtain

$$G(s) = \frac{X(s)}{U(s)} = \frac{\otimes}{\otimes_n s^n + \otimes_{n-1} s^{n-1} + \cdots \otimes_1 s + \otimes_0} \tag{11.5}$$

where $L(x(t)) = X(s)$ and $L(u(t)) = U(s)$. Equation (11.5) is known as a grey transfer function, which is the ratio of the Laplace transform of the response $x(t)$ of the $n$th order grey linear control system and the Laplace transform of the driving term $u(t)$. In fact, the transfer function represents a fundamental relationship between the input and output of a first order grey linear control system. From the following theorem, it follows that each $n$th order grey linear system can be reduced to an equivalent first order grey linear system.

**Theorem 11.3.1** *For an nth order grey linear system as shown in Eq. (11.4), there is an equivalent first order grey linear system.*

***Proof*** Assume that the given $n$th order grey linear system is

$$\otimes_n \frac{d^n x}{dt^n} + \otimes_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + \otimes_0 x = \otimes \cdot u(t)$$

Let

$$x = x_1; \ \frac{dx}{dt} = \frac{dx_1}{dt} = x_2; \ \frac{d^2 x}{dt^2} = \frac{dx_2}{dt} = x_3, \ldots, \ \frac{d^{n-1} x}{dt^{n-1}} = \frac{dx_{n-1}}{dt} = x_n$$

Therefore, we have

$$\frac{dx_n}{dt} = -\frac{\otimes_0}{\otimes_n} x_1 - \frac{\otimes_1}{\otimes_n} x_2 - \frac{\otimes_2}{\otimes_n} x_3 - \cdots - \frac{\otimes_{n-1}}{\otimes_n} x_n + \frac{\otimes}{\otimes_n} u(t)$$

and the $n$th order system is reduced to the following first order system

$$\dot{X}(t) = A(\otimes) X(t) + B(\otimes) U(t)$$

where $X(t) = [x_1, \ x_2, \ \ldots, \ x_n]^T$, $U(t) = u(t)$,

$$A(\otimes) = \begin{bmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \cdots & \cdots & \cdots\cdots & 0 \\ 0 & 0 & \cdots\cdots & 1 \\ -\frac{\otimes_0}{\otimes_n} & -\frac{\otimes 1}{\otimes_n} & \cdots\cdots & -\frac{\otimes_{n-1}}{\otimes_n} \end{bmatrix} \text{ and } B(\otimes) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\otimes}{\otimes_n} \end{bmatrix}$$

This ends the proof.

## 11.3.2 Transfer Functions of Typical Links

A grey control system that is symbolically written in an equation is also known as a grey link. When the transfer function of a link is known, from the relationship $X(s) = G(s) \cdot U(s)$ and the Laplace transform of the driving term, one can obtain the Laplace transform of the response. Then, by using the inverse Laplace transform, one can produce the response $x(t)$. The relationship between the driving and response terms is depicted in Fig. 11.3.

In the following definition, let us look at the transfer functions of several typical links.

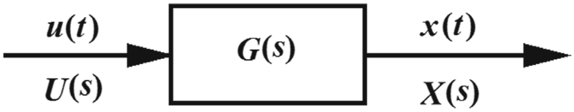



**Fig. 11.3** The driving and response terms

**Definition 11.3.2** The link between driving term $u(t)$ and response term $x(t)$ satisfying

$$x(t) = K(\otimes)u(t) \tag{11.6}$$

is known as a grey proportional link, where $K(\otimes)$ is the grey magnifying coefficient of the link.

**Proposition 11.3.1** *The transfer function of a grey proportional link is*

$$G(s) = K(\otimes) \tag{11.7}$$

*The characteristics of a grey proportional link are that when a jump occurs in the driving quantity, the response value changes proportionally. This kind of change and relationship between the drive and response are depicted in Fig. 11.4.*

**Definition 11.3.3** When driven by a unit jump, if the response is given by

$$x(t) = K(\otimes)\left(1 - e^{-tT}\right) \tag{11.8}$$

then the link is known as a grey inertia link, where $T$ stands for a time constant of the link.

**Proposition 11.2.2** *The transfer function of a grey inertial link is given by*

$$G(s) = \frac{K(\otimes)}{T \cdot s + 1} \tag{11.9}$$

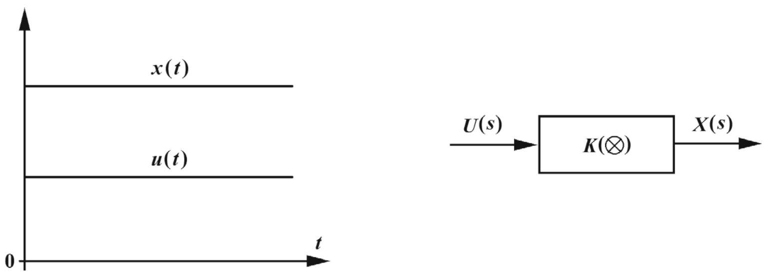



**Fig. 11.4** The grey proportional link

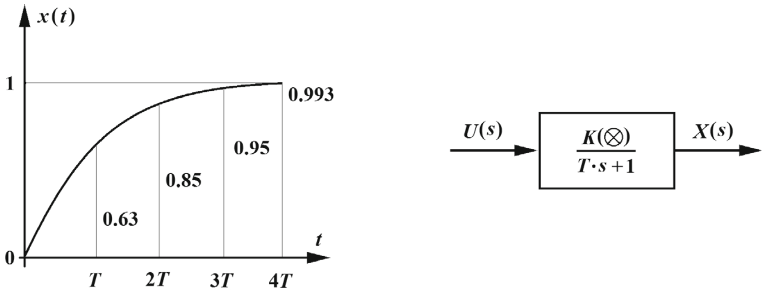



**Fig. 11.5** The grey inertia link

*The characteristics of a grey inertia link are that when a jump occurs in the driving quantity, the response can reach a new state of balance only after a period of time. Figure 11.5 provides a block diagram and the curve of change of the response of a grey inertia link when* $\tilde{K}(\otimes) = 1$.

**Definition 11.3.4** When the drive and response are related as follows, the link is known as grey integral link:

$$x(t) = \int K(\otimes)u(t)dt \tag{11.10}$$

**Proposition 11.3.3** *The transfer function of a grey integral link is given below*:

$$G(s) = \frac{K(\otimes)}{s} \tag{11.11}$$

*For a grey integral link, when the drive is a jump function, its response is* $x(t) = K(\otimes)ut$, *as shown in Fig. 11.6*.

**Definition 11.3.5** If the response and the drive are related as follows, the link is known as a grey differential link:

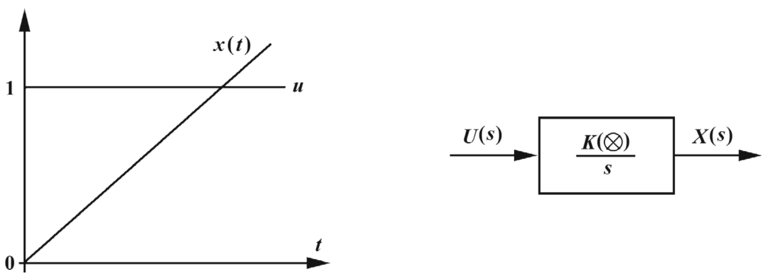



**Fig. 11.6** The grey integral link

$$x(t) = K(\otimes)\frac{du(t)}{dt} \tag{11.12}$$

**Proposition 11.3.4** *The transfer function of a grey differential link is given as follows:*

$$G(s) = K(\otimes)s \tag{11.13}$$

*The characteristics of a grey differential link are that when the drive stands for a jump, the response becomes an impulse with an infinite amplitude.*

**Definition 11.3.6** If the drive and response are related as follows, the link is known as a grey postponing link, where $\tau(\otimes)$ is a grey constant:

$$x(t) = u(t - \tau(\otimes)) \tag{11.14}$$

**Proposition 11.3.5** *The transfer function of a grey postponing link is given below*:

$$G(s) = e^{-\tau(\otimes)s} \tag{11.15}$$

*For a grey postponing link, when the drive is a jump function, it takes some time for the response to react accordingly. For details, see Fig. 11.7.*

The figure above represents some typical links met in practical applications. Many complicated devices and systems can be treated as combinations of these typical links. For instance, when the grey proportional link is combined with a grey differential link, one can obtain a grey proportional differential link. When a grey integral link is connected with grey postponing link, one establishes a grey integral postponing link. Along the same lines, multi-layered combinations can be developed for practical purposes. One of the purposes of studying grey transfer functions is that we can investigate the stabilities and other properties of systems by looking at the extreme values of relevant transfer functions.

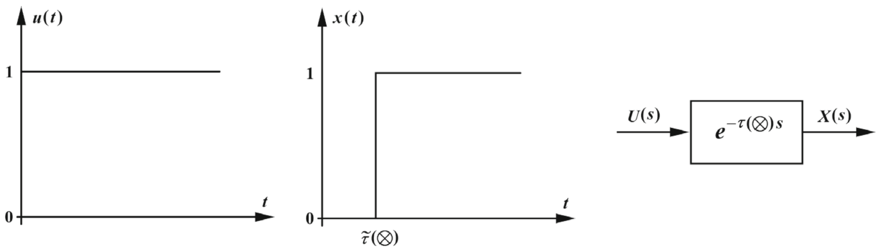



**Fig. 11.7** The grey postponing link

### 11.3.3 Matrices of Grey Transfer Functions

Matrices of grey transfer functions can be employed to express a fundamental relationship between the multi-inputs and multi-outputs of grey linear control systems. In particular, for the following grey linear control system

$$\begin{cases} \dot{X}(t) = A(\otimes)X(t) + B(\otimes)U(t) \\ Y(t) = C(\otimes)X(t) \end{cases}$$

Employing Laplace transforms produces

$$\begin{cases} sX(s) = A(\otimes)X(s) + B(\otimes)U(s) \\ Y(s) = C(\otimes)X(s) \end{cases}$$

and

$$\begin{cases} (sE - A(\otimes))X(s) + B(\otimes)U(s) \\ Y(s) = C(\otimes)X(s) \end{cases}$$

If $(sE - A(\otimes))$ is invertible, then we can further obtain

$$\begin{cases} X(s) = (sE - A(\otimes))^{-1}B(\otimes)U(s) \\ Y(s) = C(\otimes)X(s) \end{cases}$$

That is, we have $Y(s) = C(\otimes)(sE - A(\otimes))^{-1}B(\otimes)U(s)$.

**Definition 11.3.7** The $m$ $n$ matrix below is known as the matrix of grey transfer functions:

$$G(s) = C(\otimes)(sE - A(\otimes))^{-1}B(\otimes) \tag{11.16}$$

**Definition 11.3.8** For an $n$th order grey linear system, if the state grey matrix $A(\otimes)$ of the corresponding equivalent first order system is non-singular, then

$$\lim_{s \to 0} G(s) = -C(\otimes)A(\otimes)^{-1}B(\otimes) \tag{11.17}$$

is known as a grey gain matrix. If the grey gain matrix $-C(\otimes)A(\otimes)^{-1}B(\otimes)$ is used to replace the transfer function $G(s)$, then the system is reduced into a proportional link. Because $Y(s) = G(s)U(s)$, when $m = s = n$, if $G(s)$ is non-singular, we have the following:

$$U(s) = G(s)^{-1}Y(s) \tag{11.18}$$

**Definition 11.3.9** The following matrix is known as a grey structure matrix:

$$G(s)^{-1} = B(\otimes)^{-1}(sE - A(\otimes))C(\otimes)^{-1} \tag{11.19}$$

When the grey structure matrix is known, to make the output vector $Y(s)$ meet or close to meet a certain expected objective $J(s)$, one can determine the system's control vector $U(s)$ through $G^{-1}(s)\cdot J(s)$. Additionally, we can also discuss the controllability and observability of systems by using matrices of grey transfer functions.

## 11.4 Robust Stability of Grey System

Stability is a fundamental structural characteristic of systems. It stands for an important mechanism for a system to sustain itself and is a prerequisite for the system to operate smoothly. This is why stability is studied in systems control theory and it is a key objective in relevant engineering designs. Each physical system has to be stable before it can be employed in practical applications.

The stability of grey systems focuses on the investigations of informational changes. It also focuses on whether or not the grey system of concern stays stable or can recover to its stability when the whitenization value of a grey parameter moves within the field of discourse. The existence of grey parameters complicates the study of grey systems stability, and puts them at the center of attention of control theory and control engineering.

In grey systems modeling, there is a distinction between having a postponing term and not having such a term; there is also a difference between having a random term and not having such a term. Ordinarily, grey systems without involving any random and postponing term are known as grey systems; those involving postponing terms without any random terms are grey postponing systems, and those involving random terms are known as grey stochastic systems. In this section, we will study the problem of robust stability of these three kinds of systems.

### *11.4.1 Robust Stability of Grey Linear Systems*

The study of systems' stability is often limited to systems without the effect of any external input. This kind of system is known as an autonomous system. A simple grey linear autonomous system can be written as follows:

$$\begin{cases} \dot{x}(t) = A(\otimes)x(t) \\ x(t_0) = x_0, \forall t \geq t_0 \end{cases} \tag{11.20}$$

where $x \in R^n$ stands for the state vector, and $A(\otimes) \in G^{n\times n}$ is the matrix of grey coefficients.

**Definition 11.4.1** If $A(\tilde{\otimes})$ is a whitenization matrix of the grey matrix $A(\otimes)$, then

$$\begin{cases} \dot{x}(t) = A\left(\tilde{\otimes}\right)x(t) \\ x(t_0) = x_0 \end{cases} \tag{11.21}$$

is referred to as a whitenization system of the system in Eq. (11.20).

Ordinarily, we assume that the matrix $A(\otimes)$ of grey coefficients of the system in Eq. (11.20) has a continuous matrix cover:

$$A(D) = [L_a, U_a] = \left\{A\left(\tilde{\otimes}\right) : \underline{a}_{ij} \le \tilde{\otimes} \le \overline{a}_{ij}, i, j = 1, 2, \cdots, n\right\}$$

where $U_a = (\overline{a}_{ij}), L_a = (\underline{a}_{ij})$.

**Definition 11.4.2** If any whitenization system of the system in Eq. (11.20) is stable, then the system in Eq. (11.20) is referred to as robust stable.

The ordinary concept of a system's (robust) stability represents the (robust) asymptotic stability of the system.

**Theorem 11.4.1** *If there is positive definite matrix P such that*

$$PL_a + L_a^T P + 2\lambda_{\max}(P)\||U_a - L_a\|I_n < 0$$

*Then the system in Eq. (11.20) is robust stable* (Su & Liu, 2009).

***Proof*** Let us take the Lyapunov function $V(x) = x^T Px$. For any whitenization matrix $A(\tilde{\otimes}) \in A(D)$, let us compute the derivative of $V(x)$ with respect to $t$ along the trajectory of the whitenization system and obtain.

$$\dot{V}(x) = 2x^T PA\left(\tilde{\otimes}\right)x = x^T\left(PL_a + L_a^T P\right)x + 2x^T P\Delta Ax$$

$$\le x^T\left(PL_a + L_a^T P + 2\lambda_{\max}(P)\|U_a - L_a\|I_n\right)x < 0,\ \forall x \ne 0$$

This implies that the system in Eq. (11.20) is robust stable. QED. If in Theorem 11.4.1 we let $P = I_n$, then we have the result shown below.

**Corollary 11.4.1** *If*

$$\|U_a - L_a\| < -\lambda_{\max}\left(\frac{L_a + L_a^T}{2}\right) \tag{11.22}$$

*holds true, then the system in Eq. (11.20) is robust stable. If we employ another form of decomposition* $A\left(\tilde{\otimes}\right) = U_a - \Delta A$ *of the whitenization matrix* $A\left(\tilde{\otimes}\right)$ *to study the* *robust stability of the system in Eq. (11.20), then much like in Theorem 11.4.1 and Corollary 11.4.1 we can obtain the following results.*

**Theorem 11.4.2** *If there is a positive definite matrix P such that*

$$PU_a + U_a^T P + 2\lambda_{\max}(P)\|U_a - L_a\| I_n < 0$$

*then the system in Eq. (11.20) is robust stable* (Su & Liu, 2009).

**Corollary 11.4.2** *If*

$$\|U_a - L_a\| \le -\lambda_{\max}\left(\frac{U_a + U_a^T}{2}\right) \tag{11.23}$$

*holds true, then the system in Eq. (11.20) is robust stable. Both Corollaries 11.4.1 and 11.4.2 respectively provide us a meaning result, because $U_a - L_a$ in fact stands for the matrix of disturbance errors of the system in Eq. (11.20)*; *Eqs. (11.22) and (11.23) indicate that when the norm of the disturbance error matrix varies within the range of* $(0, \lambda)$*, the system in Eq. (11.20) will always be stable, where* $\lambda = \max\left\{-\lambda_{\max}\left(\frac{L_a+L_a^T}{2}\right), -\lambda_{\max}\left(\frac{U_a+U_a^T}{2}\right)\right\}$

**Theorem 11.4.3** *If $L_a + L_a^T + \lambda_{\max}[(U_a - L_a) + (U_a - L_a)]^T I_n < 0$, then the system in Eq. (11.20) is robust stable; if $U_a + U_a^T - \lambda_{\max}\left[(U_a - L_a) + (U_a - L_a)^T\right] I_n > 0$, then the system is instable*2009.

***Example 11.4.1*** Let us consider the robust stability problem of the following 2-dimensional grey linear system:

$$\dot{x}(t) = \begin{pmatrix} [-2.3, -1.8] & [0.6, 0.9] \\ [0.8, 1.0] & [-2.5, -1.9] \end{pmatrix} x(t)$$

**Solution**: Through computations, we have

$$\|U_a - L_a\| = 0.8072 < -\lambda_{\max}\left(\frac{L_a + L_a^T}{2}\right) = 1.3000$$

$$L_a + L_a^T + \lambda_{\max}\left[(U_a - L_a) + (U_a - L_a)^T\right] I_n = -1.7759 I_n < 0$$

These inequalities indicate that, by using Corollary 11.4.1 or Theorem 11.4.3, we can conclude that this given system is robust stable.

### 11.4.2 Robust Stability of Grey Linear Time-Delay Systems

The phenomena of timely postponing are very common. They are often the main reason for causing instability, vibration, and poor performance in systems. Therefore, it is very important to investigate the stability problem of postponing systems. In particular, let us look at the following $n$-dimensional linear postponing autonomous system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), \forall t \geq 0, \\ x(t) = \phi(t), \forall t \in [-\tau, 0] \end{cases} \tag{11.24}$$

where $x(t) \in R^n$ stands for the system's state vector, $A, B \in R^{n\times n}$ the known constant matrices, $\tau > 0$ the amount of time of postponing, and $\phi(t) \in C^n[-\tau, 0]$ the $n$th dimensional space of continuous functions.

**Definition 11.4.3** If at least one of the matrices $A, B$ of constants in the linear postponing system in Eq. (11.24) is grey, then this system is referred to as a grey linear postponing autonomous system, denoted as

$$\begin{cases} \dot{x}(t) = A(\otimes)x(t) + B(\otimes)x(t-\tau), \forall t \geq 0, \\ x(t) = \phi(t), \forall t \in [-\tau, 0]. \end{cases} \tag{11.25}$$

In the following equation, we assume that the constant matrices in the system in Eq. (11.25) are all grey and have continuous matrix covers; that is, $A(\otimes), B(\otimes)$ respectively have the following form of matrix covers:

$$A(D) = [L_a, U_a] = \left\{A\left(\tilde{\otimes}\right) : \underline{a}_{ij} \leq \tilde{\otimes} \leq \overline{a}_{ij}, i, j = 1, 2, \ldots, n\right\}$$

$$B(D) = [L_b, U_b] = \left\{B\left(\tilde{\otimes}\right) : \underline{b}_{ij} \leq \tilde{\otimes} \leq \overline{b}_{ij}, i, j = 1, 2, \ldots, n\right\}$$

where $U_a = (\overline{a}_{ij}), L_a = (\underline{a}_{ij}), U_b = (\overline{b}_{ij}), L_b = (\underline{b}_{ij})$.

**Definition 11.4.4** If $A(\tilde{\otimes}), B(\tilde{\otimes})$ are respectively whitenization matrices of $A(\otimes), B(\otimes)$, then

$$\begin{cases} \dot{x}(t) = A\left(\tilde{\otimes}\right)x(t) + B(\tilde{\otimes})x(t-\tau), \forall t \geq 0, \\ x(t) = \phi(t), \forall t \in [-\tau, 0]. \end{cases} \tag{11.26}$$

is referred to as a whitenization system of the system in Eq. (11.25).

**Definition 11.4.5** If any whitenization system of the system in Eq. (11.25) is stable, the system in Eq. (11.25) is referred to as robust stable.

Based on whether or not the robust stability condition of a grey postponing system depends on the amount of postponing, the robust stability condition can be divided into two classes: postponing independent and postponing dependent. In particular, the condition for a robust stable system to be postponing independent is that for any time postponing $\tau > 0$, the system is robustly asymptotic stable. Because this condition does not require the amount of postponing, it is appropriate for the study of the stability problem of postponing systems whose amounts of postponing are uncertain or unknown.

The condition for a robust stable system to be postponing dependent is that for some values of postponing $\tau > 0$, the system is robust stable, while for some other values of the postponing $\tau > 0$, the system is not stable. That is why the system's stability is dependent on the amount of postponing.

**Theorem 11.4.4** *If there are positive definite matrices $P$, $Q$ and positive constants $\varepsilon_1, \varepsilon_2$ such that the symmetric matrix*

$$\begin{pmatrix} \Xi & PL_b & P & P \\ L_b^{\mathrm{T}}P & -Q+\varepsilon_2\|U_b-L_b\|^2 I_n & 0 & 0 \\ P & 0 & -\varepsilon_1 I_n & 0 \\ P & 0 & 0 & -\varepsilon_2 I_n \end{pmatrix} < 0$$

*where $\Xi = L_a^{\mathrm{T}}P+PL_a+Q+\varepsilon_1\|U_a-L_a\|^2 I_n$, and $I_n$ stands for the identity matrix (the same symbol will be used for the rest of this chapter), then the system in Eq. (11.25) is robust stable* (Su & Liu, 2012).

**Theorem 11.4.5** *If there are positive definite matrices $P$, $Q$, $N$ and positive constants $\varepsilon_1, \varepsilon_2$ such that the symmetric matrix*

$$\begin{pmatrix} \Gamma & PL_b & \rho I_n & P & P \\ L_b^{\mathrm{T}}P & -Q+\varepsilon_2\|U_b-L_b\|^2 I_n & 0 & 0 & 0 \\ \rho I_n & 0 & -\overline{N} & 0 & 0 \\ P & 0 & 0 & -\varepsilon_1 I_n & 0 \\ P & 0 & 0 & 0 & -\varepsilon_2 I_n \end{pmatrix} < 0$$

*where $\Gamma = L_a^{\mathrm{T}}P+PL_a+Q+\varepsilon_1||U_a-L_a||^2 I_n$ and $\overline{N} = N^{-1}$, $\rho = \sqrt{\tau}$, then the system in Eq. (11.25) is robust stable.*

***Example 11.4.2*** Let us look at the following 2-dimensional grey linear postponing system

$$\begin{cases} \dot{x}(t) = A(\otimes)x(t) + B(\otimes)x(t-\tau), \forall t \geq 0, \\ x(t) = \phi(t), \forall t \in [-\tau, 0]. \end{cases}$$

Assume that the upper and lower bound matrices of the continuous matrix covers of the grey constant matrices $A(\otimes)$, $B(\otimes)$ are respectively give as follows:

$$L_a = \begin{pmatrix} -4.38 & 0.20 \\ 0.19 & -4.33 \end{pmatrix};\ U_a = \begin{pmatrix} -4.26 & 0.29 \\ 0.27 & -4.22 \end{pmatrix}$$

$$L_b = \begin{pmatrix} -0.93 & 0.21 \\ 0.23 & -0.86 \end{pmatrix};\ U_b = \begin{pmatrix} -0.88 & 0.24 \\ 0.26 & -0.82 \end{pmatrix}$$

According to Theorem 9.10, by using the solver in the LMI (linear matrix inequality) control toolbox, we obtain the behavioral solution as follows:

$$P = \begin{pmatrix} 9.4642 & 0.6983 \\ 0.6983 & 9.8228 \end{pmatrix};\ Q = \begin{pmatrix} 28.0088 & -0.0340 \\ -0.0340 & 27.8605 \end{pmatrix};\ \varepsilon_1 = 30.0826;\ \varepsilon_2 = 30.2461$$

Now from Theorem 9.11, by using the solver in the LMI (linear matrix inequality) control toolbox, we obtain the behavioral solution below:

$$P = \begin{pmatrix} 6.8592 & 0.5061 \\ 0.5061 & 7.1191 \end{pmatrix},\ Q = \begin{pmatrix} 20.2294 & -0.0246 \\ -0.0246 & 20.1920 \end{pmatrix},\ N = \begin{pmatrix} 0.0456 & 0 \\ 0 & 0.0456 \end{pmatrix}$$

$$\varepsilon_1 = 21.8024,\ \varepsilon_2 = 21.9209,\ \tau = 2.7035$$

These results indicate that the system considered in this example is robust stable. And the maximum allowed postponing length of time as obtained from Theorem 11.4.5 is 2.7035.

### 11.4.3 *Robust Stability of Grey Stochastic Linear Time-Delay System*

The mathematical model that describes a stochastic system is generally the Itto stochastic differential equation, where the often seen $n$-dimensional Itto stochastic differential postponing equation is

$$\begin{cases} \mathrm{d}x(t) = Ax(t) + Bx(t-\tau) + [Cx(t) + Dx(t-\tau)]\mathrm{d}w(t), \forall t \ge 0, \\ x(t) = \xi(t), \xi(t) \in L^2_{F_0}([-\tau, 0]; R^n), \forall t \in [-\tau, 0]. \end{cases} \tag{11.27}$$

where $x(t) \in R^n$ stands for the system's state vector, $A, B, C, D \in R^{n\times n}$ known constant matrices, $\tau > 0$ the time of postponing, and $w(t)$ a 1-dimensional Brownian motion defined on a complete probability space $(\Omega, F, \{F_t\}_{t\ge 0}, P)$. $L^2_{F_0}([-\tau\ ,\ 0];\ R^n)$ stands for the totality of all $F_0$-measurable stochastic variables $\xi = \{\xi(t)\ :\ -\tau \le t \le 0\}$ that take values from $C([-\tau\ ,\ 0]\,;R^n)$ satisfying $\sup_{-\tau\le t\le 0} E|\xi(\theta)|^2 < \infty$, while $C([-\tau\ ,\ 0]\,;R^n)$ stands for the totality of continuous functions $\phi :\ [-\tau\ ,\ 0] \to R^n$. Under the initial condition $x(t) = \xi(t) \in L^2_{F_0}([-\tau\ ,\ 0]\,;\ R^n)$, the system in Eq. (11.27) has an equilibrium point $x(t\,;\,\xi)$, and corresponds to the initial value $\xi(t) = 0$, $x(t\,;\,0) \equiv 0$.

There are several different concepts of stability for stochastic systems. In the following, we list four of the important stabilities.

**Definition 11.4.6** The equilibrium point $x(t) \equiv 0$ of the system in Eq. (11.27) is referred to as stochastically stable, if for each $\varepsilon > 0$, $\lim_{x_0 \to 0} P\big(\sup_{t>t_0} |x(t;\, t_0, x_0)| > \varepsilon\big) = 0$.

**Definition 11.4.7** The equilibrium point $x(t) \equiv 0$ of the system in Eq. (11.27) is referred to as stochastically asymptotically stable, if it is stochastically stable and $\lim_{x_0 \to 0} P(\lim_{t\to+\infty} x(t;\, t_0, x_0) = 0) = 1$.

**Definition 11.4.8** The equilibrium point $x(t) \equiv 0$ of the system in Eq. (11.27) is referred to as large-scale stochastically asymptotically stable, if it is stochastically stable and for any $t_0, x_0$, $P(\lim_{t\to+\infty} x(t;\, t_0, x_0)) = 0 = 1$.

**Definition 11.4.9** The equilibrium point $x(t) \equiv 0$ of the system in Eq. (11.27) is referred to as mean square exponential stable, if there are positive constants $\alpha > 0$, $\beta > 0$ such that $E|x(t;\, t_0, x_0)|^2 \le \alpha |x_0|^2 \exp(-\beta t)$, $t > t_0$.

A grey system is stochastic if it involves grey parameters. Concepts related to grey stochastic systems are generally introduced based on relevant concepts of conventional stochastic systems. Considering the problems we will study, let us provide the following definitions.

**Definition 11.4.10** If at least one of the matrices $A, B, C, D$ of the stochastic linear postponing system in Eq. (11.27) is grey, then the system is referred to as a grey stochastic linear postponing system, written as follows:

$$\begin{cases} \mathrm{d}x(t) = A(\otimes)x(t) + B(\otimes)x(t-\tau) + [C(\otimes)x(t) + D(\otimes)x(t-\tau)]\mathrm{d}w(t), \forall t \ge 0, \\ x(t) = \xi(t), \xi(t) \in L^2_{F_0}([-\tau, 0]; R^n), \forall t \in [-\tau, 0]. \end{cases} \tag{11.28}$$

In this section, we assume that all the coefficient matrices of the system in Eq. (11.28) are grey with continuous matrix covers. That is, the matrix covers of the grey matrices $A(\otimes)$, $B(\otimes)$, $C(\otimes)$, and $D(\otimes)$ are respectively given as follows:

$$A(D) = [L_a,\ U_a] = \{A(\tilde{\otimes}) = (\tilde{\otimes}_{aij})_{n\times n} : \underline{a}_{ij} \le \tilde{\otimes}_{aij} \le \overline{a}_{ij}\}$$

$$B(D) = [L_b,\ U_b] = \{B(\tilde{\otimes}) = (\tilde{\otimes}_{bij})_{n\times n} : \underline{b}_{ij} \le \tilde{\otimes}_{bij} \le \overline{b}_{ij}\}$$

$$C(D) = [L_c,\ U_c] = \{C(\tilde{\otimes}) = (\tilde{\otimes}_{cij})_{n\times m} : \underline{c}_{ij} \le \tilde{\otimes}_{cij} \le \overline{c}_{ij}\}$$

and

$$D(D) = [L_d,\ U_d] = \left\{D\left(\tilde{\otimes}\right) = \left(\tilde{\otimes}_{dij}\right)_{n\times n} : \underline{d}_{ij} \le \tilde{\otimes}_{dij} \le \overline{d}_{ij}\right\}$$

where $L_a = (\underline{a}_{ij})_{n\times n}$, $U_a = (\overline{a}_{ij})_{n\times n}$, $L_b = (\underline{b}_{ij})_{n\times n}$, $U_b = (\overline{b}_{ij})_{n\times n}$, $L_c = (\underline{c}_{ij})_{n\times n}$, $U_c = (\overline{c}_{ij})_{n\times n}$, $L_d = (\underline{d}_{ij})_{n\times n}$, and $U_a = (\overline{d}_{ij})_{n\times n}$.

**Definition 11.4.11** If $A(\tilde{\otimes})$, $B(\tilde{\otimes})$, $C(\tilde{\otimes})$, and $D(\tilde{\otimes})$ are arbitrary whitenization matrices of the grey matrices $A(\otimes)$, $B(\otimes)$, $C(\otimes)$, and $D(\otimes)$, respectively, then

$$\begin{cases} \mathrm{d}x(t) = A(\tilde{\otimes})x(t) + B(\tilde{\otimes})x(t-\tau) + [C(\tilde{\otimes})x(t) + D(\tilde{\otimes})x(t-\tau)]\mathrm{d}w(t), \forall t \ge 0, \\ x(t) = \xi(t), \xi(t) \in L^2_{F_0}([-\tau, 0]; R^n), \forall t \in [-\tau, 0]. \end{cases} \tag{11.29}$$

is referred to as a whitenization system of the system in Eq. (11.28).

**Definition 11.4.12** If any whitenization system of the system in Eq. (11.28) is large-scale stochastic asymptotic stable, that is,

$$\lim_{t\to\infty} x(t;\xi) = 0 \ \ a.s.$$

then the system in Eq. (11.28) is said to be large scale stochastic robust asymptotic stable.

**Definition 11.4.13** If any whitenization system of the system in Eq. (11.28) is mean square exponential stable, that is, there are positive constants $r_0$ and $K$ such that the equilibrium points of whitenization systems of the system in Eq. (11.28) satisfy

$$E|x(t,\xi)|^2 \le Ke^{-r_0 t} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2, \ \ t \ge 0$$

or equivalently

$$\lim_{t\to\infty} \sup \frac{1}{t} \log E|x(t;\xi)|^2 \le -r_0$$

then the system in Eq. (11.28) is said to be mean square exponential robust stable.

**Theorem 11.4.6** For the system in Eq. (11.28), if there is a positive definite symmetric matrix $Q$ and there are positive constants $\varepsilon_i$, $i = 1, \ldots, 6$, satisfying $M + N < 0$, then for any initial condition $\xi \in C^p_{F_0}([-\tau, 0]; R^n)$ the following holds true:

$$\lim_{t\to\infty} x(t;\xi) = 0 \ \ a.s.$$

That is, according to Su (2012), the system in Eq. (11.28) is large-scale stochastic robust asymptotic stable, where

$$\begin{aligned} M &= QL_a + L_a^T Q + (\varepsilon_1 + \varepsilon_2)Q + \varepsilon_1^{-1}\lambda_{\max}(Q)\cdot ||U_a - L_a||^2 I_n \\ &\quad + (1+\varepsilon_4)(1+\varepsilon_5)L_c^T Q L_c \\ &\quad + (1+\varepsilon_4^{-1})(1+\varepsilon_5)\lambda_{\max}(Q)||U_c - L_c||^2 I_n \end{aligned}$$

and

$$\begin{aligned} N &= \varepsilon_2^{-1}(1+\varepsilon_3^{-1})\lambda_{\max}(Q)||U_b - L_b||^2 I_n + \varepsilon_2^{-1}\cdot(1+\varepsilon_3)L_b^T Q L_b \\ &\quad + (1+\varepsilon_5^{-1})(1+\varepsilon_6)L_d^T Q L_d \\ &\quad + (1+\varepsilon_5^{-1})(1+\varepsilon_6^{-1})\lambda_{\max}(Q)||U_d - L_d||^2 I_n. \end{aligned}$$

**Theorem 11.4.7** *For the system in Eq. (11.28), if there are positive definite symmetric matrix $Q$ and positive constants $\varepsilon_i$, $i = 1, \ldots, 6$, satisfying $K + L < 0$, then for any initial condition $\xi \in C_{F_0}^p([-\tau, 0]; R^n)$, the following holds true*:

$$\lim_{t\to\infty} x(t;\xi) = 0 \quad a.s$$

*That is, the system in Eq. (11.28) is large-scale stochastic asymptotic stable, where*

$$\begin{aligned} K &= QL_a + L_a^T Q + (\varepsilon_1 + \varepsilon_2)Q + \Big[\varepsilon_1^{-1}\lambda_{\max}(Q)\mathrm{trace}\Big(G_a^T G_a\Big) + (1+\varepsilon_4)(1+\varepsilon_5)\mathrm{trace}\Big(L_c^T L_c\Big) \\ &\quad + \Big(1+\varepsilon_4^{-1}\Big)(1+\varepsilon_5)\lambda_{\max}(Q)\mathrm{trace}\Big(G_c^T G_c\Big)\Big] I_n, \end{aligned}$$

*and*

$$\begin{aligned} L &= [\varepsilon_2^{-1}(1+\varepsilon_3^{-1})\lambda_{\max}(Q)\mathrm{trace}(G_b^T G_b) + \varepsilon_2^{-1}(1+\varepsilon_3)\mathrm{trace}(L_b^T L_b) \\ &\quad + (1+\varepsilon_5^{-1})(1+\varepsilon_6)\mathrm{trace}(L_d^T L_d) + (1+\varepsilon_5^{-1})(1+\varepsilon_6^{-1})\lambda_{\max}(Q)\mathrm{trace}(G_d^T G_d)] I_n. \end{aligned}$$

If we let the matrix and constants in Theorems 11.4.6 and 11.4.7 be $\varepsilon_1 = \cdots = \varepsilon_6 = 1$ *and* $Q = I_n$, *then we can obtain the following corollaries, respectively.*

**Corollary 11.4.3** *If the upper and lower bound matrices of the continuous matrix covers of the coefficient matrices of the system in Eq. (11.28) satisfy*

$$\begin{aligned} & L_a + L_a^T + 2L_b^T L_b + 4L_c^T L_c + 4L_d^T L_d \\ & \quad < -(2||U_b - L_b||^2 + ||U_a - L_a||^2 + 4||U_d - L_d||^2 + 4||U_c - L_c||^2 + 2) I_n \end{aligned}$$

*then the system in Eq. (11.28) is large-scale stochastic asymptotic stable.*

**Corollary 11.4.4** *If the upper and lower bound matrices of the continuous matrix covers of the coefficient matrices of the system in Eq. (11.28) satisfy*

$$\begin{aligned}&L_a + L_a^T + [2\text{trace}(L_b^T L_b) + 4\text{trace}(L_c^T L_c) + 4\text{trace}(L_d^T L_d)]I_n \\ &\quad < -(\text{trace}(G_a^T G_a) + 2\text{trace}(G_b^T G_b) + 4\text{trace}(G_c^T G_c) + 4\text{trace}(G_d^T G_d) + 2)I_n\end{aligned}$$

*then the system in Eq. (11.28) is large-scale stochastic asymptotic stable.*

**Theorem 11.4.8** *For the system in Eq. (11.28), if there are positive definite symmetric matrix $Q$ and positive constants $\varepsilon_i$, $i = 1, \ldots, 3$, satisfying*

$$\begin{aligned}&QL_a + L_a^T Q + (\varepsilon_1 + \varepsilon_2)Q + \varepsilon_1^{-1}\lambda_{\max}(Q)||U_a - L_a||^2 I_n \\ &\quad < -[(1+\varepsilon_3)\lambda_{\max}(Q)\text{trace}(M_c^T M_c) + \varepsilon_2^{-1}\lambda_{\max}(Q)\,\text{trace}(M_b^T M_b) \\ &\quad\; + (1+\varepsilon_3^{-1})\,\lambda_{\max}(Q)\text{trace}(M_d^T M_d)]I_n\end{aligned}$$

*then the system in Eq. (11.28) is large-scale stochastic robust asymptotic stable. If in Theorem 11.4.8 we let $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $Q = I_n$, then we have the corollary below.*

**Corollary 11.4.5** *If the upper and lower bound matrices of the matrix covers of the grey coefficient matrices in the system in Eq. (11.28) satisfy*

$$\begin{aligned}&L_a + L_a^T + 2I_n + ||U_a - L_a||^2 I_n + 2\text{trace}(M_c^T M_c)I_n \\ &\quad < -[\text{trace}(M_b^T M_b) + 2\text{trace}(M_d^T M_d)]I_n\end{aligned}$$

*then the system in Eq. (11.28) is large-scale stochastic robust asymptotic stable.*

**Theorem 11.4.9** *For the system in Eq. (11.28), if there are positive definite symmetric matrix $Q$ and positive constants $\varepsilon_i$, $i = 1, \ldots, 6$, satisfying $\lambda_{\max}(M) + \lambda_{\max}(N) < 0$, then for any initial condition $\xi \in C_{F_0}^p([-\tau, 0]; R^n)$, the following holds true*:

$$E|x(t,\xi)|^2 \le Ke^{-r_0 t} \sup_{-\tau \le \theta \le 0} E|\xi(\theta)|^2, \; t \ge 0$$

*or equivalently*,

$$\lim_{t\to\infty} \sup \frac{1}{t} \log E|x(t;\xi)|^2 \le -r_0$$

*where the matrices $M$, $N$ are the same as in Theorem 11.4.6, $K = \frac{\tau e^{r_0 \tau}\lambda_{\max}(N) + \lambda_{\max}(Q)}{\lambda_{\min}(Q)}$, and $r_0$ is the unique real root of the following equation $r_0\lambda_{\max}(Q) + \lambda_{\max}(M) + e^{r_0\tau}\lambda_{\max}(N) = 0$, then the system in Eq. (11.28) is mean square exponential robust stable.*

## 11.5 Several Typical Grey Control Models

Grey control stands for the control of essential grey systems, including the situation of general control systems involving grey numbers, by constructing controls through employing the thinking methods of grey systems analysis, modeling, prediction, and decision-making.

### *11.5.1 Control of Redundancy Removal*

The dynamic characteristics of grey systems are mainly determined by the matrices $G(s)$ of grey transfer functions. So, to realize effect control over the systems' dynamic characteristics, one of the effective methods is to modify and correct the matrices of transfer functions and the structure matrices.

**Definition 11.5.1** Assume that $G^{-1}(s)$ is a system's structure matrix, and $G_*^{-1}(s)$ an objective structure matrix, then

$$\Delta^{-1} = G_*^{-1}(s) - G^{-1}(s) \tag{11.30}$$

is known as a structural deviation matrix (Deng, 1965).

From $G^{-1}(s)Y(s) = U(s)$ and $G_*^{-1}(s) = \Delta^{-1} + G^{-1}(s)$, we obtain $(G_*^{-1}(s) - \Delta^{-1})Y(s) = U(s)$. That is,

$$G_*^{-1}(s)Y(s) - \Delta^{-1}Y(s) = U(s) \tag{11.31}$$

**Definition 11.5.2** $-\Delta^{-1}Y(s)$ is referred to as a superfluous term. The control through a feedback of $\Delta^{-1}Y(s)$ to cancel the superfluous term is known as a control of redundancy removal (Deng, 1965). Through the effect of the feedback of $\Delta^{-1}Y(s)$, the system $G^{-1}(s)Y(s) = U(s)$ is reduced to

$$G^{-1}(s)Y(s) + \Delta^{-1}Y(s) = U(s)\big(G^{-1}(s) + \Delta^{-1}\big)Y(s) = U(s)$$

That is, $G_*^{-1}(s)Y(s) = U(s)$ has already processed the desired objective structure (Deng, 1965).

The number of entries in the structural deviation matrix $\Delta^{-1}$, used in a control with abandonment, directly affects the number of components in the controlling equipment. So, when considering the economics, reliability, and ease of application of a dynamic system, one must keep the number of elements in the deviation matrix $\Delta^{-1}$ as low as possible. That is to say, in the objective structural matrix, one should try to keep the corresponding entries of the original structure matrix. The idea of control with abandonment is depicted in Fig. 11.8.

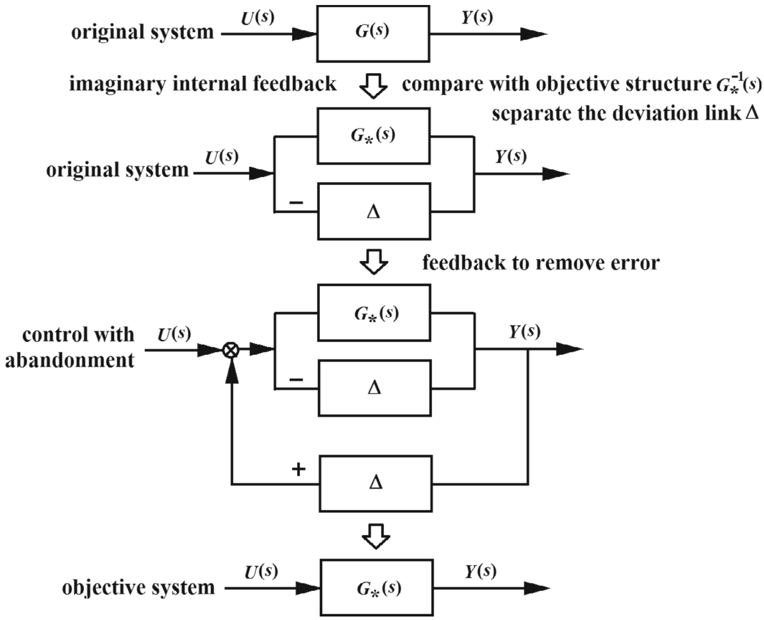



**Fig. 11.8** Control of redundancy removal

## 11.5.2 Grey Relational Control

**Definition 11.5.3** Assume that $Y = \left[y_1,\ y_2,\ \ldots,\ y_m\right]^T$ stands for the output vector, and $J = \left[j_1,\ j_2,\ \ldots,\ j_m\right]^T$ the objective vector. If the components of the control vector $U = [u_1,\ u_2,\ \ldots,\ u_s]^T$ satisfy

$$u_k = f_k(\gamma(J, Y))\ k = 1,\ 2, \ldots,\ s \tag{11.32}$$

where $\gamma(J, Y)$ is the grey relational degree between the output vector $Y$ and the objective vector $J$, then the system control is known as a grey relational control.

A grey relational control system is obtained by attaching a grey relational controller to the general control system. It determines the control vector $U$ through the grey relational degree of $\gamma(J, Y)$ so that the grey relational degree between the output vector and the objective vector does not go beyond a pre-determined range. The idea of the grey relational control system is depicted in Fig. 11.9.

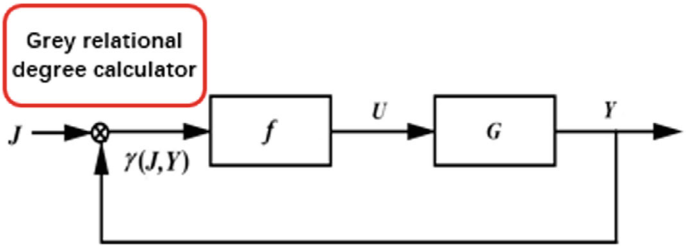



**Fig. 11.9** The grey relational control system

## *11.5.3 Control of Grey Prediction*

All the various kinds of controls studied earlier are about applying controls after first checking whether or not the system's behavioral sequence satisfies some pre-determined requirements. Such post-event controls evidently suffer from the following weaknesses:

(1) Expected future disasters cannot be prevented;
(2) Instantaneous controls cannot be done; and
(3) Adaptability is weak.

The so-called grey predictive control is designed based on the system's future behavioral tendency, which is predicted using the system's behavioral sequences and the patterns discovered from the sequences. This kind of control can be employed to avoid future adverse events from happening; it can be implemented in a timely fashion, and possesses a wide range of applicability.

The idea of a grey predictive control system is graphically shown in Fig. 11.10. Its working principle is that first one must collect and organize the device's behavioral sequence of the output vector $Y$; secondly, one must use a prediction device to compute the predicted values for the future steps; and lastly, one must compare the predicted values with the objective and determine the control vector $U$ so that the future output vector $Y$ will be as close to the objective $J$ as possible.

**Definition 11.5.4** Assume that $j_i(k)$, $y_i(k)$, $u_i(k)$ $(i = 1, 2, \ldots, m)$ are respectively the values of the objective component, output component, and control component at time moment $k$. For $i = 1, 2, \ldots, m$, let

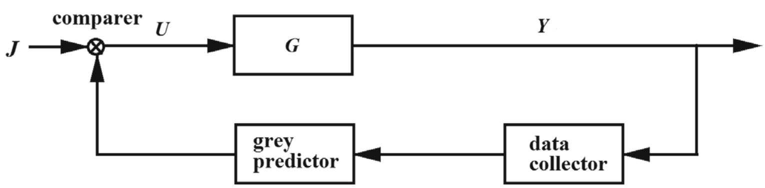



**Fig. 11.10** Grey predictive control

$$j_i = (j_i(1),\ j_i(2), \ldots,\ j_i(n))$$
$$y_i = (y_i(1),\ y_i(2), \ldots,\ y_i(n))$$
$$u_i = (u_i(1),\ u_i(2), \ldots,\ u_i(n))$$

For the control operator $f : (j_i(\lambda), y_i(\lambda)) \rightarrow u_i(k)$,

$$u_i(k) = f(j_i(\lambda), y_i(\lambda)) \tag{11.33}$$

when $k > \lambda$, the system is known as a post-event (or after-event) control; when $k = \lambda$, the system is known as an on-time control; and when $k < \lambda$, the system is known as a predictive control.

**Definition 11.5.5** If the control operator $f$ satisfies

$$f(j_i(\lambda), y_i(\lambda)) = j_i(\lambda) - y_i(\lambda) \tag{11.34}$$

That is,

$$u_i(k) = j_i(\lambda) - y_i(\lambda) \tag{11.35}$$

then when $k > \lambda$, the system is known as an error-afterward control; when $k = \lambda$, the system is known as an error--on-time control; and when $k < \lambda$, the system is known as an error-predictive control.

**Definition 11.5.6** Let $y_i = (y_i(1),\ y_i(2), \ldots,\ y_i(n))(i = 1,\ 2, \ldots, m)$ stand for a sample of output components and its GM(1,1) response formula be given as follows:

$$\begin{cases} \hat{y}_i^{(1)}(k+1) = (y_i(1) - \dfrac{b_i}{a_i})e^{-a_i k} + \dfrac{b_i}{a_i} \\ \hat{y}_i^{(0)}(k+1) = \hat{y}_i^{(1)}(k+1) - \hat{y}_i^{(1)}(k) \end{cases}$$

If the control operator $f$ satisfies

$$u_i(n+k_0) = f\left(j_i(k),\ y_i^{(0)}(k)\right), \quad n + k_0 < k_i\ i = 1,\ 2, \ldots,\ m \tag{11.36}$$

then the system control is known as a grey predictive control.

In a grey predictive control system, predictions are often done using metabolic models. So, the parameters of the prediction device vary with time. When a new data value output is produced and accepted by the sampling device, an old data value is removed so that a new model is developed. Accordingly, a series of new predicted values are provided. Doing so guarantees the strong adaptability of the system.

***Example 11.5.3*** Let us look at the EDM (electric discharge machining) grey control system (Yang & Zheng, 1996). The investigation on the control systems of EDM machines has been an important effort in the field of electric discharge machining. Each EDM can be seen as a stochastic time-dependent nonlinear system involving many parameters. Applications mainly include those situations when the conventional controls of linear, constant coefficient systems cannot produce adequate outcomes. The current commonly employed EDM control systems are established based on modern control theory. The frequently applied self-adaptive control systems generally employ mathematical models of approximation with accompanied high costs without actually realizing optimal results. Grey control is not like precise mathematical models based on complete knowledge of a system as addressed in modern control theory. It is also unlike fuzzy control, where the system is treated as a black box as all the information about the internal working of the system is disregarded, which leads to low accuracy controls. The parameters, structures, and other aspects of grey models vary with time. Such dynamic modeling can be highly appropriate for the study of EDM machines with high degrees of uncertainty and produce relatively more satisfactory control effects.

For EDM control systems, the objects of control are EDM machine tools, where outputs need signals from the testing of EDM machine tools as well as the control quantity $U$, that is, the signals about the control of the EDM machine tools. EDM control systems, in general, mean the control over systems that serve EDM machine tools. For instance, let us look at the traditional gap-voltage feedback servo control system. Due to a lack of linear relationship between the gas voltage, gap size, discharge strength, discharge state, and servo reference voltage, the effect of employing only one gap-voltage feedback servo control system is not very good.

In order to make up for the insufficiency of single loop controls, one can employ double-loop controls with the inner loop being the traditional gap-voltage feedback control and the outer loop being an impulse discharge rate feedback control that instantaneously adjusts the inner loop. The block-design chart of this control system is depicted in Fig. 11.11. Figure 11.11 shows that this control design represents a system of two loops. Based on the collected sequence of gap voltage readings $U_g(K)$, the inner loop employs the GM model to predict the next moment $\hat{U}_g(K+i+1)$. Here, $i$ stands for the prediction steps, which are then fed into the input end to determine the servo reference voltage value $U_s$, which is a proportionality coefficient. The outer loop establishes a GM model based on a sequence $Y(K)$ of output values to predict the next steps $\hat{Y}(K+i+1)$. When these predicted values are compared with requirements $Y^*$, a sequence $e(K) = \hat{Y} - Y^*$ of errors is found. These error values are then fed back into the system to adjust the proportionality coefficient $K_1$ and the servo reference voltage $U_s$, in order to adjust the inner loop. That is,

$$\Delta U = K_1\left(Y^* - \hat{Y}\right), \ U_s = K_2\hat{U}_g - \Delta U$$

Therefore, $U_s = K_2\hat{U}_g - K_1(Y^* - \hat{Y})$, where parameters $K_1, K_2$ are determined by experiments.

***Example 11.5.4*** Let us now look at the grey predictive control for the vibration of a rotor system (Zhu & Zhi, 2002). The theory and methods for active vibration

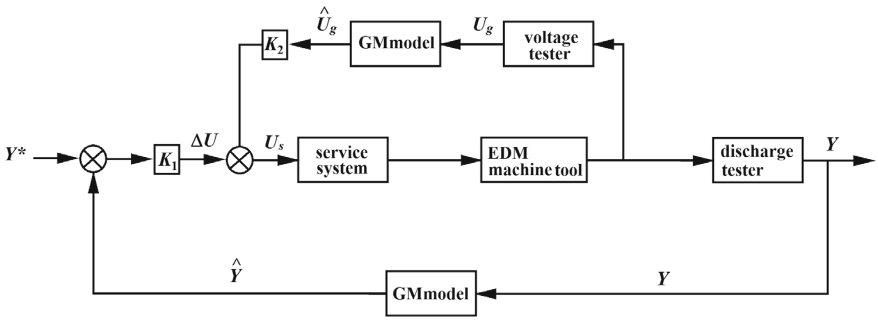



**Fig. 11.11** EDM control system

control of rotors have caught more attention in recent years. Many new control theories, such as neural network theory, time-delay theory, self-learning theory, fuzzy theory, and $H^\infty$ theory have gradually been employed in research on active control theory of rotors, leading to some good outcomes. For a Jeffcott symmetric rotor follower system with an electromagnetic damper as its executor, we employ control theory and methods of grey predictions to investigate an active amplitude control of vibration. We first establish a grey predictive control module with the GM(1, 1) as its main component. In the vibration control system of the rotor, let $I^0(k)$ and $x^0(k)$, $k = 1, 2, \ldots, n$, respectively be electric current inputting into the electromagnetic damper and the corresponding maximum output amplitude of the rotor vibration. By employing the available experimental measurement results from the literature, we obtain a set of data of $I^0(k)$ and the relevant $x^0(k)$, as shown in Table 11.1, when the sensitivity of the transducer is $10^4$ V/m.

Based on the mechanism of the GM(1, 1) model, we establish the following modification model of the system based on the errors of the grey predictions:

$$\hat{a}^{(0)}(k+1) = -a\big[x^{(0)}(1) - \beta\big]e^{-ak} + \delta(k-i)(-a\prime)\big[q^{(0)}(1) - \beta\prime\big]e^{-a\prime k}$$

where $a = 0.1862$; $x^{(0)}(1) = 1.4$; $\beta = 9.3298$; $a' = 0.14$; $q^{(0)}(1) = 0.36$; $\beta' = 3.78$, and

$$\delta(k-i) = \begin{cases} 1 & k \geq i \\ 0 & k < i \end{cases}$$

The design of our grey predictive control of the rotor system is shown in Fig. 11.12.

**Table 11.1** Sampled data of $I^0(k)$ and $x^0(k)$ when the transducer's sensitivity is $10^4$ V/m

| | | | | | |
|---|---|---|---|---|---|
| $I^0(k)$(A) | 0.1 | 0.125 | 0.175 | 0.225 | 0.325 |
| $x^0(k)$(dm m) | 1.4 | 1.35 | 1.2 | 0.9 | 0.65 |

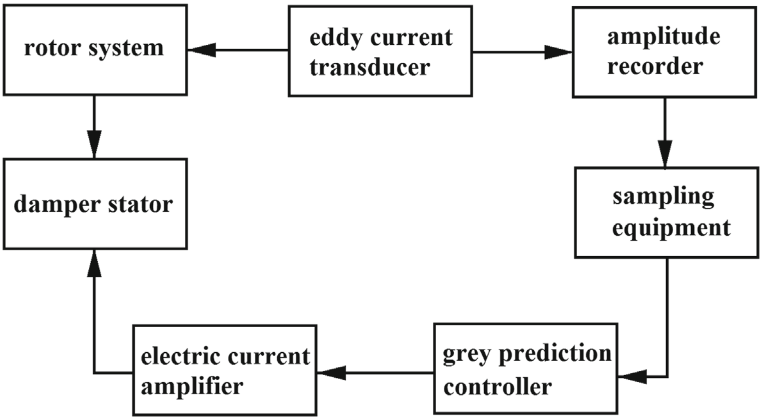



**Fig. 11.12** Grey predictive control of the rotor system

**Table 11.2** Simulations and actual measurements

| $I(A)$ | $X_{1m}$ (m) | $X_{2m}$ (m) | $X_{3m}$ (m) | $e_{12}$ (%) | $e_{13}$ (%) | $e_{23}$ (%) |
|---|---|---|---|---|---|---|
| 0.1 | $1.41 \times 10^{-4}$ | $1.4 \times 10^{-4}$ | $1.4 \times 10^{-4}$ | 0.71 | 0.71 | 0 |
| 0.125 | $1.27 \times 10^{-4}$ | $1.227 \times 10^{-4}$ | $1.35 \times 10^{-4}$ | 3.5 | – 5.93 | – 9.11 |
| 0.175 | $1.03 \times 10^{-4}$ | $0.949 \times 10^{-4}$ | $1.2 \times 10^{-4}$ | 9 | – 13.8 | – 20.92 |
| 0.225 | $0.83 \times 10^{-4}$ | $0.745 \times 10^{-4}$ | $0.9 \times 10^{-4}$ | 11.4 | – 7.78 | – 17.22 |
| 0.325 | $0.55 \times 10^{-4}$ | $0.46 \times 10^{-4}$ | $0.65 \times 10^{-4}$ | 19.57 | – 13.38 | – 29.23 |

In this control system, the displacement signal of the rotor system is measured by the eddy current transducer. The sampling equipment collects the data from the amplitude recorder, and through the effect of the grey prediction controller, controlling voltage is produced. This voltage is transformed into a controlling electric current through the current amplifier. Then, when this electric current flows through the stator coil of the electromagnetic damper, an electromagnetic force is created, which in turn controls the amplitude of vibration of the rotor within the expected range so that the system's stability is achieved.

For this grey predictive control system developed for the vibration of the said Jeffcott symmetric rotor follower, our computer simulation, when compared to the physical measurements of the amplitudes, indicates that the maximum amplitudes under the control are only about 7% of those physically observed without the control imposed on the rotor system. Table 11.2 respectively provides the results of the maximum amplitudes of two separate computational simulations and the actual measurements $X_{1m}$, $X_{2m}$, and $X_{3m}$, along with the change in the static electricity $i$ of the electromagnetic damper, when the sensitivity of the transducer is $k_1 = 10^4$ V/m, and the corresponding errors $e_{12}$, $e_{13}$, and $e_{23}$ between $X_{1m}$ and $X_{2m}$, $X_{1m}$ and $X_{3m}$, and $X_{2m}$ and $X_{3m}$.

## References

Abdulshahed, A. M., Longstaff, A. P., & Fletcher, S. (2015). The application of ANFIS prediction models for thermal error compensation on CNC machine tools. *Applied Soft Computing, 27*, 158–168.

Chen, B. W., He, T. X., Qi, C. J., et al. (2023). Spray burner swing angle reheat steam temperature control based on improved multivariate grey prediction. *Electric Power Science and Engineering, 2023, 39*(06), 55–61.

Chen, M. Y. (1982). Grey dynamics of boring machine control system. *Journal of Huazhong Institute of Technology, 1982*(06), 7–11.

Chou, J. H., Chen, S. H., & Li, J. J. (2000). Application of the Taguchi-genetic method to design an optimal grey-fuzzy controller of a constant turning force system. *Journal of Materials Processing Technology, 105*(3), 333–343.

Deng, J. L. (1965). A synthesis method of parallel correction devices for multivariable linear systems. *Acta Automatica Sinica, 3*(1), 13–26. in Chinese.

Deng, J. L. (1982). Control problems of grey systems. *Systems & Control Letters, 1*(5), 288–294.

Deng, J. L. (1985). *Grey control system.* Press of Huazhong University of Science and Technology. in Chinese.

Deng, J. L., & Zhou, C. S. (1986). Sufficient conditions for the stability of a class of interconnected dynamic systems. *Systems & Control Letters, 7*(2), 105–108.

Dinh, T. Q., Ahn, K. K., & Marco, J. (2017). A novel robust predictive control system over imperfect networks. *IEEE Transactions on Industrial Electronics, 64*(2), 1751–1761.

Dinh, T. Q., Marco, J., Yoon, J. I., & Ahn, K. K. (2018). Robust predictive tracking control for a class of nonlinear systems. *Mechatronics, 52*, 135–149.

Gao, F., Zhang, Y., & Gao, P. (2012). Research on speed controller model for high-speed train based on grey genetic algorithm. *Computer Measurement & Control, 20*(5), 1272–1275.

Gao, W., Ge, G. Y., Yang, Y., et al. (2022). Constant power cutting control method based on feedforward grey prediction and fuzzy control. *Mechanical Design and Research, 38*(03), 147–151.

Guo, L. P. (2023). Precise tracking control method for multi autonomous underwater vehicles. *Journal of Naval Science and Technology, 45*(11), 102–105.

Hu, L., Hu, Y. S., Tan, Z. Y., et al. (2024). Active control simulation and sound quality evaluation of automotive cooling fan noise. *Noise and Vibration Control, 44*(02), 202–207 (in Chinese)

Huang, H. H., Chen, Z., & Wang, H. X. (2024). Fast control power supply current control of full superconducting Tokamak nuclear fusion power plant based on improved grey prediction single neuron PID. *Journal of Electrical Engineering, 39*(06), 1886–1897.

Huang, S. J., & Huang, C. L. (2000). Control of an inverted pendulum using grey prediction model. *IEEE Transactions on Industry Applications, 36*(2), 452–458.

Lai, Y., Chen, X., Qin, Q., Li, J., & Liu, Y. X. (2004). Corrosion analysis and corrosion rates prediction of NiFe2O4 cermet inert anodes. *Journal of Center South University, 35*(6), 896–901.

Lee, W. M., & Liao, Y. S. (2003). Self-tuning fuzzy control with a grey prediction for wire rupture prevention in WEDM. *International Journal of Advanced Manufacturing Technology, 22*(7–8), 481–490.

Li, G. D., Masuda, S., & Nagai, M. (2015). Predictor design using an improved grey model in control systems. *International Journal of Computer Integrated Manufacturing, 28*(3), 297–306.

Li, H. J. Wang, L., & Kang, B. B. (2020). Research on dual hydraulic cylinder synchronous control based on grey prediction theory. *Machine Tool and Hydraulic, 48*(21), 97–100.

Li, Y. C., Zhang, F., Yang, C. Z., & Bai, W. (2001). The use of "grey system" analysis methodology for automated boiler water chemistry control in electric power plants. *Anti-Corrosion Methods and Materials, 48*(2), 96–98.

Liang, H. F., Fan, S. K., Zhang, H., et al. (2018). Simulation of grey prediction intelligent PID control in rocket launcher servo system. *Firepower and Command Control, 43*(11), 69–71+76.

Liem, D. T., Truong, D. Q., & Ahn, K. K. (2015). A torque estimator using online tuning grey fuzzy PID for applications to torque-sensorless control of DC motors. *Mechatronics, 26*, 45–63.

Liu, Y. S., Zhi, X. Z., Gu, Z. P., et al. (2004a). Research on grey relational control method of rotor system vibration. *Journal of Dynamics and Control, 2004*(03), 92–95.

Liu, Y. S., Zhi, X. Z., Gu, Z. P., et al. (2004b). Research on Grey Verhuslt predictive optimal control of rotor vibration. *Journal of Applied Mechanics, 2004*(03), 138–141+169.

Liu, Y. X., Chen, X., Zhang, G., Li, J., & Zhou, Z. (2004c). Application of mining algorithm based on grey association rule in aluminum electrolysis control. *The Chinese Journal of Nonferrous Metals, 14*(3), 494–498. in Chinese.

Liu, Z. L., Ma, S. B., He, M., et al. (2024). A control model for micro cracking degree of cement stabilized crushed stone based on CT technology. *Journal of Materials Science and Engineering, 42*(02), 238–245+283.

Lu, X., & Wang, C. (2013). Simulation on ATO speed controller based on grey prediction control. *City Track Traffic Development Research, 16*(2), 62–65.

Luo, W. (2016). Design of an intelligent control system for biomass gasification furnace. *Mechanical Manufacturing and Automation, 45*(05), 215–219.

Luo, R. C., & Chen, T. M. (2000). Autonomous mobile target tracking system based on greyfuzzy control algorithm. *IEEE Transactions on Industrial Electronics, 47*(4), 920–931.

Ni, Y. H., Zheng, L. G., & Zhou, J. L. (1994). Development of grey predictive controller for EDM servo system. *Electromachining, 1994*(06), 19–21.

Nie, X. F., Lu, G. B., & Song, H. (2016). Research on grey prediction PID control method for pressure control system of pressure regulator. *Journal of Weapon Equipment Engineering, 37*(05), 108–110+134 (in Chinese)

Pang, W., Chen, J. L., Li, K., et al. (2017). Design of a mine sweeping pear control system for a weapon based on variable step grey prediction fuzzy PID. *Mechanical Manufacturing and Automation, 46*(01), 194–196+201 (in Chinese)

Qiao, G., Zhang, W., & Xue, S. (2009). Speed control based on fuzzy PID control with grey prediction in the deep sea stepping system. *Journal of China Coal Society, 34*(11), 1550–1553. In Chinese.

Salmeron, J. L., & Palos-Sanchez, P. R. (2019). Uncertainty propagation in fuzzy grey cognitive maps with Hebbian-like learning algorithms. *IEEE Transactions on Cybernetics, 49*(1), 211–220.

Salmeron, J. L., & Papageorgiou, E. I. (2014). Fuzzy grey cognitive maps and nonlinear Hebbian learning in process control. *Applied Intelligence, 41*(1), 223–234.

Su, C. H., & Liu, S. F. (2008). On the asymptotic stability of grey stochastic linear delay systems. *Control and Decision, 23*(5), 571–574. In Chinese.

Su, C. H., & Liu, S. F. (2009). On robust stability of p-moment index of stochastic system with distributed delay and the interval parameters. *Applied Mathematics and Mechanics, 30*(7), 856–864.

Su, C. H. (2012). Robust stability of grey stochastic time-delay systems with impulsive effects. *Systems Science and Mathematics, 32*(5), 537–548.

Tian, J. Y., & Lu, Y. (2007). Research on grey prediction model of the slab temperature in heating furnace. *Journal of Northeast University, 28*(S1), 6–10. In Chinese.

Wang, L. D., Che, L., Lu, J., et al. (2015). SINS/GPS integrated navigation system based on improved grey prediction model. *Chinese Journal of Inertial Technology, 23*(02), 248–251+257 (in Chinese)

Wang, M. D., He, S. X., Li, X. L., et al. (2021). Grey prediction PI control of direct drive permanent magnet synchronous wind turbine. *Hydroelectric Energy Science, 39*(05), 189–192 (in Chinese)

Wang, W., Wu, M., Cao, W., & Lei, Q. (2010). Fuzzy-expert control based on combination grey prediction model for flue temperature in coke oven. *Control and Decision, 25*(2), 185–190. In Chinese.

Wang, X. C., & Qiu, X. J. (1999). Grey prediction guidance and simulation of air defense missile. *Journal of Institute of Command Technology, 1999*(06), 84–88 (in Chinese).

Wang, Y. T., & Xiao, J. C. (2021). Research on URV servo control system based on improved grey prediction single neuron PID. *Journal of Weapon Equipment Engineering, 42*(08), 251–257 (in Chinese)

Wei, X. Y., & Zheng, S. (2019). Reaction kettle temperature control using grey prediction and fuzzy weighted PID. *Journal of Chongqing University of Technology (Natural Science), 33*(02), 82–86 (in Chinese)

Xu, X. D., & Liao, J. W. (2024). Research on global control of solar power generation stability based on active fault tolerant synovial prediction. *Power Grid and Clean Energy, 40*(02), 142–148.

Yang, J., Zheng, L. G., Zhou, J. L, & Ni, Y. H. (1996). Research on EDM grey control system. *Electrical Processing, 5*, 22–23+45 (in Chinese)

Zhang, G., Fu, Y., & Yang, R. (2004). Novel self-adjustable grey prediction controller. *Control and Decision, 19*(2), 212–215. In Chinese.

Zhou, C. S., & Deng, J. L. (1989). Stability analysis of grey discrete-time systems. *IEEE Transactions on Automatic Control, 34*(2), 173–175.

Zhu, X. P., Zhi, X. Z., et al. (2002). Grey predictive control of the vibration of rotor systems. *Mechanical Science and Technology, 21*(1), 97–101. http://igss.nuaa.edu.cn/ (In Chinese)

# Chapter 12
# Spectrum Analysis of Sequence Operators

## 12.1 Introduction

Changhai Lin et al. introduced spectrum analysis into grey system theory firstly in 2019 (Lin et al., 2019).

Generally, system data is presented in the form of time series data. Due to the influence of system disturbance, there will be some deviation between the observed data and the original behavior data series. It is of great significance for people to understand the evolution law of system to analyze and recognize the influence of system disturbance factors correctly. The spectrum analysis of time series data provides us a new perspective to understand time series data. Some characteristics of time series data can be presented more clearly in the frequency domain.

Behavioral prediction of a system under the influence of shocking disturbances has always been a difficult problem. In this case, the available data of the system's behavior can no longer truthfully reflect the law of change of the system. At this situation, if we directly established our model and made our predictions using the severely affected data without first considering the disturbance, then our predictions would most likely fail. This is because the model would not have described the true state of change of the underlying system. Therefore, one of the main tasks of grey forecasting is to uncover the laws of change of certain system variables themselves based on the available data of the system (Liu, 1991).

As usually, a general data sequence composited by various factors of trend and noise (Fig. 12.1a), cycles (Fig. 12.1b), shock disturbance by long-duration impulse (Fig. 12.1c), shock disturbance by transient impulse (Fig. 12.1d), and some factors be ignored (Fig. 12.1e), even some factors joining with noise or be seen as noise (Fig. 12.1a). The evolution rule of data series may change at some points which are called change points (Page, 1955). Before and after the change points, people need to use different models to describe the change rule of data series. The difference may be the change of model form, or the change of one or some parameters in the model (Fig. 12.1f).



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2_12

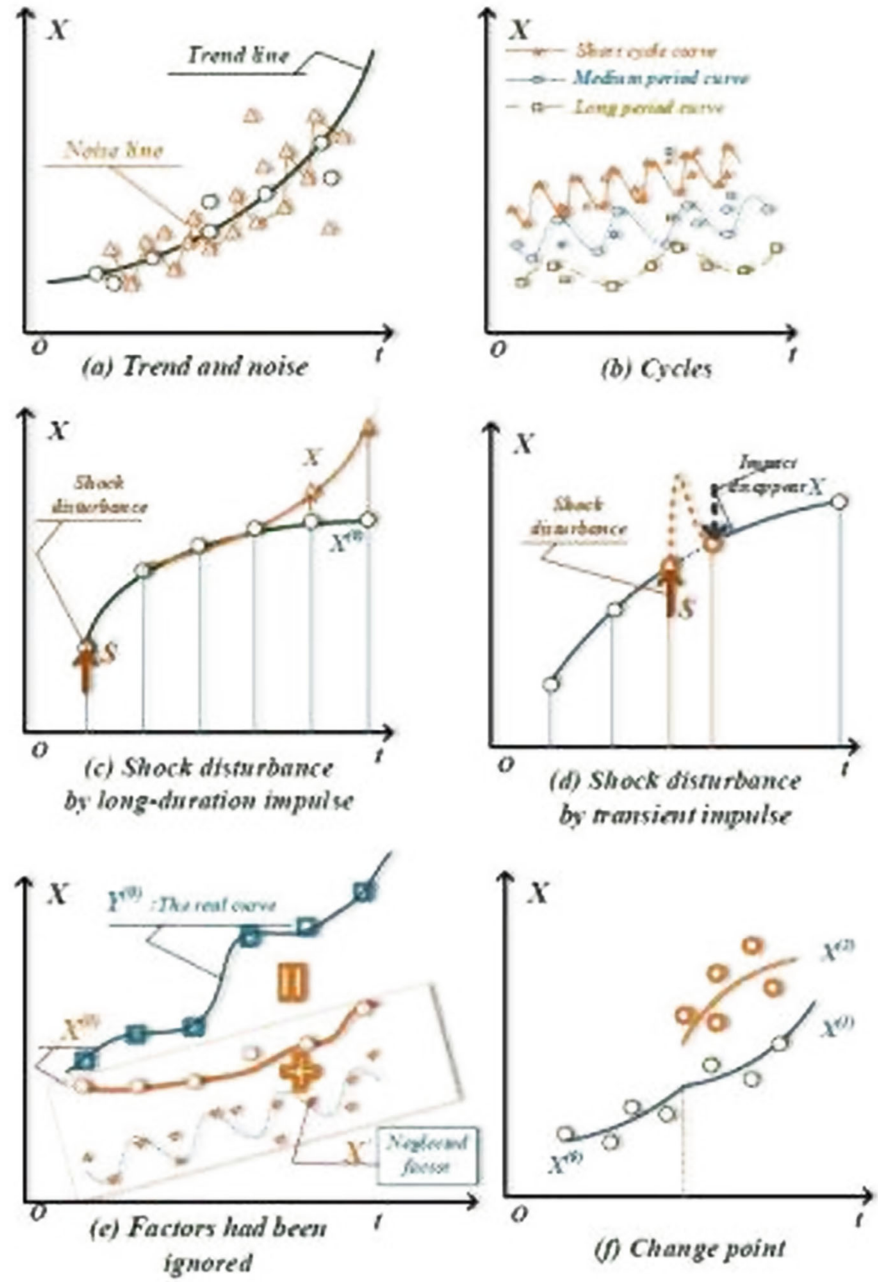



**Fig. 12.1** Various factors of a data sequence

It's very difficult to analyze and discriminate the factors of a data sequence in time domain. Thanks to spectrum analysis, we can transfer data in time domain to frequency domain by Fourier transformation. Then analyze and discriminate various factors of a data sequence in frequency domain (Liu et al., 2020).

## 12.2 Spectrum Analysis of Time Series Data

Spectral analysis is an important term proposed by Isaac Newton. He first used the concept of spectral analysis in his paper submitted to the Royal Society in 1672 (Newton, 1672). In this paper, Newton mentioned the famous prism experiment. As we all know, prism can decompose sunlight into seven colors. The principle is to decompose the light of different colors from the white light by using the different refractive index of the medium to the light of different colors. In the experiment, Newton also used two positive and negative inverted prisms. Through the first prism, white sunlight was decomposed into different colors of light, while the second inverted prism synthesized different colors of light into white sunlight. In the whole process of decomposition and synthesis, the essence of light has not changed. Prism can be seen as a conversion tool to show the characteristics of light (Newton, 1672) (Fig. 12.2).

The knowledge of physics tells us that every color of light represents a frequency range in visible light. Color analysis of light belongs to the scope of spectrum analysis. Spectrum analysis, as the name implies, its research on the object is carried out in the frequency domain. In the process of system analysis, the data of system behavior observed in the real physical world are mostly time series data recorded based on time, which can be abstracted as a function of time and belong to the scope of time domain. The spectrum analysis of time series data is based on signal decomposition. With the help of the mathematical tool of Fourier transform, the spectrum analysis regards the time series signal as the superposition of sine waves or cosine waves with different periods and amplitudes. The sine wave or cosine wave with different periods and different amplitudes is defined as the frequency content with one amplitude in frequency domain. The conversion process of time series data from time domain

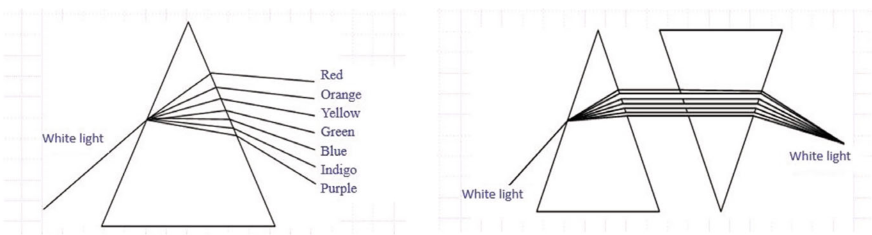



**Fig. 12.2** Decomposition and synthesis of white light

to frequency domain can also be expressed as the process of mapping different frequency content of time signal to frequency domain.

The spectrum analysis of time series data is a method of information mining. Information that is not easy to find in time domain analysis can be found through spectrum analysis. By decomposing and analyzing the time series data, we can quantitatively analyze the periodic law contained in the time series data. The magnitude and proportion of different frequency content can be quantitatively analyzed by calculating the frequency amplitude at different frequency points of time series signal.

## 12.3 Filtering Effect of Mean Operator and Accumulation Operator

The classical model of grey system—mean GM(1, 1) is based on accumulation operator and mean operator. The dual effects of accumulation operator and mean operator produce magical effects, so that people can use the mean GM(1, 1) to obtain high simulation and prediction accuracy based on few data.

In 1987, Professor Deng Julong studied the grey exponential law of the accumulation operator (Deng, 1987), and found that the the randomness of grey data sequence can be weaken under the action of the accumulation operator and show the variation law of the exponential function. Referring to the digital signal processing (DSP) system, we will study the mean operator and accumulation operator in the frequency domain through Z-transform, as well as the filtering effect of their series action. The contents and main conclusions of this section are based on the research of Lin et al. (2022).

### *12.3.1 Filtering Effect of Mean Operator*

The general 2-term weighted moving average operator can be rewritten into the following form

$$y[n] = b_0 x[n] + b_1 x[n-1] \quad b_0 + b_1 = 1. \tag{12.1}$$

Equation (12.1) can be regarded as the transfer function of DSP signal system, the following Eq. (12.2) can be obtained from Z transformation.

$$Y[z] = b_0 X[z] + b_1 X[z] z^{-1}. \tag{12.2}$$

From Eq. (12.3), it is easy to obtain the frequency domain expression of the transfer function of the digital filter system corresponding to the 2-term weighted moving average operators as follows

$$H[z] = \frac{Y[z]}{X[z]} = b_0 + b_1 z^{-1} \tag{12.3}$$

Let $b_0 = b_1 = 0.5$, the frequency domain expression of the transfer function of the digital filter system corresponding to the mean operator can be obtained as follows

$$H[z] = \frac{Y[z]}{X[z]} = 0.5 + 0.5z^{-1} \tag{12.4}$$

It can be seen from Fig. 12.3 that when the frequency content is 0, the frequency amplitude is 1, and when the frequency content is greater than zero, the frequency amplitude is less than 1. And the higher the frequency content, the smaller the frequency amplitude. That is, the mean operator has the effect of low-pass filtering, the low-frequency part (Evolution Law) of the data remains basically unchanged under the action of the mean operator, and the high-frequency part (fluctuation or disturbance) will be compressed and suppressed. Through spectrum analysis, it is further confirmed that the randomness of grey data sequence can be weaken and the real evolution law will be presented under the action of mean operator.

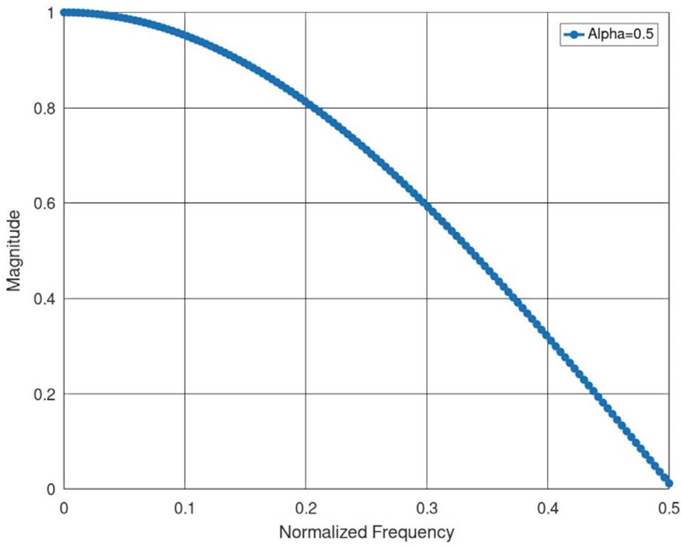



**Fig. 12.3** The frequency domain curve of mean operator equivalent filter transfer function

### *12.3.2 Filtering Effect of Accumulation Operator*

The first order accumulation operator (1-AGO) can be rewritten into the following form

$$y[n] = \mathrm{x[n]} + \mathrm{y[n-1]} \tag{12.5}$$

Equation (12.5) can be regarded as the transfer function of DSP signal system, the following Eq. (12.6) can be obtained from Z transformation.

$$Y[z] = X[z] + Y[z]z^{-1} \tag{12.6}$$

From Eq. (12.6), it is easy to obtain the frequency domain expression of the transfer function of the digital filter system corresponding to 1-AGO as follows

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1}{1 - z^{-1}} \tag{12.7}$$

From Eq. (12.7) and $z = e^{j\omega}$, it follows that

(1) When $0 \le \omega \le \pi/3$, $|H[\omega]|>1$. The amplitude of output Y[ω] will be greater than the amplitude of input X[ω]. The system amplifies the input spectrum.
(2) When $\omega < \pi/3$, $|H[\omega]| < 1$. The amplitude of output Y[ω] will be less than the amplitude of input X[ω]. The system compresses or suppresses the input spectrum.
(3) $z = 1$ is the pole of the transfer function of the digital filter corresponding to 1-AGO.

The frequency domain curve of transfer function of 1-AGO equivalent filter as shown in Fig. 12.4.

The first-order accumulation operator equivalent digital filter belongs to low-pass filter, that is, the low-frequency content (less than a critical frequency) in the input signal can pass through or be amplified. The high frequency content (greater than a critical frequency) in the signal will be compressed or suppressed.

The data fluctuation and random disturbance of general discrete data series belong to high-frequency content. These information will be suppressed during the action of 1-AGO equivalent digital filter. Aperiodic system evolution law belongs to low-frequency signal, which can pass through or be amplified in the process of 1-AGO equivalent digital filter. It is also proved that for general non-negative quasi smooth sequences, the randomness can be reduced by the action of accumulation operator, showing an approximate exponential growth law.

Since the transfer function of digital filter corresponding to 1-AGO has pole, and the frequency content ω = 0 is its pole. This means that when the frequency content is 0, the transfer function of 1-AGO corresponding digital filter has infinite amplification effect. Furthermore, the conclusion of Theorem 4.7.3 of this book is

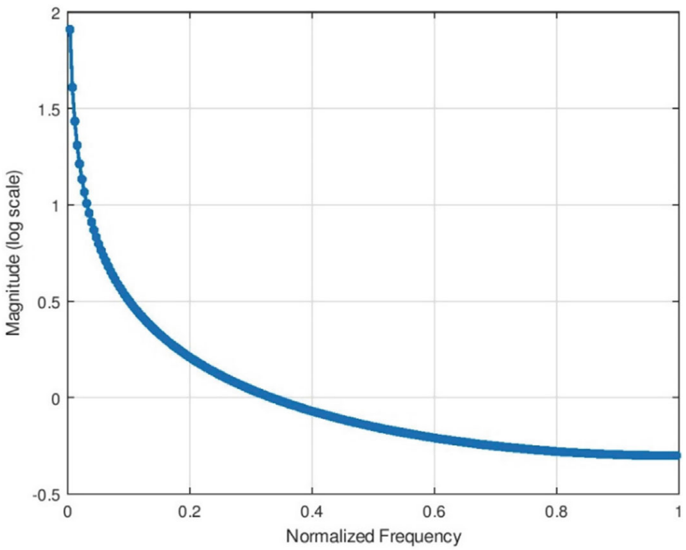



**Fig. 12.4** The frequency domain curve of transfer function of 1-AGO equivalent filter

confirmed from the mechanism: the function of accumulation operator shouldn't over. That is, if the action sequence of the r-th accumulation operator of $X^{(0)}$ has obvious exponential law, the application of AGO operator will destroy its regularity and turn the exponential law grey.

### *12.3.3 Filtering Effect of Series Operator*

Let the corresponding digital filter transfer functions mean operator (12.4) and accumulation operator (12.7) be denoted by $\mathrm{H_E(z)}$ and $\mathrm{H_A(z)}$ respectively. According to the transfer function calculation formula of series system, the transfer function of equivalent filter of the series operator of 1-AGO and mean operator can be obtained as follows

$$\begin{aligned} H[z] &= \mathrm{H_A z H_E(z)} \\ &= \frac{1}{1-z^{-1}}\left(0.5+0.5z^{-1}\right) \\ &= \frac{0.5+0.5z^{-1}}{1-z^{-1}} \end{aligned} \tag{12.8}$$

The frequency domain curve of series equivalent filter of the 1-AGO and mean operator as shown in Fig. 12.5.

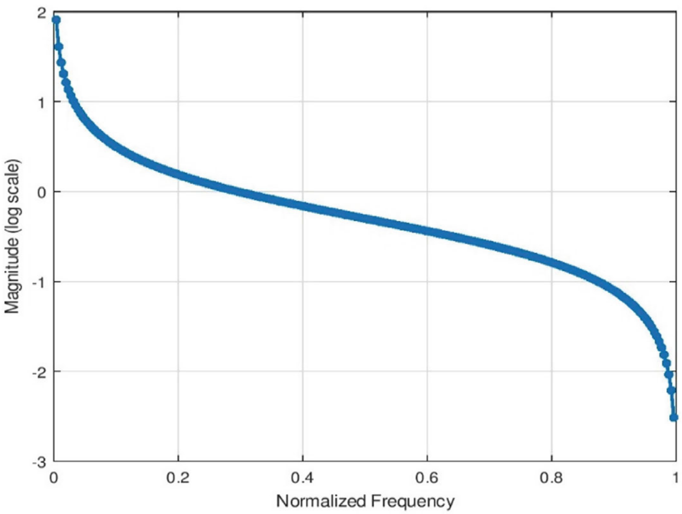



**Fig. 12.5** The frequency domain curve of series equivalent filter of the 1-AGO and mean operator

As can be seen from the comparison with Fig. 12.4 that The accumulation operator acting alone or the accumulation operator acting in series with the mean operator can produce similar amplification effect on the low-frequency part of the signal. But for the high-frequency part of the data sequence (fluctuation and noise), the series operator

$$H[z] = \mathrm{H_A(Z)H_E(Z)}$$

has stronger suppression effect than the accumulation operator acting alone. The signal-to-noise ratio of the series operator sequence is significantly improved. This also proves from the mechanism why the mean GM (1, 1) model can obtain high simulation and prediction accuracy based on small data in most cases.

## 12.4 Spectrum Analysis of Buffer Operator

In order to solve the prediction problem of impact disturbance system, Liu proposed the concept of buffer operator, established the axiom system of buffer operator, and designed the widely used average weakening buffer operator (AWBO) (Liu, 1991). Please refer to Chap. 4 of this book for details.

Let the x(k)d in the following AWBO

$$x(k)d = \frac{1}{n-k+1}[x(k) + x(k+1) + \ldots + x(n)];\ k = 1, 2, \ldots, n$$

be denoted by y(k). The AWBO can be rewritten into the following Formula (12.9)

$$y(k) = \frac{1}{n-k+1}\left[\sum_{i=1}^{n} x(i) - \sum_{i=1}^{k-1} x(i)\right] k = 1, 2, \ldots, n \tag{12.9}$$

Replace k with k − 1, we have

$$y(k-1) = \frac{1}{n-k+2}\left[\sum_{i=1}^{n} x(i) - \sum_{i=1}^{k-2} x(i)\right] k = 2, \ldots, n \tag{12.10}$$

Eliminate the denominator at the right end of Eqs. (12.9) and (12.10), we have

$$y(k)(n-k+1) = \left[\sum_{i=1}^{n} x(i) - \sum_{i=1}^{k-1} x(i)\right] k = 1, 2, \ldots, n \tag{12.11}$$

$$y(k-1)(n-k+2) = \left[\sum_{i=1}^{n} x(i) - \sum_{i=1}^{k-2} x(i)\right] k = 2, \ldots, n \tag{12.12}$$

Subtract Eq. (12.12) from Eq. (12.4), we obtain

$$y(k)(n-k+1) - y(k-1)(n-k+2) = \sum_{i=1}^{k-2} x(i) - \sum_{i=1}^{k-1} x(i) \tag{12.13}$$

Therefore

$$y(k)(n-k+1) - y(k-1)(n-k+2) = -x(k-1), k = 2, 3, \ldots, n \tag{12.14}$$

The digital signal processing expression corresponding to Eq. (12.15) as follows

$$Y(Z)(n-k+1) - Y(Z)Z^{-1}(n-k+2) = X(Z)Z^{-1} \tag{12.15}$$

$$k = 2, 3, \ldots, n \tag{12.16}$$

So, the transfer function of AWBO can be obtained

$$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{Z^{-1}}{\left[(n-k+1) - (n-k+2)Z^{-1}\right]} \tag{12.17}$$

The actual data simulation results show that the AWBO equivalent digital filter also belongs to low-pass filter. For the low-frequency part of the input signal, the amplitude of the spectrum of the AWBO action sequence is higher than that of the reference, which means that AWBO has amplification effect on the low-frequency content in the sequence. For the high-frequency part of the input signal, the amplitude of the spectrum of the AWBO action sequence is lower than the reference amplitude, indicating that AWBO has the effect of restraining, attenuating or blocking the high-frequency content in the sequence. The high-frequency part of the input signal of the impact disturbance system is mainly composed of impact disturbance components. Therefore, AWBO can weaken the impact disturbance (Lin et al., 2021).

## References

Deng, J. L. (1987). Grey exponential law of accumulating generation: The optimization information problem of grey control systems. *Journal of Huazhong University of Science and Technology, 12*(5), 11–16.

Lin, C. H., Wang, Y., & Liu, S. F. et al. (2019). On spectrum analysis of different weakening buffer operators. *The Journal of Grey System, 31*(4), 111–121.

Lin, C. H., Song, Z. Y., Liu, S. F., et al. (2021). Study on mechanism and filter efficacy of AGO/IAGO. Grey Systems: Theory and Application, 11(1), 1–21.

Lin, C. H., Liu, S. F., Fang, Z. G., & Yang, Y. J. (2022). Spectrum analysis of moving average operator and construction of time-frequency hybrid sequence operator. Grey systems-theory and application, 12(1), 101–116.

Liu, S. F. (1991). The three axioms of buffer operator and their application. Journal of Grey System, 3(1), 39–48.

Liu, S. F., Lin, C. H., Tao, L. Y., et al. (2020). On spectral analysis and new research directions in grey system theory. The Journal of Grey System, 32(1), 108–117.

Newton, I. (1672). A letter to the royal society.

Page, E. S. (1955). A test for a change in a parameter occurring at an unknown point. *Biometrika, 42*(3–4), 523–527.

# Appendix
# Introduction to Grey Systems Modeling Software

## A.1 Introduction

In 1982, Julong Deng initiated and established grey systems theory. Currently, grey systems theory is widely applied in areas such as social sciences, economics, agriculture, meteorology, military and science, providing solutions to a large number of practical problems and challenges met in everyday life. Various versions of grey system modeling software have played a very important role in such large scale practical applications of grey systems theory. Along with the rapid development of information technology, high level programming languages have gradually matured, applications of computing packages have been routinized, and grey systems modeling programs have also become sophisticated.

In 1986, Xuemeng Wang and Jiangjun Luo created their grey systems modeling software using BASIC language and published *Programs of Grey Systems' Prediction, Decision-Making, and Modeling*. In 1991, Xiuli Li and Ling Yang respectively developed grey modeling software using GWBASIC and Turbo C. In 2001, Xuemeng Wang, Jizhong Zhang, and Rong Wang published the book entitled "Computer Procedures for Grey Systems Analysis and Applications," in which they listed the structure and procedure codes established for grey modeling. All of these computer software packages were developed on the DOS platform and have become obsolete in the more user-friendly Windows framework.

In 2003, Dr. Bing Liu developed the first grey systems modeling software for Windows using VisualBasic6.0. As soon as this package was available, it was most welcomed in the community of scholars and practitioners of the grey systems research, and became the first choice of application in the field of grey systems modeling.

With the rapid development of software development technology, the constant changes in people's operating habits, and the continuous development and improvement of grey system theory itself, people's requirements for grey system modeling software are also constantly increasing. In 2009, Dr. Zeng Bo developed a new grey system modeling software based on the object-oriented programming language



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2

Visual C #, which greatly promoted the application and popularization of grey system theory. The software has now been updated to version 10.0 and has reached hundreds of thousands of users.

## A.2 Software Features and Functions

On one hand, an ideal grey systems modeling software package needs to have the computational power to handle practical models, and on the other hand it has to deal with user confirmation, registration, and other functionalities. The software system accompanying this book sufficiently combines the capabilities of the C/S (client/server) and B/S (Browser/Server) modules, where the C/S part completes computational functions, while the B/S part handles the relevant operations that serve the user and his communication with the server. With a view to improve existing systems, the design of this package focuses more on the reliability, practicality, compatibility, upgradability, accuracy, operational convenience, visual appeal and user friendliness. This package has the following characteristics:

- Data entry is convenient and fast.

For data sequences of the same kind, the package provides a rectangular window into which the user can simply copy the sequences with one operation. For grey clustering and grey decision-making modules that involves large amounts of data, it is evidently inconvenient to employ the traditional way of entering data values. In such instances, the user can enter data in an Excel document and then open the data file into this package system. This software system makes use of the powerful data entry ability of Excel while making data entry convenient for the user.

- Modules are designed according to functionalities.

In software engineering, a module is a relatively independent system unit of procedures. Each such unit of procedures handles and materializes a relatively independent task. In other words, it contains a group of independent procedures. Each program module has its own external characteristics, such as its own name, label, and interfaces. During the design of this software package, the developer scientifically organized the contents of grey systems theory, defined the relevant functions and related modules.

- This system provides operational details as well as periodic results.

For modules with complicated computational procedures where intermediate results are also important, the system provides a textbox that can store and show multi-line operational details. The user can monitor data changes in each computational step so that he can further understand how the model operates. Also, the software interface provides relevant information to remind the user of the relevant formulas employed in the model.

- The functionalities of the modules are greatly expanded.

Based on current practical applications of grey systems theory, combined with the most recent research results, this software system is the most up-to-date system available in the market. It includes: weakening operators (mean weakening buffer operators, geometric mean weakening buffer operators, weighted mean weakening buffer operators), strengthening operators (mean strengthening buffer operators, geometric mean strengthening buffer operators, weighted mean strengthening buffer operators), grey incidence analysis (relative degree of incidence, closeness degree of incidence), clustering analysis (based on center-point triangular whitenization weight functions), grey prediction (GM$(1, n)$ and DGM$(1, 1)$ models), grey decision analysis (intelligent grey target decision making), among other contents.

- The degree of accuracy of the computational results can be adjusted.

The computation precision of different systems is different. In this software system there is a ComboBox, which can select of computational precision. Therefore, the user can choose the desired degree of accuracy for his work.

- The operation of the software system is convenient and easy to learn.

This software system is based on the Windows interface using pull-down menus, where the commonly employed modeling techniques of grey systems theory are effectively gathered together. The user only needs to have an elementary understanding of how a desktop PC works to successfully use this software system. At the same time, this system has a relatively strong ability to locate and correct mistakes. When an illegal operation is performed, the system will provide an accurate and detailed hint.

- This system is developed using Visual C#.

C# is an object-oriented programming language created by Microsoft and an important part of Microsoft's .NET development environment. Also, Microsoft Visual C# is an integrated development environment (IDE) constructed on C# by Microsoft. It is designed for the operation of many application software packages created on the .NET framework. C# possesses powerful capabilities, type safety, object orientation, and other superb functions. It is currently the main development tool of C/S software architecture.

## A.3 Main Components

The new edition of the grey system modeling software consists of five modules including grey sequence operators, grey incidence analysis models, grey clustering evaluation models, grey forecasting models and grey models for decision-making,

**Table A.1** The basic constitution of the grey system modeling software

| | | |
|---|---|---|
| Grey system modeling software | B/S part | User info |
| | | Statistics |
| | C/S part | Grey sequence operators |
| | | Grey relational analysis |
| | | Grey clustering evaluation |
| | | Grey forecasting |
| | | Grey decision-making |

**Table A.2** Grey sequence operators

| | | |
|---|---|---|
| Grey sequence operators | Weakening operators | Average weakening buffer operator (AWBO) |
| | | Weighted average weakening buffer operator (WAWBO)) |
| | | Geometric average weakening buffer operator (GAWBO) |
| | | Weighted geometric average weakening buffer operator (WGAWBO) |
| | Strengthening operators | Even strengthening buffer operator (ESBO) |
| | | Average strengthening buffer operator (ASBO) |
| | | Weighted average strengthening buffer operator (WASBO) |
| | Information mining operators | Accumulating generation operator |
| | | Inverse accumulating generation operator |
| | | Even operator by adjacent neighbor |
| | | Operator of stepwise ratio |

given the currently available research on grey systems theory and its practical applications. The software system modules are shown in Tables A.1, A.2, A.3, A.4, A.5 and A.6.

**Table A.3** Grey relational analysis models

| | |
|---|---|
| Grey relational analysis models | Deng's model of degree of grey relation |
| | Absolute degree of grey relation |
| | Relative degree of grey relation |
| | Synthetic degree of grey relation |
| | Closeness degree of grey relation |
| | Similitude degree of grey relation |

**Table A.4** Grey clustering evaluation models

| | |
|---|---|
| Grey clustering evaluation models | Grey clustering model of variable weight |
| | Grey clustering model of fixed weight |
| | Grey clustering model using center-point mixed triangular possibility functions |
| | Grey clustering model using end-point mixed triangular possibility functions |

**Table A.5** Grey forecasting models

| | | |
|---|---|---|
| Grey forecasting models | Singular variable models | Even GM(1,1) |
| | | Original difference GM (1, 1) |
| | | Even difference GM (1,1) |
| | | Discrete grey model |
| | | Grey Verhulst model |
| | Multi-variable models | Model GM(0, N) |
| | | Model GM(1, N) |

**Table A.6** Grey models for decision-making

| | |
|---|---|
| Grey models for decision-making | Weighted multi-attribute grey target decision |
| | Two stages model for decision-making |

## A.4 Operation Guide

### *A.4.1 The Confirmation System*

To verify legal ownership, the user needs to enter his account number and password before he can actually start using the system. However, if every time the user uses the system he has to confirm his legal ownership of the software package, it will become an annoyance. So, to guarantee the legal ownership of the user and maintain the operational simplicity of the system, the system applies the XML-based client


**Fig. A.1** The confirmation window

programming technique. When the user attempts to run the program for the first time, the system will prompt him to provide the needed account number and password. The provided data will then be delivered through the internet to the database located at the server to verify their legality. When the user attempts to use the program on different, subsequent occasions, he can directly enter the main interface window without having to enter his account number and password again.

On the first time of confirmation, if the user does not have an account number or password, he needs to click on the "User registration" button (see Fig. A.1) to register for a free user account (B/S). If the user forgets his password, he can click on the "Recall password" button to retrieve his password. Figure A.2 is the flow chart of confirmation.

### *A.4.2 Using the Software Package*

After successful confirmation of legal ownership, the user will enter the system's main interface window, as shown in Fig. A.3. Various grey systems theory modules (and their sub-modules) are administrated through menus.

Figure A.4 provides the flow chart of various sub-modules of the system

I. **Data Entering**

Before running the program, one needs to first enter data into the software package and specify the system parameters. As mentioned earlier, there are two ways to input data. One can directly enter data into the provided text box or import data from an external Excel document. For those modules that require large amounts of data, the

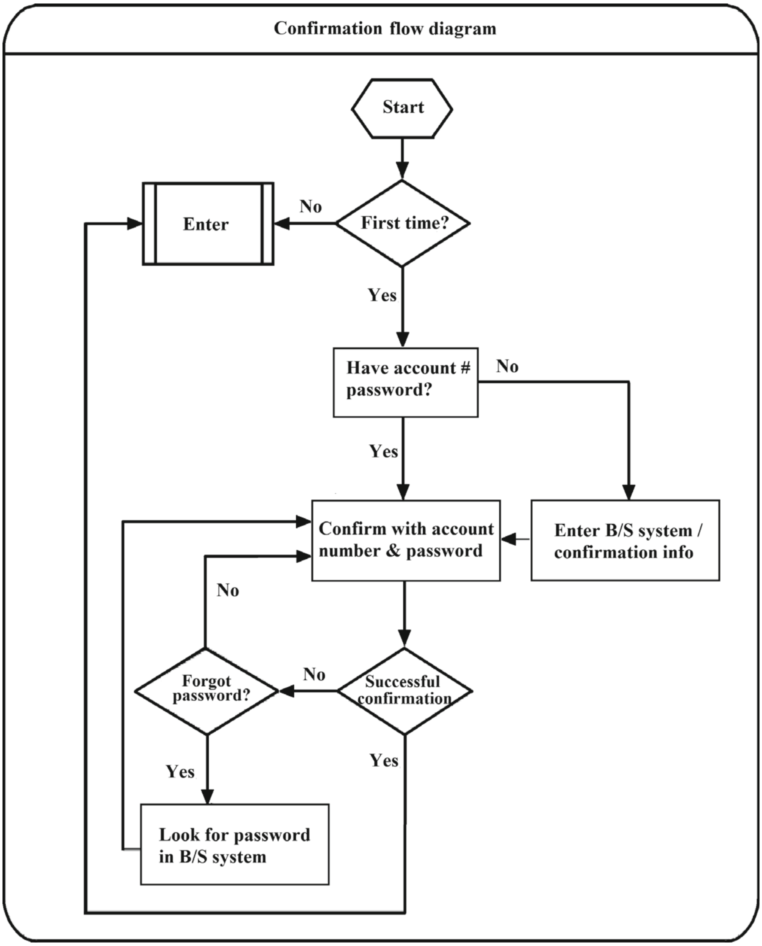



**Fig. A.2** The confirmation flow chart

only way provided for entering data is through importing data from Excel documents. The following sections look at the details of these two data entering methods.

- Enter data directly into the provided text box.

With VisualC#, there are two kinds of controllers available for direct data entry. One is the TextBox controller, and the other the ComboBox controller. The former controller is used to develop the standard Windows' editing controller of the text-box, which is used to acquire the user's input or show what is already stored in the storage space. When entering data into the text-box, right click the mouse inside the text box. When the cursor blinks inside the text box, one can start entering data. The ComboBox of the Windows' window group is mainly used to show data in a down-drop list box. As a default, ComboBox consists of two parts: the top is a text

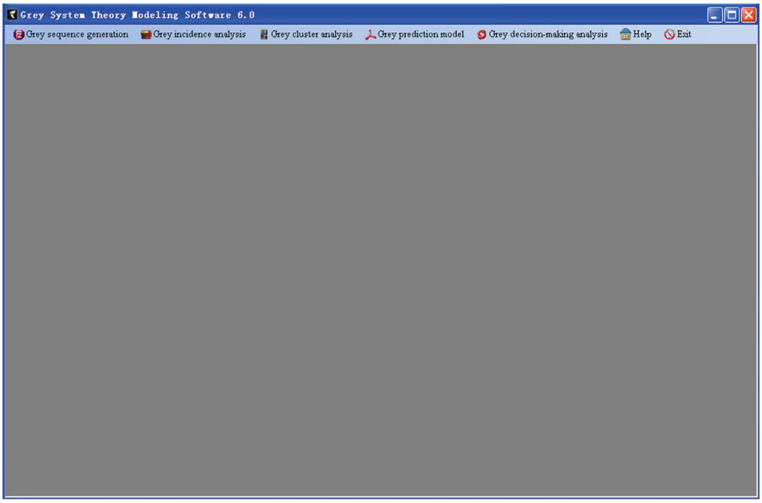



**Fig. A.3** The main interface window

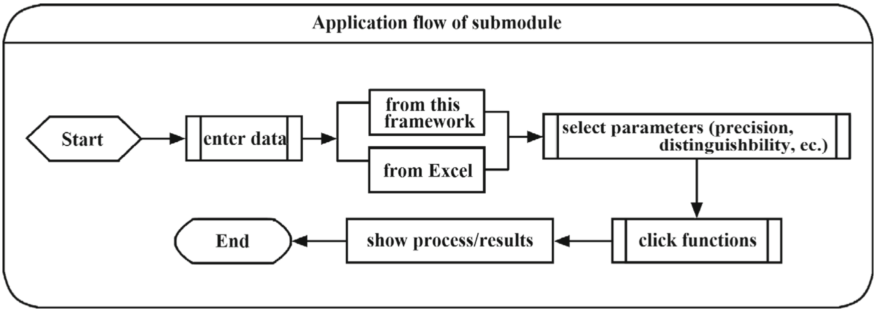



**Fig. A.4** Sub-modules

box in which the user is allowed to enter data, and the bottom is a list box where the user can make selections. It is because the ComboBox consists of the text box on the top and the list box at the bottom that it is named a ComboBox. When using the ComboBox to enter data, the user needs to first check whether or not the list box contains the data he wants. If so, he can simply use the mouse to directly make the selection; otherwise, he needs to enter data into the text box on the top. The detailed procedure for entering data in the ComboBox is similar to that of operating the TextBox and is therefore omitted here.

Note: When entering data using either the TextBox or the ComboBox, the state of entry method needs to be adjusted to half-angle. Data values entered in the state of full angle will be treated by the program as illegal data, which will directly affect the normal operation of the program and potentially lead to unexpected outcomes.

- Import data from an Excel document.

Both the TextBox and ComboBox can only accept small amounts of data values. For entering large sums of information, the use of either the Textbox or the ComboBox is inefficient, and can also lead to errors. To resolve the problem of entering large sums of data values when dealing with grey clustering and grey decision-making, for instance, it is very often the case that large amounts of information are involved, and this software system makes use of the powerful Excel. First enter and edit the needed data in Excel, and then use the provided interface to import the Excel data into the software system. Excel is one of the components of Microsoft Office. It is a tabulated testing and computing software developed for Microsoft Windows and Apple's Macintosh. Its straightforward interface, excellent capabilities of computation and graphics make it the most widely employed PC software used for data analysis. Through Excel, our software package system can conveniently acquire data.

Each Excel document generally contains three tables, respectively labeled as Sheet1, Sheet 2, and Sheet3. When an Excel document opens, it generally shows Sheet1. When entering data according to the system's requirements, one can directly type in the corresponding values in the relevant rows and columns. Upon finishing data entry into the Excel document, one can employ our system's input function to import the Excel data. When importing an Excel data document, first select the path from which the Excel file is located. As soon as the importing path and location of the file is confirmed, the data will be successfully imported. In fact, the process of importing data connects the Excel file to our system through a specific path so that the data in the Excel file can be mapped into the database controller DataGridView.

DataGridView is a database controller of VisualC#, which can exactly and entirely reveal data from a source file. Through the DataGridView controller of VisualC#, data can be acquired from an Excel file. However, our system does not provide any of the editing capabilities of DataGridView. In other words, if it is found in DataGridView that there is an error in the data, this error cannot be corrected directly within DataGridView. Instead, one has to return to the original Excel file to make the correction and then reimport the entire corrected file back into the grey systems modeling package.

Notes:

- The DataGridView controller does not have any editing capability. To make changes in the data, one has to do it in the original Excel file.
- When entering data into an Excel document, one needs to do so in the mode of "half-angle." All data entered in the mode of "full-angle" will be treated as illegal entries, which will directly affect the normal operation of the grey systems modeling package, and potentially lead to unexpected outcomes.
- The table names of the Excel file have to be the default Sheet1, Sheet2 and/or Sheet3 without any modification, otherwise the import of data will be affected.
- The data entry field of Excel is very large. However, one often needs only a few rows and columns. Make sure that there are no symbols or blank cells accidently entered into other area of the field. Otherwise, the data transfer will be affected.

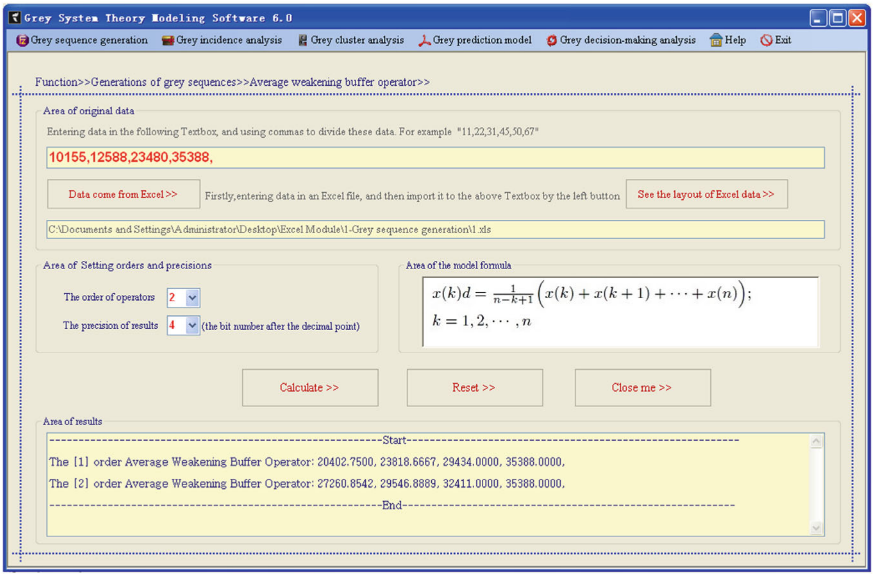



**Fig. A.5** The interface of the mean weakening buffer operator

## II. **Model Computations**

(1) Grey sequence operators

Click on "Sequence generation." From the pull-down menu that appears, select the module according to the practical modeling need. The corresponding detailed modeling interface appears. Let us use the "average weakening buffer operator" as an example to illustrate how to apply grey sequence generations. What is shown in Fig. A.5 is the interface of the mean weakening buffer operator.

This interface window contains three main areas: the first shows the original data sequence, which is the area for data entry or importing data; the second area shows the "order and outcome precision," in which it is possible to adjust the order of the operator being applied and the corresponding precision of the computational outputs based on one's modeling needs; and the third the area is where computational results are shown. After the data entry is completed, click on the "mean weakening buffer operator (AWBO)" button. Immediately, the generated sequence will appear in the generated sequence window. Figure A.6 shows a work sheet of an Excel document. When applying this Excel capability and importing data from Excel, the user has to follow this shown format exactly.

(2) Grey relational analysis models

Similar to the generation of grey sequences, there are two ways to input data for all parts of relational analysis, so such data entry details are omitted here. However, this is not valid for Deng's degree of grey relation due to the need for a large amount of

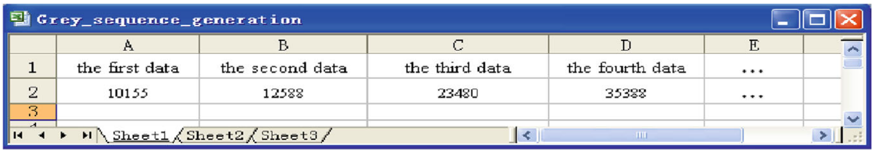



**Fig. A.6** The required Excel file format

data. For Deng's degree of grey relation, this software system allows only data entry through Excel documents without the option of direct data entry. Figure A.7 shows the editing format of an Excel document, while Fig. A.8 shows the complete work interface.

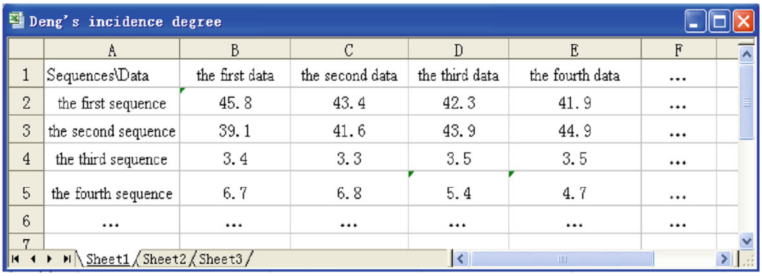



**Fig. A.7** The required Excel file format

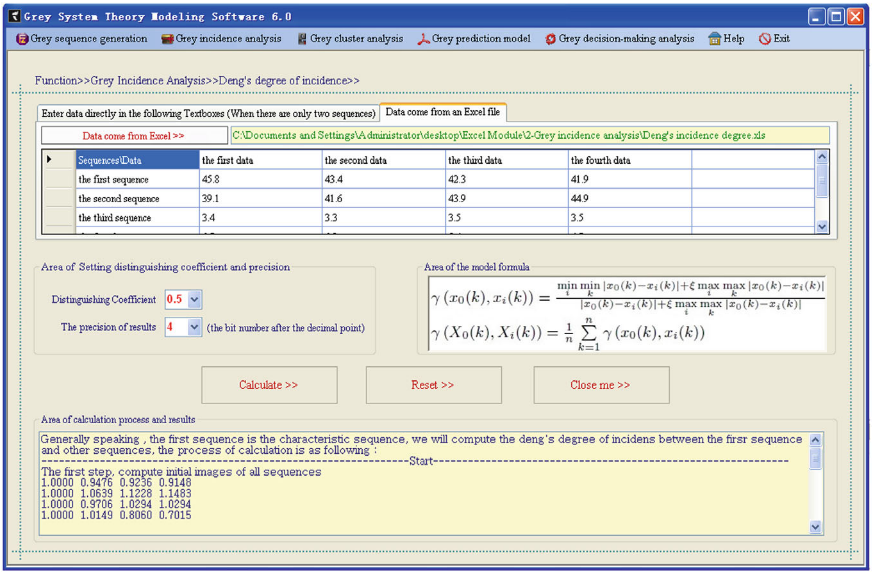
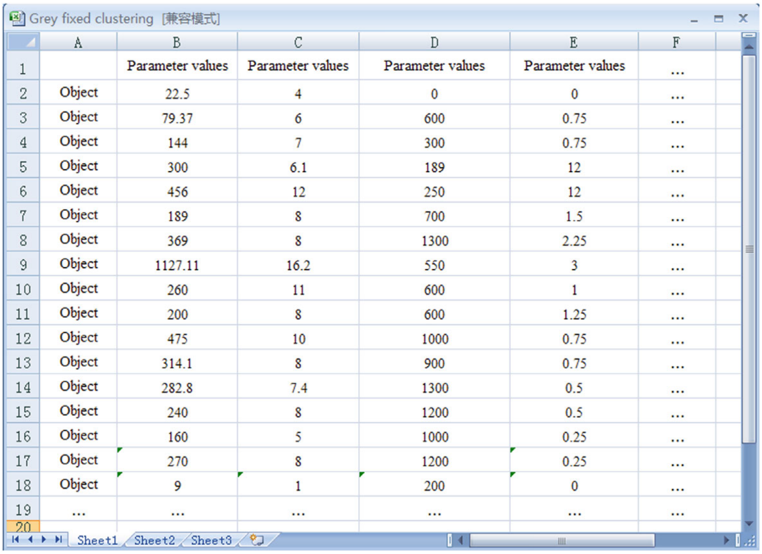



**Fig. A.8** The complete work interface of Deng's degree of grey relation

(3) Grey clustering evaluation models

Similar to Deng's degree of grey relation, grey clustering evaluation also requires a large amount of original data. However, in grey clustering evaluation it is possible to have several different types of data, including objective-criterion data, possibility functions, and criteria weights. Therefore, for grey clustering evaluation, the system again provides only one way to enter data, which is by importing Excel files. The key to using this part of the functions is to correctly edit the different types of data in the Excel documents. Sheet1 contains the objective-criteria data (Fig. A.9), Sheet2 the corresponding possibility functions (Fig. A.10), and Sheet3 the weights of the criteria (Fig. A.11).

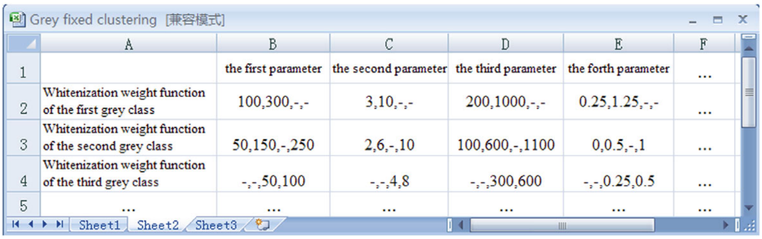
Figure A.12 shows the operating interface of a grey clustering analysis. As for how to apply grey variable weight clustering and analysis based on center-point mixed


Grey fixed clustering [兼容模式]

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | Parameter values | Parameter values | Parameter values | Parameter values | ... |
| 2 | Object | 22.5 | 4 | 0 | 0 | ... |
| 3 | Object | 79.37 | 6 | 600 | 0.75 | ... |
| 4 | Object | 144 | 7 | 300 | 0.75 | ... |
| 5 | Object | 300 | 6.1 | 189 | 12 | ... |
| 6 | Object | 456 | 12 | 250 | 12 | ... |
| 7 | Object | 189 | 8 | 700 | 1.5 | ... |
| 8 | Object | 369 | 8 | 1300 | 2.25 | ... |
| 9 | Object | 1127.11 | 16.2 | 550 | 3 | ... |
| 10 | Object | 260 | 11 | 600 | 1 | ... |
| 11 | Object | 200 | 8 | 600 | 1.25 | ... |
| 12 | Object | 475 | 10 | 1000 | 0.75 | ... |
| 13 | Object | 314.1 | 8 | 900 | 0.75 | ... |
| 14 | Object | 282.8 | 7.4 | 1300 | 0.5 | ... |
| 15 | Object | 240 | 8 | 1200 | 0.5 | ... |
| 16 | Object | 160 | 5 | 1000 | 0.25 | ... |
| 17 | Object | 270 | 8 | 1200 | 0.25 | ... |
| 18 | Object | 9 | 1 | 200 | 0 | ... |
| 19 | ... | ... | ... | ... | ... | ... |
| 20 | | | | | | |

Sheet1 Sheet2 Sheet3

**Fig. A.9** The objective-criteria data


Grey fixed clustering [兼容模式]

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | | the first parameter | the second parameter | the third parameter | the forth parameter | ... |
| 2 | Whitenization weight function of the first grey class | 100,300,-,- | 3,10,-,- | 200,1000,-,- | 0.25,1.25,-,- | ... |
| 3 | Whitenization weight function of the second grey class | 50,150,-,250 | 2,6,-,10 | 100,600,-,1100 | 0,0.5,-,1 | ... |
| 4 | Whitenization weight function of the third grey class | -,-,50,100 | -,-,4,8 | -,-,300,600 | -,-,0.25,0.5 | ... |
| 5 | ... | ... | ... | ... | ... | ... |

Sheet1 Sheet2 Sheet3

**Fig. A.10** The corresponding whitenization weight functions

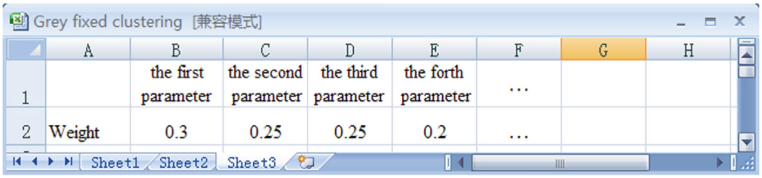



**Fig. A.11** The weights of the criteria

triangular possibility functions, the operational details are similar and therefore omitted

(4) Grey prediction models

Grey prediction models stand for an important part of grey systems theory. The operation of each individual prediction model is roughly the same. So, let us use the EGM(1, 1) model to illustrate how to use the software system. The main steps include: enter or import data; click the "computation, simulation, prediction" button to compute the model parameters and the simulated values, and select the simulation accuracy; enter the desired number of predicted values, then click "prediction results." Figure A.13 shows the operational interface of the EGM(1, 1) model.

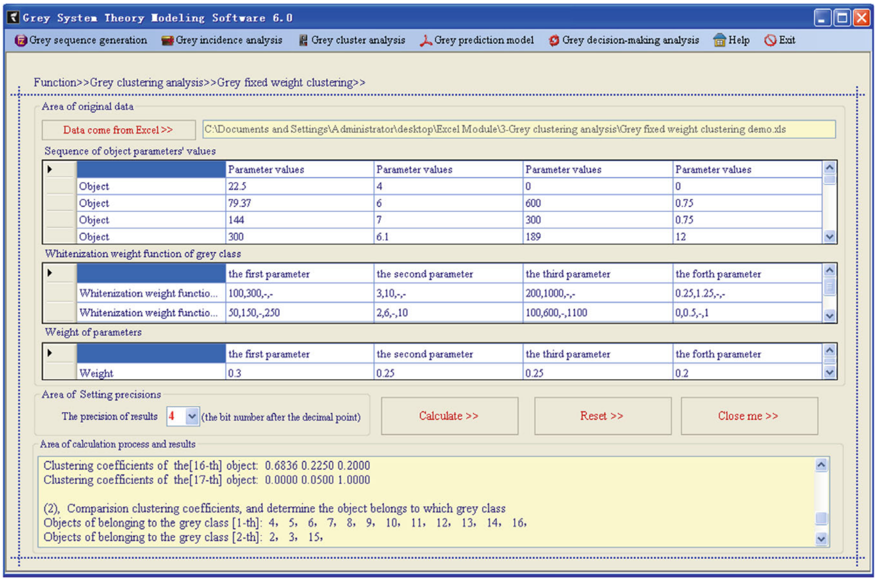



**Fig. A.12** The operating interface of a grey clustering evaluation

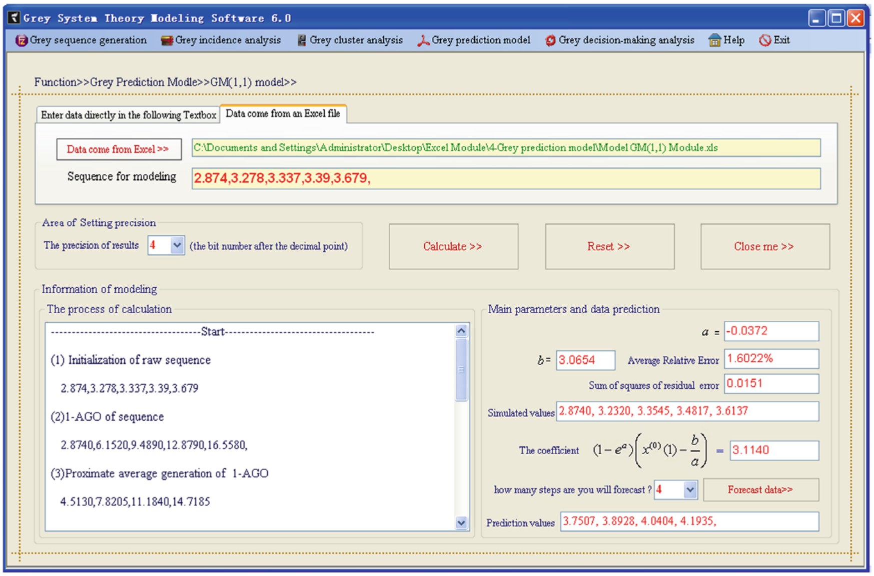



**Fig. A.13** The operational interface of the GM(1,1) model

## (5) Grey decision-making

This part of the software package contains two modules, namely the multi-attribute grey target decision-making model and the two-stage model for decision-making.

The data layout of an intelligent grey target decision-making model is the same as that of any synthesized objective decision-making model, except that there is an additional column of threshold value, as shown in Fig. A.14, where the interval (Liu, 2017; Liu et al., 2004) means that the lower effect threshold value is 14 and the upper effect value is 18. Figure A.15 shows the entire operating page of an intelligent grey target decision-making model.

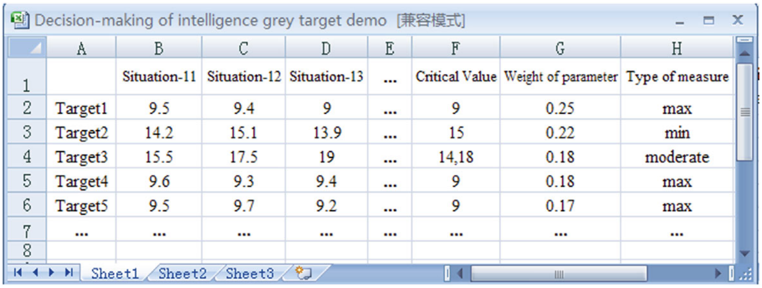



**Fig. A.14** The exact layout of data page for multi-attribute grey target decision-making

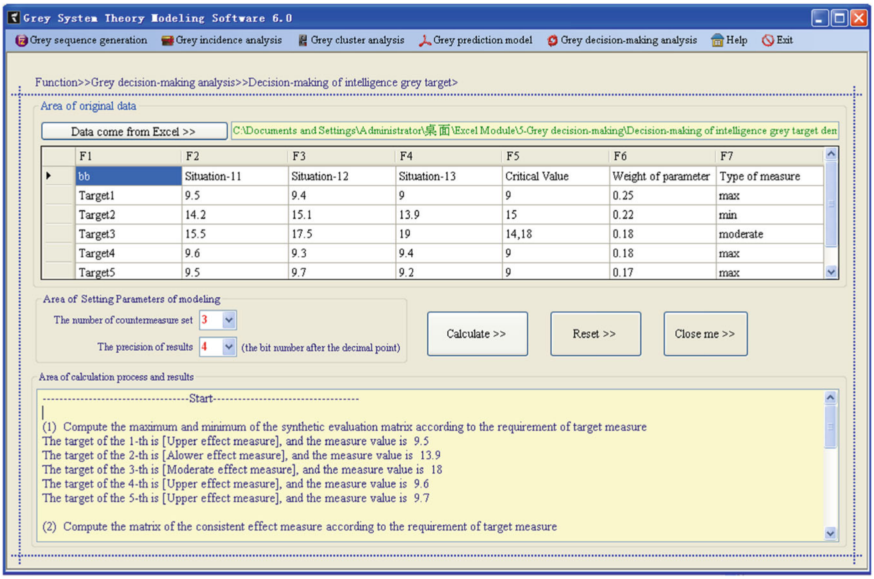



**Fig. A.15** The entire operating page of multi-attribute grey target decision-making

# Memorabilia of the Establishment and Development of Grey System Theory (1982–2024)

1. **Initial stage (1982–1987)**

Based on data retrieved from the China National Knowledge Infrastructure (CNKI) database. The number of papers on Grey System Theory included by CNKI increased from 6 in 1982 to 82 in 1987. The establishment of Wuhan (National) Grey System Research Association marks the end of the initial stage of grey system theory.

In 1982, professor Julong Deng has published the first paper on Grey System Theory in System and Control Letters (Deng, 1982a, 1982b).

In 1984, the first national academic conference on grey system theory and applications which chaired by professor Julong Deng was held in Taiyuan, Shanxi Province, China (Wang & Luo, 1985).

In 1985, Wuhan grey system consulting department was established (Xiao, 1990).

In 1985, professor Julong Deng has published the first book on Grey System Theory by National Defense Industry Press (In Chinese) (Deng, 1985).

In 1986, the course of grey system theory is included in the postgraduate training plan at both of Huazhong University of Science and Technology and Henan Agricultural University (Liu et al., 2022a, 2022b)

In 1987, Wuhan (National) Grey System Research Association which professor Julong Deng served as the president was established (Liu, 2024a, 2024b).

2. **Developmental stage (1988–2000)**

During this period, a professional international journal- The Journal of Grey System has been established. Professor Julong Deng was rated as a doctoral supervisor. Ten doctoral students in the field of grey system theory have graduated, becoming the first generation successors of the original theory of grey system theory.

In 1989, the first Journal of "The Journal of Grey System" which edited by professor Julong Deng was released by "Research Information Ltd" in the UK (Liu et al., 2022a, 2022b).

In 1990, a research direction for training doctoral students in the field of grey system theory has set up at system engineering discipline of Huazhong University of Science and Technology (Liu et al., 2022a, 2022b).



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2

In 1991, the grey system research office which professor Julong Deng served as the director was established in the Automation Department of Huazhong University of Science and Technology (Liu et al., 2022a, 2022b).

In 1995, Professor Sifeng Liu passed the entrance examination and entered Huazhong University of Science and Technology to pursue a doctoral degree, under the guidance of Professor Julong Deng (Liu et al., 2022b).

In 1995, Professor Sifeng Liu was elected as vice president of Wuhan (National) Grey System Research Association (Website of GSSC, 2024).

In 1996, The 9th national grey system academic conference attended by scholars from both sides of the Taiwan Strait was held in Wuhan (Liu & Xu, 1996).

In 1997, Regional Grey System Society of Taiwan was established (Liu et al., 2004).

In 1998, An Introduction to grey systems: foundations, methodology and Applications was published by IIGSS Academic Publisher in USA (Liu & Lin, 1998).

In 2000, with the approval of the Academic Degrees Committee of the State Council, Nanjing University of Aeronautics and Astronautics has established a doctoral degree authorization point for management science and Engineering which Professor Sifeng Liu serves as the Chief Discipline Leader (Liu et al., 2022b).

In 2000, Institute for Grey Systems Studies was established at Nanjing University of Aeronautics and Astronautics (IGSS-NUAA). Over the past more than 20 years, IGSS-NUAA has trained 200 doctoral students, postdoctoral fellows, and visiting scholars in the field of grey system theory, forming a main force engaged in grey system theory research (Xie et al., 2022).

3. **Rapid growth period (2001–2010)**

Grey System Theory is listed as the first leading research direction of doctoral program in the discipline of Management Science and Engineering at Nanjing University of Aeronautics and Astronautics. IGSS-NUAA has trained over 100 doctoral students, postdoctoral research fellows, and visiting scholars in the field of grey system theory, formed the main force engaged in grey system theory research. The number of papers on Grey System Theory included by CNKI increased from 1856 in 2001 to 11,900 in 2010.

In 2001, Grey System Theory is listed as the first leading research direction of doctoral program in the first level discipline of Management Science and Engineering at Nanjing University of Aeronautics and Astronautics (Xie et al., 2022).

In 2002, the first Doctor's Forum on Grey System Theory was held in Nanjing (Liu et al., 2022b).

In 2004, Kybernetes: The International Journal of Systems & Cybernetics published a special issue on grey systems theory which edited by Mianyun Chen, Sifeng Liu and Yi Lin (Liu & Lin, 2004).

In 2004, both the number of publications on grey system theory and the number of citations of IGSS at NUAA are ranked No.1 in web of science. This record is consistently maintained up to now (Xie et al., 2022).

In 2005, the Grey System Society of China (GSSC), CSOOPEM, was approved by the Chinese Association for Science and Technology and Ministry of Civil Affairs, China.

In 2005, Professor Sifeng Liu was elected as the president of GSSC. Professor Julong Deng served as Honorary President of GSSC (Website of GSSC, 2024).

In 2006, for the first time, the national grey system academic conference was supported by China Higher Science and Technology Center (CHSTC) which Nobel laureates Tsung-Dao Lee served as the director. CHSTC's support for grey system academic activities has lasted for 16 years (Xie et al., 2022).

In 2007, the first IEEE International Conference on Grey Systems and Intelligent Services was held in Nanjing (Liu, 2007).

In 2007, the Technical Committee of IEEE SMC on Grey Systems was established (Webpage of IEEE SMC, 2024).

In 2007, The Journal of Grey System is included in the SCI database (Website of JGS, 2024).

In 2008, the course of grey system theory of Nanjing University of Aeronautics and Astronautics was rated as a National Excellence Course (Website of IGSS, 2024).

In 2009, academician Xuesen Qian, the first winner of the National Highest Science and Technology Award of China, sent professor Sifeng Liu and Dejin Song a letter to encourage their research work (Qian, 2009).

In 2010, the team with professor Sifeng Liu as the chief expert was rated as a National Excellence Teaching Team of China (Website of IGSS, 2024).

In 2010, the Journal of "Grey Systems-Theory and Application" which established and edited by professor Sifeng Liu was launched by Emerald Group in the UK (Website of GS, 2024).

4. **Globalization period (2011–2024)**

International academic organizations, international academic journals, and a series of international academic conferences are the main symbols of internationalization in a newly established discipline field. Scholars from over 130 countries around the world have published papers on grey system theory. More than 18,000 grey system related papers indexed in the Web of Science database. A large number of grey system theory papers have been rated as highly cited papers by various databases and journals. Many grey system theory researchers have been rated as "Global top 2% scientists" or highly cited scientists.

In 2011, the 2011 IEEE International Conference on Grey Systems and Intelligent Services was held in Nanjing (Liu, 2011).

In 2012, the course of grey system theory of Nanjing University of Aeronautics and Astronautics was selected as a National Excellence Sharing Course of China (Website of IGSS, 2024).

In 2012, the first Workshop of European grey system research collaboration network chaired by professor Yingjie Yang was held at De Montfort University (Website of DMU, 2012).

In 2013, professor Julong Deng, the founder of Grey System Theory, passed away (Website of GS, 2024).

In 2013, the 2013 IEEE International Conference on Grey Systems and Intelligent Services was held in Macau (Liu, 2013).

In 2013, Professor Sifeng Liu was selected for a Marie Curie International Incoming Fellowship (FP7- PEOPLE- IIF-GA-2013-629051) of the European Union(Zheng, 2014).

In 2014, an international network project entitled "Grey Systems and Its Applications" (IN-2014-020) directed by professor Yingjie Yang was funded by The Leverhulme Trust (Xie et al., 2022).

In 2014, a book Series of Grey Systems in Chinese which edited by professor Sifeng Liu were launched by Science Press (Liu et al., 2022b).

In 2015, the International Association of Grey System and Uncertainty Analysis (GSUA) was established in UK (Xie et al., 2022).

In 2015, the first International Congress of GSUA was held in Leicester, UK (Liu et al., 2022b).

In 2017, Grey Systems-Theory and Application is included in the ESCI database (GS News, 2017).

In 2017, Polish Scientific Association of Grey Systems which Dr. Rafał Mierzwiak served as the founding president was set up (Website of IAGSUA, 2024).

In 2017, the 2017 IEEE International Conference on Grey Systems and Intelligent Services was held in Stockholm, Sweden (Website of IGSS, 2017).

In 2017, according to a citation report by China National Knowledge Infrastructure, the book Grey System Theory and Its Applications, which authored by Prof. Sifeng Liu and published by Science Press, is identified as the No.1 top sited books in the pandect of natural science of China (CNKI, 2017).

In 2017, Professor Sifeng Liu has been selected to be one of the top 10 shortlisted promising scientists in the MSCA 2017 Prizes (An, 2022).

In 2018, the course of grey system theory of Nanjing University of Aeronautics and Astronautics was selected as a National Excellence Online Open Course (Xie et al., 2022).

In 2018, Grey Systems Society of Pakistan which Dr. Saad Ahmed Javed served as the founding president was established (Website of IAGSUA, 2024).

In 2018, Academician Jinpeng Huai, The Minister of Education of China, then the Executive Vice President and Secretary of the China Association for Science and Technology sent a letter of condolence to Professor Sifeng Liu, praised his important contributions to the development and dissemination of grey system theory (Xie et al., 2022).

In 2019, Grey Systems-Theory and Application is included in the SCI database (Website of GS, 2024).

In 2019, Turkish Association of Grey Systems Theory which professor Erdal Aydemir served as the founding president was established (Website of IAGSUA, 2024).

In 2019, the 2019 International Congress of GSUA was held in Bangkok, Thailand (Website of IGSS, 2024).

In 2019, Angela Dorothea Merkel, then German Chancellor, praised professor Julong Deng, the founder of grey system theory, and professor Sifeng Liu, a developer of grey system theory (Yu & Le, 2019).

In 2020, Grey Systems-Theory and Application is included in the Scopus database (Website of GS, 2024).

In 2020, the 2020 International Congress of GSUA was held in Nanjing (Website of IAGSUA, 2024).

In 2020, the course of grey system theory of Nanjing University of Aeronautics and Astronautics was selected as a National first class offline course (Xie et al., 2022).

In 2020, the course of grey system theory of Nanjing University of Aeronautics and Astronautics was selected as a National first class online course (Xie et al., 2022).

In 2021, a book Series of Grey Systems which edited by professor Sifeng Liu, professor Yingjie Yang and professor Jeffrey Forrest were launched by Springer-Nature Group (Dong & Tao, 2021).

In 2021, Grey Data Analysis has been put into service on iCourse International (Tao & Rui, 2021).

In 2021, the 2021 International Congress of GSUA was held in Nanjing (Website of IAGSUA, 2024).

In 2021, Grey Systems-Theory and Application is included in the Ei Compendex database (Hu, 2022).

In 2022, the 40th Anniversary Commemorative Exhibition of the Founding of Grey System Theory was held in Nanjing University of Aeronautics and Astronautics. Many renowned scholars and Leaders of Professor Yong'an Zheng, then the Secretary of NUAA, Academician Zhongde Shan, then the president of NUAA, Academician Haiyan Hu, the former president of BIT, Academician Baozhu Guo, former Chief Designer of Satellite Engineering of China, Ms Suping Hu, former deputy director of the Standing Committee of the Shanxi Provincial People's Congress, professor, Professor Yingjie Yang, the Executive President of GSUA, Professor Hong Chi, the President of Chinese Society of Optimization, Overall Planning and Economic Mathematics and Professor Xiaochang Ding, the President of Jiangsu Higher Education Association attended the conference and delivered congratulatory messages (Zhou & Dong, 2022).

In 2023, Research Center for Grey Systems and Uncertainty Analysis which professor R. M. K. T Rathnayaka serve as the director was established at Sabaragamuwa University of Sri Lanka (Rathnayaka, 2023; Website of IAGSUA, 2024).

In 2023, Professor Sifeng Liu received the "Global Excellence Award" from the Grey Systems Society of Pakistan (GSSP) (GS News, 2023).

In 2023, the 2023 International Congress of GSUA was held in Zhengzhou (HAU News, 2023).

In 2024, the Center for Grey System Studies was established at Northwestern Polytechnical University (CGSS-NPU).

In 2024, Professor Sifeng Liu is listed as a global top 0.05% Highly Ranked Scholar-Lifetime by Scholar GPS (Website of Scholar GPS).

## Farewell to Our Tutor

The memorial speech by Sifeng Liu on behalf of students of Prof. Deng.

At 12:15 in the afternoon on June 22, 2013, our most beloved tutor, Professor Deng Julong, saw the end of his eighty-year life journey and left us forever. Professor Deng Julong was a tireless lifelong pioneer and founder of Grey System Theory, so the world has lost a visionary. In recent days, dark clouds gathered and there was a long period of wet weather: it was God's crying for the death of a great scholar.

Prof. Deng graduated from the Department of Electrical Engineering of the Huazhong Institute of Technology in 1955 and then taught at the Department of Automatic Control. In the 1960s, he put forward the idea of control with abandonment. In the 1970s, the method of control with abandonment became a typical control method internationally. In 1982, he pioneered Grey System Theory and created a brand new subject area in the history of science. His academic achievements were highly respected by many in the scientific circles.

In Prof. Deng's academic career of 60 years, there existed neither holidays, weekends nor a line between service and retirement. Prof. Deng was the editor of The Journal of Grey System, an international journal, for 24 years. In his capacity as editor he screened articles, checked the contents of their experiments and edited them in English; he was dedicated and tireless. Until the last moment of his life, he was still working on publishing a scholarly book.

I still remember 1983, when I first participated in a course on Grey System Theory. The mimeographed teaching materials had a blue cover and were presented as a book. It was like finding a treasure, as the book attracted me deeply. The book really inspired me, as I was a young scholar who was going through a period of confusion and lack of direction for academic study. The book shone with sparkles of wisdom and built a lighthouse for a knowledge seeker who was in the mist of trying to find his way in academic research. This book became the light in my life's journey. From then onwards I forged an indissoluble bound with Grey System Theory.

My most unforgettable memory is the first time I joined Prof. Deng's comprehensive course on Grey System Theory in Yu Jiashan's air-raid shelter in 1986. The course deepened my understanding and awareness of many scientific problems in Grey System Theory. In 1995, when I was about 40 years old, I formally became a disciple of Prof. Deng, and from then on I took on the mission of disseminating and developing Grey System Theory.

Today, Grey System Theory is accepted by academics worldwide. A variety of academic works on grey system theory have been published in different languages, including English, Japanese, Korean and Romanian. In 1989, the British journal entitled The Journal of Grey System was launched. In 1997, a Chinese publication named Journal of Grey System was launched in Taiwan. Later in 2004, this same publication started to be published in English. In 2011, Emerald launched a new journal entitled Grey system: Theory and application.

Since November 2007, the biennial IEEE International Conference on Grey System and Intelligent Services has been successfully held in Nanjing, China, attracting scholars from different parts of the world. The fourth IEEE International Conference on Grey System and Intelligent Services will be held at the University of Macau, in November 2013.

Currently, a significant number of scholars from China, United States, England, Germany, Japan, Australia, Canada, Austria, Russia, Turkey, the Netherlands, Iran, and others, have been involved in the research and application of Grey System Theory. Many countries have begun to recruit and cultivate doctoral and other postgraduate students in the area of Grey System Theory. To date, more than 100 students have graduated and received Ph.D.s in this area, and tens of thousands of graduate students have carried out their research on Grey System Theory.

At the Nanjing University of Aeronautics and Astronautics, Grey System Theory has become an important course for undergraduate, masters and doctoral students from many different colleges.

In 2008, Grey System Theory was selected for a national course award in China. In 2013, it was selected as an open learning course, free of charge to all lovers of Grey System Theory.

Currently, there are more than 70 projects on Grey System Theory research and applications funded by the National Natural Science Foundation of China. Many projects have been supported by the European Union, the United Kingdom, USA, Canada, Spain, Romania and other countries.

In 2012, De Montfort University in the UK funded and organized the first European collaboration network for Grey System research, and representatives from 14 European Member States attended the session.

Grey System Theory, as an emerging discipline, has carved its place in science and demonstrated strong vitality.

Farewell to our tutor!

God bless you!

(Prof. Sifeng Liu is a former Ph.D. Student of Prof. Julong Deng. Currently Prof. Liu works at the Nanjing University of Aeronautics and Astronautics. He also serves as the founding director of the Institute for Grey Systems Studies, the founding chair of the IEEE SMC Technical Committee on Grey Systems, and the president of Grey Systems Society of China.)

# References

An, J. (2022). Promoting grey system theory to the world-professor Liu Sifeng from Nanjing University of Aeronautics and Astronautics. *People's Weekly*, 12-05-2022.

CNKI. (2017). Annual report of highly cited Books in China.

Deng, J. L. (1982a). Control problems of grey systems. *Systems & Control Letters, 1*(5), 288–294.

Deng, J. L. (1982b). Grey control system. *Journal of Huazhong University of Science and Technology, 10*(3), 9–18.

Deng, J. L. (1985). *Grey system (social, economic).* National Defense Industry Press.

Dong, W. J., & Tao, L. Y. (2021). IGSS-NUAA signs a publishing agreement for the "Grey System Series" with Springer Nature Group. https://www.mbachina.com/html/nuaa/20211024/366874.html

GS News. (2017). Grey systems-theory and application is included in the ESCI. https://www.emeraldgrouppublishing.com/journal/gs

GS News. (2023). Professor Liu Sifeng won the Global Excellence Award. https://www.emeraldgrouppublishing.com/journal/gs

HAU News. (2023). HAU successfully held the 2023 GSUA International Congress and the 37th National Grey System Academic Conference. https://skc.henau.edu.cn/a/xueshujiaoliu/20231016/703.html

Hu, M. L. (2022). Grey systems-theory and application is included in EI Compendex. http://www.scope.org.cn/newsMes.aspx?id=7789&t=52

Liu, S. F., & Xu, Z. X. (1996). *New progress in grey system research.* Huazhong University of Science and Technology Press.

Liu, S. F., & Lin, Y. (1998). *An Introduction to grey systems: Foundations, methodology and Applications.* IIGSS Academic Publisher.

Liu, S. F., & Lin, Y. (2004). Guest edited. *Kybernetes: The International Journal of Systems & Cybernetics, 33*(2).

Liu, S. F., Dang, Y. G., & Fang, Z. G. (2004). *Grey systems theory and its application* (3rd ed.). Science Press

Liu, S. F. (Ed.) (2007). *Proceedings of IEEE international conference on grey systems and intelligent services.* IEEE Press.

Liu, S. F. (Ed.) (2011). *Proceedings of IEEE international conference on grey systems and intelligent services.* IEEE Press.

Liu, S. F. (Ed.) (2013). *Proceedings of IEEE international conference on grey systems and intelligent services.* IEEE Press.

Liu, S. F. (Ed.) (2017). *Proceedings of IEEE international conference on grey systems and intelligent services.* IEEE Press.

Liu, S. F., Tao, Y., Xie, N. M., Tao, L. Y., & Hu, M. L. (2022a). Advance in grey system theory and applications in science and engineering. *Grey Systems: Theory and Application, 12*(4), 804–823.



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2

Liu, S. F., Tang, W., Xie, N. M., & Tao, L. Y. (2022b). The establishment, development and international communication of Chinese Original grey system theory. *Journal of Nanjing University of Aeronautics and Astronautics (Social Sciences Edition), 24*(4), 1–10.

Liu, S. F. (2024a). *Grey systems theory and its application* (10th ed.). Science Press.

Liu, S. F. (2024b). *Grey systems analysis* (2nd ed.). Springer.

Qian, X. S. (2009). A congratulatory letter to Liu Sifeng and Song Dejin at NUAA. In S. F. Liu et al. (Ed.), *Grey systems theory and its application* (5th ed.), Science Press

Rathnayaka, R. M. K. T. (2023). Research center for grey systems and uncertainty analysis was established at Sabaragamuwa University of Sri Lanka.

Sifeng Liu Center for Grey Systems Studies Northwestern Polytechnical University.

Tao, L. Y., & Rui, H. D. (2021). Grey data analysis has been put into service on iCourse International. https://www.emeraldgrouppublishing.com/journal/gs

Wang, X. M., & Luo, J. J. (1985). The first academic symposium on "Grey System and Agriculture" was held in Taiyuan. *Shanxi Province. Agricultural Zoning, 03*, 65.

Webpage of IEEE SMC. (2024). Grey systems. https://www.ieeesmc.org/technical-activities/systems-science-and-engineering/

Website of DMU. The first workshop of European grey system research collaboration network was held at De Montfort University. https://www.dmu.ac.uk/home.aspx

Website of GSSC. (2024). Nurturing talents, building organizations, and promoting Chinese original theories to the world. http://gssc.nuaa.edu.cn/2023/0613/c18012a312911/page.htm

Website of IGSS. (2024). Institute for Grey Systems Studies. http://igss.nuaa.edu.cn/

Website of JGS. (2024). *The Journal of Grey System*. https://jgrey.nuaa.edu.cn

Website of GS. (2017). *Grey Systems-Theory and Application*. www.emeraldinsight.com/gs.htm

Website of GS. (2024), *Grey Systems-Theory and Application*. www.emeraldinsight.com/gs.htm

Xiao, K. J. (1990). Wuhan grey system consulting department. In T. H. Lu (Ed.), *1990 Wuhan Yearbook*, Wuhan University Press.

Xie, N. M., Tao, L. Y., Wang, J. J., Dong, W. J., & Zhang, W. L. (2022). *40th Anniversary commemorative book of the founding of grey system theory*. Institute for Grey Systems Studies.

Yu, P., & Le, W. Y. (2019). Close up: German Chancellor Merkel Visits Wuhan. *The Xinhua News Agency*, 07-09-2019.

Zheng, J. M. (2014). Professor Liu Sifeng at NUAA selected for the Marie Curie Program of European Union. *Guangming Daily*, 04-03-2014.

Zhou, N., & Dong, W. J. (2022). The opening ceremony of the 40th anniversary commemorative exhibition of the establishment of grey system theory was grandly held. http://www.scope.org.cn/newsMes.aspx?t=52&id=9822

# Index


**A**
Accuracy, 23, 26, 27, 31–33, 36, 37, 66, 75, 90, 185, 187, 198, 199, 201, 205–207, 210–212, 214, 215, 230, 231, 233, 243, 247, 257, 261, 272–274, 281, 346, 356, 360, 364, 365, 375
After-event control, 344
All-data GM (1, 1), 212
Average image, 93, 96, 300, 302
Average operator, 75, 76, 356
Average weakening buffer operator, 69, 71, 73, 229, 360, 366, 372
Axioms for buffer operator
 analytic representation, 67
 fixed points, 67
 information basis, 14, 15

**B**
Background value, 186, 195
Basic predicted value, 278
Basic value, 49, 163
Behavioral sequence
 economic sequence, 92
 horizontal sequence, 92
 time sequence, 92
Black number, 46, 260
Bull's-eye-distance, 298, 299

**C**
Candidates, 265, 314
Capital elasticity, 251
Capital input, 250, 251
Characteristic
 most favorable, 128, 131
 quasi-favorable, 128
 quasi-preferred, 128, 132, 133
 variable, 59, 63, 92, 127–130, 133, 219, 220
Characteristic vector, 324
Coefficient
 clustering of variable weight, 164
 constraint, 306
 decision, 267, 292, 306, 312–314, 318
 development, 179, 181, 195, 199, 204–206, 211, 220, 227, 272, 305
 equal weight cluster, 167
 fixed weight cluster, 167, 168, 263
 generation, 205
 grey evaluation, 90, 179
 grey impact, 63
 grey influence, 90, 186
 grey responsibility, 5
 inverse matrix, 164
Combined model
 prediction model, 26
Component
 maximum component, the, 292, 313, 314
Consecutive neighbor, 279
Constraint condition, 22
Contour line, 282, 283
Contour moment, 282, 283
Contour point, 282
Control
 closed loop, 321
 open loop, 321
Control by remove redundant, 2
Control of grey predictions, 347
Control vector, 285, 324, 332, 343, 344
Countermeasure
 desirable countermeasure, 297



S. Liu, *Grey Systems Analysis*, Series on Grey System,
https://doi.org/10.1007/978-981-97-8727-2

set, 293–295, 300, 301, 304, 309, 311
Critical value, 46, 291
Cross-sequences, 15, 120

**D**
Decision making
coefficient for decision-making, 292
coefficient vector for decision-making, 292
Decision scheme
decision scheme set, 293–295, 298, 300, 301, 304
desirable scheme, 296, 297
Derivative, 186, 187, 207, 209, 210, 218, 322, 333
Development domain, 276, 277
Differential restored value, 207
Discrete grey model, 185, 188, 190, 216, 217, 272, 367
Distance, 45, 89, 101, 108, 109, 283, 298, 305, 307, 310
Distance function, 310
Distinguishing coefficient, 95
Distortion date sequence, 278, 279
Distortion prediction, 16, 278
Domian
grey number, 15
grey sequence, 13

**E**
Economic reform, 21, 22, 28
Effect, 297, 309
Effect mapping, 295
Effect measure
effect measure for benefit type objective, 292
effect measure for cost type objective, 292, 310
lower effect measure, 292, 307, 308, 310
lower effect measure for moderate objective, 292
moderate effect measure, 310
synthetic effect measure matrix, 307, 310
uniform effect measure, 306, 309, 310, 312
upper effect measure, 292, 307, 308, 310, 312
upper effect measure for moderate objective, 292
Effect value, 291, 295–297, 300, 303, 305, 307–312, 376
Efficiency matrix, 30
Entropy
evaluation coefficient vector, 156
information, 156
Equal grey matrices, 45
Equal time interval sequence, 100, 101, 212
Equation
shadow, 189, 191, 222
whitenization, 191, 222–224, 229
Equivalent scheme, 298
Error-afterward control, 345
Error-on-time control, 345
Error-prediction control, 323
Error sequence
modelability, 207, 209
Estimation operator, 63, 76
Even difference GM (1, 1), 186, 367
Even difference model, 193–196, 198, 199, 201, 202
Even GM (1, 1), 185, 190
Event, 4, 5, 13, 23, 27, 271, 293–298, 300, 301, 304, 309, 311, 344, 345
Exponential function
homogeneous exponential function, 81, 196
non-homogeneous exponential function, 81
Exponential region, 271
Exponential sequence
homogeneous exponential sequence, 81, 195–199, 201
non-homogeneous exponential sequence, 81

**F**
Feedback loop, 321
Five-step modeling, 283, 284
Fluctuation term, 251
Four elements for decision-making, 260
Fractional accumulation, 185
Fractional grey model, 186, 215
Frequency domain, 353, 355–360
Function
distance, 310
grey transfer, 326, 330, 331, 342
homogeneous exponential, 81, 196
inverse, 327
smooth continuous, 274
Fuzzy mathematics, 1, 2, 6, 8, 25–27, 48

**G**
General grey number, 45, 53, 55–59

Grey action quantity, 195
Grey characteristic value, 156
Grey clustering evaluation
basic value of grey class, 156
cluster based on grey relational analysis, 155
clustering decision-making, 36
equal weight clustering coefficient, 167
fixed weight clustering coefficient, 167
variable weight clustering coefficient, 15, 156, 374
Grey control, 15, 16, 33, 321, 322, 324–327, 342, 346
Grey derivative, 187
Grey differential link, 329, 330
Grey distortion prediction, 278
Grey distortion sequence
lower distortion sequence, 278
upper distortion sequence, 278
Grey forecasting model, 271, 365, 367
Grey gain matrix, 331
Grey induction coefficient, 242
Grey inertia link, 328, 329
Grey influence coefficient, 90, 186
Grey integral link, 329, 330
Grey linear space, 59
Grey Markov chain, 255, 256
Grey matrix
diagonal, 325
difference, 332
inverse, 116
non-singular, 331
power, 242
product, 242
scalar, 53
scalar multiplication, 50
sum, 49
symmetric, 336
transposition, 192
triangular, 156
unit, 59
Grey number
conceptual type, 48
continuous, 46
discrete, 46, 54, 271
domain, 356
essential type, 46, 48, 278, 342
information type, 15
kernel, 50, 53–60
layer type, 293
non-essential type, 46
non-synchronous, 47
reduced form, 56, 58–60
synchronous, 47, 59
wish type, 48
with only lower limits, 55
with only upper limits, 55
with zero center, 50
Grey postponing link, 330
Grey prediction control, 323, 348
Grey prediction model, 16, 26, 31, 32, 35, 36, 185, 187, 188, 190, 233, 242, 272, 322–324, 375
Grey proportional link, 328, 330
Grey relation
cluster, 15
matrix, 15, 110, 127, 128, 130, 131, 157, 159
order, 89, 110, 113, 133
Grey relational cluster, 15, 157
Grey relational control, 322, 343, 344
Grey relational degree
absolute relational degree, 89, 102, 124, 125, 129–131, 135–138, 232, 300, 301
deng's relational degree, 95–97, 119, 138
nearness relational degree, 88
negative relational degree, 116
relative relational degree, 89, 103, 105, 118, 124–126, 138
similitude relational degree, 103, 105–108, 119, 122, 125, 131, 132
synthetic relational degree, 103, 105, 106, 119, 131, 132
three-dimensional relational degree, 106
Grey relational factor
set of grey relational factor, 92, 94
Grey relational operator
average image, 93
averaging operator, 93
initial image, 96, 103
initialing operator, 104
interval image, 94
interval operator, 93
set of grey relational operator, 92, 94
Grey target
bull's-eye-distance, 299
one-dimensional decision-making, 15, 16, 296
s-dimensional decision-making, 297, 298
spherical grey target, 291
Grey target decision
intelligent grey target decision, 16, 291, 292, 306, 376

multi-attribute grey target decision, 309, 376, 377
multi-attribute weighted grey target decision, 291, 292, 306
Grey verhulst model, 30, 186, 222–224, 227, 228

**H**
Homogeneous Markov chain, 255

**I**
Imagined optimum effect vector, 300, 301
Imagined optimum model, 301
Imagined optimum scheme, 300
Information
density, 2
fuzzy, 27
grey, 1, 9, 13–15
new, 9, 14, 212–214
old, 14, 214
stochastic, 28
unascertained, 272
Information content, 13
Initial image, 92, 104, 105, 130
Input-output, 59
Internal point, 24, 253
Internal subsystem, 346
Intersection, 51, 120, 261, 282
Interval grey number, 31, 45–47, 52, 54, 55, 185, 261, 262, 271, 293
Interval prediction
basic predicted value, 278
development domain, 276, 277
highest predicted value, 276
linear domain, 274
lower bound function, 274
lowest predicted value, 277
predicted domain, 276
proportional domain, 274
trumpet-like domain, 274
uniform domain, 274
upper bound function, 274
value domain, 274
wrapping domain, 275
Interval value image, 93, 94

**K**
Knowledge base, 1, 24, 64
K-zigzagged line, 94, 98

**L**
Labor elasticity, 251
Labor input, 250
Laplace transformation, 217, 326, 327, 331
Law of grey exponent
law of negative grey exponent, 82
law of positive grey exponent, 82
law of quasi grey exponent, 82, 84
Length
information field, 103
Length of a sequence, 103, 105
Levels of accuracy, 75, 90, 199
Limit value, 31
Linear region, 64
Log-linear form, 251
*Lower bound*, 340, 341
Lower bound, 45, 46, 54, 274, 276, 277, 336, 340, 341
Lower effect measure, 307, 308, 310

**M**
Markov chain, 243, 255, 256
Mathematical model of linear control system
controllable, 325
observable, 325
output equation, 324
output grey matrix, 15, 59, 242, 322
state equation, 324
state grey matrix, 331
Matrix
cluster coefficient, 164, 166
control, 285, 325
development, 285
grey characteristic, 156
mean-value, 53
moving probability, 254
nth step moving probability, 254
objective structure, 342
perturbation grey, 185
relational, 127, 128, 157, 159
scalar matrix, 104
state, 285, 324, 325
synthetic effect measures, 307, 309, 310
uniform effect measures, 306, 310, 312
Mean, 13, 15, 47, 59, 63, 76, 78, 97, 230, 273, 279, 338, 339, 341, 346, 356, 357, 359, 360, 365, 372
Mean-generating operator
adjacent neighbor, 222, 227, 279
Mean slope, 187
Membership function, 27, 48, 258–260

Metabolic GM (1, 1), 212
Model
Cobb—Douglas, 16, 25, 242, 250
combined prediction, 26, 36, 90, 186, 187
comprehensively balanced, 37
critical, 46, 261, 358
distortion GM (1, 1), 278, 279
error modification, 208, 209
error-satisfactory, 273
general mathematical, 1
GM (0, N), 33
GM (1, 1), 271
GM (1, 1) power, 271
grey econometrics, 16, 243–245
grey input-output, 242
grey production function, 251
grey state, 256
grey Verhulst, 30, 186, 222–224, 227, 228
ideal, 307, 364
optimal grey input-output, 242
relation-satisfactory, 14, 273
remnant GM (1, 1), 207–209
Modeling
five-step, 283, 284
grey systems, 6, 16, 82, 195, 251, 332, 342, 363, 364, 371
Model simulation, 242
Moderate effect measure, 310
Monotonic sequence
decreasing sequence, 66, 68, 69, 72
increasing sequence, 66, 68, 72, 114
Most favorable factor, 128, 132
Moving average denoise operator, 75, 77
Moving probability, 254

**N**
New data, 75, 215, 345
New information GM (1, 1), 186, 212
Non-uniqueness principle, 251
Norm
continuous function, 335
Normality, 67, 95, 308

**O**
Objective, 15, 16, 24, 28, 35, 37, 91, 112, 247, 272, 278, 291–293, 295–298, 300–312, 324, 332, 342–344, 374, 376
Objective function, 16, 292, 298, 310, 342
Old data, 215, 345
On-time control, 345
Operations of grey number
addition, 226
division, 54
multiplication, 57
multiplication by a scalar, 50, 105
negative element, 261
power, 292
reciprocal, 116
subtraction, 57
Operator
averaging, 93, 167
buffer, 229, 360, 365, 366, 372
first order, 66, 67, 80, 216, 217
grey relational, 92
strengthening, 15, 67–69, 72–74, 365
weakening, 31, 67–71, 73, 75, 365, 366, 372
zero starting point, 98
Optimum
countermeasure, 298
decision scheme, 298–301, 310
effect vector, 299–301
event, 298
Original difference GM (1, 1), 186, 367

**P**
Paradox of decision-making, 313
Parameter, 16, 24, 25, 30, 31, 35, 75, 90, 91, 186, 187, 189, 190, 201, 203, 204, 220, 224, 242, 245–249, 251, 252, 254, 261, 275, 277, 284, 304, 322, 326, 332, 345, 346, 353, 368, 375
Partial-data GM (1, 1), 212
Partition, 155
Positioned coefficient
positioned coefficient of grey number, 47
synchronous, 324
Possibility function
lower measure, 156, 161–163, 172, 174
moderate measure, 156, 161, 163
typical form, 160–163
upper measure, 156, 161–163, 174, 181
Power model, 186, 222, 223, 225
Prediction control, 323, 348
Prediction value, 276, 277
Preference, 48, 263, 265, 267, 268
Principle of absolute greyness
informational differences, 13
minimal information, 14
new information priority, 14, 292

non-uniqueness, 13, 14, 251
recognition base, 14
Probability, 26, 27, 48, 55, 56, 88, 231, 255–257, 273, 274, 337
Progress in technology, 12
Property closeness
maximum valuation, 89, 116, 120, 292, 308, 312, 313
non-negativeness, 80

**Q**
Quasi-favorable factor, 133
Quasi-optimal countermeasure, 301
Quasi-optimal event, 298
Quasi-optimal scheme, 301
Quasi-preferred factor, 132, 133
Quasi-smooth sequence, 77, 80, 82, 84

**R**
Ratio inverse, 366
Reciprocal image, 116
Relative errors, 202, 203, 209–211, 219, 222, 230, 254, 273, 280
Relative simulation error, 273
Remnant GM (1,1) model, 207
Restored value by inverse accumulation, 207
Restored value through derivative, 207
Reverse image, 116
Root-mean-square deviation, 273
Rule of maximum value, 312, 313

**S**
Satisfactory effect, 296
Self-memory grey model
other effective item, 226
self-memory item, 226
Sequence
behavioral, 64, 92, 93, 95, 97, 98, 119, 127–130, 133, 195, 246, 344
behavioral horizontal, 92
behavioral time, 92
equal time interval, 100, 212
errors, 196, 200, 202, 230, 254, 280
first order, 66, 67, 80, 248, 249
grey derivative, 187
homogeneous exponential, 81, 195–199, 201
length, 99–101, 103, 105, 107, 127
monotonously decreasing, 66, 68, 72, 73, 113, 114, 118, 305
monotonously increasing, 72, 73, 113, 114, 118, 305
non-dimensional, 92
non-homogeneous exponential, 81
quasi-smooth, 77, 80, 82
relative errors, 280
representative, 244
smooth, 77, 79, 186, 358
smooth ratio, 77
stepwise increasing, 78
upper distortion, 278
vibrational, 195, 196, 199–201
Sequence affected by operator, 78
Sequence operator
accumulating, 84
average operator, 75
inverse accumulating operator, 210
nth order, 326
smooth ratio, 77
stepwise ratio, 77, 78
unbiased operator, 31
Set of events, 293, 294, 300, 304
Shock wave
shock disturbed sequence, 66
shock disturbed system, 75
term of shock disturbance, 353
Simulation error
mean relative, 206, 273
relative, 202, 203
Simulation value, 208, 210, 233, 253
Slope
average slope, 94
Space
Euclidean, 297
grey relational factors, 94
Spectrum analysis, 15, 353, 355–357, 360
Spherical grey target, 291, 297, 298
State equation, 324
State transition probability, 255–257
State vector, 285, 324, 333, 335, 337
Stationary point, 114
Stochastic process, 64, 254
Superfluous term, 342
Superior
class, 295, 296
countermeasure, 295, 296
event, 295, 296, 309
object, 295, 296
scheme, 296, 298, 299
Superiority analysis, 127
Synthetic effect measure, 309, 310, 312
System
asymptotically stable, 338

closed loop control, 321, 322
controllable, 325
grey, 73, 75, 78, 82, 87
grey linear control, 324, 326, 331
observable, 325, 326
open loop control, 321, 322
prediction models, 188
stable, 332–334, 336–341
uncontrollable, 69
Systems prediction, 1, 2, 15, 21, 187, 250, 272, 285, 286, 345

**T**
Tendency term, 304
Trap in prediction, 65
Triangular possibility function
center-point triangular possibility function, 156, 157, 160, 179
end-point triangular possibility function, 23, 157
mixed possibility functions, 156, 157
True behavioral sequence, 65, 66
Trumpet-like region, 274
Truth value, 65
Turning point, 160, 161, 172–175, 178–180
Two stages decision model, 314

**U**
Uncertain, 1, 4, 5, 13, 14, 21–24, 26, 27, 113, 155, 185, 201, 255, 257, 260, 262, 291, 336
Uncertainty
fuzzy, 26, 27
grey, 1, 27, 28
rough, 26, 28
stochastic, 27
Uniform domain, 274
Union, 4, 51, 55, 260, 267, 382, 385
Unitized
clustering coefficient, 313
clustering coefficient vector, 313
Upper domain, 274
Upper effect measure, 292, 307, 308, 310, 312
Useless predicted moment, 346

**V**
Value domain, 274
Variance, 35, 59, 88, 91, 242, 254, 273
Vector
cluster coefficient, 164
decision, 292, 297–300, 302, 303, 306, 309, 312–315, 317
decision coefficient, 292, 306, 313
grey constraints, 22
grey n-dimensional, 335
uniform effect measures, 312
Vibration sequence
amplitude, 66

**W**
Wave form prediction, 281
Weighted average weakening buffer operator, 366
Weighted coefficient of kernel clustering for decision-making, 314
Weighted moving average operator, 76, 356
Weight of criterion, 306
Weight vector group with kernel, 312–318
Whitenization of a grey number
mean whitenization, 47
whitenization equation, 223
whitenization matrix, 333
White number, 46, 49, 50, 55, 57, 64, 259, 260
Wrapping domian, 275

**Z**
Zigzagged line, 94, 98, 99, 105, 274, 281, 282