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Gloria Ann Stillman
Jill P. Brown *Editors*

Lines of Inquiry in Mathematical Modelling Research in Education

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Preface

Almost from the beginning of human existence, mathematics has been developed to describe the world around us. As humans became more sophisticated, so too, did the mathematics. Furthermore, the use of mathematics extended beyond describing the world and was used to make sense of the world and subsequently support actions. These actions include explaining (e.g. the best location for an international meeting), deciding (e.g. the guilt or innocence of an accused or to wear a seatbelt whilst driving), designing (e.g. bridges, pyramids, bar codes) and predicting (e.g. planetary alignment, if a tsunami is likely to impact a given location). Sometimes, the decision may be to desist from taking a particular action (e.g. to decide not to leave a small child in a car on a hot day).

This essential interrelationship between the real world and mathematics has been recognised by many in education and educational research as of critical importance and has given rise to a sub-field of educational research related to the teaching and learning of mathematical applications and mathematical modelling. Arguments continue as to the importance and placement of modelling and applications in school mathematics. The chapters in this book generally follow the view that even young students should be challenged to solve real-world problems. Across the levels of schooling and into tertiary, modellers will use the mathematical knowledge and tools they have at their disposal to solve a given problem. Such engagement with real problems will motivate students to learn mathematics and appreciate its usefulness and importance. Through solving real-world problems, students will come to appreciate the importance of simplifying the complex and messy real world. This simplification in order to find a first solution, which is then validated and revisited with added complexity, will support the same approach in pure mathematics problems. The use of collaborative groups, and the subsequent interthinking to solve real-world problems, enhances student engagement with mathematics and increases the capacity of students to solve tasks.

However, society, in general, still too often holds mathematics in low esteem and this in turn impacts on how mathematics is taught and learnt from the early years through schooling and in universities and other tertiary educational institutions. Applications continue to have a presence in mathematics curricula. In application tasks, the task setter begins with the mathematics and determines a real situation where this mathematics is used. For example, with a focus on volume, an application might be how many trips with a given sized truck are needed to transport cartons of given dimensions. With a focus on quadratic functions, an application is the trajectory of a cricket ball when hit for a six. Alternatively, in modelling tasks, the task setter begins with reality then looks to the mathematics that might be useful and then returns to reality to determine if the mathematical model or subsequent analysis actually answers the real-world problem. Task solvers may find an alternative approach or use different mathematics but are still expected to take the real world into account as being critical to the solution. They will discover, over time, that some mathematical solutions are not, in fact real-world solutions. Sadly, some students only ever experience application tasks during their school mathematics experiences or are not given the opportunity to become independent modellers able to solve real-world problems that interest them.

Nonetheless, the importance of applications and modelling has been continuing to grow in recent decades. In particular, every 4 years, ICMEs include regular working or topic study groups and lectures on the topic. ICME proceedings indicate the state-of-the-art at the time. Biennial International Conferences of the Community of Teachers of Mathematical Modelling and Applications (ICTMA) have been held since 1983 and the books published following these continue to provide a valuable source of research and other activity in the field.

This book is a collection of chapters, the core ideas of which were originally presented at the Topic Study Group 21, *Mathematical applications and modelling in the teaching and learning of mathematics*, at the Thirteenth International Congress on Mathematics Education, ICME-13, in Hamburg, Germany (24–31 July 2016); but they are extended and have undergone a rigorous review process. Co-chairs of the Group were Jussara Araújo (Brazil) and Gloria Stillman (Australia) with topic group organising team members Morten Blomhøj (Denmark), Dominik Leiß (Germany) and Toshikazu Ikeda (Japan). An outline of the papers presented, and discussion can be found in the main congress proceedings.

A state-of-the-art overview was presented by Gloria Stillman at ICME-13, which forms the basis for Chap. 1 and suggests future theoretical and empirical lines of inquiry in mathematics education research related to teaching and learning of mathematical applications and mathematical modelling. The subsequent chapters cover a variety of issues across all levels of schooling, primary and secondary, tertiary mathematics and teacher education. The chapters include tasks used with students and teachers, teaching ideas developed, experiences gained, empirical results and theoretical reflections. In the final chapter, Jill P. Brown and Toshikazu Ikeda overview the contributions along the lines of inquiry suggested, emphasising the shared view of mathematical modelling as solving real-world problems, and conclude with suggestions for further research.

Thanks to all contributors to this book. Thanks also to our institution, The Australian Catholic University, for the support during our work on the book, and to the publishers, Springer, and Series Editor, Gabriele Kaiser, for making it possible for this work to be shared widely.

Ballarat, Australia Gloria Ann Stillman
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Contents

Chapter 1
State of the Art on Modelling in Mathematics Education—Lines of Inquiry

Gloria Ann Stillman

Abstract This chapter provides a brief overview of the state of the art in research and curricula on mathematical modelling and applications of mathematics in education. Following a brief illustration of the nature of mathematical modelling in educational practice, research in real-world applications and mathematical modelling in mathematics curricula for schooling is overviewed. The theoretical and empirical lines of inquiry in mathematics education research related to teaching and learning of mathematical applications and mathematical modelling regularly in classrooms are then selectively highlighted. Finally, future directions are recommended.

Keywords Mathematical applications · Mathematical modelling · Theoretical lines of inquiry · Empirical lines of inquiry

1.1 What Is Mathematical Modelling?

Mathematical modelling conceived as real-world problem solving is the *process* of applying mathematics to a real-world problem with a view to understanding it (Niss et al. 2007). It is *more than applying mathematics* where we also start with a real-world problem, apply the necessary mathematics, but after having found the solution we no longer think about the initial problem except to check if our answer makes sense (Stillman 2004). With mathematical modelling the use of mathematics is more *for understanding the real-world problem/situation*. The modeller also poses the problem(s) and questions (Christou et al. 2005; Stillman 2015). To illustrate what this means in educational practice, a modelling task from a university teacher education course follows.

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1.1.1 An Example from Teacher Education

This task was used in a university mathematics unit for primary pre-service teacher education students. It was one of three choices (the others being dust storms and the spread of HIV/AIDS). The students had 4 weeks to work on the task independently out of class. The task is about the felling of a eucalypt forest on the edge of the freeway between Melbourne and Ballarat. The trees were not particularly old and not mature enough for harvesting. This context was used to ask students to pose a problem based on the logging of the forest as a modelling task. There was little to no information in the local press and the local council was less than helpful to students who enquired as to why the forest was removed. The following task stimulus was given to the students. All students were in the first semester of the first year of a 4-year education degree to become primary school teachers (teaching Preparatory year to Year 6).

The Task—Harvesting the Eucalypt Forest: Those of you who drive the Western Freeway between Ballarat and Ballan will have noticed that a large plantation of Eucalypts has been felled and the logs transported away. Using mathematical modelling pose a problem related to removal of the forest that can be mathematised and solved. [The task was accompanied by several photographs taken before, during and after the felling of the trees.]

Many mathematically tractable problems were posed by the students who worked on the task individually in their own time. An example from one student, Hannah (a pseudonym), follows:

> I will be researching and investigating the effects of human logging and deforesting of the Eucalypt forest on the Western freeway between Ballarat and Ballan.
>
> The problem I pose is this: At what rate would replanting need to occur for it to be sustainable with the rate of deforestation, and what percentage of the forest needs to remain 'untouched', either entirely or for a period of time, to maintain a viable habitat to creatures it may be home to?
>
> In order to come up with a reliable conclusion I will need to research the following:
>
> What was the original size of the forest?
>
> Why and for what purpose is it being logged?
>
> What age does the timber need to be for it to be commercially useful?
>
> Growth rate of the Eucalypt? [from Hannah's Modelling Task Report]

To begin she needed to know the initial number of trees. To work this out she firstly determined the area of the forest. Using a Google map aerial view, she divided the forest into four common shapes to best cover the entire area (Fig. 1.1). The shaded green in the top right corner is where trees had already been felled. This area was also included to determine how many trees were in the forest to begin with. Using scaling and area formulae she determined the forested area was 1,587,000 m^2. Assuming

Fig. 1.1 Finding area of original forest beside highway near Ballan (used with permission)

trees could be planted at the rate of 1000 per hectare this gave 158,700 trees as the size of the original forest.

Next she assumed a growth rate of 1.2 m per year and that the trees were being harvested with 15 years growth of useable timber, that is, trees with 18 m useable logs.

To transport the logs from the site she used 5 B-double logging trucks for 5 days for 46 weeks per year (allowing for 6 weeks holiday/annual leave). Each truck consisted of two trailers that could carry twenty-two 6 m logs in each. This meant that the trees were cut into three 6 m logs and 366.66 trees trucked per week (16,866.66 annually). If the trees were logged continually at this rate and not replenished, the forest planation would be removed within 9.4 years of commencement of logging.

She then re-assessed her modelling as she had yet to incorporate sustainability. She realised that she had to determine the rate of logging to achieve her goal, not use existing rates. She decided that she would log 158,700 trees over 16 years so at the rate of 9919 trees annually and this would use 3 B-double trucks a day. She would then, at the same time, need to be planting 9919 trees annually and harvesting these after they had produced 15 years growth of useable timber. She did not continue on to answer other parts of her question posed.

The task and Hannah's modelling is an example of *descriptive modelling*, the most common form of modelling (Niss 2015). The purpose of the mathematical modelling was to analyse an existing real world situation (the felling of a forest) as a means of answering a practical question (what rate to (log and) replant so as to sustain the forest). Both mathematical and extra-mathematical knowledge were needed to answer this question. This is also an example of using *modelling as content* "empowering students to become independent users of their mathematics" (Galbraith 2015a, p. 342) rather than as a means to serve other curricular requirements such as teaching mathematical content (i.e. *modelling as vehicle*).

1.2 Real-World Applications and Mathematical Modelling in Curricula

Uptake and implementation of real-world applications and mathematical modelling in curricula in school and university vary widely. At ICME-7 in Quebec in 1992, Blum lamented in Working Group 14 on *Mathematical Modelling in the Classroom*,

> there is still a substantial gap between the forefront of research and development in mathematics education, on the one hand, and the mainstream of mathematics instruction, on the other hand. In most countries, modelling (in the broad and, even more so, in the strict sense) still plays only a minor role in everyday teaching practice at school and university. (1993, p. 7)

Fortunately, there has been some change in the intervening years with Maaß (2016) noting at ICME-13 in Hamburg:

> Nowadays in Germany Mathematical Modelling is part of the national standards of mathematics education and in consequence is part of many professional development courses, also addressing topics like differentiation and assessment when modelling. Textbooks include modelling tasks (to a different degree) and many teachers (though maybe not the majority) do include modelling in their mathematics classes. Of course, this has not always been the case.

Most implementations in individual mathematics subjects align with expressed goals of modelling and/applications in curriculum documents but this is not always borne through in the overall structure of the curriculum where there are alternative mathematical offerings or alternative pathways (e.g. academic versus vocational) (Smith and Morgan 2016). The goals are roughly equivalent to the five arguments that Blum and Niss (1991) present as those given for support of real world applications and mathematical modelling in curricula. In the following, research and evaluation studies where the particular curricular goal underpins the approach taken to modelling are shown in brackets. From a *mathematical point of view* such goals could be:

- To be a vehicle to teach mathematical concepts and procedures (e.g. Lamb and Visnovska 2015);
- To teach mathematical modelling and ways of applying mathematics as mathematical content (i.e. as an essential part of mathematics) (e.g. Didis et al. 2016; Tekin Dede 2019; Widjaja 2013);
- To promote mathematics as a human activity answering problems of a different nature giving rise to emergence of mathematical concepts, notions and procedures (e.g. Rodríguez Gallegos 2015).

From an *informed citizenry perspective,* goals include:

- To provide experiences that contribute to education for life after school such as looking at social problems (e.g. Yoshimura 2015);
- To promote values education (e.g. Doruk 2012);
- To question the role of mathematical models in society and the environment (e.g. Biembengut 2013; Ikeda 2018);
- To ensure or advance “the sustainability of health, education and environmental well-being, and the reduction of poverty and disadvantage” (Niss et al. 2007, p. 18) (e.g. Luna et al. 2015; Rosa and Orey 2015; Villarreal et al. 2015).

Smith and Morgan (2016) reviewed curriculum documents in 11 education jurisdictions identifying three main rationales in orientations of curricula to use of real-world contexts in mathematics, namely:

(1) “mathematics as a *tool* for everyday life,
(2) the real world as a *vehicle* for learning mathematics, and
(3) engagement with the real-world as a *motivation* to learn mathematics” (p. 40).

In Australia, they examined state curricula in Queensland where there has been mathematical modelling and applications in the senior curriculum for many years and New South Wales where there is no modelling and a very traditional mathematics curriculum. In Canada, they looked at curricula in Alberta and Ontario where modelling was reported in the latter as “embedded as a system-wide focus in secondary school mathematics education” (Suurtamm and Roulet 2007, p. 491). Other curricula examined came from Finland, Japan, Singapore, Hong Kong, Shanghai and the USA southern states of Florida and Mississippi.

In seven of these eleven educational jurisdictions, alternative pathways were offered, with more [mathematically] advanced pathways having less emphasis on real-world contexts. Such findings raise questions for those charged with overseeing curriculum implementation to consider in relation to the espoused goals of curricular embedding:

- If mathematics is seen as a tool for everyday life—why is the real-world given less emphasis for students studying more advanced mathematics?
- If the purpose was as a vehicle for learning, or motivation, why is there less focus on real-world contexts in the years of schooling prior to pathway options?

Changing the emphasis for different year levels or by nature of mathematics studied *conflicts* with all three of the espoused rationales.

1.3 What Do We Know?

Since the late 1960s, researchers in mathematics education have increasingly focussed on ways to change mathematics education in order to include mathematical applications and mathematical modelling regularly in teaching and learning in classrooms. This was in response to the dominance in many parts of the world of the school mathematics curricula by an abstract approach to teaching focusing on the

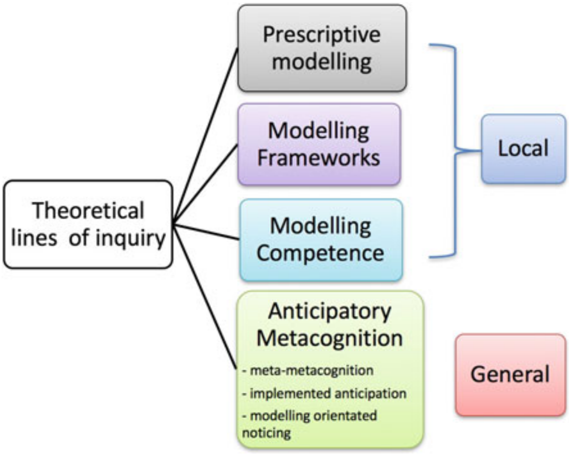

Fig. 1.2 Focuses of theoretical lines of inquiry

teaching of algorithms divorced from any applications in the real world. The focus of this research has been both theoretical and empirical. Within mathematical modelling and applications educational research, there has been an on-going building of analytical theories establishing foundational concepts and categories and interpretative models and theories for interpreting and explaining observed structures and phenomena which have been organized into stable, consistent and coherent systems of interpretation (Niss 1999). Constructs from these are claimed to meet particular theoretical or empirical evidence. This has led to many viable lines of inquiry over the years and the purpose of this chapter is to highlight some of these that are current within the field. To select examples I have surveyed the literature in the more recent books in the ICTMA series and the major mathematics education research journals.

1.3.1 Theoretical Focuses—Lines of Inquiry

In research into the teaching and learning of mathematical modelling there is a strong emphasis on developing "home grown theories" where the focus is on "particular *local theories*" such as the modelling cycle and modelling competencies rather than *general theories* from outside the field (Geiger and Frejd 2015). As the extent of theoretical developments in this field is extensive, four examples of current theoretical lines of inquiry—three local theories (prescriptive modelling, modelling frameworks/cycles and modelling competencies) and one general line of inquiry (anticipatory metacognition)—will be used to give a flavour of current thinking and work (Fig. 1.2). Some of these have been the subject of empirical testing or confirmation whilst others await such work.

1.3.1.1 Prescriptive Modelling

The first local theory is *prescriptive modelling*. The terms *descriptive model* and *prescriptive model* have been used previously by Meyer (1984) to describe models used for different modelling purposes: "A descriptive model is one which describes or predicts how something actually works or how it will work. A prescriptive model is one which is meant to help us choose the best way for something to work" (p. 61). According to Niss (2015), the modelling cycles used in theoretical and empirical research are limited with regards to adequately capturing all processes involved in prescriptive modelling. Descriptive modelling is usually the focus of practice as it is used to *understand* an existing part of the world. However, it is not the modelling cycle as such that is different in prescriptive modelling. What has happened is that historical development in keeping with the types of problems used has coupled the modelling cycle with descriptive modelling, so that features of descriptive modelling have become misleadingly assigned as intrinsic to the modelling cycle. In contrast, what happens within different phases of the cycle can differ stemming from the differing purposes of prescriptive and descriptive modelling.

An example comes from Galbraith (2009, pp. 58–62) where he worked on the question: Is the method for scoring points in the heptathlon fair? 'Fairness' was interpreted with respect to strengths in track (e.g. 100 m hurdles) or field (e.g. javelin) events. Galbraith began to answer this question by evaluating the outcome of an earlier unknown (to him) modelling process by looking first at existing formulae and their implications for fairness. The modelling develops from there. A major difference is the essential role of sensitivity testing within the evaluation of prescriptive modelling. This ensures a cyclic dimension to the modelling process as it involves assessing the impact of changes in assumptions (e.g. world records in all contributory events should have similar weighting on the respective points scored in an event) or changing parameter values (e.g. a 1% increase in performance at the 1000 point mark of excellence in the different events) on the initial solution.

Niss (2015) points out that prescriptive modelling has little purchase in mathematics education, rarely being a focus. It would therefore follow that mathematics educators are less interested in modelling to take action based on decisions resulting from mathematical considerations so as to *change* the world. Niss (2015) advocates strongly for a greater focus in both theoretical and empirical research on prescriptive modelling in mathematics education using tasks of higher complexity than have been used in the limited work in this area to date.

1.3.1.2 Modelling Frameworks/Cycles

On the other hand, much work has been done on the second local theory to be highlighted—various modelling frameworks/cycles. Borromeo Ferri (2006), Czocher (2013), Doerr et al. (2017), and Perrenet and Zwaneveld (2012), amongst others, provide overviews of exemplars of these theoretical lines of inquiry in more recent years.

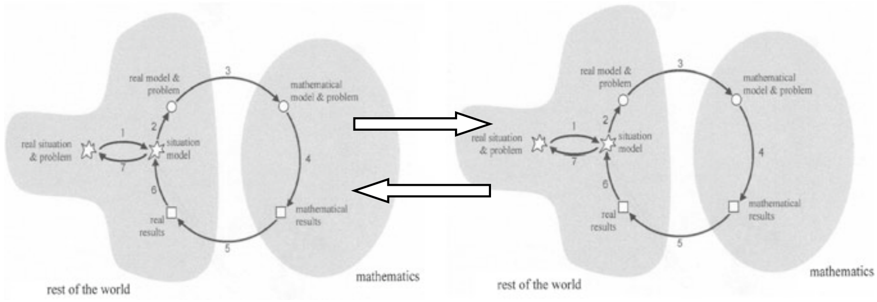

Fig. 1.3 Dual modelling cycle framework (Saeki and Matsuzaki 2013, p. 91)

The cycles/frameworks serve the researchers' purposes as is illustrated in the following example. A recent Japanese development in this area is the Dual Modelling Cycle Framework (Fig. 1.3) which combines two representations of the modelling cycle as depicted by Blum and Leiß (2007).

Sometimes, when modellers are unable to anticipate a model or solve a modelling task, they imagine models from a similar task in their prior experience to help progress the solution of the first task. Saeki and Matsuzaki (2013) used this idea to design two similar tasks that could be used in teaching to scaffold such a process for struggling modellers. By solving the analogous second task using a second modelling cycle, the modellers are, theoretically at least, able to apply the results to the location on the modelling cycle for the first task where they were struggling, forming linked dual modelling cycles (see Fig. 1.3). This theoretical work has been the subject of empirical testing and confirmation with both Japanese students (e.g. Kawakami et al. 2015) and Australian students (Lamb et al. 2017).

Fundamentally, the modelling cycle is a logical progression of problem-solving stages as the mathematical model, for example, cannot be solved before it has been formulated or the interpretation of outputs from the mathematical work before it has been done, etcetera. It is a theoretical description of what real-world modelling involves. Empirical data confirm its global structure; they do not give rise to it. Both the Blum and Leiß (2007) and the Saeki and Matsuzaki (2013) approaches elaborate this essential cycle with enhanced pedagogy in mind but not all cycles have been constructed with the logic of the modelling process in mind. Do we really need separate cycles for modelling with technology, say? Why would we expect the process to be different? Isn't the logic of the use of technology in these circumstances driven by the logic of the modelling process?

1.3.1.3 Modelling Competence/Competencies

The last local theory to be dealt with is related to one of the most important goals for student modellers in any curricular implementation which is to develop "modelling competence" (Blomhøj and Højgaard Jensen 2003) or "modelling competency" (Niss et al. 2007). “Competence is someone’s insightful readiness to act in response to the challenges of a situation” (Blomhøj and Højgaard Jensen 2007, p. 47) and was introduced in the context of the Danish KOM project (Niss 2003) which focussed on mathematical competencies and the learning of mathematics and created a platform for in-depth reform of Danish mathematics education at all levels. Readiness to act is not the same as the ability to act on this readiness, however. Modelling competency, on the other hand, refers to an individual’s ability to perform required or desirable actions in modelling situations to progress the modelling (Niss et al. 2007). Kaiser (2007) would call this “modelling abilities” and would insist modelling competency includes a willingness to want to work out real world problems through mathematical modelling.

Each of the following *modelling competencies* based on phases in the modelling cycle can be subdivided into lists of sub-competencies:

- competencies to understand real-world problems and to construct a reality model;
- competencies to create a mathematical model out of a real-world model;
- competencies to solve mathematical problems within a mathematical model;
- competency to interpret mathematical results in a real-world model or a real situation
- competency to challenge solutions and, if necessary, to carry out another modelling process (Kaiser 2007, p. 111)

In addition, *metacognitive modelling competencies* have been proposed by both Maaß (2006) and Stillman (1998). However, metacognition was linked to modelling much earlier by McLone (1973) and Lambert et al. (1989). Competence in modelling would thus involve an ability to orchestrate a set of sub-competencies in a variety of modelling situations.

Several aspects of theoretical work in the area of modelling competence and modelling competencies are currently the subject of empirical testing and confirmation. Kaiser and Brand (2015) provide an insightful overview of the main theoretical lines of inquiry within the International Conferences on the Teaching of Mathematical Modelling (ICTMA) research community since the 1980s. Further work in this area is described in Kaiser et al. (2018).

1.3.1.4 Anticipatory Metacognition

Metacognition is considered important by several researchers in the research and practice of mathematical modelling especially reflection on actions when addressing a real world problem (Blum 2015; Vorhölter 2018). In reality metacognition is essential to properly conducted modelling as evaluation of the partially complete model(s) should be occurring through verification and the final model needs to be validated against the problem situation to see if it produces acceptable answers to the question posed. The focus of the reflection on actions is on the mathematics employed and the modelling undertaken. A new development in this area is anticipatory metacognition. *Anticipatory metacognition* is about reflection that points *forward* to actions yet to

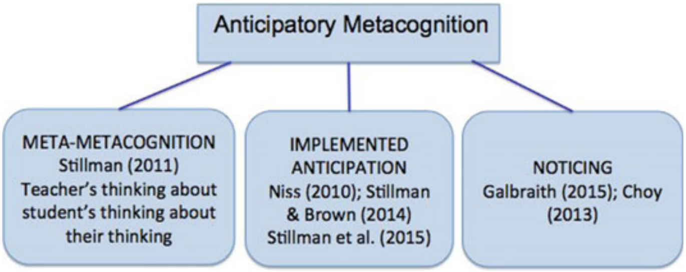

Fig. 1.4 Proposed dimensions of anticipatory metacognition

be undertaken, that is, noticing possibilities of potentialities. These reflections can arise from prior progress or lack of it. Anticipatory metacognition encompasses three distinct dimensions (see Fig. 1.4): meta-metacognition, implemented anticipation, and modelling oriented noticing (Galbraith et al. 2017).

Meta-metacognition results from teachers thinking about, that is, reflecting on, the appropriateness or effectiveness of their students' metacognitive activity during mathematical modelling and subsequently acting bearing this in mind (see Stillman 2011). Implemented anticipation is Niss's notion (2010) of successful implementation of anticipating in ideal mathematisation of a modelling situation. It results from the successful use of foreshadowing and feedback loops to govern actions in decision making during mathematisation (Stillman et al. 2015).

Modelling oriented noticing involves 'noticing' how mathematicians as well as educators act when operating within the field of modelling, from both *mathematical* and *pedagogical* points of view (Galbraith 2015b). It provides a way to study aspects central to modelling, for example, problem finding and problem posing as well as conducting modelling. For both there is cognitive involvement. Modelling oriented noticing also facilitates study of task design and study of support for student activity by teachers.

From a teaching viewpoint, to carry out tasks successfully requires more than just observing. Discernment of the *relevance* of what is observed is essential, followed by appropriate action. The term 'noticing' as employed in Galbraith (2015b) encapsulates these components. Choy (2013) came up with the notion of *productive mathematical noticing* by combining the notion of mathematics teacher noticing, involving the generating of new knowledge through selective attending and knowledge-based reasoning to develop a repertoire of alternative strategies, with Sternberg and Davidson's (1983) processes of insight. The latter are selective encoding, selective comparison and selective combination. By extending this idea to modellers (who can be students), Galbraith et al. (2017) proposed the notion of *productive Modelling Oriented Noticing* (pMON). For modellers, pMON involves the processes of (a) sifting through information to notice what is relevant and what is irrelevant (i.e. selective encoding), (b) comparing and relating relevant information with prior experiences and knowledge (i.e. selective comparison), and (c) combining the relevant infor-

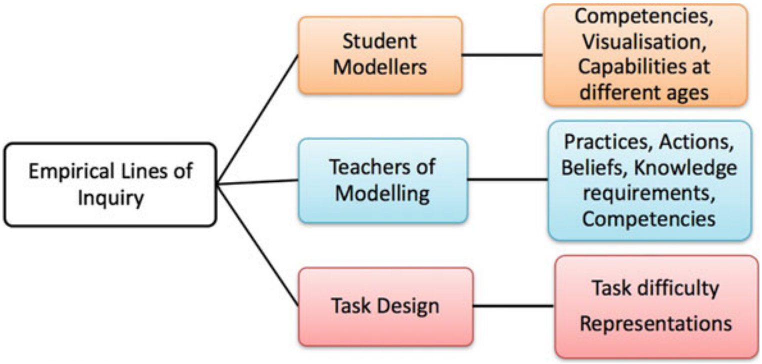

Fig. 1.5 Focuses of empirical lines of inquiry

mation (i.e. selective combination) to *generate productive alternatives* for decision making when responding to events as they carry out a modelling activity.

Aspects of the theoretical dimensions of anticipatory metacognition have been, or are currently, the subject of empirical testing and confirmation (Geiger et al. 2018; Stillman and Brown 2014).

1.3.2 Empirical Lines of Inquiry

Focuses of empirical lines of inquiry in mathematical modelling research are many and varied. Given the space available, I will focus on just three: student modellers, teachers of modelling and task design (Fig. 1.5). Within each of these foci, a small subset of exemplar studies and the major findings from these will be overviewed.

1.3.2.1 Empirical Results of Studies Focusing on Student Modellers

Prominent lines of inquiry focussing on students concern their modelling and mathematical competencies, visualisation and their capabilities at different ages.

Quantitative research by Kaiser and Brand (2015) has confirmed that modelling competency of student modellers appears to consist of a global overarching modelling competency and several sub-competencies, namely, simplifying/mathematising, working mathematically and interpreting/validating. Overall modelling competency was defined in Brand's study (2014) as the ability to solve complete modelling tasks as well as use metacognitive abilities to monitor the modelling process. Fifteen classes of Year 9 students from 4 higher-track and 2 comprehensive secondary schools in Hamburg, Germany, took part. However, these results need to be replicated in other contexts to show they are independent of the examples, intervention approach and test instruments used by Brand. In contrast, when Zöttl et al. (2011) applied Rasch modelling (Rasch 1960) to data in the KOMMA research project[1] in an attempt to capture the essential components of modelling competency (in keeping with Kaiser 2007) of Year 8 students, a sub-dimensional model proved superior to a uni-dimensional model. Thus, the results related to structure of the modelling competency differ with respect to the role played by the overall modelling competency from those of Kaiser and Brand (2015). The role of a global overarching modelling competency remains an open question. Further work on the conceptualisation of modelling competency and sub-competencies is presented in this volume by Hankeln et al.

A technology based study by Brown (2015) focussed on the visualisation tactics (i.e. employing either mental images or technology-generated images or both) of Year 11 Australian students attempting to solve a real world task involving platypus numbers in the wild. Unfortunately, students did not appreciate the cognitive role played by visualisation in supporting refinement of models and mathematisation in modelling. The potential of graphing technology to facilitate this process was thus not realised. In contrast, Villarreal et al. (2015, 2018) reported how pre-service education students in Argentina used the visual affordances of digital tools to represent their data in a visual manner to analyse the situation they were modelling and to communicate their results in an impactful manner.

English (2013) has shown that complex modelling tasks relating to engineering-based experiences can be handled by Years 7–9 Australian students. Such experiences target future competencies in the mathematical sciences, connecting learning across disciplines and involving the student modellers in planning, designing, constructing, testing and refining a life based model to solve problems of the built environment such as transport. In subsequent work, English and Watson (2018) have reported on how the statistical literacy of Year 6 students can be enhanced through modelling with data by developing shared problem spaces between mathematics and statistics.

1.3.2.2 Empirical Results of Studies Focusing on Teachers of Modelling

Empirical lines of inquiry that take teachers as their focus have focused on teacher practices, actions, beliefs, knowledge requirements and competencies, amongst other characteristics and influencing factors when planning, preparing, engaging in, assessing and reflecting on their facilitating of student modelling in and outside classrooms.

Two different approaches that teachers can take in the classroom in supporting the development of modelling competencies are *atomistic* where the focus is on mathematising processes and analysing models mathematically and *holistic* where the focus is on the modelling process as a whole with all phases expected to play a part. Further results from Kaiser and Brand (2015), for example, confirmed that both atomistic and holistic approaches fostered students' modelling competency in all sub-competencies mentioned above. The holistic approach promoted overall modelling competency more effectively. The hypothesis that the sub-competencies connected

[1]KOMMA was a research project funded by the German Federal Ministry for Education and Research (PLI3032).

to the sub-processes of the modelling cycle would be fostered more effectively by experiencing different modelling sub-tasks in an atomistic approach was not confirmed. However, as pointed out above, these results need further testing with broader samples of teachers and classes and tasks.

Czocher (this volume) raises the interesting question with respect to competencies for facilitating student modelling: How does a facilitator aid a student in moving from a nonmathematical interpretation of a problem situation to a mathematical interpretation of that same problem situation? In other words, how do teachers bring students to the realization that the crux of modelling is to reduce the complexity of a real-world situation so models can be applied or constructed, not to keep that complexity of the situation so the "model" is an exact image of reality?

A study by Kuntze et al. (2013) investigated Austrian teachers' self-perceptions of their pedagogical content knowledge with respect to diagnostic knowledge related to the modelling process and to providing modelling specific feedback. In particular, both pre-service and in-service teachers in the sample focussed on general suggestions rather than specific support related to the modelling process in reacting to potential difficulties students might experience when modelling. Results showed that these teachers needed professional development related to both modelling specific Pedagogical Content Knowledge and self-efficacy as teachers of modelling. Blomhøj (this volume) argues that there is also a need for the development of tools that allow teachers to make better use of theories of learning of mathematical concepts and to view modelling activities as a didactical means for supporting students' learning of mathematics not just to develop students' modelling competency.

1.3.2.3 Empirical Results of Studies Focusing on Task Design

Task design in educational modelling contexts is a fruitful area for research as specifications for suitable problems for the classroom need to be based on some sort of theoretical or empirical evidence. It seems wise that the essential elements of tasks used successfully in modelling implementations in research studies for different purposes be captured in design criteria that can be used for both classroom and research purposes in the future. However, it must be borne in mind how such tasks are implemented is a bigger issue than task design per se.

Reit and Ludwig (2015) have used simple modelling tasks in their work that are designed for an holistic approach to both teaching and assessment. The tasks were designed to meet the following criteria: authenticity of context, realistic numerical values, possession of a problem solving character, a naturalistic format for the question and openness of solution approaches. The degree of difficulty of these tasks was conjectured to be able to be determined using order of thought operations and cognitive demand from the perspective of cognitive load theory (Sweller 2010). Empirical results with Year 9 students confirmed thought structure complexity was related to solution rate with more sophisticated thought structure lowering solution rate of tasks.

The design of multiple choice items to test first year educational science students' ability to connect written descriptions of realistic situations to linear and almost linear models when presented in different representations (symbolic, tabular or graphical) underpins the study by Van Dooren et al. (2013). The representational mode in which an item was presented had a high impact on students' modelling accuracy and on the tendency to inappropriately connect non-linear situations to linear models. The students were proficient in connecting descriptions to models when the situation was linear. However, when the situation was almost linear they also connected these erroneously to a linear model. The authors point out that whilst the use of such testing can be for diagnosis and rectifying errors with respect to identifying suitable models, hopefully with the intended purpose of being able to do this in more in-depth modelling situations, it should also be interspersed with the use of more authentic real-world situations in tasks. Extrapolation of findings from insights obtained by use of multiple-choice items to modelling expertise to solve extended problems still presents as a credibility gap.

1.4 Future Directions

From this brief overview of current lines of inquiry in the field of mathematics education research related to teaching and learning of mathematical applications and mathematical modelling, a number of questions arise that could seed future research projects. Some of what challenges our current thinking in theoretical lines of inquiry are opportunities to advance knowledge. In particular, one might ask:

- What are the similarities, differences and relationships between descriptive and prescriptive modelling?

Similarly, issues that have arisen above with respect to particular theoretical frames or empirical studies give rise to a number of potential empirical lines of inquiry. Generative questions for these could be:

- How does activity within phases of a prescriptive modelling problem differ from its descriptive counterpart and what are the implications for scaffolding?
- What scaffolds would ensure meta-validation when prescriptive modelling is conducted? Fully?
- What is the role of a global overarching modelling competency in modelling?
- Should particular sub-competencies or global modelling competencies be the focus of teaching in regular classrooms?
- What is the role of anticipatory metacognition (especially pMON) by teachers and student modellers in ensuring technology is used in a transformative manner in modelling?
- How do teachers come to realise that by not offering young students challenging situations to model, we are not realising the potential of both students and teaching in the classroom?

- How is self-efficacy as a teacher of modelling different from self-efficacy as a teacher of mathematics? At secondary level? At primary level? At tertiary level?
- What is the structure of professional learning for teachers needed to enhance modelling specific Pedagogical Content Knowledge and self-efficacy as teachers of modelling?

It must be emphasised that this list is not meant to be exhaustive and is very much influenced by the particular selection of studies, within the categories, I have highlighted in the various lines of inquiry that have "caught my eye", as it were, in my surveying of the literature at the time of ICME-13.

1.5 Final Considerations

In this chapter a brief overview of the state of the art in curricula and research on mathematical modelling and applications of mathematics in education has been provided. The theoretical lines of inquiry in mathematics education research related to the teaching and learning of mathematical applications and mathematical modelling regularly in classrooms, selectively highlighted, have been the local theories of prescriptive modelling, modelling frameworks/cycles and modelling competencies and the potentially more general theory of anticipatory metacognition. Modelling frameworks/cycles and modelling competencies have received quite a deal of attention from scholars and researchers from both a theoretical and empirical perspective. The notions underpinning prescriptive modelling, on the other hand, have been in existence for some time but have not really been central to the modelling debate but Niss's (2015) drawing the attention of the field to them could arouse sufficient interest for them to be pursued further and brought to realization within classrooms and be the subject of future research. The ideas underpinning anticipatory metacognition have also been around for some time, albeit in other fields, but they have not been combined until now. Although some beginning work has been done with some of the dimensions of anticipatory metacognition, this is an area where there is a lot more empirical work to do.

The empirical lines of inquiry have taken as their focus student modellers, teachers of modelling and task design. This selection is in keeping with general major emphases in the field. The examples overviewed for lines of inquiry focussing on students concern their modelling and mathematical competencies, visualisation and their capabilities at different ages. All of these are fertile ground for further study. Prominent empirical lines of inquiry that take teachers as their focus concern teacher practices, actions, beliefs, knowledge requirements and competencies, The third area, task design, has been less of a focus at the time of surveying in studies of actual classroom practice and more of a focus for good instruments to assess modelling. This is however an area that changes emphases rapidly depending on who is researching in the field at the time of surveying the field.

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Chapter 2
Toward a Framework for a Dialectical Relationship Between Pedagogical Practice and Research

Jussara de Loiola Araújo

Abstract In this chapter, I present initial steps towards a framework for a dialectical relationship between pedagogical practice and research in the field of modelling in mathematics education. These methodological reflections have arisen from the development of research on modelling guided by critical mathematics education, and are grounded in a socio-political perspective of research. I will develop the idea that pedagogical practice and research should be seen as part of a single unit, mutually developing and influencing each other; that they are different, have different purposes, and may be incompatible, but one presupposes and constitutes the other. When a pedagogical practice is taking place at the same time as a research study, the researcher can have a double role, as a researcher and as a teacher, and the participants of the research can also have the role of students, in that pedagogical practice.

Keywords Pedagogical practice of modelling · Socio-political research · Dialectic · Critical mathematics education

2.1 Introduction: Setting the Scene and Presenting the Objective

Mathematical modelling as an endeavour in mathematics teaching and learning gained more strength in the 1960s. However, according to Niss et al. (2007), it was just after the 1990s that mathematical modelling in mathematics education, as a field of research, reached what the authors call its "maturation phase", in which empirical studies started to be developed. In that period, the community became more organised with the foundation of the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) and its affiliation as a Study Group of the International Commission on Mathematical Instruction (ICMI), in 2004. The

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authors claim that the ICMI Study Volume (Blum et al. 2007) "might be said to formally mark the maturation of applications and modelling as a research discipline in the field of mathematics education" (Niss et al. 2007, p. 29).

This very short summary of its history shows that the research field of modelling in mathematics education is relatively young. One characteristic of the maturation phase of the field is the growth of the amount of research being developed. A natural consequence is the emergence of studies aiming at understanding the nature of the research being developed in the field. Questions such as: "What are the research studies about?" or "How have the designs of research studies been characterized?" are more and more common. Examples of these kinds of studies can be found in the international literature (e.g., Geiger and Frejd 2015) as well as in the Brazilian literature (e.g., special issue of *Revista Eletrônica de Educação Matemática*[1]).

In this sense, the inspiration for this chapter is a particular overview of the research studies being developed in the young field of modelling in mathematics education. As a member of ICTMA and of the Brazilian community of mathematical modelling in mathematics education, I have observed that it has been common for research to be conducted in a context in which a pedagogical practice takes place at the same time. We can find reports of the simultaneous occurrence of pedagogical practice and research, for example, in the volumes of the ICTMA book series (e.g., Villarreal et al. 2015), or in academic journals (e.g., Albarracín and Gorgorió 2015).

A clear example of this situation can be found in the work of Schwarzkopf (2007). The author presents results of an empirical study guided by the following question: "How can we describe from a theoretical point of view the interplay between the real-world and mathematics, realised in everyday mathematics classroom interaction?" (p. 210). The study followed an interpretative research paradigm and data were gathered from "regular mathematics lessons of a fourth grade class of primary school (10 year old) students in Germany" (p. 211), where students were involved in the search for solutions to a modelling task and the teacher was the mediator of the discussion. The activity performed by students and teacher in the mathematics lesson is an example of what I call a *pedagogical practice* (see Sect. 2.2). It is the context of the *research* developed by Schwarzkopf (2007). The researcher depicted two episodes from the pedagogical practice as data to be analysed with the support of the theoretical framework in which the research was grounded.

Sometimes, the distinction between pedagogical practice and research in the literature of modelling in mathematics education is not so clear. In Swan et al. (2007)'s work, for example, the intention was to "illustrate how modelling promotes the learning of mathematics." (p. 276). Based on the description of some examples of pedagogical practices, the authors claimed that modelling helps in the development of mathematical language and tools and promotes the asking and answering of mathematical questions. Promoting the learning of mathematics is clearly an objective of pedagogical practices in mathematics classrooms. However, probably some research was necessary to support the assertions made by the authors. The concept

[1]The journal is published in Portuguese and is available at https://periodicos.ufsc.br/index.php/revemat/issue/view/2153.

of "learning", for example, can be understood in different ways, according to different theoretical frameworks. Swan et al. (2007) probably based their remarks in evidence from empirical studies to make their assertions about the development of mathematical language and tools and the promotion of questioning. Finally, they were supposed to relate the described competencies with the learning of mathematics, since the objective of the chapter was the promotion of learning of mathematics. All these procedures are proper for the development of research.

My main interest in the research studies being developed in the field of modelling in mathematics education comes when a pedagogical practice is specially created for the conducting of a research study, as was done by Maaß (2010). Maaß integrated six modelling units into the teaching of mathematics classes for two parallel classes (13–14 years old) over a 15 month period, being both their teacher and researcher. This is also the case in the research developed by myself and in the ones developed by my masters and doctorate students, which evoked the need for methodological reflections on the conducting of our own research (Araújo et al. 2012, 2015; Campos and Araújo 2015). Undertaking reflections on the act of researching is a desirable practice when one adopts a socio-political perspective in mathematics education, as is the case in our group:

> A socio-political perspective in mathematics education does not only offer possible theoretical tools and interpretations, but also emphasises the researcher's awareness of the research process and on how he/she privileges – and silences – diverse aspects of the research activity. In this sense, examining the process of research and its elements – and evidencing the power relationships involved in them – becomes one of the central features of socio-political approaches in mathematics education research. (Valero and Zevenbergen 2004, p. 3)

Considering the co-existence of pedagogical practice (sometimes specially created for the conducting of the study) and research in the young field of modelling in mathematics education and the socio-political perspective in mathematics education assumed by us, my objective in writing this chapter is to describe the current state of the reflections carried out by ourselves, with the aim of designing a framework for understanding the dialectical relationship between pedagogical practice and research, which I represent as *pedagogical practice|research*.[2]

When a pedagogical practice is taking place at the same time that a research study is, the researcher can have a double role, as a researcher and as a teacher, and the participants of the research can also have the role of students, in that pedagogical practice.

In the next sections, I will present the main elements of the pedagogical practice|research dialectic clarifying the concept, and discuss how the relationships

[2]I have represented this dialectic by a vertical bar between its two components. The symbol "|" used in the expression is called the Sheffer stroke. Any expression p|q is true if and only if not both p and q are true. "In other words, the total expression is true only if it contains a contradiction. In a materialist dialectical approach, such *inner contradictions* […] represent cultural-historical processes and entities and, in fact, constitute the driving forces underlying individual|cultural development." (Roth 2005, p. xviii, emphasis in the original). In my use of the symbol, I mean that in a pedagogical practice|research it is not true that only a pedagogical practice is happening nor is it true that only a research is happening.

between the *teacher|researcher* and the *student|participant* can be described and understood by means of this dialectic. Some examples of studies will be given to illustrate these dialectical relationships.

2.2 Pedagogical Practice|Research

Before reflecting on the dialectic itself, it is important to present how I understand the two elements that it is composed of: pedagogical practice and research.

Research involves a systematic investigation supported by rigorous theoretical and methodological references, accepted by the community in which the research is taking place (see also Niss 2001). According to Bicudo (1993, p. 21), it is an "inquiring search conducted from the [research] question". The author draws attention to the importance of not confusing research with pedagogical practice. The latter is an activity that clearly aims to put people in touch with the cultural arsenal already produced by the society in which they were born. *Pedagogical practice* involves a subject or content, a teaching methodology, materials, some theoretical perspective guiding the development of the practice and of course, the teacher and the students. Therefore, research and pedagogical practice are two different activities; they have different purposes and are organised in different ways.

The pedagogical practices where I normally conduct research involve modelling activities guided by critical mathematics education (Alrø et al. 2010; Skovsmose 2005). These practices can be classified in the socio-critical perspective of modelling in mathematics education (Kaiser and Sriraman 2006). In these practices, students are invited to use mathematics to investigate situations with reference to reality and, at the same time, to reflect on (and to question) how mathematics is used as a tool and language of power (Araújo 2007; Barbosa 2006). Such pedagogical practices ask for "not only mathematical (conceptual) understanding, but also contextual knowledge, political awareness and judgements based on values" (Jablonka 2007, p. 197). The intention is not only to develop mathematical calculus skills, but also to promote the critical participation of students/citizens in society, discussing the political, economic, environmental issues, etcetera, in which mathematics serves as a form of technological support. Pedagogical practices of mathematical modelling guided by critical mathematics education play an important role in the socio-political perspective in mathematics education because of its ties to the students' everyday lives, their experiences, their places in society with all its political, economic and environmental conditions.

It is important that the methodological approach of research is in harmony with its theoretical guidelines. In this sense, my studies are based on a critical research paradigm that, according to Skovsmose and Borba (2004), makes use of procedures that go beyond the observation and recording of situations that actually happened. For these authors, "doing *critical research* also means to explore *what is not there* and *what is not actual*", which implies an investigation into "what could be. Critical research pays special attention to hypothetical situations, although still considering what is actual. Critical research investigates alternatives" (p. 211, emphasis in the original). The authors claim that critical research should encompass a mixture of proper research and activities of educational development. They give an example of a researcher being engaged in a development project, working together with a teacher. They worked co-operatively and developed a research study and a pedagogical practice (the activity of educational development) at the same time.

In other words, the main reason for the creation of a pedagogical practice to conduct the research is not the absence or scarcity of contexts in which to do it, but the intention to provoke "the *critique and the transformation* of the social, political, cultural, economic, ethnic, and gender structures that constrain and exploit humankind, by engagement in confrontation, even conflict" (Guba and Lincoln 1994, p. 113, emphasis in the original). Mathematical modelling guided by critical mathematics education can be a feasible way to put these ideas into practice, which is illustrated in the following example.

2.2.1 An Example

The development of modelling projects guided by critical mathematics education is not common in day-to-day mathematics classes at the university where I am a teacher. Pedagogical practices there are characterised by the common practice of mathematics classes in higher education, with lectures, exercises and tests. This educational context, existing prior to the research, is what Skovsmose and Borba (2004) call the *current situation* (CS).

In fact, this is the current situation in most mathematics classrooms. According to Pais (2010, p. 134), "in school life an environment of criticism and questioning is absent. The teacher has a mission consisting of transmitting [to the students] a particular body of knowledge". However, "the relationship between the mathematical knowledge that is expected to be developed by teachers and students in classrooms and the knowledge developed and used in other mathematical practices" (Jablonka 2010, p. 89) is something that we should be concerned with. Such ideas, based on critical mathematics education, help to ground a critique of the current situation, seeking to transform it, taking into account an *imagined situation* (IS) (Skovsmose and Borba 2004, p. 213), in which modelling pedagogical practices guided by critical mathematics education would be part of mathematics classes at university. Imagined situations are "vision about the possibilities of alternatives" to the current situation.

Up to this point in the example, I have spoken of two pedagogical practices: both the current and the imagined situations describe activities involving (or that would involve) teachers and students, a discipline (mathematics), teaching methods, and a theoretical perspective influencing (or that would influence) each one. In my case, the process of critiquing the current situation while envisioning an imagined situation is strongly linked to the development of research. This is a characteristic of the pedagogical practice|research dialectic.

A research study developed by myself (Araújo 2013) aimed to characterise and analyse students' learning (understood according to a specific theoretical framework) while developing modelling projects guided by critical mathematics education. Two components in the description of the objective of the research can be highlighted: (i) the modelling projects guided by critical mathematics education, that are the *pedagogical practice*, the context of the research; and (ii) the characterisation and analysis of students' learning within this context, which is the objective of the *research*.

Since this intended context for the research does not exist in the current situation at the university, and as the result of a critique of this situation, it would be necessary to pursue the imagined situation. However, imagining alternatives to a current situation does not necessarily mean that they will be implemented in the way we imagined. I therefore invited former students of mine to participate voluntarily in the research study, developing a modelling project outside the context of formal university courses, which gave rise to what Skovsmose and Borba (2004, p. 214) call an *arranged situation* (AS): a "practical alternative that emerges from a negotiation involving the researchers and teachers" as well as students. "The arranged situation may be limited by different kinds of structural and practical constraints. But it has been arranged with the imagined situation in mind".

Five students, coming from three different courses within the Exact Sciences area, accepted the invitation to participate in the research: Alberto, from the Systems Engineering course; Pedro and Rafael, from the Physics course; Natália and Débora,[3] from the Mathematics course. In addition, three researchers also participated in the meetings to develop the research: Jussara Araújo (researcher in charge), Ana Paula Rocha and Ilaine Campos (assistant researchers and, at the time, Masters students in education, supervised by myself). The development of the research occurred over nine meetings from October 2012 to June 2013. This group of eight people worked on the modelling learning project, guided by issues of critical mathematical education, which constituted the *pedagogical practice* in which the research was carried out.

The theme of the modelling project developed by the group was the purchase of property (real estate). It was chosen by the five students, members of the group. After they had chosen the theme, they delimited the objective of the project, which was to establish a ranking of the factors that might influence choice when purchasing property. To achieve this objective, the group designed a questionnaire, which included more demographic questions, referring to name, city in which the person resides, civil status, etcetera; and questions related to the importance that the person attached to living close to hospitals, schools, city centre, workplace, leisure areas; importance of the financial value of the property; and easy payment methods for the acquisition of the property, among others. The questionnaire was conducted with 163 people and, after receiving the responses, the group constructed a mathematical model to assign a degree of relevance to each of the factors that were included in the questionnaire and then, to establish a ranking of these factors.

Before describing the research, observations about the pedagogical practice is pertinent. The purchase of property is not necessarily a theme that would provoke the

[3]The names of the participants of the research are fictitious.

emergence of ideas of critical mathematics education. Then, to carry out my plans as researcher, I had to make these ideas arise from the discussion along with the pedagogical practice. Since the pedagogical practice was taking place in an arranged situation, with a small group of students and with the help of two assistant researchers, it was possible for us to be attentive and stimulate critical mathematical education discussion throughout the pedagogical practice. In others words, critical themes do not belong necessarily to the day-to-day mathematics classes at the university so it was necessary to stimulate them. This is an insight that has arisen from the pedagogical practice being integrated into the arranged situation.

To develop the *research*, which had this pedagogical practice as context, three different methodological procedures were adopted: Firstly, participant observations (Adler and Adler 1994) were used as the researchers acted and intervened throughout the modelling project created for the development of the research. These observations were recorded through field notes and filming. Secondly, a record of the activities was kept in an online text editor. The research participants were asked to register the development reports of the project in a shared document—which both researchers and participants had access to—stored by the application package Google Drive (2017), a file storage and synchronization service, developed by Google. This way, the participation in the research could happen in person or virtually, in the virtual space created for the group. Thirdly, interviews, both individually and as collectives, were conducted by me to build deeper insights into the episodes created from the videos (the first procedure) or into the report produced online (the second procedure). The interviews were semi-structured (Fontana and Frey 1994), since there were some questions previously planned, but several others were elaborated according to the progress of the interview. The recording of the interviews was also done through filming.

Although the intention was to do modelling grounded in the ideas of critical mathematics education, Araújo et al. (2014) concluded that, according to a proposal made by Rafael (a participant of the research study), the role of mathematical models was to make predictions about the future, more accurately, with fewer mistakes. However, such a proposal was in conflict with the ideas of critical mathematics education. Thus, an analysis of data gathered for the research study shed more light on conceptions of mathematics rooted in the school mathematics tradition (Skovsmose 2005), which are questioned by critical mathematics education. Based on our study, we were able to conclude that having the intention to develop modelling guided by critical mathematics education is not sufficient to do so and, thus, the research study carried out in the pedagogical practice gave new insights into the current situation.

The descriptions of the pedagogical practice and of the research highlight two sets of procedures, one set corresponding to each practice, which, however, occurred simultaneously:

- Pedagogical practice: choice of a theme for the modelling project; definition of the objective of the project; searching for information; elaboration of a questionnaire; data collection; building model(s); seeking solutions.

- Research: participant observations recorded by field notes and filming; a record of the activities in an online text editor; collective or individual interviews, recorded on film.

With regard to these procedures, we had to be very clear about the differences and relationships between pedagogical practice and research (Bicudo 1993), in order not to state that a pedagogical procedure is a research procedure, or vice versa. In fact, we had to be very clear the whole time, since the relationship and mutual influence between these two practices were present from the beginning, as can be seen in the previous description.

As I have explained in the introduction to this chapter, we have used a vertical bar to represent the pedagogical practice|research dialectic (Araújo et al. 2012, 2015). Goulart and Roth (2006) use this notation to represent the dialectical union between margin and centre (margin|centre), which was used to describe the participation of children in science education activities. According to these authors, this participation may be marginal or central to the activity planned and proposed by the teacher. However, Goulart and Roth (2006, p. 682) argue that the margin and centre are “two mutually presupposing but incompatible aspects of the same unit of analysis”. These ideas fit nicely into the way I and my colleagues understand the relationship between pedagogical practice and research: “they are part of a single unit, mutually developing and influencing each other. They are different, have different purposes, and may be incompatible, but one presupposes and constitutes the other” (Araújo et al. 2012, p. 10).

The pedagogical practice and the research, evolving dialectically, can give new insights into the current situation, offering new inspirations for the imagined situation, and, at the same time, giving rise to new research questions, since “dialectics deals with systems in movement through time. The elements of a dialectical contradiction relate to each other within the moving structure, historically” (Engeström and Sannino 2011, p. 370).

To this point, I can summarize three characteristics of the dialectical relationship between pedagogical practice and research:

- The process of critiquing the current situation while envisioning an imagined situation is strongly linked to the development of research.
- Pedagogical procedures are different from research procedures; however, the relationship and mutual influence between pedagogical practice and research are very tight.
- The pedagogical practice and the research, evolving dialectically, can give new insights into the current situation, offering new inspirations for the imagined situation, and, at the same time, giving rise to new research questions.

I have focused my attention, particularly, on the people involved in the pedagogical practice|research dialectic: the teacher|researcher and the students|participants, as I describe in the following two sections.

2.3 Teacher|Researcher

When pedagogical practice and research occur in a dialectical relationship, the researcher may also be acting as a teacher, giving rise to the teacher|researcher dialectic (Campos and Araújo 2015). This dialectic can be described based on the Masters research conducted by Campos (2013). Her focus was the involvement of the students (the participants of the research) in a modelling activity which aimed to relate students' involvement in the modelling activity to their life experiences and expectations for the future.

I was the supervisor for Campos' study and the research was conducted in a discipline taught by myself, in which modelling projects guided by critical mathematics education were developed, in a way similar to that described in the previous section. The participants of the research were two groups of students of the discipline while they developed their modelling projects. Therefore, Campos and Araújo (2015) reconsidered data from the *research* carried out by Campos (2013) in the context of the modelling project (i.e., the *pedagogical practice*) developed by one of these groups. This group was composed of seven students: Amanda, Carlos, Catarina, Eduardo, Emanuel, Fernanda and Rodrigo.[4]

Brazil is a federal republic formed of 26 states plus the Federal District, and Minas Gerais is one of the states. The theme of the modelling project developed by the group was "The Public-Private Partnership (PPP) in the Brazilian prison system in the State of Minas Gerais". A PPP is a set of activities performed in a shared way by the State and the private sector, for the development of services of the public sector. The objective of the group's modelling project was to calculate how much the State would cease to spend within the prison system if it was to establish a PPP.

Campos and Araújo (2015) rely on four aspects to characterise the teacher|researcher dialectic. These will now be outlined.

2.3.1 Aspect 1: From Researcher to Teacher

During one of the group meetings to develop the project, the researcher asked the group questions regarding their motives for choosing the theme of the modelling project and forming groups. These questions were closely related to the objective of the research of Campos (2013). One of her questions was: "Carlos has proposed the theme; then, you were joining because you would work with people with whom you have already developed other works, and, therefore, have greater affinity?" (Campos and Araújo 2015, p. 331). Carlos showed that he had understood her purpose as a researcher by saying that the objective of the research would be, then, the motivation of the group. However, the researcher is also a mathematics teacher, and in making contact with the information that the group had, regarding its modelling project, exposed this side by asking, "How will mathematics be linked here?" (p. 332) and

[4] The names of the participants of the research are fictitious.

went on guiding the group in the development of the project. In Campos and Araújo's analysis (2015, p. 332),

> Initially, the researcher was acting and, spontaneously, on impulse, the teacher came on the scene. That is, at that moment, it happened a change of roles in the actions of the researcher|teacher [from researcher to teacher]. However, this was only possible due to her actions as a researcher, since her initial questions provided greater insights on the theme of the modelling project.

2.3.2 Aspect 2: Research Participants Are Students, so the Researcher Is the Teacher

The participants of the research, who were students and knew that the researcher was a mathematics teacher, spontaneously requested that she assume this role to guide them in the development of the modelling project. A dialogue involving two participants and the researcher illustrates this situation. Emanuel, addressing the researcher, stated: "So, if we do it like this, you can help us. I think you get the idea, right?" (Campos and Araújo 2015, p. 333). The researcher answered affirmatively and Fernanda stated, immediately: "Ah! So let's speed it up here. What variable will we work on then?" (Campos and Araújo 2015, p. 333). Here, the students induced and encouraged the researcher to act as a teacher.

2.3.3 Aspect 3: The Teacher Acts on Her Own Initiative

The group had difficulties with the mathematical content and could not move forward in the mathematical modelling. Faced with the group's lack of progress, the teacher decided to help them, guiding them in the construction of the model, as, for example, when she said: "So, this seventy will be multiplied by the coefficient that we'll call C [she wrote it down in the notebook]." (Campos and Araújo 2015, p. 334). Carlos made a comment about this: "This is the most difficult in math: making the formula, to represent the data, put the bracket, seventy. This is too hard! The C cannot be negative, is it right?" (Campos and Araújo 2015, p. 334).

Therefore, in some moments, the teacher acted on her own initiative and put in the background her role of researcher. Thus, "the demands [of the] modelling environment and the researcher|teacher's formation led her to act as a mathematics teacher" (Campos and Araújo 2015, p. 335).

2.3.4 Aspect 4: The Teacher's Reflections Favouring the Performance of the Researcher

Having faced the difficulties that the students were having with mathematics, the researcher realised that each member of the group was involved with the modelling project in a different way, depending on his/her familiarity with mathematics. At this moment, it is important to remember that the objective of the research was to relate the involvement of the students in the modelling project with their life experiences and expectations for the future. The experience as the teacher and her researcher's reflections led her to devise new interview questions: "The activity had several moments, okay? Specifically, about the time dedicated to putting the ideas into mathematical terms, how do you analyse your relationship with math and your involvement?" (Campos and Araújo 2015, p. 335). In other words, the typical actions of a researcher "were planned and guided under the influence of her actions as a teacher in the context of the research" (p. 335).

In the end, from the analysis of these four aspects, Campos and Araújo (2015) concluded that the teacher|researcher could be described by an alternating between the roles of the teacher and the researcher, in which one always influences and is influenced by the other, in a dialectical way. Therefore, when pedagogical practice and research occur in a dialectical relationship, the teacher|researcher acts sometimes as a teacher and at other times as a researcher, and each role is highlighted or put aside according to the necessities of the development of the pedagogical practice|research.

2.4 Students|Participants

The role of the participants, throughout research, is influenced by the relationship established between them and the researcher (Bogdan and Biklen 1994). From the study of Campos and Araújo (2015), we can see that the teacher|researcher is constituted as a result of her relationship with the participants, who, at that time, were also students developing a modelling activity. It makes sense then, to talk about the students|participants dialectic. I start characterising the students|participants dialectic from an interpretation of each one of the four aspects described in the previous section, now focusing on the role of the students|participants of the pedagogical practice|research. The four aspects may be rewritten from the point of view of students|participants in the following way:

- When the researcher started to act as a teacher, the participants began to act as students;
- The researcher is a teacher, so the research participants are students;
- If the teacher acted on her own initiative, then the students, who needed to develop the modelling project, took the opportunity to receive guidance;
- Because they behaved as students, revealing their low familiarity with mathematics, the participants provided information relevant to the researcher.

Thus, the first characteristic of the dialectic is that the students|participants are constituted in relation to the teacher|researcher.

In light of these reflections, and because of my experience, another point that should be considered is the ethical care that the teacher|researcher must take with the students|participants. Pedagogical practice and research, when each one occurs in isolation, already have their own rules to ensure the ethical care with students and the research participants, respectively. However, when we consider the pedagogical practice|research dialectic, new questions concerning ethical care may arise. Some issues such as ensuring rigorous data collection procedures, in the research, combined with others of pedagogical practice, such as ensuring high quality mathematical education, can give rise to conflicts, which the teacher|researcher must think and make decisions about in the heat of the moment during the pedagogical practice|research.

In the research described by Campos and Araújo (2015), for example, when the teacher acted on her own initiative (Aspect 3), perhaps the ethical care as a teacher has been dominant in order for her to ensure the students' learning and success in the development of the modelling project. Another example comes from a previous study (Araújo et al. 2010), in which I was a teacher and a researcher at the same time. In that study, while students developed a mathematical modelling project, I collected data for the research. When analysing the data, I realised that the methodological procedures of the research were less apparent than those of the pedagogical practice. For ethical reasons I should have focused on acting as the teacher at that time.

It is worth noting, however, that the predominance of the role of the teacher in the teacher|researcher dialectic in relation to the researcher, for ethical reasons, does not mean that the research will be hampered, due to a possible relaxation of the methodological rigour. As described by Campos and Araújo (2015), the teacher|researcher, by acting as teacher, led new insights for the researcher, requiring a reorganization of the methodological procedures, which is typical of the socio-political perspective of research in mathematics education. Thus, a second characteristic of this dialectic is that the ethical concerns regarding the students|participants help to constitute the methodological rigour of the research, which in turn, is related to the educational quality of the pedagogical practice.

2.5 Final Remarks

In this chapter, I sought to present the initial design of a framework for a dialectical relationship between pedagogical practice and research in the field of modelling in mathematics education, guided by the ideas of critical mathematics education, as well as some characteristics of the people involved in these practices: the teacher|researcher and the students|participants. The framework is being designed by means of my participation as a member, both locally and internationally, of the community of modelling in mathematics education and as I become more experienced as a researcher and advisor of master and doctorate students. In the chapter, I based my argument mainly in two examples of such research studies.

I understand that this discussion is important not only for the socio-critical perspective of modelling in mathematics education, but also for the other perspectives. As discussed in the Introduction to this chapter, modelling in mathematics education is a relatively young research field that is experiencing a maturation phase. Thus, it is important that our community looks at the corpus of research studies undertaken in order to organize them and to understand characteristics of practices that have been legitimized by this community (see Galbraith 2013).

One of these practices is the simultaneous occurrence of pedagogical practice and research. In literature, there is work, for example, in which procedures that are proper to pedagogical practices are mistakenly called research procedures and vice versa. The research field needs to have a clear distinction between these two practices. In this chapter, I sought to go forward in this direction.

In the specific case of modelling guided by critical mathematics education, "examining the process of research and its elements" (Valero and Zevenbergen 2004, p. 3) is paramount, since this is desirable in a socio-political approach in mathematics education. Having the objective of critiquing and transforming the social structures that constrain and exploit humankind, research studies about modelling according to critical mathematics education are being developed in pedagogical practices specially created for this purpose. The socio-critical perspective in mathematics education is rooted in transforming mathematics pedagogical practice by taking into account all of the student's background and foreground, and by guiding the students to become independent decision makers and critical users of (mathematical) information. Mathematical modelling, because of its ties to the real world, strongly depends on the situation of the students, in comparison to other parts of mathematical education. These characteristics reinforce the necessity of being very clear about the proper procedures of each—pedagogical practice and research – since there are objectives of different natures to be reached.

As I stated at the outset, the ideas presented here are initial steps towards the framework. Further studies, reflections and examples are needed for a greater maturity of this framework and I hope that our community of modelling in mathematics education feels encouraged to join me in this challenge.

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Chapter 3
Towards Integration of Modelling in Secondary Mathematics Teaching

Morten Blomhøj

Abstract The inclusion of models and modelling in mathematics curricula has been a major trend internationally in recent decades. This has taken place in interplay with research on the teaching and learning of modelling and applications. However, it is still a pending challenge for research how to support real integration of modelling and applications into mathematics teaching. At the secondary school level in particular, the duality between the aim of developing students' modelling competence and that of supporting their learning of mathematics through modelling activities is essential for understanding and furthering the integration. The interplay between research and the development of teaching practices with regard to these two aims is discussed. In particular, the potential and challenges of using theories on the learning of mathematics to support the integration of modelling as a didactical approach will be illustrated and discussed in relation to two examples of mathematical modelling of dynamical phenomena at secondary level.

Keywords Secondary mathematics · Integration of modelling · Modelling competence · Conceptual learning · Modelling dynamical phenomena

3.1 Introduction

Research on the teaching and learning of mathematical modelling has developed to a level where it constitutes its own field of research within the mathematics education community (see Niss et al. 2007, pp. 28–32). Through this research, a coherent theory consisting of four main elements has been established: (1) A set of potential and actually used justifications for including modelling and application at different levels; (2) Conceptions of a mathematical model, a modelling process and of modelling competence, and related well-argued for and empirically tested ways of supporting the students' development of modelling competency and their learning

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of mathematics through modelling activities; (3) Experiences and theoretical based knowledge about opportunities and challenges in teaching, learning and assessing of modelling competency; and (4) Theoretical based methodologies for developing teaching practices through (in-service) education and collaboration between teachers and researchers in developmental projects.

In his plenary address at ICME-12, Werner Blum surveyed the achievements in the field from the perspective of what it tells us about quality in the teaching of applications and modelling at secondary level. Based on empirical findings, Blum identified ten important aspects in a teaching methodology integrating modelling and applications (Blum 2015, pp. 83–86). However, he concluded with the following remark:

> I would like to emphasise that all these efforts will not be sufficient to assign applications and modelling its proper place in curricula and classrooms and to ensure effective and sustainable learning. The implementation of applications and modelling has to take place systemically, with all system components collaborating closely: curricula, standards, instruction, assessment and evaluation, and teacher education. (p. 87)

One important contribution from research to such a systemic approach is to develop further the interplay with teaching practices. This challenge, however, stands differently with regard to the two ends of the dual aim for integrating applications and modelling in secondary mathematics teaching; namely to support the students' development of modelling competence or to enhance the students' learning of mathematics by means of modelling and applications. Therefore, for analytical purposes, in this chapter a distinction is made between these two aims, although they are closely connected in a duality.

With regard to the first aim, the research has developed in interplay with the practices of teaching modelling in specific courses of lessons or as part of developmental projects. The research surveyed by Blum (2015) provides a strong basis for a teaching practice aiming at developing the student's modelling competence. In Maaß (2006) the concept of modelling competence is unfolded with respect to the modelling cycle and related reflections. In Blomhøj and Højgaard (2007) modelling competency is discussed as a main justification for secondary mathematics in general education. Working with the entire process of mathematical modelling—the full modelling cycle—is here seen as the natural and necessary constituent of the development of modelling competency in teaching (pp. 48–49). Research has pinpointed theoretically and empirically learning difficulties related to the different phases in the modelling process, see for example Borromeo Ferri (2006). Moreover, research has developed different ways of conceptualising progress in the (individual) students' modelling competency. Blomhøj and Højgaard (2007) described progress in modelling competency using the notions from the Danish KOM-project (Niss and Højgaard 2011) and wrote about the development of modelling competency in three dimensions, namely the degree of coverage with respect to the modelling cycle, the technical level—mathematically and/or in modelling techniques, and the domain of action—meaning the domain of extra-mathematical situations in which the modelling competency can be put into action. Other researchers define different levels of modelling competency by means of the students' understanding of, and reflection on, their modelling work, see for examples vom Hofe et al. (2005). Kaiser and Brand (2015) analyse how the concept of mathematical modelling competencies has developed in research during the latest three decades. With regard to the aim of developing students' modelling competencies, it is still a challenge for research to establish a basis for conceptualising students' progress in and level of competency in modelling.

However, in general, research on the teaching and learning of modelling provides theory based designs for, and investigations of, many different ways of organizing mathematical modelling activities in classrooms with the aim of developing the students' modelling competencies. A rich and extensive documentation thereof is found in the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA) biennial international conferences and in the related volumes in the Springer book series *International perspectives on the teaching and learning of mathematical modelling*, see http://www.ictma.net/.

The theories developed are to a high degree aligned with the development of practices of teaching modelling with the aim of developing the students' modelling competencies. The basic concepts, notions and theories about models and modelling are developed in close interplay with the development of teaching practices, and are therefore, in principle, quite easily applied in designing and/or analysing mathematical modelling activities in classrooms.

With regard to the second main aim of teaching mathematical modelling at secondary level, namely to support the students' learning of mathematics the situation is different. The theoretical foundations for the potential of using modelling as a didactical vehicle for supporting the students' learning of mathematical concepts and methods are to be found not only in research on the learning of applications and modelling, but also in research on the learning of mathematics in general. In the field of research on the teaching and learning of modelling there are three frameworks, which have a particular focus on modelling as a means for supporting the students' learning of mathematics. Two of them can be characterized as having an epistemological perspective on modelling, namely Realistic Mathematics Education (RME) and The Anthropological Theory of Didactics (ATD). These are both comprehensive frameworks covering mathematics teaching and learning in general. The third framework is the Models and Modelling Perspective (MMP) and can be characterized as having a contextual problem solving perspective on modelling (Kaiser and Sriraman 2006).

In RME and ATD, modelling is subordinated amongst more general theories on the teaching and learning of mathematics. In RME the learning process is understood as the learners' dynamical reinvention of mathematical knowledge through the process of mathematisation (Freudenthal 1983). The process starts with students' mathematising their experienced reality in different contexts (horizontal mathematisation). Thereby, the students develop a foundation for acquiring the theoretical meaning of the concepts and methods used. This learning process is conceptualized in RME as vertical mathematisation. Through mathematizing with an increasing level of abstraction the mathematical concepts gradually obtain their theoretical meaning. The notion 'from model of [some particular type of situation] to model for [understanding a mathematical concept]' is essential in RME for understanding the changing role of models in the process of learning mathematical concepts (Gravemeijer and Doorman 1999).

ATD offers a general and strong theory for understanding and designing mathematics teaching. The basic assumption is that mathematical knowledge—and human knowledge in general—is developed with the aim of contributing to the answering of some specific type of questions, and that mathematics teaching therefore should identify and use as its point of departure for teaching such generating questions for the mathematical knowledge in the curricula. The didactical research under ATD therefore develops and investigates implementations of what is called Activity of Study and Research organized around a question with strong generative power (Chevallard 2011). To the degree that the questions are referring to extra-mathematical situations and contexts—and that is typically the case—the ATD approach places mathematical modelling in the centre of the learning of mathematics emphasizing modelling as a didactical means for learning mathematics rather than modelling competency as an aim.

Within MMP, the research often focuses on what it means to understand particular important mathematical concepts and methods in different situations and contexts, and on how to design modelling activities where the students can activate the pinpointed aspects of the concepts in their modelling activities (Doerr and Lesh 2011). These didactical activities are called Modelling Eliciting Activities (MEA), and a set of design principles for MEAs has been developed and tested in many projects (e.g., Lesh and Doerr 2003). Points of departure are taken in everyday real-life contexts, in meaningful contexts established in the teaching, or in authentic applications in other disciplines or professions. Emphasis is placed on the students' construction of meaningfulness in the modelling process and through related reflections in the support of the students' learning of mathematics. However, research within MMP is also concerned with developing the students' competencies in problem solving and mathematical modelling. Both ATD and MMP take a systemic approach and include the interplay between the activities in class, the teachers' activities before, under and after the teaching, the researchers' activities and the interaction between the teachers and the researchers.

As a common core, all three frameworks build on the assumption that the learning potentials of modelling lie in the fact that the students' learning of mathematical concepts can be anchored and given cognitive roots through the students' modelling activities. The students' conceptual understanding can be challenged and developed further through working with modelling and applications in a variation of contexts.

Research has long since identified, investigated empirically, and explained theoretically learning difficulties principally related to the learning of mathematical concepts. These theories are developed independent of the frameworks for modelling-based teaching described above. However, as illustrated in the sections to follow, these theories pinpoint learning difficulties, which can be brought into light and helped overcome by means of modelling activities.

In traditional forms of mathematics teaching it is indeed possible to overlook or to disregard the fact that many of the students do not learn the key mathematical concepts at secondary level. Many students complete their secondary education with a rudimentary understanding of important concepts such as rational and real numbers, variable, equation, function, rate of change, derivatives and integrals. The problems become evident at tertiary level, where the students' mathematical conceptions are too fragile to form a basis for further education with mathematics.

From a systemic point of view, full integration of modelling and application in secondary mathematics teaching requires that the drawbacks of traditional forms of teaching when it comes to supporting the students' learning of key mathematical concepts are realized and that modelling is seen as a didactical means for overcoming such learning difficulties. Therefore, it is crucial that teachers can build upon an understanding of the theories pinpointing and explaining such learning difficulties. This is, however, quite demanding, since the relevant theories are difficult for teachers to relate to their practice. Accordingly, there is a challenge for research to find ways to support the interplay between theories on the learning of mathematical concepts and the development of teaching practice in modelling.

3.2 Learning Mathematics Through Modelling in Practice

Together with colleagues, and in collaboration with teachers during the years, I have been involved in developing and researching mathematical modelling as a means for supporting students' learning of mathematics from lower secondary level (Grade 7) to early university level. Typically, the projects or in-service courses have involved the teachers' planning, teaching, and evaluating modelling lessons or courses in their own practice. In general, the aim of these activities has been to support the integration of modelling in mathematics at lower secondary (Grades 7–9 in the Danish comprehensive school) and at upper secondary level (Grades 10–12 in the Danish gymnasium). In both systems modelling is included, but not really *integrated*, in the curriculum.

These experiences show that it is much easier for teachers to work with the aim of developing the students' modelling competence than to deliberately plan for modelling activities to support the students' learning of particular mathematical concepts. Typically, during courses, the teachers make use of the modelling cycle as a tool for planning a modelling course and as a tool for analysing the students' modelling work. However, it is more difficult for teachers to see and pinpoint for the students, the learning potentials in their modelling activities and to use such situations for challenging and developing the students' conceptual understanding. In particular, the teachers often find it difficult to draw on the students' different experiences and results from modelling activities in building a shared understanding in the class of the concepts or methods involved. It is difficult for the teachers to connect the students' modelling anchored understanding to the mathematical knowledge in the curriculum.

In developmental projects and in-service courses aiming at helping teachers integrate modelling as a means for supporting the students' learning of mathematics, we have been facing the challenge of how to make better use of theories on the learning of mathematical concepts (Blomhøj and Kjeldsen 2013b).

In particular, we have used theoretical ideas related to:

- the important role of representations for the learning of mathematical concepts (Steinbring 1987, 2005);
- concept images (Vinner and Dreyfus 1989);
- the process-object duality in concept formation (Sfard 1991); and
- the previously mentioned RME notion of the development from a model *of* some type of situation to a model *for* the understanding and learning of a mathematical concept (Gravemeijer and Doorman 1999).

All these theories have proven helpful for analysing the students' learning during their modelling work; for developing and improving modelling problems; and as a resource for supporting and challenging the students during their work with modelling projects in a first-year university course (Blomhøj and Kjeldsen 2010).

However, the theories are not easy to apply for teachers. Therefore, there is a need for tools, which can help teachers to see modelling activities as a didactical means for supporting the students' learning of key mathematical concepts. An example is the schema used in the following two examples for spanning the possible use of different forms of representations of process and object aspects of key mathematical concepts involved in a modelling process in a particular context.

The divide between process and object aspects is according to the model for formation of mathematical concepts developed by Sfard (1991). Of course the schema primarily makes sense in relation to concepts, which have clear process and object aspects, but even in such cases—as indicated in the two examples below—it is not a simple task to distinguish between representations of process and object aspects of mathematical concepts—the same representation (the same signs) can often be interpreted as referring to both process and object aspects of a concept.

For a given modelling activity, the schema can be filled out a priori in order to uncover the potentials for supporting the students' work with, and sense making of, representations of the mathematical concepts involved. Also, the schema can be used to structure and analyse evidence for the students' work with the different representations and their mutual connections in actual modelling activities.

In each cell it is possible to distinguish between the concrete model or modelling situation on the one hand, and a generalised model on the other hand; that is to emphasize the possible change of perspective from seeing the model at hand as a model of a particular situation to seeing it as an emerging generalised model for understanding the mathematical concept in focus (Gravemeijer and Doorman 1999). So far the schema has been used in courses and developmental projects in relation to: the function concept, linear functions, exponential functions, the derivative concept, and the integral concept. In the following two examples the schema is used to summarise the potentials for supporting the students' conceptual learning in relation to the modelling activities.

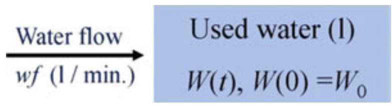
3.3 Modelling Dynamical Phenomena

Two modelling situations at Grade 8 and 9 are presented. They can be seen as possible early elements in a longitudinal teaching and learning trajectory on the modelling of dynamical phenomena by means of compartment modelling, difference (and later differential) equations supported by the use of digital technology. The two examples include key mathematical concepts such as variable, function, rate of change and integrals. Of course, the concepts of rate of change and integrals are only present in the modelling activities as contextualized intuitive ideas. However, experiences from such modelling activities can provide the students with a foundation for learning calculus, and to develop gradually competence for modelling of dynamical phenomena during the upper secondary level. Blomhøj and Kjeldsen (2010) present and discuss a modelling project, which can be seen as a possible continuation of this trajectory at university level. This section illustrates how the previously discussed theories on the learning of mathematical concepts can be used to pinpoint potentials in modelling situations for supporting the students' conceptual learning.

3.3.1 The Morning Shower

A morning shower is a rich context for modelling a simple dynamical phenomenon, namely the use of water depending on the showering time. It can be used in lower secondary mathematics teaching from Grade 7 as a context for introducing linear functions. The idea originates from a project called *Mathematical Mornings* (Blomhøj and Skånstrøm 2006). In that project, the main idea was to challenge the students to use mathematics to describe and analyse some phenomena from their everyday morning life. The objectives were to: (1) motivate mathematical work, (2) establish stable cognitive roots for the students' conceptions of basic mathematical concepts, and (3) to provide the students with experiences of mathematics as a means for describing, analysing and understanding everyday life situations.

The task for the students was to produce an A3 poster of his or her mathematical morning based on observations and data. The students were expected to make a poster each of their own mathematical morning, but they were encouraged to help each other. This idea has been used many times in later projects and in-service courses and also by other educators. In many of these courses and projects the morning shower has proven to be a fruitful modelling task for supporting the students' learning of linear functions at lower secondary level. The students are (in our privileged culture) indeed very familiar with the real situation and they can quite easily see the connection between the duration of their shower and the amount of water used. From experience, many students can make the reasonable simplifying assumption that the flow of water can be assumed constant after the shower has begun. Also, quite often students notice themselves that there is some cold water in the pipeline, which has to run off before they enter the warm shower. Many students—especially girls—also notice that the duration of their showers depend on how much time they have and whether or not they wash their hair. A typical boy's remark is: "Coming after my sister I may not need to wait for the cold water to run off, but my shower will be short since she always uses almost all the warm water." In general, the students can easily activate their everyday experiences in relation to the real situation. This is of course a wonderful opportunity for the mathematics teacher to motivate the students' modelling work: "So you need a table, a graph or a formula, to give you the amount of water used for your shower depending on its duration". Based on such assumptions and reflections it makes sense for the students to measure the water flow in their own shower and the time they wait before entering the shower (or the amount of cold (not warm) water in the pipe, W_0). From a modelling perspective, it is a central point that meaningful measuring of the specific magnitudes has a conceptual model of the real situation as its prerequisite. In subsequent teaching the situation can be represented by a compartment diagram stressing that it is the amount of used water, $W(t)$, which we want to keep track of, while increasing by means of the constant water flow, *wf*:



As illustrated in the photo in Fig. 3.1a it is not necessarily very easy in practice to measure the water flow in the shower. Students may need to repeat the measurement because they forget to measure the time needed to fill the bucket or because not all of the water goes into the bucket the first time. However, it is doable for most students, and with thoughtful planning of the course of lessons nearly all students can be expected to produce relevant data and to calculate the water flow for their own morning shower. From here they can produce tables, a graph and maybe even an algebraic representation of the amount of used water as a function of the duration of the shower. However, even if students can write an equation for a linear function modelling their shower, it is not at all necessary that they perceive this as a representation of a mathematical object.

Students can use a spreadsheet (*Excel* or *GeoGebra*) to produce a table by starting with the amount of cold water and adding the calculated water flow per minute for each consecutive minute and produce a graph by hand or by means of a spreadsheet. In this way the students work with representations of the process aspect of a linear function. Figure 3.1b shows a graph drawn using *GeoGebra* to represent the situation where a student has measured the water flow to be 6 L per minute and the amount of cold water as 3 L. The points corresponding to time of showering and water used (t, W) are calculated in a spreadsheet in *GeoGebra*, plotted and connected to form the stippled line in Fig. 3.1b. Based on concrete calculations of the amount of water used for showers of different duration, that is: 5, 10, 15 min, the students can be challenged individually to set up an equation using t (or x) as a variable for the time in minutes and W for the water used in litres: $W = 6\ \text{L/min} \cdot t + 3\ \text{L}$. Thereby, the students can gain support for developing their concept images of a graph of a function

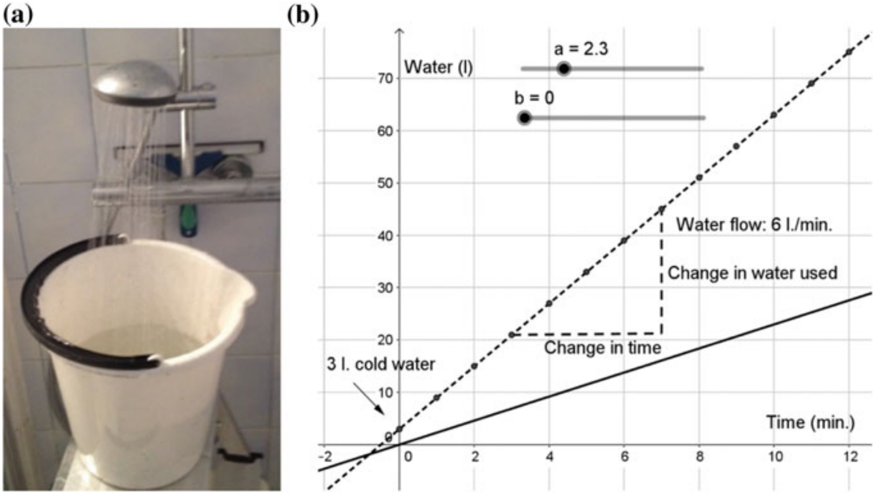


Fig. 3.1 **a** Measuring the flow in the shower and **b** representing the model graphically

as consisting of exactly the coordinate points created by the function relationship and therefore—in this case—fulfilling an equation defining the function.

The notion of students' concept images as developed by Vinner and Dreyfus (1989) pinpoints exactly the necessity for the students to work with all important aspects of a mathematical concept. It is easy to find students in Grade 10 or later, who cannot really make sense out of the fact that a given point belongs to the graph of a function in a given context. According to Vinner and Dreyfus, the explanation is simply that this relation is not part of their concept image of a function. In traditional mathematics teaching, it is possible to learn about functions and be able to draw graphs of functions and solve standard tasks, without understanding important connections between different representations of the function concept.

Individually, and in subsequent teaching for the whole class, the students can be challenged to write the general equation for a straight line shown in *GeoGebra*, $y = ax + b$, where the value for the parameters a and b can be changed by means of sliders as shown in Fig. 3.1b. By developing and experimenting with such an interactive sketch, the students can experience that the straight line through their points has a unique representation by the parameters a and b. These can then be given their natural interpretations as the slope of the line (and in the real situation as the flow of water or the rate of change measured in litres/minute) and as the intersection with the y-axis (the cold water, the initial value of the state variable in litres).

Through such activities, the students can obtain support for taking the reification step in their formation of their concept of a (linear) function (Sfard 1991). They can experience a linear function as a representation of the process of calculating the amount of used water for a variation of the time of showering and see how this

	Natural language	Diagrammatic	Numerical	Algebraic	Algorithmic	Graphical
Process	The shower as a process and the meaning of wf and W_0	The compartment diagram	t | 0 | 1 W | 3 | 9 wf | +6	$W(t{+}1) = W(t) + 6$; $W(0) = 3$	Spreadsheet representation of $W(t{+}\Delta t)$	A plot with corresponding points
Object	Shower duration and used water relation and the meaning of wf and W_0	Not in use	Function table possibly with $\Delta t = 0.5$ for $t \in [0,15]$	$W(t) = wf \bullet t + W_0$ *or* $W(t) = 6t + 3$	A DG or Spreadsheet representation of $W(t)$ with wf and W_0 as parameters	A graph of the function $W(t)$

Fig. 3.2 Representations of a linear function in the *Morning Shower*

function can be seen as a mathematical object with certain qualities and different representations. As pinpointed by Sfard, this is a crucial step in the formation of mathematical concepts in general. Students' modelling work in situations that make sense for them has the potential to support this essential step in the concept formation process.

Through their modelling of the morning shower situation, students can construct a stable cognitive root for the concept of function as a process connecting to variables as well as a mathematical object, which can be represented in different forms: natural language, a diagram, a numerical table, an algebraic equation, an algorithmic representation, and a graphical representation. Thereby, the epistemological triangle (Steinbring 1987) for the concept of function can be spanned in different ways all referring to the morning shower situation with which the students have concrete experiences.

Moreover, through such activities the students may develop a model for understanding linear functions in general, that is, change the perspective from working with a model of some real situation to see the model at hand as a means for understanding a mathematical concept (Gravemeijer and Doorman 1999). In this case, it is even relevant to talk about the embodiment of the slope of a straight line and the rate of change, since the students after having modelled their shower, now have the power to change the slope of the line—within certain limitations of course—by turning the flow of water up or down during their showers. Such experiences provide a strong cognitive root for learning the concepts involved.

Figure 3.2 shows different forms of representations of a linear function that might come into play in the *Morning Shower* modelling activity.

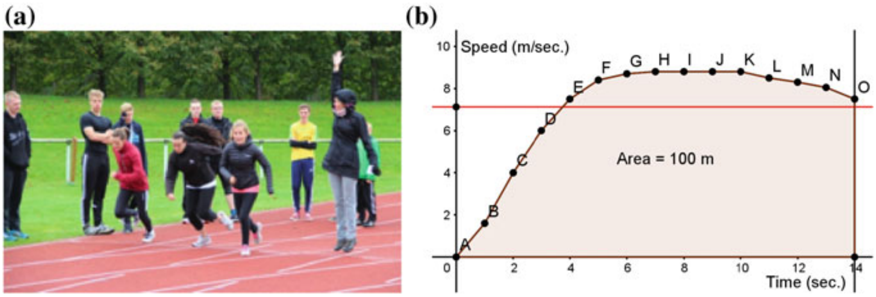


Fig. 3.3 **a** A 100 m sprint and **b** speed graph in *GeoGebra* for a sprint in 14 s

3.3.2 The 100 m Sprint

The second example has its origin in a project where students in Grades 8–10 had the physical experience of running a 100 m sprint (see Fig. 3.3a). All should have had their individual time recorded. The aim of the activity is for each student to model his or her personal sprint in terms of describing mathematically how the speed and the distance changed during their sprint. The learning aim is to support the students' understanding of speed as the rate of change of the distance and establish a cognitive root for understanding the integral concept. The students' activity can be framed by the following connected tasks:

(1) Calculate your average speed in metres per second for your 100 m sprint. Make a coordinate system in *GeoGebra* in which you can depict your speed in m/sec from the time you started to the end time of your sprint.

(2) Start by imaging that you had run with the same speed from the start to the end. Draw the speed graph for that situation. (Of course, you did not, since you stood still at the start of the race otherwise, there would have been a false start.)

(3) How can you calculate the distance that you have run from this graph? (Of course, you already know that it is 100 m.)

(4) Assume that you have run with constant acceleration starting from still. Draw the graph for your speed from the start to your end time in this situation. With what speed would you have crossed the finish line in that situation? Is that a realistic possibility?

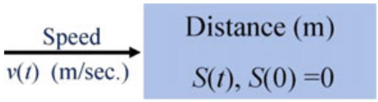
(5) Use the spreadsheet in *GeoGebra* to fill in for each second during your sprint, your best estimate for your speed at that exact moment of your sprint. Draw this speed graph in the same coordinate system. Adjust your speed estimates so as you reach the 100 m in the time from your real sprint.

From their experiences running the sprint and supported by their answers to the tasks (1)–(4), Grade 8–10 students can make reasonable estimates for their speed during their sprint in task (5). By making use of the dynamical interplay between the spreadsheet, the graph window in *GeoGebra*, and the calculation of the area under the speed graph, the students can adjust their estimated speed for each second during their sprint, so it fits with the 100 m in exactly their time.

The graph for a case where the 100 m sprint is done in 14 s is shown as a *GeoGebra* figure in Fig. 3.3b. The figure also includes the graph for the constant speed equal to the average speed of 7.14 m/s as the horizontal line. The points A, B,…, O are taken from the spreadsheet in *GeoGebra* and the area (the distance) is calculated dynamically by means of the tool ‘Area of polygon’. If a point is changed, say point E (4, 7.5) to (4, 6.5), the area will change from 100 to 99 m because 1 m/s for 1 s is missing in order to reach the 100 m in 14 s.

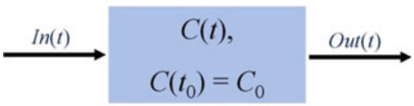
These features in *GeoGebra* enable the students to estimate realistic speed graphs for their sprint and to experience the direct relationship between the speed in each second and the distance (the area) covered.

The situation can also be represented by means of a compartment model:



The compartment model of the 100 m sprint can be seen as a generalization of the model for the shower, since in this case the rate of change is a non-constant function. The speed function, $v(t)$, cannot be given by a simple algebraic expression. However, from the students’ experiences with the real situation it is conceptually clear for most students that their 100 m sprint can be modelled by a speed function represented by a graph. Hence, the modelling of the 100 m sprint can contribute to the extension of the students’ concept image of a function to include relationships, which is not defined by an algebraic expression. In addition, this activity has the potential for developing into a “model for” the students’ understanding of how the distance is determined by the speed and how it can be calculated by means of summing up (integrating) the speed.

The compartment approach to the modelling of dynamical phenomena can be continued in upper secondary level and at university level supporting the students’ modelling competence as well as their mathematical understanding. The compartment representation provides—at least for some students—a foundation for understanding the fundamental theorem of calculus:



$$C'(t) = In(t) - Out(t); C_0 = C(t_0)$$

$$C(t_1) = C_0 + \int_{t_0}^{t_1} In(t) - Out(t)\,dt$$

The net rate of change for a compartment is the sum of inflows minus the sum of outflows. The level of a compartment at time t is the value at previous t_0 plus the integral of the net rate of change from t_0 to t. Together with experiences from modelling dynamical phenomena as in the two examples of the compartment formulation constitutes an intuitive explanation of the fundamental theorem of calculus. Hereby, it is also indicated how a longitudinal learning trajectory within the modelling of dynamical phenomena by means of compartments and difference equation can help

	Natural language	Diagrammatic	Numerical	Algebraic	Algorithmic	Graphical
Process	Description of the change of speed during a sprint	The compartment diagram for the change in distance	t | 0 | 1 v | 0 | 1.8 S | 0 | 0.9 Table with units, $v(t)$ could be estimated and/ or measured	$S(t{+}1) = S(t) + ½(v(t{+}1){+}v(t))$ with $v(0) = 0$ and $S(0) = 0$	*GeoGebra* representation of $v(t)$ and $S(t)$ enables estimation of $v(t)$ to yield 100m in the right time	Dynamic graphs of $v(t)$ as in Fig 3.3b and $S(t)$
Object	Description of the general pattern of the time – speed relation	Not in use	The function table for $v(t)$ and $S(t)$ for the whole sprint	Only relevant for constant or linear time-speed relations. At higher levels actual time-speed relations could be fitted and integrated to yield $S(t)$	Representation which enables calculation of $v(t)$ (interpolation) and $S(t)$. $S(t)$ as area function for $v(t)$	Corresponding graphs for $v(t)$ and $S(t)$. $S(t)$ as area function for $v(t)$

Fig. 3.4 Representations of the time-speed-distance relationships in *The* 100 m *Sprint*

overcome learning difficulties connected to key mathematical concepts (Blomhøj and Kjeldsen 2010, 2013a).

The possible representations of the process and object aspects of the relationships between time, speed and distance, which can come into play in this modelling activity are summed up in the schema in Fig. 3.4.

3.4 Conclusion

Mathematics education research has a lot to offer for helping the integration of mathematical modelling in secondary mathematics teaching. In general, the theory-practice relation stands differently with respect to the educational aim of developing the students' modelling competencies and the aim of supporting the students' learning of mathematics through modelling activities respectively (Blomhøj and Ärlebäck 2018).

With regard to the objective of developing the students' modelling competencies research is already quite well aligned with the development of practice. Projects and courses in classrooms are already to some degree based on theoretical notions and ideas developed in research. Of course, there are still challenges for research, such as conceptualising—also for assessing—the students' progress in modelling competency.

In order for modelling and applications to be fully integrated in secondary mathematics teaching, modelling should also be seen and understood as a didactical means for supporting the students' learning of mathematics. Here also, theories are available in the form of the frameworks mentioned and in the form of theories on difficulties related to the learning of mathematical concepts. These theories can serve as a basis for the justification of modelling as an integrated element in secondary mathematics teaching in curricula reforms. The theories also provide a basis for designing teaching, where modelling is used as the didactical means for supporting the students' conceptual learning. However, in order to be helpful for teachers the theories need to be concretized and re-contextualised in developmental projects or in-service education. The schema of representations shown here is one example of how research can develop tools, which can help researchers and teachers to connect theories on the learning of mathematical concepts to concrete modelling activities. Designing and investigating longitudinal learning trajectories for the students' learning of important mathematics through modelling activities as illustrated above could be another possible research approach for furthering integration of modelling in secondary mathematics teaching.

Of course, even though research may provide a necessary basis for the integration of modelling in curricula and in teaching practices at secondary level, it is not in any way sufficient for ensuring the integration in practice. As pointed out by Blum (2015), systemic approaches are needed in order to really support the integration of modelling in the practice of secondary mathematics teaching. Here, a main challenge for research is to develop and test methodologies for collaboration between researchers and teachers in various institutional contexts. Araújo (2019) develops and discusses a framework for a dialectical relationship between pedagogical practice and research in mathematical modelling, which may serve as a basis for developing such methodologies.

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Chapter 4
Real-World Task *Context:* Meanings and Roles

Jill P. Brown

Abstract This chapter details results of a study intended to increase understanding of the myriad meanings of real-world task context in mathematics education and their relevance to modelling. The research aim was to ascertain how context is viewed within the broader mathematics education community. Data analysis reported here followed an examination of use of the terms: context, task context and real-world in four mathematics education journals. Four samples, one from each journal, two in 2014 and two in 2017 where all papers using the term *real-world*, comprised the purposive sample used for the in-depth investigation. Whilst, often not defined by the authors, in most papers the context was *real-world task context* and, in the majority, this played an *essential*, rather than *incidental*, role.

Keywords Real-world · Context · Task context

4.1 Introduction

> That applications and modelling have been, and continue to be, central themes in mathematics education is not at all surprising. Nearly all questions and problems in mathematics education, that is questions and problems concerning human learning and the teaching of mathematics, influence and are influenced by relations between mathematics and some aspects of the *real world* [emphasis added]. (Blum et al. 2007, p. xii)

Within the mathematical modelling and applications community, the term *context* often implies a *real-world context* is being assumed. Blum et al. describe this extra-mathematical world as including the broad contexts of “the world around us,” “everyday problems” and “preparing for future professions” (p. xii). However, such a meaning is not always evident both within and beyond this mathematics education community. In mathematics education research ‘context’ has an even greater variety of meanings—explicitly stated or not. Boero (1999), in the guest editorial for an

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ESM Special Issue on 'Teaching and Learning Mathematics in Context', noted the varied meanings of the term and, in particular, *situation context* or context for "learning, using and knowing mathematics" (p. 207) versus *task context* as articulated by Wedege (1999) as "representing reality" (p. 206). Boero describes the former as "workplace, classroom social context, computer learning environments, etc. … [and *task context*] as everyday life situations evoked in a problem-solving task" (p. vii). Wedege described *situation context* as circumstances (historical, social, psychological, etc.) in which "something happens, or, … is to be considered" (p. 206). Busse and Kaiser (2003), writing within the modelling community, describe context as "a rather nebulous concept, used by many authors in different meanings and ways" (p. 3) although its importance was not in question according to these authors. Whilst the importance of situation context is acknowledged, the focus in this chapter is task context.

In characterising the relationship between task context and the real-world, Stillman (1998) distinguished "three levels of embeddedness of context" (p. 246). These describe the extent to which the real situation remains as the situation is simplified for use in the classroom. She describes three types of problems where this embeddedness varies from almost non-existent to pseudo-real to real and the problem can be characterised as *border*, *wrapper* or *tapestry*. In *border* problems, the mathematics and task context are entirely separate. The real-world context can be ignored by the task solver. Knowing about the context is of no help to understanding or solving the problem or interpreting or validating the solution. In *wrapper* problems, the task solver must engage with the real-world context to 'find' the mathematics which is hidden within the context. Beyond that, the real-world can be ignored, or discarded as only the mathematics is needed for solving (Stillman 1998) although context can be used for checking if a solution makes sense. The third level, *tapestry*, occurs when the real-world task context and mathematics are interwoven, and task solvers need to move between the two continually crossing the boundary between the real-world and the mathematical world (Stillman 1998) throughout the solution process.

Context is often claimed to help learning, usually via fostering active engagement (Stillman 2004). A recent large study of year 2 students (50 schools) in Alaska showed that implementation of "the reform-oriented and culturally based Maths in a Cultural Context (MCC) teacher training and curriculum … significantly improved students' mathematical performance" (Kisker et al. 2012, p. 74). Previously, Langrall et al. (2006) examined the role of context knowledge in solving statistical tasks by Grade 6 Australian students, finding several important uses by students including supporting their interpretation of the data and in taking a critical stance to the data.

Smith and Morgan (2016) reviewed curriculum documents from 11 jurisdictions to ascertain the relationship between the real-word and school mathematics. They identified three orientations to real-world contexts in mathematics, a tool for everyday life, a vehicle for learning, and engagement with the real-world motivating learning. In four jurisdictions, a single main pathway was followed with variation in speed and extent of progress. However, in the other seven jurisdictions alternative pathways were offered, "with the less [mathematically] advanced pathways having a stronger emphasis on real-world contexts" (p. 42) including in assessment tasks. In these jurisdictions, if mathematics is seen as a tool for everyday life then why is this given less emphasis for students studying more advanced mathematics? If the purpose was as a vehicle for learning, or for motivation, then why is there less focus on real-world contexts in the years of schooling prior to students needing to select or embark on particular pathway options? As Smith and Morgan noted, changing the emphasis for different year levels or by nature of mathematics studied conflicts with all three of the espoused purposes.

Others claim or posit that use of real-world contexts can, or may, hinder understanding. Dapueto and Parenti (1999) note that students may face extra challenge "in relation to knowledge of the context" (p. 15). Wroughton et al. (2013) add it might be distracting to students in a statistical sampling context, whilst Zevenbergen et al. (2002) claim "there is considerable cause for concern when such a strategy [the use of contexts in school mathematics] is used simplistically" (p. 8). Cooper and Dunne (2000) have suggested that students from working class backgrounds can be misled by school mathematics questions set in everyday contexts because they misread the task as calling for an everyday response. They suggest middle class students tend to ignore the context and focus on the (esoteric) mathematical calculation required. Wijaya et al. (2014) reported that 38% of errors made by Year 9–10 Indonesian students on released PISA items were related to "understanding the context-based task". Huang (2004) found 48 Grade 4 Taiwanese students were more successful on tasks related to unfamiliar context than familiar contexts, and (perhaps not surprisingly) took longer to solve tasks with familiar contexts, suggesting that unfamiliar contexts are ignored whilst familiar ones take more time to make sense of.

The *ICMI Study on Modelling and Applications in Mathematics Education* was held in 2004 with Niss et al. (2007) suggesting the Study might "formally mark the maturation of applications and modelling as a research discipline in the field of mathematics education" (p. 29). Niss et al. define applications as being when mathematics is applied to some aspect of the extra-mathematical world for some purpose including "to understand it better, to investigate issues, to explain phenomena, to solve problems, to pave the way for decisions, …. The term 'real-world' is often used to describe the world outside of mathematics" (p. 3) and this can be in another school subject or related to personal or social issues.

The purpose of this chapter is to analyse how 'real-world' context is used or understood 'today', given 10 years have passed since the study volume was published. To achieve this, the author sampled leading mathematics education journals to ascertain what these meanings are and their purposes for different researchers. The overarching research question that is the focus of the study is: How is context viewed in the broader mathematics education community as evident in research publications? More specifically, this entailed answering for each published paper: *What are the meanings and roles of real-world task context in the learning of mathematics according to mathematics education research?*

4.2 Method

Document analysis is an analytical qualitative research method requiring "data be examined and interpreted in order to elicit meaning, gain understanding, and develop empirical knowledge" (Bowen 2009, p. 27). It can be used to complement other methods or as a sole method. In this study, the intention is to better understand how context is used in research reported in journal publications so document analysis will be used as a stand-alone method. As with all qualitative research data, "detailed information about how the study was designed and conducted should be provided" (p. 29) as will be the case here.

4.2.1 Journal Selection

In attempting to ascertain the view of context in the mainstream mathematics education research community, a review of literature was called for with a reliable method for choice of sample. Noting the variety of ways to assess the quality of academic journals (e.g., acceptance rates, prestige of editors, citations), Nivens and Otten (2017) used two journal metrics (Scopus's SCImago Journal Rank and Google Scholar Metrics h5-index) to compile a ranking of 69 mathematics education journals, after discounting Web of Science's Impact Factor as few mathematics education journals are in the relevant database. The journals considered explicitly focused on mathematics and/or statistics education. This metrics approach overcomes some limitations, such as personal opinion in earlier work by Toerner and Arzarello (2012) who compiled a ranking after surveying experts in the field.

Nivens and Otten (2017) found reasonable agreement that the top eight mathematics education journals are: *Educational Studies in Mathematics* (ESM), *International Journal of Science and Mathematics Education* (IJSME), *Journal of Mathematical Behavior* (JMB), *Journal of Mathematics Teacher Education* (JMTE), *Journal for Research in Mathematics Education* (JRME), *Mathematical Thinking and Learning* (MTL), *School Science and Mathematics* (SSM), and *ZDM: Mathematics Education* (ZDM). However, the ranking within these is less clear, although ESM was ranked in the top two in both. Six journals were in the top seven by both measures, with JRME first and fourth. MTL was in the top seven on one list but does not appear on the GSM ranking with too few papers (<100 papers in 2011–2015). These eight journals formed the original list considered for sampling and analysis.

From these journals, two were eliminated from the analysis on the basis of their focus being broader than mathematics education or having a narrower focus eliminating IJSME and SSM that include science education, and JMTE which focuses on mathematics teacher education. A fourth journal, ZDM, was eliminated on the basis that, unlike the other journals access to authors is by invitation only. Thus, a selection of four journals was determined. As ESM and JRME are the oldest journal in the sample, it was decided to begin with these and use that analysis to inform the subsequent analysis of JMB and MTL.

4.2.2 Initial Analysis

A text content search for each journal was undertaken electronically using the proprietary/available search engine for the terms, *context, task context*, and *real-world*. For ESM this was via Springer Link (1968–2017), JRME via JSTOR (1970–2017), MTL via Taylor and Francis Online (1999–2017), and JMB via Science Direct (1995–2017, i.e., not available for all years of publication). In addition, data about the number of papers published was also collected.

4.2.3 Detailed and In-Depth Analyses

It was decided to begin with an in-depth analysis of ESM. As 2014 was a decade after the ICMI Study on Modelling and Applications in Mathematics Education was held, it was deemed appropriate to use 2014 for an in-depth study of ESM and JRME, noting the former is based in Europe and the latter in USA. Coincidentally, 2014 provided the largest sample possible from ESM which was then used to inform the subsequent analysis. This was followed with a 2017 sample, a decade since the Study Volume was published, in each of MTL and JMB, more recently established journals, providing the most recent samples possible.

Purposeful sampling was adopted in order to find information-rich cases rather than representative cases (Patton 2002). For the years targeted, for each journal, papers that included search items context, task context and real-world/real world were selected for detailed analyses highlighting these focuses. Each paper was read in full.

Following, the detailed analyses of papers, a further in-depth analysis followed to produce an analytical summary matrix (Miles et al. 2014). Firstly, papers were classified by *type*—theoretical, commentary, document analysis, or research-based. Secondly, the paper *focus with respect to the context* being situational or a task context, or in some cases other (culture/religion) was ascertained. Thirdly, for papers with a task context focus, the *role of the real-world* was classified as incidental, pseudo-real or essential. Finally, where the role of the real-world task context was classified as essential, this was then further classified as being of a minor or major focus. Where actual tasks were included, the degree of *embeddedness of the task context* was categorised as border, wrapper or tapestry. A summary of this analysis process is presented in Fig. 4.1. The purpose was to facilitate assessing of the sub-questions by journal before aggregating into a meta-analysis across samples in Sect. 4.7.

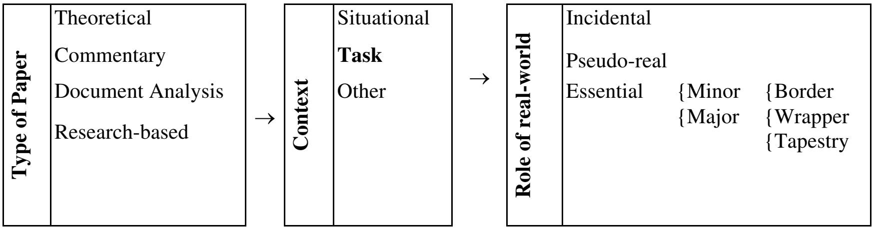

Fig. 4.1 Overview of analysis process

The role of the real-world task context was categorized as *incidental* when (i) it was one of many considerations of study either related to data collection or analysis, (ii) it was a natural part of the mathematics focus (e.g., speed) or (iii) it arose in the findings. The role of the real-world was described as *pseudo-real* when the task solver had to "suspend reality and ignore common sense" (Boaler 1994, p. 554). The role was categorised as *essential* where it played an important part in the study. However, as this importance varied, two levels *minor* or *major* were used to distinguish between being essential in the study but of low importance to being not only essential, but also intrinsic to the study. All PISA-related studies were categorized as major as the intention of PISA (even if disputed) is to assess students' mathematical literacy in a variety of contexts, which are mainly real-world contexts.

For papers where task context was *essential* the embedding of the real-world in the task context was characterized, following Stillman (1998), as *border*, *wrapper* or *tapestry*. Where multiple tasks were presented, there may have been a range of embeddedness across different tasks. For PISA-related studies, the degree of embeddedness may vary across all levels from task to task, so these papers were excluded from this level of analysis. Although Stillman's (1998) characterisations of contextualization were developed to describe more substantive tasks than appear in some of the literature surveyed, it was apparent they would be useful in distinguishing differences in the task contexts identified in the literature.

4.3 Content Analysis: ESM

4.3.1 Initial Analysis and Sample Selection

During 1968–2017 (Volumes 1–96), 2277 papers were published in ESM but the number published per volume and year varies (average 27 issues/year). The search for *context* identified 1566 papers in these years, and 595 papers in the years 2008–2017 (i.e., post 2007 ICMI Study Volume publication). Not surprisingly, there was a greater frequency of the term *context* (see Tables 4.1 and 4.2) with many uses of the term context referring to an *educational context* or *social context* (as will be discussed) rather

Table 4.1 Occurrences of search terms ESM

Search term	All years	Years by 'decade'				
	1968–2017	68–77	78–87	88–97	98–07	08–17
Context	1566	97	166	266	443	595
Task context	1277	51	120	216	373	515
Real-world	390	73	40	63	95	119
No. of papers	2277	350	322	402	531	672

Table 4.2 Search terms by year (last eight years) ESM

Search term	2010	2011	2012	2013	2014	2015	2016	2017
Context	46	57	68	79	66	59	60	60
Task context	39	44	60	73	56	48	51	58
Real-world	7	10	14	17	20	13	13	6
No. of papers	56	65	68	86	71	75	77	66

than a *real-world or extra-mathematical context* as may be expected in mathematical modelling or application specific literature.

The rate of use of the terms *context* and *task context* have steadily increased in ESM since 1968. This is evident even when the increase in papers per year and the variation in the number of papers per year are accounted for. Similarly, the term *real-world* has shown a generally increasing trend, although its use, and rate of increase are much lower. The search results for real-world resulted in 390 instances of the term *real-world* for the years 1968–2017, and 119 for the last decade (2008–17). Not only is 2014 one decade on from the ICMI Study, but also it has the maximum number of results for the term, which tail off after this year. Twenty papers (E1–E20) were in the sample. (See Appendix in electronic supplement for full details of papers sampled.)

4.3.2 Detailed Analysis

Initial exploratory analysis considered the country of author and location of study. Authors were based in 13 different countries with studies based in 11 different locations showing that the author demographic was not Euro-centric, despite the location of the publishing house. Keyword analysis showed none of the papers had real-world or context as a key word. Keywords suggestive of real-world contexts were Critical mathematics (E3), Medication dosage calculation problem-solving (E5), Drug errors, (E5), Authentic (E5) and perhaps Map tasks (E10), PISA or Mathematics literacy (E1, E7, E14), and In and out of school (E18). This should indicate a note of caution for content analyses that only look for key words.

In nine papers, the *real-world* was mentioned only once, six papers contained 2–3 mentions and in the remaining five papers 4–7 occurrences were found. The number of mentions of the term was however, not sufficient, to determine the emphasis or importance of the real-world in the paper, as is illustrated by the papers of Bratlinger (E3) and Roth (E16), both with only one mention. Bratlinger's study of high school students excluded from mainstream schooling, emphasised the real-world as he focuses on how critical mathematics, especially through classroom discourse patterns, can increase student awareness or understanding of factors impacting on their lives, that is their lived real-world. Similarly, with a significant focus on the real-world, Roth (E16) highlights the disparities between mathematics in the workplace (the real-world) and school mathematics as he reports an ethnographic study involving apprentice electrical engineers. Approaches to mathematics of conduit bending in the field, using rules of thumb, were distinctly different from trigonometry approaches in the apprentice classroom although both locations were guided by the country electrical code.

In contrast, in other papers with few mentions, the use of real-world was almost incidental, as expected. In E12 the real-world was used only to differentiate between using dynamic digital artefacts to solve abstract algebraic exercises and describing real world relationships. Similarly, in E11 McCloskey argues that the rituals of performing in school mathematics are sometimes distinct from ways of performing mathematics in the real-world. Whilst important, this received little attention in the paper. In a study of Year 7 Spanish students, the E17 authors describe a 'realistic context' of a breakfast held in the school gym with students to be seated on chairs in rows of equal length. The upper stream class students are described as using "a real world context that was exchanged for mathematical meanings". Clearly, the real-world was not needed to make sense of the task, nor was the solution reviewed in light of the real-world situation.

With similar tenuous links to the real-world, Jiang et al. (E7) analysed responses to test items by approximately 350 Grade 6 students, from China and Singapore. The use of speed was said to be, in part due to its connection between the mathematics and real world. Two questions are shown here:

> Q1. A man drove at 72 km/h for 2 h, then the distance he travelled was ______km.
> Q9. On Sunday, Judy went to see her grandma who lives 150 km away. After cycling at an average speed of 15 km/h for a few hours, she got tired and took a lift from a passing truck. The truck's average travelling speed is 75 km/h. When she got to her grandma's house, she checked the time and knew that the trip took her 6 h. Find the time she cycled.

These tasks raise questions of task authenticity. Palm (2006) describes authentic tasks as those representing a real-life situation or problem, whilst Van den Heuval-Panhuizen (2005) argues authentic tasks (should) require students to think about, or imagine themselves in, the context. For Q1 the real-world could be used for checking. For Q9, we ask—is it realistic for Judy to plan a 150 km bike ride to visit her grandma? Perhaps it is in China. Certainly, in Singapore a country with approximate ‘dimensions’ 50 km East to West and 27 km North to South and a coastline of 193 km (source: Wikipedia), it is not. A third task where distance to a bookshop was 72 km was similarly not realistic in Singapore.

In contrast, two papers had the maximum of seven explicit references to the real-world (E6, E8). Ding and Li (E6) undertook an analysis of how the distributive property is presented (319 instances) in a Chinese textbook series. Their main focus was on ascertaining how the transition from concrete (physical or visual) to abstract occurred. They claim activating real-world knowledge or experiences can increase solving and sense-making opportunities but warn “perceptually rich but irrelevant information may distract learners’ attention or may be interpreted as an essential part of the intended concepts” (p. 103). The authors convey their view of ‘real world contexts’ in mathematics as being dispensable. For example (p. 107):

> Find the total cost for five jackets priced at ¥65 each and five pants priced at ¥45 each. The textbook provided two solutions $(65 + 45) \times 5$ and $65 \times 5 + 45 \times 5$ to this word problem, which together illustrated the distributive property $(65 + 45) \times 5 = 65 \times 5 + 45 \times 5$.

The context is irrelevant to the task solution and its use as a *border* (Stillman, 1998) can simply be ignored and the solution is not related to cost of clothing. In E6, the use of context was generally limited to introductory tasks and portrayed very much as allowing initial activation of student knowledge and as a necessary but minimised means to accessing abstract representations of the mathematics, seen as the aim of learning.

A distinctly different view of the real-world is presented in E8. This theoretical paper is a critique of PISA. Kanes et al. argue that whilst the domain of Mathematical Literacy highly values the real-world, a student who drew on additional knowledge of the real-world, outside that provided in the question item, would receive no credit and this is contrary to what PISA claims to assess. This paper resonates with the perspective of Andrews et al. (E1) who suggest that the reason Finnish students perform well on PISA, compared to TIMMS results, is not due to an increased emphasis in teaching and learning using real-world context, but rather to students’ high literacy skills allowing them to interpret what a question is asking and undertake the required calculations.

Cleary, frequency of use of the term *real-world* was no indicator of its importance or role in the papers sampled.

4.3.3 In-Depth Analysis of the ESM Sample

A summary matrix of the in-depth analysis for the ESM 2014 sample is presented in Table 4.3. Column one presents the *type* of paper, column two identifies each paper and its *context focus*. The final column classifies the *role of the real-world* for those

Table 4.3 Context focus of sample papers and categorization of task contexts (ESM)

Type of paper	Paper (context focus)	Role of real-world task context
Theoretical (4)	E2 (RW tool for analysis, mainly situation) E11 (Situation context) E12 (Situation, using digital artefacts to bridge RW and abstract MW) E19 (Situation/historical, calculus to solve RW tasks)	– – – –
Commentary (1)	E15 (Critical commentary, situation context)	–
Document analysis (3)	E6 Text book (Task context, concrete (incl. RW) → abstract)	Minor: Border
	E8 PISA (Task context, challenging authenticity of PISA)	Major: PISA
	E9 Policy (Situation and task context—curriculum focus (PS/MM/skills) impacts task type/context)	(Incidental)
Research-based (12)	E1 (Task context, based on PISA)	Major: PISA
	E3 (Task context, critical mathematics)	Major: Tapestry
	E4 (Cultural context—religion)	–
	E5 (Task context, medicine dosage)	Major: Tapestry
	E7 (Task context, speed)	Minor
	E10 (Task context, RW application of way/path finding—'navigation of map tasks')	Pseudo-real
	E13 (Task context, using RW to illustrate concept (∟—plumb bob—Pythagoras teaching experiment)	Minor
	E14 (Task context, PISA based, graphical items)	Major: PISA
	E16 (Task context, conduit bending, classroom v workplace)	Major: Tapestry
	E17 (Mainly situation—found use of RW part of discourse expectations for high ability students)	(Incidental)
	E18 (Task context, RW of leisure/work DARTS amateur/professional)	Major: Tapestry
	E20 (Task context, RW 1 of 2 dimensions in lesson observation tool)	Minor

Note In E9 and E17 the real-world was mainly situational, but there was some incidental real-world task context focus

papers identified as having a task context focus and where this is essential (minor or major), a further categorisation by the embeddedness of task contexts presented by authors.

As shown in Table 4.3, the twenty papers were theoretical (4), commentary (1), document analyses (3), and eleven had a task context focus. In most papers, the term context was not defined, but its meaning, as operationalised by the author(s), could be inferred. In eight papers (all four theoretical papers: E2, E11, E12, E19; the commentary paper: E15; one document analysis: E9, and two research papers: E4, E17), context referred exclusively, or mainly, to a situation context (including digital, historical and cultural environments) rather than to a task context, even though the sample was selected based on the term real-world. In E4, the real-world focus was religion or culture.

In the remaining 12 papers (two document analyses and 10 research), for one the task context was *pseudo real* (E10), and for the remaining 11 it was *essential*. For four of the essential, the real-world task context had a *minor* focus. In E6 (document analysis) the role of the real world was classified as *border* and for the remaining three research papers (E6, E13, E20) the embeddedness of the real-world was unable to be further classified as actual tasks were not provided. The additional seven papers had the real-world as a *major* focus. Three of these focussed on PISA tasks (E1, E8, E14) and four (E3, E5, E16, E18) used *task context* as tapestry.

Three of the four studies where the task context was *tapestry* related to the world of work or leisure (drug dosages in nursing, conduit bending in electrical work, and dart scoring). All focussed on learning mathematics in vocational education. The fourth study was a teaching experiment from a reformist critical mathematics perspective where active engagement with 'real' mathematics by students was viewed as partly empowering marginalized students.

With respect to context being seen as a help or a hindrance, no study claimed it to be a hindrance. Some authors (e.g., E9) in their literature reviews presented previous claims to this effect, but none did so as a result of the study being reported. For example, the authors of E9 cited research by Cooper and Dunne (2000) (see Sect. 4.1). Others, such as E14 noted that success rates on more challenging questions are lower than on less challenging questions, as one would expect. Level of challenge directly correlated with the degree of contextualization or interaction of task solver with the context.

4.4 Content Analysis: JRME

4.4.1 Initial Analysis and Sample Selection

Journal for Research in Mathematics Education was first published in 1970 with one volume per year until 1997 (Vol. 28) with six issues. Since 1998 there have been five issues published each year. A search for context, task context, and real-world identified 906, 582, and 241 instances, respectively, over the life of the journal. For 2008–2017 (i.e., *post ICMI Study Volume*), the same search terms resulted in 244, 153, and 53. See Tables 4.4 and 4.5 for additional data.

Table 4.4 JRME frequency of search terms overall and by decade

Search term	All years 1970–2017	Decades[a]				
		70–77[a]	78–87	88–97	98–07	08–17
Context	906	49	129	262	222	244
Task context	582	30	81	168	150	153
Real-world	241	17	45	122	75	53
No. of papers	2121	320	505	508	405	383

[a]*Note* 1970–1997 is less than a decade as decades calculated from 2017 back in time

Table 4.5 JRME frequency of search terms by year for recent years

Search term	2010	2011	2012	2013	2014	2015	2016	2017
Context	22	22	26	30	23	21	21	20
Task context	14	15	17	19	15	14	14	14
Real-world	4	4	6	10	7	3	4	7
No. of papers	38	31	40	45	40	34	37	36

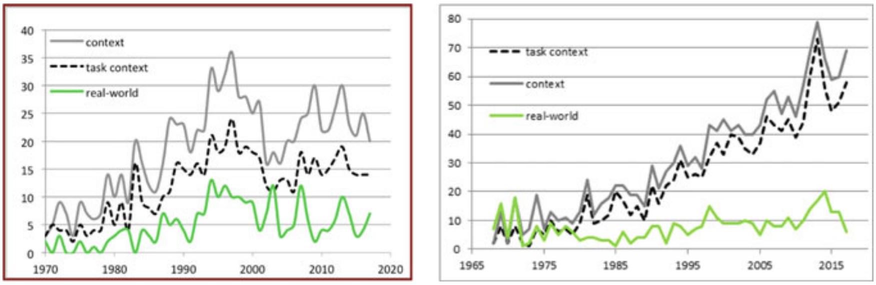

Fig. 4.2 Occurrence of search terms per year in JRME (left) and ESM (right)

Many patterns identified in the ESM data were not reflected in the JRME data. The term *context* was more frequent than *task context* and since the mid-1990s neither show evidence of the general increasing trend evident in the ESM data. Further patterns can be seen in Fig. 4.2 and clearly compared to the data from ESM. Note the vertical scale for JRME is half that for ESM. There is an increased use in terms *context* and *task context*, but substantially lower in JRME than in ESM. For the term *real-world*, both samples show a similarly low rate of increase over the journal history (approx. 0.13 per year based on linear trend line).

In 2014, the search for *real-world* identified seven papers from a total of 23 papers (30%, excluding book reviews). This proportion was similar to that of ESM (28%). Seven JRME papers were sampled (J1–J7) (see Appendix).

4.4.2 Detailed Analysis

In contrast to the geographical diversity shown by the study and author location in the ESM sample, in JRME 2014 six of the seven papers were written by authors based in USA (15 authors plus the NCTM committee of six with Lesh, a total of 22) and the research of US students or teachers. The remaining paper was written by two German researchers reporting a study of German secondary students. Hence, at least in the selected sample, the JRME data are almost exclusively from and about the USA. None of the papers had *real-world* or *context* as a key word. The only key words suggestive of real-world contexts were mathematical models, statistical models, and modelling—all in J7. The number of mentions of *real-world* was low (1–3), except for J7 with 38 instances.

Larsen et al. [J2] use the term real-world to describe the university environment where four IBL courses in which the students were taught, as they argue research-based student-centred learning can be the reality at universities [situation context]. Similarly, in J3 Mesa et al. provide a commentary on problems of mathematics instruction at US community colleges and note the disconnect between learning in class and real-world experiences with concepts. Munter [J5] details an interview-based instrument to characterize high quality mathematics instruction. The task dimension has five levels [0–4] with levels two and three referring to the real world. From level two, tasks focus beyond practising procedures and the real world can engage students, whilst problem solving and applications at level three emphasize real-world connections or prior knowledge.

For J1 the authors saw lack of explicit real-world context for negative integers as contributing to difficulties in understanding. They argue that one cognitive obstacle (subtrahend < minuend), identified both historically and in current student thinking, is in part related to the lack of real-world sense making of the notion of "removing more than one has" (p. 52). Contexts (e.g., money, elevation differences) are used in clinical interviews to provide a sense-making situation for 6–10-year-old students to develop conceptual understanding of integers—to overcome cognitive obstacles.

Moore [J4] presents one student's understanding of angle measure and trigonometric functions during participation in a teaching experiment. Tasks used included a person riding on a Ferris wheel and a bug riding on a fan blade. Real-world contexts provided a sense-making situation for the student to develop conceptual understanding of angle and the sine function (e.g., why position of bug on fan should be described relative to the length of fan blade). The author was clearly of the view that real world contexts would support student understanding, however, this was not explicitly discussed, nor was it part of the analysis reported.

In J6, the NCTM research committee report from an analysis of NCTM annual conference research pre-sessions that these sessions do not give enough attention to mathematical thinking "experiences that focus on mathematizing reality" (p. 169) from multiple areas of mathematics. They acknowledge that some such research is reported at more specialist biennial conferences such as ICTMA. To move forward, the authors propose research addressing the nature of problem-solving situations requiring mathematical thinking beyond school.

Schukajlow and Krug (J7) report on a teaching experiment to determine if encouraging multiple solutions impacted on student interest, competence, and autonomy. Students were prompted to provide multiple solutions to ill-defined real-world problems with vague conditions (e.g., not enough information). The authors clearly define the real-world as being outside the mathematical world. The vague conditions led to differing assumptions and hence different solutions. They argue that not only does solving real-world problems assist students in understanding the mathematics better, but also it allows students to "learn how they can apply mathematics and build mathematics models in their current and future lives" (p. 499). Encouraging multiple solutions had a positive effect on student interest, autonomy and competence.

4.4.3 In-Depth Analysis of JRME Sample

Table 4.6 is a summary matrix of the in-depth analysis for the JRME sample. For three papers, a *situational context* was the focus [J2, J3, J5]. Both J2 and J3 were commentary papers whilst J5 was research based. In J6 the real-world focus was *incidental* arising in the recommendations following the document analysis. The remaining three papers [J1, J4, J7] were research based with a *task context* focus.

The role of the real-world task context was categorised as *incidental* in J5, as this arose from the analysis of 900 interviews and J6 where clearly, the committee see the importance of the types of mathematical thinking inherent in solving real-world tasks, but the real-world focus was *incidental* in the arising recommendations. In both J1 and J4 the real-world task context was *minor*. Whilst J1 posited that real-world contexts would be helpful for young learners in providing integer related context, their study found that the students did not interact with the task in such a mathematical way. Rather the students reasoned about the absolute values related to the situation not using negative integers in their task solving. The teaching experiment in J7 was designed on the premise that the more realistic the task, the greater student interest and competence.

Similarly, to the ESM sample, papers in JRME, with the exception of J7 left it to the reader to infer what was implied by the real-world. All papers with a *task context* focus provided examples to illustrate their explanation. This allowed the researcher, and thus the reader, to easily infer if the role of the real-world was a major or minor focus of the tasks used and subsequently the level of embeddedness of the real-world in the task context following Stillman's categories of border, wrapper and tapestry.

Table 4.6 Context focus of sample papers and categorization of task contexts [JRME 2014]

Type of paper	Paper (context focus)	Role of the real-world task context
Commentary (2)	J2 (Brief report, situation context—undergraduate mathematics education)	–
	J3 (Research commentary, situation context community colleges)	–
Document Analysis (1)	J6 NCTM pre-session papers (Recommends increased research on RW mathematical thinking)	Incidental
Research–based (4)	J1 (Task context, learning about integers, contextual tasks one task type used)	Minor: border
	J4 [Task context, use RW (Ferris wheel, bug on fan) to illustrate concepts (radian, sine functions)]	Minor
	J5 (Focus on situation context of quality teaching, RW tasks 1 of 4 dimensions)	Incidental
	J7 (Task context, teaching experiment with RW tasks)	Major: wrapper

4.5 Content Analysis: MTL

4.5.1 Initial Analysis and Sample Selection

Mathematical Thinking and Learning (MTL), was first published in 1999 with three issues per year. A search for context, task context and real-world identified 249, 235 and 63 instances respectively over the life of the journal. Table 4.7 presents additional data. The term *context* and *task context* are found in the majority of papers. In contrast, *task context* is also found in most papers. The term *real-world* was found at lower rates (19% overall, 22% last decade) but higher than the rates for both ESM and JRME for the same time periods.

Table 4.7 shows in 2017 of 13 MTL papers, three (≈23%) included the term real-world. It must be noted that this journal published far fewer papers per year (in the last decade 183 papers, compared to 340 for JMB, 383 for JRME and 672 for ESM). Three papers (M1–M3) were sampled (see Appendix).

Table 4.7 MTL frequency of search terms

Search term	All years	2008–17	2013	2014	2015	2016	2017
Context	249	146	12	11	12	12	10
Task context	235	131	13	11	12	15	12
Real-world	63	40	5	3	0	6	3
No. of papers	330	185	13	12	15	12	12

Note Some cumulative years may include book reviews in addition to papers

4.5.2 Detailed Analysis

The authors of all papers were located in the USA as were the participants in their studies. Key words are not included on MTL papers. The papers had one (M2), four (M1) and 25 instances (M3) of the term real-world.

In M2, Stephens et al. investigated the functional thinking of 100 students, beginning in Grade 3 over three years. The authors draw on literature noting the importance of context in functional thinking, however, 'real-world context' used involved finding a relationship between the number of seats and number of desks being arranged at school for a party. Bargagliotti and Anderson (M1) describe statistical modelling as analogous to mathematical modelling. Solving real-world problems was one of the guiding principles for the professional learning, however, teachers used the available data to focus on developing key statistical understandings rather than solving real-world problems.

In M3, with 25 instances of *real-world* indicative of the major focus on real-world tasks, Wernet investigated interactions around context, especially those in the written curricula, in three Grade 8 classrooms. Mathematical modelling was central in the curriculum. Contextual tasks included realistic or imaginary situations whereas modelling tasks begin in the non-mathematical world and required mathematics to simplify, structure and solve the problem, which is then interpreted. Wernet classified tasks as displaying low authenticity, medium authenticity, or full alignment between the task and real-life scenario following Palm (2006). When implemented, tasks with low authenticity tended to stay low whereas those with at least medium authenticity tended to generate more discussion about context. Contrary to what is often claimed, Wernet reports that, in no instance were students observed to struggle with contextual understanding and drew appropriately upon their own everyday experiences. In fact, students mathematized with little difficulty, attributed to three years' experience with contextual tasks in the curriculum including opportunities to discuss the contexts.

Table 4.8 Context focus of sample papers and categorization of task contexts [MTL 2017]

Type	Paper (context focus)	Role of real-world task context
Research-based (3)	M1 (Task context, RW one guiding principle for tasks developing statistical understanding)	Minor: Wrapper
	M2 (Task context, RW one considerations in task design for functional thinking)	Minor: Wrapper
	M3 (Task context, analysis of task written and enacted for real world authenticity)	Major

4.5.3 In-Depth Analysis of MTL Sample

All papers in the MTL sample were research-based and all had a *task context* focus. For two papers, the real-world task context had a *minor* focus and in both cases, the embeddedness of the real-world in the tasks was classified as *wrappe*r. For M3, where the real-world has a *majo*r focus, the author was analysing tasks used with respect to their authenticity. A summary matrix of the in-depth analysis is presented in Table 4.8.

4.6 Content Analysis: JMB

4.6.1 Initial Analysis and Sample Selection

Journal of Mathematical Behavior (JMB) was first published in 1990, but only available to search from 1995. A search for context, task context and real-world identified 621, 8 and 160 instances respectively (see Table 4.9). The term *context* is found in the majority of papers. In contrast, *task context* was rarely found. The term *real-world* was found at a similar rate (18% overall, 22% last decade) to MTL, higher than the corresponding rates for ESM and JRME.

Table 4.9 JMB frequency of search terms

Search term	All years[a]	2008–17	2013	2014	2015	2016	2017
Context	621	319	47	41	37	33	53
Task context	8	4	0	0	2	1	1
Real-world	160	74	12	10	9	5	12
No. of papers	876	340	51	43	40	39	53

[a]*Note* Data accessible 1995–2017. Some cumulative years may include book reviews

The search for *real-world* identified only 160 instances of the term 1994–2017 with 12 in 2017. A trend in this sample is difficult to discern. Twelve papers (B1–B12) were in the sample (see Appendix).

4.6.2 Detailed Analysis

The majority of authors and location of the studies were in the USA. Nine of the 12 papers had both authors and participants based in the USA. The theoretical paper (B2) had one author from Turkey and one from the USA. An additional research paper (B8) had one author and participants from Italy and two authors from Belgium. B7 had all authors and participants from Israel. As with JRME, this sample is almost exclusively from and about the US. Only one paper had *real-world* as a key word (B6) and none had *context* as a key word. The only other keywords indicative of real-world contexts were applications (B6) and mathematical modelling (B3) and possibly ‘word problem solving’ (B8).

In nine papers, the *real-world* was mentioned 1–3 times and in three papers (B2, B6, B8) 4–7 instances. Again, this frequency was not sufficient to determine the importance of the real-world to the authors. For four papers with few mentions the real-world was incidental. Hopkins et al. (B5) undertook a study of the role of coaches in a school district (14 primary schools) undergoing reform. Whilst arguing that ‘ambitious mathematics teaching’ includes providing opportunities for students to solve real-world problems, no analysis was reported specifically linked to solving real-world problems. Smith et al. (B10) researched ‘instructional teacher leadership’ and it was a participant who emphasised the real-world, noting she was now focussing on “making it real world to them” (p. 276). B1 reports 251 secondary mathematics teachers’ “meanings for slope, measurement, and rate of change” (p. 168). In B7, the study involved 60 Grade 9 Israeli students and the extent of surprise in the solution of two abstract geometry problems. The sample lesson snippet used a real-world context of bicycle riders.

Similarly, with few mentions of the real-world, three papers had this as a minor focus. Harel (B4) undertook a teaching experiment with in-service secondary mathematics teachers on the theory of systems of linear equations. Although tasks used in the introductory unit include real-world contexts (e.g., traffic flow) and the teachers “indicated that they felt that the engagement in the unit’s ‘real-world’ scenarios” (p. 79) enhanced their understanding, no real-world scenarios are reported as being presented in the main unit. The literature review in B11 included how understanding of whole numbers and negative integers can be grounded in real world contexts; but, in the clinical interviews, none of the questions reported were set in a real-world context, although analysis identified task solvers invoking the real-world. Wickstrom et al.’s (B12) study of pre-service primary teachers’ conceptions of area, drew on literature that noted, “they demonstrate a procedural understanding of area often limited to memorized formulas disconnected from real-world applications” (p. 112) and the premise that such understanding is not sufficient for future teaching. The pre-task was abstract, but the post task was set in a real-world context, namely tiling a shower floor. Despite the authors drawing on literature related to understanding in real-world settings, this was not discussed in their analysis.

Two papers with few mentions of the real-world had this as a major focus. Paoletti and Moore (B9), undertook a teaching experiment with two pre-service undergraduate secondary mathematics teachers exploring covariational reasoning. Tasks used included bottle filling and emptying (see Swan 1985) and travelling between two towns using an applet. Results suggest real-world situations such as a Ferris Wheel moving in different directions or a car travelling to and from school will support students' parametric reasoning. Czocher (B3) compared two approaches to teaching undergraduate engineers, one emphasising decontextualized techniques to solve differential equations, whilst the other "emphasised modelling principles to derive and interpret canonical differential equations as models of real world phenomena" (p. 78). Her statistically significant results showed the modelling approach aided student learning. Data came from extensive classroom observation and three common problems on the final examination involving contextualized examples. Czocher noted the students who experienced the modelling perspective were more flexible in their thinking and better able to handle initial conditions.

The papers with more mentions of the real-world also varied in emphasis with one (B2) dismissing its usefulness. Cetin and Dubinsky's theoretical paper (B2) discusses decontextualization as one meaning ascribed to reflective abstraction. They dismiss the argument that the absence of context is what makes abstraction difficult and question use of real-world contexts to teach mathematical concepts for three reasons: "what is 'real-world'" (p. 71) varies for the individual; there is a danger students might learn something about the context but little about mathematics; and claim there is little research showing that realistic contexts help students learn decontextualized mathematics.

In contrast, in B6 and B8, the real-world was of major importance. Jones (B6) reports an exploratory study in first year calculus, arguing the majority of research in the area, focuses on kinematics and seeks to address this gap in the literature. Jones reports "applied contexts seem to bring out covariation-based thinking more than pure mathematics contexts" (p. 107). The tendency for some students to invoke time, in timeless contexts, to help with sense making, whilst sometimes helpful became problematic. Clearly, more experiences with contexts where time is not a variable would be helpful. Mellone et al. (B8) investigated whether there is a relationship between Grade 5 students' situation models and the realistic nature of their answers to problems. Clearly defining modelling as the process of creating a mathematical model from a situation model, they found working in pairs and rewording then solving led to an increase in realistic responses but for only one problem.

4.6.3 In-Depth Analysis of the JMB Sample

Eleven of the papers in the sample were research-based and the remaining paper theoretical. With regard to the context focus, they were more challenging to categorise than in the other two samples. For nine papers, the context was clearly a real-world task context (B1, B3, B4, B6, B7, B8, B9, B11 and B12) however for three papers (B2, B5, B10) the classification of situational or task context was not possible. The reasons for this varied, in B2 the real-world is dismissed, in B5 it relates to the goal of teaching, whereas in B10 it arose in the data collected. A summary matrix of the in-depth analysis is presented in Table 4.10.

For the nine papers, able to be classified by context focus, this was clearly on a *real-world task context* in all papers. For two, this was *incidental* (B1, B7) and the other seven *essential* (four *major* focus, three *minor* focus). For all four where the real-world context was a major focus, the embeddedness of the real-world was categorised as *tapestry*. For the three with a minor focus, one was classified as *border*. The remaining two were not classified further, as in B4 no actual tasks were reported and in B11 the real-world was evoked by the task solvers rather than the task setters who presented abstract tasks.

For B1, the real-world was classified as *incidental* as it was the teacher participants who used real-world examples (i.e., inclined planes, ski slopes) where steepness could be visualised. Similarly, in B7, the real-world was *incidental*, arising when the author compared the real-world to the mathematical world in discussing surprising situations in mathematics.

Task context was classified as having a minor focus in three papers. In B12, although the authors drew on relevant literature and had one of two tasks with a real-world context, there was no analysis or discussion related to the real-world. Similarly, in B4, the real-world was used only in the introductory unit of their study and although found helpful by teacher participants played no part in the majority of this research. The study by Whitacre et al. (B11) of school students' reasoning about integer comparisons was the only example from all samples, where the real-world context was evoked by the task solver as described by Boero (1999, p. vii). In all other cases, the real-world was *evoked by the task setter*, but here, although the task was abstract, the task solver brought in the real world to support problem solution.

Four papers (B3, B6, B8, B9) were classified as having a major focus on real-world task context, all with the embeddedness of the real-world as tapestry. Three of these had a focus at university undergraduate level, B3 with two classes of engineering students, B6 first year calculus students and B9, pre-service undergraduate secondary mathematics teachers. In contrast, B8 reported a study of Grade 5 school students.

Table 4.10 Context focus of sample papers and categorization of task contexts [JMB]

Type	Paper (context focus)	Role of real-world task context
Theoretical (1)	B2 (dismiss use of RW as they focus on abstraction)	–
Research-based (11)	B1 (Task context, inclined plane, ski slope suggested by teacher participant in study of rate of change)	Incidental
	B3 (Task context, comparison of content vs. context approach to teaching DEs)	Major: Tapestry
	B4 (Task context, real-world contexts for initial units about systems of linear equations)	Minor[a]
	B5 (Task context, goal of ‘ambitious teaching includes solving real world problems)	Incidental
	B6 (Task context, moving beyond kinematics context for applications of derivatives)	Major: Tapestry
	B7 (Task context, abstract geometry problems, lesson illustrated used real-world task)	Incidental
	B8 (Task context, pair work and student rewording of tasks to increase rate of realist solutions)	Major: Tapestry
	B9 (Task context, covariational reasoning, bottle problem, car problem—driving between 2 cities)	Major: Tapestry
	B10 (Participant notes importance of real-world)	Incidental
	B11 (Task context, evoked by task solvers)	Minor
	B12 (Task context, post task item shower tiling)	Minor: border

[a]*Note* Tasks not given so no further classification possible

4.7 Discussion: Looking Across the Samples

As shown in Fig. 4.3, across the four journal samples, most times the *real-world* was mentioned (34 of 42 papers, approx. 81%) this was in reference to the *task context* rather than situation context. However, the author’s purpose in just over 25% (nine papers) was *incidental* and arose in a review or discussion of the literature or in the data or its analysis or recommendations. This ranged from dismissing the

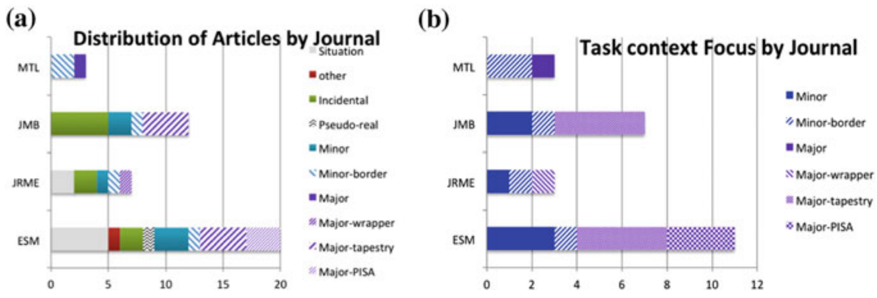

Fig. 4.3 Distribution of categories. **a** All categories, **b** task context classification only

'use of real-world contexts' to focus on abstraction, as a part of defining ambitious teaching, arising in the data or its analysis to recommendations for the use of real-world contexts. In the remaining 25 papers, one had a *pseudo-real* context, and 24 the real-world task context was *essential* with 11 having a *minor* focus and 13 a *major* focus on the real-world task context.

To answer the overarching research question, *How is context viewed?* It is helpful to consider the type of paper. Excluding the 10 non-research papers (i.e., theoretical papers or commentaries of which nine had a situation focus) and consider the 34 papers reporting research (including document analyses), all but one (E4 Cultural context) had a *task context* focus. Hence, for almost all authors reporting research, context was viewed as the *real-world task context* whereas for non-research papers, the real-world was part of the situation context. As noted, context was most often not defined although its meaning could be inferred.

In determining, *What are the meanings and role of real-world task context?* three overarching categories (incidental, pseudo-real, and essential) were defined and used in the analysis of the papers with a real-world task-context focus. Of the 33 research papers with a focus on *real-world task context*, this focus was *incidental* in eight, *pseudo-real* in one, and for the majority (24) *essential*. For 11 of these 24 research papers, the focus was *minor* and for 14 a *major* focus. So, in considering the research papers, not only is the context most likely to be a real-world task context, this focus on the real world is more likely to be *essential* than not. Furthermore, when task context had a *minor* focus and tasks could be further characterized, this tended to be as *border* or *wrapper* (not tapestry). In contrast, where the real-world task context was a *major* focus, tasks were almost exclusively classified as *tapestry* (or PISA).

Of the papers where the focus in the *real-world task context* was *essential*, seven papers (6 of 11 minor, 1 of 13 major) were unable to be further classified in terms of task context embeddedness (border, wrapper, or tapestry). The reasons for this varied. In one paper, the researchers deliberately used real-world contexts to illustrate key mathematical ideas. In another, the real-world was one dimension of the analysis but gave no further details, and in a third, the task solver(s) evoked the real-world in solving abstract tasks.

Context was at times portrayed as a hindrance, however, this only occurred when authors referred to other studies (usually very selectively) or were theorising. These authors also tended to see the real-world as (only) a pathway to the abstract mathematical world. In the actual research reported in these four purposive samples, in no study was a *real-world context* found to hinder learning. In contrast, the opposite was reported, the *real-world* helped in teaching and learning (four studies) and one reported mixed findings.

The four papers reporting positive outcomes include the teaching experiment comparing modelling versus decontextualized approaches to teaching differential equations in first year calculus in terms of performance on the final examination. Both low and high achievers performed significantly better in the class with the modelling perspective, being more flexible in their thinking and better able to handle initial conditions. At the secondary level, two papers reported *real-world context* as helpful. In Grade 8, rich contexts, particularly when teachers supported sense-making discussion about the context and the mathematics, supported student engagement with tasks of high cognitive demand. Requiring Grade 9 students to provide multiple solutions to authentic real-world problems had a positive effect on student interest and competence. In the fourth paper, it was the secondary teacher participants in the study who reported the usefulness of the *real-world contexts* in supporting their understanding.

A further 12 research studies had *real-world task contexts* as an inherent part of their study, from which it is inferred the authors had the expectation that real-world contexts are supportive of teaching and/or learning. For some, this was an integral part of the mathematics that was the focus of the study (e.g., primary: speed, mapping; secondary: trigonometry; tertiary: (first year calculus) derivatives, (nurse education) drug dosages, (teacher education) co-variational reasoning; and in-service teachers: statistics). Given over half of all papers and over 70% of the research papers sampled considered the *real-world task context* as playing an *essential role*, this author concurs with Niss et al. (2007) suggesting the maturation of the applications and mathematical modelling research discipline.

4.8 Concluding Remarks

It appears the nature of the construct: *context* previously described as nebulous (Busse and Kaiser 2003) has become more focussed in recent times. Although, drawing on the analysis of the overall data and the purposive samples, the construct *context* is still used in multiple ways as previously noted by Boero (1999). At times the construct was not explicitly defined although its meaning in the sample analysed could be inferred. It is incumbent on the modelling and applications community and in fact all mathematics education researchers to clearly articulate when the real-world is an important aspect of their research.

Stacey (2015) in articulating the way PISA "theorises and operationalises the links between the real world and the mathematical world" (p. 57) notes that using real-world contexts is considered essential in the teaching and learning of mathematics. Context in PISA "refers specifically to those aspects of the real world that are used in the item" (p. 74). This *essential* use of context was evident in the majority of papers in the purposively selected samples reported in this chapter. What constituted the *real-world* (Niss et al. 2007), the authenticity of the context (Palm 2006; Van den Heuvel-Panhuizen 2005), and the degree of embeddedness of the real-world task context (Stillman 1998) varied greatly. Clearly, when researchers had the real-world context as a *major* focus, this degree of embeddedness was higher with tasks characterised as *tapestry* (or PISA) whereas if only a *minor* focus, the embeddedness tended to be lower, tasks characterised as *border* or *wrapper.* As the level of challenge for students generally directly correlated with the degree of contextualization or interaction of task solver with the context, it is important all students have opportunities to interact with 'tapestry type tasks' (Stillman, 1998). Notwithstanding the challenges inherent in solving tasks of high cognitive demand, in part due to the real-world task context (Dapueto and Parenti 1999), no studies reported findings where the real-world context hindered learning. Researchers focusing on *real-world task contexts* do consider these as critical and hence need to be understood, at least in order to understand the problem, if not throughout the solution process. In contrast, a minor focus on the real-world generally saw trivial contexts or those that the task solver could ignore entirely, showing that this essential use of real-world contexts is not accepted by all in the mathematics education community.

Whilst some papers reported research where context helped learning, none concluded context was a hindrance, and rather more papers were not even considering this question as important. Perhaps this question has, for most, become too simplistic to consider as the complexities of learning, particularly when engaging with real-world tasks, are well understood by researchers who see this engagement as essential and are more focused on other aspects of learning assuming the real-world is an intrinsic part of this process.

Knowing mathematics means learners can use their mathematics to solve real-world problems (e.g., Freudenthal 1973; Gravemeijer et al. 2017; Pollak 1969). Further research is recommended in school mathematics classrooms, ascertaining ways in which teachers should be aspiring to support learners in knowing more about the world in which they live and analysing how the real-world contexts support student learning of mathematics and maintaining the high cognitive demand of such tasks. The real-world is a complex and messy place, thus real-world task contexts should reflect this reality and the embeddedness of the task should, following Stillman (1998), be at least *wrapper*—where task solvers must consider the context—if not at the highest level of *tapestry*—where the real-world and mathematics are interwoven, and both must be engaged with throughout the solution process. Finally, researchers must acknowledge that such tasks involve higher order thinking and are necessarily more challenging and demanding of learners. Engagement by learners with such tasks is a critical part of mathematics for all learners at all levels of schooling and beyond.

Appendix/Online Supplementary Material

ESM Sample

E1 Andrews, P., Ryve, A., Hemmi, K., & Sayers, J. (2014). PISA, TIMSS and Finnish mathematics teaching: An enigma in search of an explanation. *Educational Studies in Mathematics, 87*, 7–26.

E2 Artigue, M., & Mariotti, M. A. (2014). Networking theoretical frames: The ReMath enterprise. *Educational Studies in Mathematics, 85*, 329–355.

E3 Brantlinger, A. (2014). Critical mathematics discourse in a high school classroom: Examining patterns of student engagement and resistance. *Educational Studies in Mathematics, 85*, 201–220.

E4 Chan, Y.-C., & Wong, N.-Y. (2014). Worldviews, religions, and beliefs about teaching and learning: Perception of mathematics teachers with different religious backgrounds. *Educational Studies in Mathematics, 87*, 251–277.

E5 Coben, D., & Weeks, K. (2014). Meeting the mathematical demands of the safety-critical workplace: Medication dosage calculation problem-solving for nursing. *Educational Studies in Mathematics, 86*, 253–270.

E6 Ding, M., & Li, X. (2014). Transition from concrete to abstract representations: The distributive property in a Chinese textbook series. *Educational Studies in Mathematics, 87*, 103–121.

E7 Jiang, C., Hwang, S., & Cai, J. (2014). Chinese and Singaporean sixth-grade students' strategies for solving problems about speed. *Educational Studies in Mathematics, 87*, 27–50.

E8 Kanes, C., Morgan, C., & Tsatsaroni, A. (2014). The PISA mathematics regime: Knowledge structures and practices of the self. *Educational Studies in Mathematics, 87*, 145–165.

E9 Lerman, S. (2014). Mapping the effects of policy on mathematics teacher education. *Educational Studies in Mathematics, 87*, 187–201.

E10 Logan, T., Lowrie, T., & Diezmann, C. (2014). Co-thought gestures: Supporting students to successfully navigate map tasks. *Educational Studies in Mathematics, 87*, 87–102.

E11 McCloskey, M. (2014). The promise of ritual: A lens for understanding persistent practices in mathematics classrooms. *Educational Studies in Mathematics, 86*, 19–38.

E12 Morgan, C., & Kynigos, C. (2014). Digital artefacts as representations: Forging connections between a constructionist and a social semiotic perspective. *Educational Studies in Mathematics, 85*, 357–379.

E13 Moutsios-Rentzos, A., Spyrou, P., & Peteinara, A. (2014). The objectification of the right-angled triangle in the teaching of the Pythagorean Theorem: An empirical investigation. *Educational Studies in Mathematics, 85*, 29–51.

E14 Olande, O. (2014). Graphical artefacts: Taxonomy of students' response to test items. *Educational Studies in Mathematics, 85*, 53–74.

E15 Radford, L. (2014). On the role of representations and artefacts in knowing and learning. *Educational Studies in Mathematics, 85*, 405–422.

E16 Roth, W.-M. (2014). Rules of bending, bending the rules: The geometry of electrical conduit bending in college and workplace. *Educational Studies in Mathematics, 86*, 177–192.

E17 Straehler-Pohl, H., Fernández, S., & Gellert, U. (2014). School mathematics registers in a context of low academic expectations. *Educational Studies in Mathematics, 85*, 175–199.

E18 Swanson, D., & Williams, J. (2014). Making abstract mathematics concrete in and out of school. *Educational Studies in Mathematics, 86*, 193–209.

E19 Tall, D., & Katz, M. (2014). A cognitive analysis of Cauchy's conceptions of function, continuity, limit and infinitesimal, with implications for teaching the calculus. *Educational Studies in Mathematics, 86*, 97–124.

E20 Walkowiak, T. A., Berry, R. Q., Meyer, J. P., Rimm-Kaufman, S. E., & Ottmar, E. R. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity. *Educational Studies in Mathematics, 85*, 109–128

JRME Sample

J1 Bishop, J. P., Lamb, L., Philipp, R. Whitacre, I., Schappelle, B., & Lewis, M. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education, 45*(1), 19–61.

J2 Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. *Journal for Research in Mathematics Education, 45*(4), 406–418.

J3 Mesa, V., Wladis, C., & Watkins, L. (2014). Research problems in community college mathematics education: Testing the boundaries of K-12 research. *Journal for Research in Mathematics Education, 45*(2), 173–192.

J4 Moore, K. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102–138.

J5 Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584–635.

J6 NCTM Research Committee. (2014). The NCTM research presession: A brief history and reflection. *Journal for Research in Mathematics Education, 45*(2), 157–172.

J7 Schukajlow, S. & André Krug. (2014). Do multiple solutions matter? Prompting multiple solutions, interest, competence, and autonomy. *Journal for Research in Mathematics Education, 45*(4), 497–533

MTL Sample

M1 Bargagliotti, A. E., & Anderson, C. R. (2017). Using learning trajectories for teacher learning to structure professional development. *Mathematical Thinking and Learning, 19*(4), 237–259.

M2 Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Gardiner, A. M. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning, 19*(3), 143–166.

M3 Wernet, J. L. (2017). Classroom interactions around problem contexts and task authenticity in middle school mathematics. *Mathematical Thinking and Learning, 19*(2), 69–94.

JMB Sample

B1 Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior, 48*, 168–193.

B2 Cetin, I., & Dubinsky, E. (2017). Reflective abstraction in computational thinking. *Journal of Mathematical Behavior, 47*, 70–80.

B3 Czocher, J. A. (2017). How can emphasizing mathematical modeling principles benefit students in a traditionally taught differential equations course? *Journal of Mathematical Behavior, 45*, 78–94.

B4 Harel, G. (2017). The learning and teaching of linear algebra: Observations and generalizations, *Journal of Mathematical Behavior, 46*, 69–95.

B5 Hopkins, M., Ozimek, D., & Sweet, T. M. (2017). Mathematics coaching and instructional reform: Individual and collective change. *Journal of Mathematical Behavior, 46*, 215–230.

B6 Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *Journal of Mathematical Behavior, 45*, 95–110.

B7 Koichu, B., Katz, E., & Berman, A. (2017). Stimulating student aesthetic response to mathematical problems by means of manipulating the extent of surprise. *Journal of Mathematical Behavior, 46*, 42–57.

B8 Mellone, M., Verschaffel, L., & Van Dooren, W. (2017). The effect of rewording and dyadic interaction on realistic reasoning in solving word problems. *Journal of Mathematical Behavior, 46*, 1–12.

B9 Paoletti, T., & Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *Journal of Mathematical Behavior, 48*, 137–151.

B10 Smith, P. S., Meredith L. Hayes, M. L., & Lyons, K. M. (2017). The ecology of instructional teacher leadership. *Journal of Mathematical Behavior, 46*, 267–288.

B11 Whitacre, I., Azuz, B., Lamb, L. L. C., Bishop, J. P., Schappelle, B. P., & Philipp, R. A. (2017). Integer comparisons across the grades: Students' justifications and ways of reasoning. *Journal of Mathematical Behavior, 45*, 47–62.

B12 Wickstrom, M. H., Fulton, E. W., & Carlson, M. A. (2017). Pre-service elementary teachers' strategies for tiling and relating area units. *Journal of Mathematical Behavior, 48*, 112–136.

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Chapter 5
Approaches to Investigating Complex Dynamical Systems

France Caron

Abstract This chapter reports on a short experiment conducted in Canada to explore the potential, and feasibility, of introducing complex dynamical systems into mathematics curricula, both in schools and in university undergraduate programs. In particular it aimed to identify those mathematical habits of mind that are (or could be) developed in schools, universities and outside these traditional learning environments through exploring complex systems when approaching real life situations. The use of game design to engage members of the Canadian mathematics education community in the modelling of a real-life ecosystem brought to light different mathematical habits of mind and provided a snapshot of where we are with respect to modelling in mathematics education. Put in perspective with current lines of inquiry in the modelling of complex dynamical systems, in integrating modelling in mathematics education, and in developing computational thinking, the experiment opened a reflection on the possibilities and feasibility of helping tackle the complexity of our world's most pressing challenges through mathematics education.

Keywords Dynamical systems · Complex systems · Curriculum · Habits of mind · Structuring

5.1 Introduction

More and more, and despite its relatively modest presence in today's mathematics curricula (Stillman and Kaiser 2017), modelling is being recognised as an important element of the learning of mathematics. Not only is it a means for students to understand and integrate the mathematics they learn, but also it is, more fundamentally, a goal in itself of mathematics education. As a goal of a mathematics curriculum its purpose is for students to develop competencies that enable them to tackle situations

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from a mathematical perspective, whether these situations come from everyday life, science, the workplace or the complex challenges our world currently faces (Niss et al. 2007). Many studies (e.g. Kaiser et al. 2011) suggest that modelling as a learning goal is best served with authentic situations and open-ended problems, where students are free to invoke whatever mathematical knowledge they feel serves their purpose. Under such conditions, students may experience the various phases of the modelling cycle; including simplifying and structuring a real world situation into a specific problem with its goal, components and relationships; mathematising the problem (i.e. translating it into a mathematical model); working mathematically to reach a solution; interpreting and validating the results (and the model or the data) with respect to the original situation; and repeating the whole process if need be (Blum et al. 2002).

To assist students in developing powerful models, one could think of revisiting curriculum content so as to provide a richer mathematical toolbox for addressing real life needs of today and tomorrow. Yet, on that front, the following warning was expressed more than twenty years ago:

> Given the uncertain needs of the next generation of high school graduates, how do we decide what mathematics to teach? Should it be graph theory or solid geometry? Analytic geometry or fractal geometry? Modeling with algebra or modeling with spreadsheets? These are the wrong questions, and designing the new curriculum around answers to them is a bad idea. (Cuoco et al. 1996, p. 375)

Instead of merely "replacing one set of established results by another one (perhaps newer or more fashionable)", Cuoco et al. suggested organising curriculum design around *habits of mind*, to allow students "to become comfortable with ill-posed and fuzzy problems, [...] to look for and develop new ways of describing situations" (p. 373) and "to close the gap between what the users and makers of mathematics do and what they say" (p. 376).

Some general habits of mind described as useful for doing mathematics include pattern-sniffing, experimenting, formulating, tinkering, inventing, visualizing, and conjecturing. More specific mathematical habits of mind would include approaches that either rely on a particular mathematical idea or reflect "the way mathematicians approach things" (Cuoco et al. p. 384). Contrary to the notion of *mathematical thinking styles* (Borromeo Ferri 2010) or the more general notions of *thinking styles* (Sternberg 1997) and *learning styles* (Felder and Brent 2005), the consideration of mathematical habits of minds was not brought forward to capture individual differences in preferred ways of thinking, understanding or doing things, nor to examine their influence in teaching and learning. Rather, it was meant to serve as a framework to organise a mathematics curriculum, where each of these habits of mind as a research method becomes a learning goal of mathematics education, and the content to be learned is identified on the basis of its potential contribution to developing these goals. Wu et al. (2015) take a similar viewpoint when they use the development of the thinking underpinning mathematical modelling as the teaching and the learning goals in a tertiary mathematical modelling course. Rather than focusing on methods to be learned, they promote activities that build on intuition, help make connections and innovation through critical thinking, cultivate inductive thinking, etcetera.

Yet, as proposed by Hunt (2007), there may be the need for citizens in general, and policy makers in particular, to develop a better appreciation of some key ideas of the mathematical sciences that can help us understand and deal with society's major challenges. Addressing issues of greatest public concern often depends on the "reliability of prediction of multi-component complex systems" and the inclusion in the analysis of "possible sudden changes and isolated events" (Hunt 2007, p. 3).

Nature, human organisations, as well as the multiple layers where the two meet are indeed often best described as complex systems.

> Complex systems are systems that comprise many interacting parts with the ability to generate a new quality of collective behavior through self-organization, e.g. the spontaneous formation of temporal, spatial or functional structures. […] This recognition, that the collective behavior of the whole system cannot be simply inferred from the understanding of the behavior of the individual components, has led to many new concepts and sophisticated mathematical and modeling tools for application to many scientific, engineering, and societal issues that can be adequately described only in terms of complexity and complex systems. (Meyers 2011, p. v)

The behaviour of a complex system is intrinsically hard to predict, as the effect can be very distant from the cause in both space and time; what may look like a radical new rule may have little long-term effect while relatively small perturbation can ultimately result in a major impact. The consequent conflict between long-term and short-term goals (Forrester 1996) may partly explain the challenges our world is currently facing.

If developing a better understanding of complex systems should be regarded as one of the goals of education, and if the study of complex systems has to call upon "new concepts and sophisticated mathematical and modelling tools" (Meyers 2011, p. v), this may bring us back to the apparent dilemma of having to choose between adding content to the general mathematical toolbox and developing the mathematical habits of mind "to close the gap between what the users and makers of mathematics do and what they say" (Cuoco et al. 1996, p. 376).

In an attempt to examine how mathematics education could better equip students to approach ill-defined problems of increasing complexity, colleagues and I explored the opportunity for, and feasibility of, introducing complex dynamical systems in the mathematics curricula, both in schools and university undergraduate programs. This was done through activities and discussions within the context of a working group at the Canadian Mathematics Education Study Group (CMESG) annual conference (Caron et al. 2015). The CMESG meetings are a place where people from mathematics and mathematics education across Canadian universities share and explore issues and ideas on the teaching and learning of mathematics. In the various activities and discussions that took place in the group, one modelling experiment proved particularly interesting, as it generated sustained engagement among the participants, allowed a variety of models to emerge and revealed different and complementary mathematical habits of mind. Although some connections could be made with individual thinking, or learning styles, the observed differences appeared to have more to do with the different formal training and experiences our participants brought to the table.

While providing an interesting snapshot of where we are in Canada with respect to modelling in mathematics education, this experiment opened a reflection on the possibilities and feasibility of tackling the complexity of our world's most pressing challenges through mathematics education. This reflection led to examining the different lines of inquiry for exploring complex systems and the way these lines have been reflected in curriculum and resource development, school experiments and research on modelling in mathematics education. In particular the aim here is to identify those mathematical habits of mind that are (or could be) developed in schools, universities and outside these traditional learning environments through exploring complex systems when approaching real life situations.

5.2 The Experiment

In planning the working group, the use of games quickly imposed itself as a potentially rich and accessible approach to engage students and teachers at different levels of mathematics in experimenting with the dynamics of complex systems. The game context later revealed its value for revisiting our conception of mathematical habits of mind, with the current state and ubiquity of technology.

The working group consisted of 14 individuals who had elected to explore over three half-day sessions the topic of complex dynamical systems. The group showed a rich diversity of profiles, with school mathematics teachers (elementary and secondary), along with university professors and graduate students in either mathematics or mathematics education.

The first of the three sessions was dedicated to playing with simulators and games (built on cellular automata or agent-based models) in order to bring forward some of the characteristics of complex dynamical systems (with their different patterns for long-term behaviour) and their typical applications (social contexts and environmental situations).

In the second session, participants regrouped in three teams (A, B and C) to design a game to replicate the recent change in the dynamics of the Yellowstone ecological system, after wolves had been reintroduced. The situation, for which they could find additional information on the internet, was described as follows:

The Yellowstone Game

In the 1990s, wolves were re-introduced into Yellowstone National Park. A couple of decades later, the grizzly bear population in the park increased significantly. Why did that happen? Researchers hypothesised that a chain of interactions was at play. As wolves prey on elk, they decrease their population. All kinds of berries, previously consumed by elk (which also destroyed berry shrubs), are now able to recover, thus providing an abundant source of food for bears, especially in fall, when they prepare for hibernation. Build a game that will allow you to verify researchers' claims about the chain of events that led to the increase in the bear population. Then use your game to predict what will happen with the wolf, elk, bear and berries populations in the long run, in particular if the elk population declines to very small numbers.

To anchor the activity in a game environment, participants were provided with concrete materials (i.e. many small coloured square tiles, spinners and dice, biased and unbiased) that they were free to use.

In Session 3, an alternative model of the same situation was presented to the participants who were invited to explore further the modelling software (Stella) with which it had been built. A discussion followed where the participants shared their view on the value and feasibility of introducing dynamical complex systems in school mathematics, and on the resources that could help moving in that direction. The remarkable variety and evolution in the games and models that emerged with the Yellowstone situation gave a unique perspective into how to reconcile, within a modelling exercise approachable by all, complementary or even conflicting habits of mind. This will be the focus of the next session.

5.3 Habits of Mind at Play

All teams engaged with great enthusiasm in the game design activity, well beyond the time that had been planned for. Within each of the three teams, the diversity of the participants' background resulted in an interesting negotiation of the game structure, format and rules.

Teams A and B, included mathematicians who rapidly led their team onto the road of compartmental models or differential equations *à la* Lotka-Volterra. Very little time was initially spent on structuring the situation, as models seemed readily available, only requiring adaptation. With such entry into the task, the mathematicians in the group displayed their habit of *thinking of change analytically, in terms of functions and differential equations* to capture continuous variations (especially over time) of variables and to look for dependences that might describe causal phenomena. The proposal of a compartmental model, in Team A came from a mathematician working with biologists, and reflected a particular way of modelling often used in biology: *thinking of systems in terms of flows*. From their non-verbal behaviour, we could perceive some frustration among other team participants who did not feel at ease with the proposed equations or could not envision how the compartmental description could be used. They disengaged from the conversation until they found a way of regaining access, either by reaffirming the game design objective or by proposing an alternative approach.

In Team A, the diagram associated with the proposed compartmental model appeared to have facilitated a shift of representation. The compartments associ-

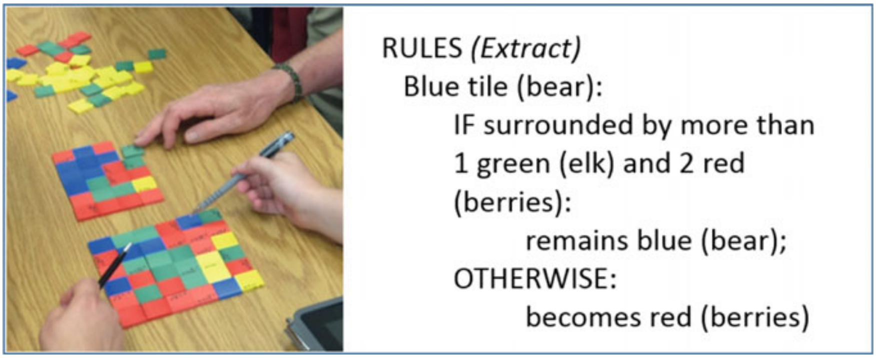

Fig. 5.1 Integrating spatial considerations with a cellular automaton

ated with the variables representing the species populations became distinct piles of tiles, enabling some physical enactment of compartmental models to take place as tiles were sequentially added or removed from piles, according to *rules that were being defined*, making the members *think iteratively* of the system. Perceiving a difficulty in capturing the various interactions (predation and competition) that characterise the Yellowstone situation with the addition and removal of tiles, Team A later moved to a *spatial representation* of the problem. The *domain was made discrete* as a 6×6 grid with coloured square tiles as cells where the colour indicated the dominating species in each small part of the territory (Fig. 5.1). The game would unfold like a cellular automaton similar to one they had experienced the day before, thereby maintaining an *iterative* view of the system.

In designing their cellular automaton for modelling the Yellowstone situation, the participants had to *define rules* that would reflect interactions between species. That was done by specifying when and how, with the *use of conditional statements*, the dominating species in one area (represented by a cell) would change in relation with the dominating species in the surrounding eight cells; one of these rules made use of randomness, to break a tie in competing species of surrounding cells. Generating the configuration for each iteration, one cell at a time, made the team appreciate the implications of the rules they had created. Despite the repetitive nature of the task, which seemed to call for a computer implementation, the team shared great enthusiasm in observing their game unfold and assessing the value of their model in capturing the evolution of the ecosystem. It turned out, to their great amusement, that within only 6 iterations, the berries ended up dominating the whole park, which could only *raise questions regarding the adequacy of the model*, its granularity and its rules. The team thus had gone through the whole modelling cycle and was in a position to reiterate.

A similar approach was eventually adopted by Team B, with a substantial increase in motivation from where they were when differential equations were being discussed. Their game had the extra feature of *keeping a record* of the changes in the dominating species for each cell, by superimposing tiles (i.e., stacking) for implementing change of colour. Their larger grid and high consumption of tiles prevented them from reaching a point in the simulation where they could assess the value of their model.

Structuring with cellular automata had brought forward this *algorithmic and iterative way of thinking* and made it a workable compromise for two of our three teams. While most team members could recognise mathematical structures, all could make sense of the model they were building and using for *simulating the dynamics* of the ecosystem. The fact that all participants were in the driving seat for *describing the interactions*, *inventing the rules*, and *experimenting* with them made them engage actively with these very useful habits of mind in mathematics. Their engagement may also have been favoured by the fact that a grid made out of colour tiles appealed to both the *analytical* and the *visual* thinkers in the teams.

Team C, where no member was a mathematician, went in a completely different direction. They elected to go for the most realistic representation of the situation they could think of, by *modelling at the level of the individual*, animal or plant, by *integrating movement* (a key feature of animals!) and by *addressing interactions as encounter events*. They *defined rules* such as the following:

> If a bear has not eaten in 10 turns, it disappears.
>
> After a tenth encounter between a bear and a wolf, a bear is replaced by a wolf. Otherwise, they bounce off.
>
> If a bear encounters berries, the bear sticks beside the berries for one step and then moves on. The bear gets power points. Every 10 cells, the bear drops a berry seed that will grow the next season.

This direction was inspired both by the exploration the previous day of an agent-based simulator (Caron et al. 2015), aimed at reproducing the movement and behaviour of two species (in a predator-prey relationship) within a given bounded region, and their experience of today's highly realistic and sophisticated computer games. The level of detail at which they positioned their modelling (the individual) prevented them from playing their game without a computer implementation. On this issue, Team C shared a very interesting viewpoint; they saw their role was to specify the components and rules of their game, and that they could turn to a programmer to implement them in a simulator. By 'outsourcing' the implementation of their model they had no means of validating yet, they were implicitly agreeing to their dependence on a potentially large black box.

After having presented their own games, the participants were shown a Stella model of the Yellowstone situation, equivalent to the one shown in Fig. 5.2, and later invited to explore the features of this modelling software. The working group came to appreciate how *modelling interactions as inflows and outflows* of reservoirs could support the development of skills for capturing the dynamics of a system. The ease of *building, defining and combining the relations* between system components, through graphical description and limited use of symbols, made it more appealing and accessible than the differential equations to participants with either less experience of such mathematical tools or an admittedly more visual learning style. The possibility to simulate directly from that description provided new meaning to the

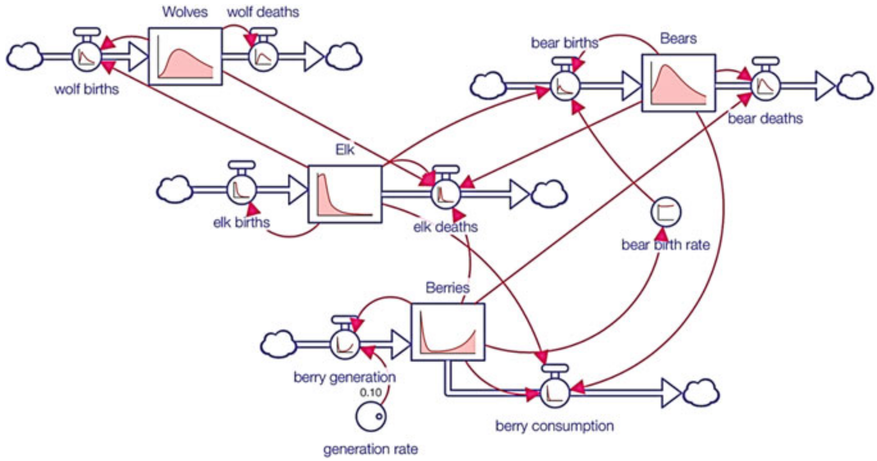

Fig. 5.2 Modelling and simulating with Stella

compartmental model that had been proposed in Team A. However, the perceived difficulty of *integrating space and movement* in the model made it less interesting than some of the other models that had been explored.

As the complexity of the situation and its perceived approachability within a game environment gave rise to radically different approaches, the experiment provided some insight into some mathematical habits of mind that are (or could be) developed in schools, universities and outside these traditional learning environments when approaching real life situations. This observation led to taking a closer look at each approach (where they emerged, where they are currently applied) as well as the initiatives for transposing them into mathematics or science education. Putting the activity in greater perspective allowed the author to connect some of the reflections that emerged in the working group discussion with affordances and obstacles that had been identified in those initiatives.

5.4 Modelling Complex Systems

As the study of complex systems is still quite recent and the tools and methods for doing so have evolved over the last decades, it is worth taking a look at what has been done, both inside and outside educational settings. Bosch et al. (2005) have stressed the importance of knowing the mathematical activities outside the school environment that motivate and justify the teaching and learning of mathematics with a given task.

5.4.1 Functions and Differential Equations

Differential equations constitute the classical tool for describing change in modelling nature and mechanisms. Dating back to Newton and Leibnitz, they are still at the core of the study of physics and its various applications, and have been used in many other disciplines: chemistry, biology, economics, etcetera. In a first course on differential equations, students are often presented with classical situations such as spring-mass systems, electric circuits, cooling of objects, where such equations are derived. With these fairly simple situations, the differential equations that describe the system can be solved analytically to produce a solution that can be expressed as a closed-form function that very closely approximates reality. In many ways, these are not complex systems.

A system does not have to be very complex to defeat classical *analytical methods* for finding the solution. In order to solve the typical prey-predator problems, even with only two species, one has to turn to techniques other than traditional calculus algebraic manipulations to capture the behaviour of the solution of the associated system of differential equations: *phase plane analysis*, *discretisation*, *numerical methods* and *computer simulation*. These ecology problems, along with those that come from a wide variety of domains (fluid dynamics, pharmacology, etc.) and that do not lend themselves to analytical solutions are typically not addressed before the third year of undergraduate mathematics, in mathematical modelling or numerical analysis courses. But they have contributed to making mathematicians, computer scientists, engineers, physicists, biologists and other users of mathematics also *think of functions algorithmically*, with recurrence and iteration. Solving the associated equations to simulate the evolution of such systems typically requires translating the mathematical model into a computer model (Greefrath et al. 2011). Depending on the complexity and peculiarities of the problem, and on the resources accessible to the user, this may entail possibly creative *use of a specialised software*, some *programming* and even new *algorithm design*. The actual mathematical work on the model, which is often reduced to a little box or a thin arrow in the different cyclical representations of the modelling process, can actually become a major endeavour on its own, requiring many steps, iterations and internal verification.

Even if a solution cannot be expressed in terms of known functions, this certainly does not mean that reasoning using functions does not contribute to the modelling of a complex situation and analysis of a model. But functions are used as building blocks to structure the system, to describe relations between variables and/or their rates of change, using known principles or formulating new assumptions. This *use of functions as building blocks* for modelling is quite different from the one that is typically promoted in the different Canadian school curricula, where, by the end of secondary school, learning to model with functions is often associated to “graphing data and determining the [specific] function that best approximates the data” (e.g. Western and Northern Canadian Protocol (WNCP) 2008). While curve fitting and regression may help approach unknown relationships between two variables and develop initial intuitions, they can only bring limited contribution to understanding of the situation, as no profound structuring is involved (Doerr et al. 2017; Galbraith 2007).

The learning of differential equations can help *structuring a situation using rate of change.* But as these equations are usually introduced after one or two courses of calculus, a relatively small portion of the population gets to develop and maintain such habit of mind.

5.4.2 System Dynamics Software

Enabling more people to *think in terms of structural relationships, flows, accumulations* and *feedback mechanisms* to explore the evolution of complex systems has been the driving force behind the development of system dynamics software. These ideas come from Jay W. Forrester, the founder of system dynamics, who transferred his engineering knowledge to study the dynamics behind a variety of industrial, urban and global situations (Forrester 2007). He was active in promoting system dynamics for K–12 education, encouraged by the emergence in the 1980s of user-friendly simulation software with advanced graphical user interfaces, such as *Stella, Powersim* and *Vensim.*

These tools, and the more recent web-based *Insight Maker*, provide an icon-based modelling environment where key aggregate variables (stocks represented by rectangular reservoirs) are defined by the user and made subject to inflows and/or outflows that will define their rate of change. These flows can be made to depend on the value of various variables and parameters. Running a simulation shows the evolution of the different variables over time; the resulting functions are constructed dynamically by the software, through numerical integration, for every simulation the user decides to run, and with the time step specified, allowing relatively smooth passage between discrete and continuous models.

With the possibility of combining multiple interactions, randomness and the systematic use of numerical integration, these environments lend themselves naturally to the modelling and simulation of progressively more complex systems. The ease of modifying or refining structural relationships allows for some *experimenting* and *tinkering*. Such modelling environments have been around for more than thirty years, and have been used in research in a variety of fields to model the behaviour of different phenomena or processes (e.g. spread of a virus and immunisation, carbon cycle and global warming, fisheries management, effect of taxes or social policies).

Doerr (1996) provided a rich retrospective of the first ten years of use of *Stella* in various mathematics or physics classes. She reported that early experiments in the USA in the late 1980s had led to recognition of the importance of computer modelling and system dynamics for secondary education. Modelling with diagrams was considered to favour a conceptual representation of the system that allowed students to both communicate their understanding of a system and experiment with it. With the system taking charge of the calculation, it was expected that students would engage in a more *qualitative analysis of problems*, focusing on principles and feedback mechanisms. Although the hydraulic metaphor was revealed to be generally more difficult for students to grasp than anticipated, there was some evidence of improvements in students' qualitative reasoning about problem situations, and in identifying structural similarities between them.

Despite the development of some resources (e.g., Fisher 2001; Kreith and Chakerian 1999), a perceived lack of alignment to school curriculum resulted in little implementation of system dynamics modelling in North American school mathematics. Apart from enthusiastic pioneers, mathematics teachers have not felt as comfortable with the approach as their science colleagues (Doerr 1996; Fisher 2011). It may be that this modelling approach cannot be associated with the mathematical habits of mind that they have developed. Indeed, some of the topics brought by Fisher (2011) in the lists she proposes to develop progressively systems thinking in K–12 (flows, feedback loops, equilibrium, tipping points, etc.) appear more solidly anchored in the science curriculum than in the current mathematics curriculum. It can also be that the teachers of mathematics are hesitant in having their students rely on a black box for time integration. Using a spreadsheet variant with direct access to the discretised integration formula, for example, was much better received by mathematics teachers than the use of stock-flow diagrams (Tinker 1993 as cited in Doerr 1996).

Resistance against the use of a black box also appeared in our working group discussion. Participants commented that the proper use of a tool like Stella would have to rely on the knowledge of calculus and the numerical methods responsible for allowing the simulation to unfold. But then, one participant suggested that we might have held initially similar views about dynamic geometry software (DGS) and the necessary geometry knowledge to use them; this was until experimentation with younger students showed that well designed exploration activities with DGS could contribute to developing new intuition and motivate the learning of concepts, properties and proofs. Similarly, he said, we could envision developing intuition with respect to functions and calculus through modelling with system dynamics software.

5.4.3 Cellular Automata

The distance between school mathematics and cellular automata may appear even greater than with system dynamics. Conceived in the 1940s by physicist, mathematician and computer scientist John Von Neumann, a cellular automaton is an array of "cells" (arranged in one or more dimensions) in which the state or behaviour of a cell in some generation is determined by rules that involve the states of neighbouring cells in the previous generation. Simulations consist of iterative application of the rules at each time step. Cellular automata are thus discretised models of dynamical systems, both in time and space. Instead of working with aggregate variables, they propose a distributed representation of a system, where one or many state variables are assigned to each of the cells of the domain. Changes to these variables are made locally, at the cell level, based on the exclusive knowledge of the states of immediate neighbours and on rules, which act as conditional statements.

As the focus on immediate neighbours in generating the future impedes us from envisioning and appreciating the system as a whole, it is mainly through running the system (typically via a computer implementation) that one can gain insight into its collective behaviour. This emphasises the notion that the "collective behavior of the whole system cannot be simply inferred from the understanding of the behavior of the individual components" (Meyers 2011, p. v), which is a defining feature of a complex system.

Cellular automata were popularised in 1970 through John Conway's "Game of Life", aimed at emulating the evolution of a community of living organisms (Gardner 1970). Since then, they have been used as models in a large array of applications: wildfire propagation, voting dynamics, land use and urbanisation, dynamics of infection, etcetera. In each area of application, experimentation and research have led to more sophisticated versions of cellular automata. In urban simulation, for instance, the notion of neighbourhood has been extended to allow action-at-a-distance; transition rules have been adapted to accommodate hierarchy, self-modification and stochasticity (Torrens and O'Sullivan 2001). As simple models of complex systems, cellular automata also have given rise to radically new approaches for solving problems of fluid dynamics in porous media and complex geometries: movement of the fluid has been modelled as pseudo-particles moving on a grid, from a node to one of its neighbours, while the rules for handling collisions are defined so as to *preserve conservation principles* (Raabe 2004).

Some cellular automata simulators were developed (but not necessarily maintained) as outreach material on the web. Spreadsheet versions have been documented (Catterall and Lewis 1985; Hand 2005), with the claimed advantages that the workings of the model are made explicit to students and that they can be implemented easily without the need for programming skills.

Participants of the working group noted that using tiles, as was done with the Yellowstone cellular automata, could help introduce grade 8 students to recursion. They valued recursion as a "big idea" that captures the notion of transformation, much better than the function does in the way it is usually taught. The possible progression from playing a game with predefined rules to changing the rules, and then to designing completely new rules was perceived to offer a realistic development perspective for engaging students in modelling complex systems.

5.4.4 Agent-Based Models

An agent-based model of a system is a collection of autonomous and adaptive entities called agents. Each agent individually assesses its situation and makes decisions for itself on the basis of a set of rules. *Algorithmic thinking* still applies, but it explicitly, and more effectively, addresses learning and adaptation that characterise individual behaviour. Well suited to capture the complexity of biological systems, agent-based modelling has also been used, amongst other things, to approach human behaviour in

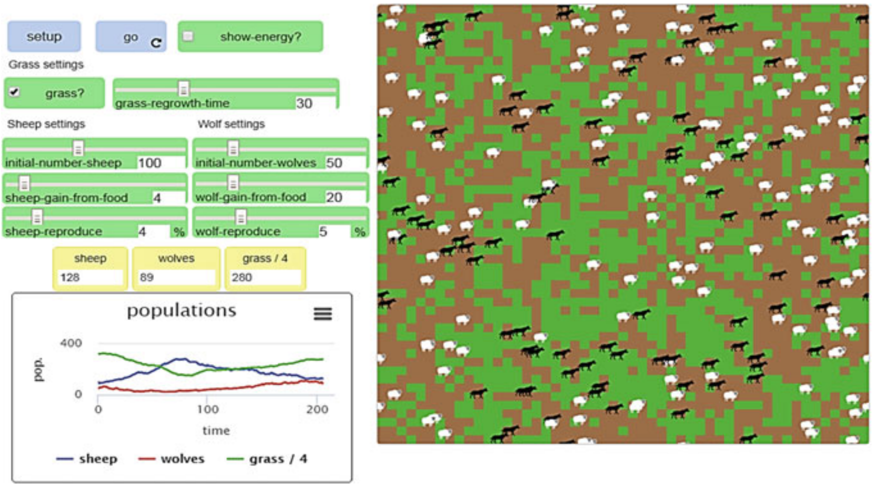

Fig. 5.3 A prey-predator simulator with NetLogo. http://www.netlogoweb.org/

collective settings (e.g. emergency evacuation) or human systems and organisations (e.g. finance markets).

As agent-based models aim to describe the behaviour of the system's constituent units (the agents) and their interactions, their coding and simulation can capture emergent phenomena, which may be hard to predict otherwise (Bonabeau 2001; Meyers 2011). Such emergent phenomena will be perceptible at a *level* higher than that of the agents, as in a flock of birds. The emerging phenomenon may be counterintuitive. For instance, students may be surprised to observe from a simulation that when too many cars try to move forward on the same highway, the resulting traffic jam will go backwards (Wilensky and Resnick 1999).

In order to favour the development of *multilevel thinking* and appreciation of how order can emerge from randomness, Wilensky and Resnick (1999) have each developed educational agent-based modelling environments for exploring complex dynamical systems. Named after the original Logo environment created by Seymour Papert, *StarLogo* and *NetLogo* were conceived as extensions of this pioneering microworld. Instead of instructing a single turtle to move, the users program the rules of movement for the agents, when and where they are generated or deleted, as well as their actions when they meet or collide. The ground where the agents move can also be organised as a cellular automaton (Fig. 5.3).

While *StarLogo* is aimed at K–12 students with a programming environment similar to the one of *Scratch*, *NetLogo* with its relatively simple programming language has been designed also for another type of user: the expert in his field with limited or no programming experience. It seems to have met a need, as the use of *NetLogo* in research papers has grown to a point where an extension has been developed to interface with R statistical software for more sophisticated interactive analysis of simulation results (Thiele and Grimm 2010).

From various experiments where he analysed student reasoning in exploring complex system simulations, Wilensky has come to the conclusion that in order to make sense of these systems, students need to use two complementary forms of reasoning: the "*agent-based*" *reasoning* to think in terms of properties and behaviour of the individual entities of the system, and the "*aggregate*" *reasoning* to think about the properties and rates of change of populations and higher level structure (Jacobson and Wilensky 2006).

In the working group, the team that went for an agent-based model of the Yellowstone situation mainly stayed at the agent level, and did not address the aggregate level explicitly. As a result, this team's game gave the impression that the potential for increased realism came with reduced opportunity for mathematical analysis.

5.5 Discussion

As can be gathered from this brief overview, research has developed and used a variety of approaches for investigating complex systems in nature and society. Although quite different, these approaches share some common features. For one thing, they all make extensive use of *computer simulation* to follow the evolution of such systems, through the *iterative application of rules or numerical schemes that describe change or interactions*. These rules or schemes are designed so as to *reflect the invariants in situations of change,* with possible adaptation. As another fundamental element, the passage to a computer treatment involves one or several kinds of *discretisation*: of time, of space, of matter.

Frejd (2017) argued that modelling as a professional activity is different from modelling as a school activity, as it is often "based on knowledge and experiences reaching far beyond what can be found in a secondary mathematics classroom" (p. 377), with computers, programming and specialised software playing a major role in the development of models and their use. That being said, there have been attempts at reducing the knowledge gap. Some of the models, techniques and underlying ideas involved in investigating complex dynamical systems appear within reach of secondary students and the development at school of such practices would expand substantially the class of problems that citizens could explore and potentially solve. Consequently, there has been sustained effort from dedicated individuals and teams in allowing students to develop these skills and notions, through the careful design and classroom use of modelling and simulation tools and learning activities.

Despite this long lasting commitment and promising beginnings in secondary education in the 1990s, the presence in school of these approaches has remained relatively marginal, at least in Canada. The same can be said within the literature on modelling in mathematics education; papers that report on experiments with students investigating dynamical complex systems in school mathematics are relatively few, and often have been written by the people that have developed resources for such investigation. Trying to explain this limited transfer of modelling practices raises questions and issues that align along different lines of inquiry regarding mathematical modelling integration in the teaching of mathematics. These will be addressed in turn.

5.5.1 Epistemological Issues

In applied mathematics, for numerical simulations based on mathematical techniques (i.e. discretisation, numerical methods), which by their very nature can only approximate the solution, a case must be made, when solving numerically a system of equations, that the error has been controlled and that the equations have been solved correctly; this is what is known as *verification* (Roache 1998). When these equations, or any other type of mathematical model, are meant to represent a situation, there is also the need for external consistency, that is, ensuring the model and its associated solution adequately represent the situation. This is the *validation* part of the modelling process, which is shared with science and engineering (Roache 1998), and for which simulations, as a new form of scientific production, have blurred the line between theorising and experimentation (Greca et al. 2014).

Verification and validation techniques for models, simulations and scientific theories are not part of the typical repertoire of teachers of mathematics, who are more familiar with proof. For them to engage with their students in simulation-based modelling activities and exploration of complex systems may require letting go of their attachment to rigour (with the risk of sending the unfortunate message that anything goes when it comes to modelling, simulation or science) or to define heuristics for the class that might help control the quality of the models and simulations. Alternatively, it could be an opportunity for interdisciplinary work with their colleagues of science.

5.5.2 Interdisciplinary Collaborations

Judging from the literature, there have been more classroom experiments with the investigation of dynamical complex systems in the teaching of science than in the teaching of mathematics. Many factors seem indeed to favour science education over mathematics education for adopting such a paradigm: the very visible presence of simulation in science, the central notion of model, the tradition with experimentation and validation, etcetera. In addition, familiar scientific principles, that are the objects of study in a science course, can be used when structuring a real-life situation. Using such principles (including those from geometry), pre-existing models, and deductive reasoning, one can build a theoretical model that not only *describes* the situation but also contributes to *explaining* it (Doerr et al. 2017).

Yet, as “a rather small number of relatively simple structures appear repeatedly in different businesses, professions, and real-life settings” (Forrester 1993, p. 189), there is an opportunity, and maybe even a need, for mathematics education to become involved. Some elements of these structures can already be mapped with mathemat-

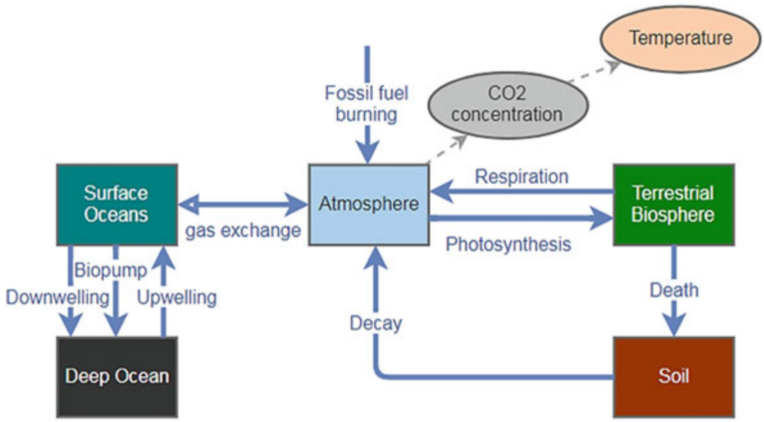

Fig. 5.4 Structure of a simple model of the carbon cycle with Insight Maker. https://insightmaker.com/insight/79473/Global-Carbon-Cycle

ical content (e.g., flow with rate of change or derivative). While simple generic structures such as feedback loops may not (yet) be part of the mathematics curriculum, they can still be used as a learning context to introduce prescribed content (e.g. exponential functions) and to provide opportunities for additional complexity, possibly in collaboration with science teachers.

Such collaboration can bring to the fore the role of mathematics in assessing the validity of assertions. For example, as some like to reduce climate change to a matter of belief, we could move the discussion down to simple models (Fig. 5.4) that can be used to both explain the phenomenon and predict its evolution. This would have students look into the assumptions that are made, the principles that are used, their translation into mathematical equations and their logical implications.

Another potential contribution of mathematics education to the study of complex systems would be to develop an understanding of the mathematical work involved in the simulations. If we are to teach modelling of complex systems to raise critical awareness in future citizens on important issues, then they must be equipped with the knowledge of how such models can produce results. This would mean crossing into algorithms and computational thinking.

5.5.3 Technology and Computational Thinking

In her seminal paper, Wing (2006) pleaded for the importance, in this 21st century, of developing *computational thinking* as a fundamental skill for everyone. Several elements she described can be associated with the modelling of complex dynamical systems or with the habits of mind that were unveiled within our experiment (Sect. 5.2):

reformulating a seemingly difficult problem into one we know how to solve, perhaps by reduction, embedding, transformation, or simulation

thinking recursively

choosing an appropriate representation for a problem

modeling the relevant aspects of a problem to make it tractable

using invariants to describe a system's behavior succinctly and declaratively. (p. 33)

As summarised by Wing (2006, p. 34), "computational thinking is more than being able to program a computer. It requires *thinking at multiple levels of abstraction.*" It encompasses "the thought processes involved in formulating a problem and expressing its solution(s) in such a way that a computer—human or machine—can effectively carry out" (Wing 2014, p. 8). As the development of such a wide-ranging habit of mind may very well be influenced by the tools we use, and as the "practice of choosing effective computational tools" can also be considered part of computational thinking (Weintrop et al. 2016, p. 139), there may be a need to take a closer look at the tools and tasks with which K–12 students learn to model. The objective should be for students to grow with these tools, as they learn to explore, modify or build their own models of increasing complexity, and to develop through the process *conditional, recursive* and *multi-level thinking*. Some of the black boxes hidden in those tools should be opened progressively, as students learn the mathematics involved.

To support all of this, there was a general consensus within the working group that it would be beneficial to have students learn programming at school. But mathematics education could also help by building on concepts and approaches that are the most effective at opening the class of problems that can be solved. This brings us back to the mathematical content.

5.5.4 Curriculum and Mathematical Content

It may still be a question of perceived distance between the knowledge that appears useful to approach dynamical complex systems and the mathematical content to be taught, as prescribed by the curriculum. Yet, there are some elements in today's mathematics curriculum with which connections could be established or reinforced for the modelling and simulation of dynamical complex systems.

Prior learning of probability diagrams (grids, sets, trees,…) could be built upon to recognise and value such discrete structures as powerful models: not just for describing the situation and communicating the variables of interest (which already is an important asset for interdisciplinary collaborations), but also for suggesting the algorithm to do the mathematical work required to generate the solution.

Recurrence relations may be present in school mathematics, but they are sometimes limited to an auxiliary status to introduce or to characterise functions (e.g. Ministry of Education, Ontario (MEO) 2005). In the ICTMA book series, chapters addressing the use of discrete models are much fewer than those addressing the use of continuous functions. While there is a recognition that discrete models are rarely taught in school, and that students, when modelling a situation, go directly to well-known functions without much reflection (Kaiser et al. 2011), we see at the same time other researchers (e.g. Amit and Neria 2010) consider the recursive approaches that students naturally tend to adopt as a weaker form of generalisation than a functional approach. Without denying the importance of the concept of function, we may have to acknowledge that the attention it receives tends to eradicate other valid alternative modelling approaches, which could prove more effective in producing a solution. Encouraging students to describe patterns in a recursive fashion opens the door to computation on a spreadsheet and simulation. It may lead later to a better appreciation of how, when applied on a progressively smaller time scale, such change descriptions develop into differential equations, and how these can be integrated numerically, using the same recursive approach. With the ease of understanding Euler's method, the box of numerical integration of system dynamics software does not have to be black.

We could also value from a mathematical perspective explorations done with cellular automata; not only can they be seen as a generalisation in more dimensions of recursive formulas, where stability and periodicity can be simultaneously observed, but also they can be considered a valid initiation to the learning of multi-variable functions, where the space domain is discretised and where the table of values is stored in a matrix. Computing the state of a new cell based on the state of its neighbours is actually similar to what is done in numerical schemes that are used to solve partial differential equations. For instance, the partial differential equation that models 2D heat equilibrium can be discretised and transformed into an iterative numerical scheme in which the temperature at a point on a grid is repeatedly adjusted to the average of the temperature of its four closest neighbours on that grid. As numerical methods can sometimes be intuitive and disconcertingly easy, with only a few arithmetic operations involved, it may seem regrettable that these simple iterative schemes are not made part of the students' toolbox before they have managed to master all advanced calculus techniques.

Agent-based modelling may be harder to connect to specific mathematical content, but it could well be an interesting entry into programming, which constitutes, in itself, a very rich toolbox to tackle complexity and a limitless playground for using mathematics. If programming ever gets to be thought of as a mathematical activity, its learning and its use for integration of modelling in mathematics education could build on some mathematically productive habits of mind students may have developed in playing computer games.

5.6 Conclusion

Organising a mathematics curriculum around *habits of mind*, so that students "become comfortable with ill-posed and fuzzy problems" (Cuoco et al. 1996) appears more than relevant in our increasingly complex and vulnerable world. There is indeed a need both for future professionals to deal with our most pressing environmental and societal challenges, and for citizens in general to have a better understanding of what is at play when predictions, assessments or decisions are made, as "models have a huge impact on our world" (Doerr et al. 2017). Yet, some of the key mathematical habits of mind that are required for approaching complex systems are anchored in concepts, techniques and tools that, though accessible and sometimes even natural to secondary students, have received little attention in general mathematics education. If we consider that "mathematics is about the study of pattern and structure, and the logical analysis and calculation with patterns and structures" (Brown and Porter 1995), then we should be looking for a set of structures and approaches that, while allowing appreciation of the internal coherence of mathematics, could increase our capacity to understand our world and tackle its key challenges.

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Chapter 6
Precision, Priority, and Proxies in Mathematical Modelling

Jennifer A. Czocher

Abstract In recent years, scholars have moved away from "modelling as a vehicle" to learn mathematics approaches and have instead emphasized the value of modelling as content in its own right. This shift has raised tensions in how to reconcile authentic mathematical modelling with curricular aims. The aim of the research study reported in this chapter is to explore one aspect of this tension: the divergence of student thinking from the task-writer's intentions. Analysis of task-based cognitive interviews led to two interrelated findings: participants' choices did not lead to intended solutions (nor to curricular objectives) and participants' choices were guided by their giving priority to variables and assumptions that aligned with their desire to reflect precision and complexity of their lived experiences of the task situations being modelled. Two common interpretations of such findings are to fault the participants as incapable of applying their knowledge to solve the problems or to fault the tasks as being inauthentic. I use actor-oriented theory of transfer to reconcile these opposing views.

Keywords Actor-oriented theory · Mathematizing · Student cognition

6.1 Introduction

Historically, scholars understand there are "two fundamentally different purposes when teaching mathematical modelling" (Stillman et al. 2016, p. 283) in the classroom (Julie and Mudaly 2007; Niss et al. 2007). One is to use "modelling as a vehicle for facilitation and support of students' learning of mathematics as a subject" (Niss et al. 2007, p. 5). The other is to learn mathematics "so as to develop competency in applying mathematics and building mathematical models" (Niss et al. 2007, p. 5). These authors stressed that these approaches are not a dichotomy, meaning neither tasks nor facilitators' intentions in using the tasks must be classified as one or the other. Though the role of mathematical modelling in achieving curricular aims has

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both amplified in recent years (e.g. National Governors Association Center for Best Practices and Council of Chief State School Officers 2010; Niss and Hojgaard 2011; OECD 2017) and undergone attempts at standardization (e.g., Bliss et al. 2016), in many classrooms the emphasis remains on the teaching of modelling as a vehicle for teaching mathematical concepts and processes.

A wide variety of tasks are used to further curricular aims, ranging from word problems, to application problems, to original projects (Blum and Niss 1991). Using such tasks to plan and sequence learning trajectories for students means the tasks must have *intended solutions*, a predetermined strategy, heuristic, process, or outcome that aligns with mathematical learning objectives. In this chapter, I focus on tasks which were designed to further specific curricular aims and therefore have intended solutions from the task setter's perspective.

However, one challenge in using modelling in classrooms is that the solution of a modelling task is not inherent to the task itself (Czocher 2015; Murata and Kattubadi 2012; Schwarzkopf 2007). For example, Manouchehri and Lewis (2017) reported on 1000 middle school students' solutions to the word problem *Which is the best job option, one that pays \$7.50/hour or one that pays \$300/week?* The task is used to address the topic of linear equations. The intended solution is to formulate two linear equations, $y = 7.5x$ and $y = 300$ and seek their intersection. Since $x = 40$ at the intersection, the two job offers are supposed to be equivalent. However, the intended solution only makes sense under two implicit assumptions: (i) only the number of hours worked per week matters (ii) 40 h per week is expected. The students in the study did not operate under these assumptions. They considered issues like the cost of transportation, health care benefits, and whether or not full-time employment was feasible. These considerations do not lead to the intended solution, but they are not "wrong."

The difference between student reasoning and the intended solution can be accounted for in terms of socio-mathematical norms developed at school. Watson (2008) argued that school mathematics is its own discipline, and therefore is apart from professional mathematics. For example, some word and applications problems can be solved by referring to semantic cues, without any reference to mathematics or the story in the problem (Martin and Bassok 2005). For students and teachers this may mean that modelling devolves into a search for official formulas, recalling a similar problem from class, or attending only to keywords. Even in a laboratory setting, students working on problems couched in a real-world context can be influenced by the expectations of school mathematics to give "more legitimate" solutions based on known formulas (Schoenfeld 1982b). Similarly, Julie and Mudaly 2007) hypothesized that teachers express a preference for models that are relevant to their immediate circumstances. Thus, a typical response to students using their "real world" reasoning, like those in Manouchehri and Lewis's (2017) study, might be to dismiss it as incorrect in order to refocus the student toward the intended solution. While this option may lead to short-term success, it can also have long-term consequences. Students who receive consistent negative feedback may learn to respond to problems in ways consistent with the expectations of "school mathematics" rather than with their own reasoning (see, for example, Engle 2006). Indeed, the literature is full of examples of students who generalize their own (non-mathematical) rules based on this kind of feedback or who do not check whether their own responses make sense (Erlwanger 1973; Greer 1997; Schoenfeld 1982a, 1991; Verschaffel et al. 2000).

These lines of inquiry have influenced research into the teaching and learning of modelling. Scholars have shifted their focus onto students' current knowledge and understanding (Blum and Borromeo Ferri 2009; Doerr 2006; Schukajlow et al. 2015; Stender and Kaiser 2015; Wischgoll et al. 2015). While a socio-mathematical perspective articulates the tension between school mathematics and student thinking, it does not yet account for how students might make sense of modelling tasks that are used to further curricular aims. Some work still needs to be done on how to anticipate what students might suggest and how to productively interpret those suggestions. The purpose of this study was to explore the ways in which students' ways of reasoning might diverge from the intended solutions of the task setter who aims to provide students with experiences addressing particular curricular objectives.

6.2 Empirical and Theoretical Background

There are many theoretical perspectives on the nature of mathematical modelling and what it entails. Kaiser (2017) provides a recent and comprehensive survey. One perspective is termed a cognitive approach because it foregrounds mathematical thinking and emphasizes analysis of students' modelling processes. Since the main goals of the cognitive approach are to reconstruct individuals' modelling routes or to identify difficulties encountered by students during their modelling activities (Kaiser 2017), it is an appropriate approach for studying how student reasoning diverges from intended solutions while working on tasks with intended curricular aims.

In the cognitive view, modelling is a process that transforms a non-mathematical question into a mathematical problem to solve. A model is then a conceptual correspondence between real-world entities and phenomena and a mathematical expression. The modelling process can be decomposed into a series of cognitive and mathematical activities (e.g. Blum and Leiß 2007; Maaß 2006) which replace a real-world system with a mathematical interpretation that can be analysed mathematically. Results are then interpreted in terms of real-world constraints and assumptions and the model is modified if necessary. Simplifying/structuring and mathematizing are central to setting up the mathematical problem to solve. They are most challenging to carry out (Galbraith and Stillman 2006; Stillman et al. 2010). Simplifying/structuring includes identifying conditions and assumptions from the real-world context, establishing variables, and acknowledging that some variables or constraints are unimportant. Mathematizing refers to introducing conventional representational systems (e.g., equations, graphs, tables, algorithms) to present mathematical "properties and parameters that correspond to the situational conditions and assumptions that have been specified" (Zbiek and Conner 2006, p. 99).

The cognitive approach highlights the role individuals' prior knowledge and decision-making play in mathematical modelling. Stillman (2000) reported on a tripartite framework distinguishing three knowledge sources used by secondary students during mathematical modelling: academic, episodic, and encyclopaedic. Each knowledge source derives from the individuals' prior experiences. *Academic knowledge* derives from the study of academic topics (e.g. linear equations, kinematics). *Encyclopaedic knowledge* is general knowledge about the world (e.g. that one ought to check for traffic before crossing a road). *Episodic knowledge* is truly personal and experiential (e.g. recalling a ride to the top of the Empire State Building on a recent trip to New York City). Most reasoning during mathematical modelling occurs as a blend (Fauconnier and Turner 2003) of real-world knowledge and mathematical knowledge (Czocher 2013). Yet, Stillman (2000) found that episodic knowledge has a stronger influence on mathematical modelling than the other two forms of knowledge, suggesting that students draw more from their personal experiences than from what they learn in other subject areas or general world knowledge. Therefore, how individuals engage in modelling depends as much on their prior non-mathematical experiences as an on their mathematical knowledge.

Yet knowledge on its own is not a good predictor of task performance. Research from a long line of inquiry into transfer of knowledge has demonstrated that possessing relevant knowledge of mathematics or of the modelling task context is not sufficient for addressing the task (Nunes et al. 1985; Verschaffel et al. 2000). Equally important are whether the individual brings her knowledge to bear on the task and the decisions she makes about how to use that knowledge. Specifically, because modelling involves generating idealizations of the real world situation (Borromeo Ferri 2006), any decision made by the modeller to simplify the problem filters, and is filtered by, the individual's knowledge sources. As the study of Manouchehri and Lewis (2017) shows, differences between the intended solution and the students' ideas are not limited to the peculiarities of school mathematics—they depend on students' encyclopaedic and episodic knowledge. In their Job Problem, the intended solution assumes that the only meaningful variable is number of weekly hours worked. The students raised issues based on their encyclopaedic and episodic knowledge; they wished to consider health care benefits and ease of transportation. Considering these important variables necessarily changes the mathematics used. For example, if transportation is the most important factor (rather than hours worked) an individual should choose the job she can get to reliably rather than the job she cannot get to at all.

The interdependency of phases of modelling with individuals' knowledge leads to idiosyncratic and non-linear individual modelling routes (Ärlebäck 2009; Borromeo Ferri 2006, 2007; Czocher 2016). The term idiosyncratic responses means that making sense of, or responding to, student work on modelling tasks, even in tasks purportedly as straightforward as those with intended solutions, is difficult. For example, Schoenfeld (1982b) asked undergraduate mathematics majors to estimate the number of cells in an adult human body. The intended solution was a "ballpark estimate," based on the assumption that a human is shaped roughly like a cylinder and crude estimates of the cylinder dimensions. Instead, the participants sought increasingly finer estimates of the volume of the human body, without pausing to evaluate their own productivity. Since the marginal increases in precision for measurements of human volume would not have impacted the cell estimate substantially, Schoenfeld interpreted the students' work as an example of metacognitive failure.

A teacher's response in this situation might also have been to classify the student's work as incorrect because the student did not use the intended strategies. However, given the relative scale of human volume to cell size, the students' activity can be understood as sensible. Likewise, a student who answers the Job Problem with the question "Is there a bus stop at both jobs?" might be considered to be evading the mathematical problem.

The foregoing discussion raises questions about how to interpret students' thinking on modelling tasks productively. In particular, we can wonder: How does learners' real-world knowledge guide their selection of relevant variables and assumptions? and Are there productive ways to frame students' choices that can guide facilitators? To answer such questions, it is necessary to examine students' modelling behaviour within the task environments that they may encounter in classrooms from a perspective that assumes the students' responses are sensible.

To study how student work diverges from intended solutions, I selected Lobato's (2006, 2012) *actor-oriented theory* of transfer as a theoretical lens. This is an appropriate choice because from a cognitive perspective, the modelling process is conceived as a blend of disparate knowledge bases, implying that some form of transfer of knowledge to a novel setting occurs. Viewing individuals' knowledge as experiences then allows examination of how "rational operations emerge from experience" (Jornet et al. 2016, p. 290). That is, actor-oriented theory begins from the perspective that students' activities are sensible.

As a premise, actor-oriented theory distinguishes between an actor's perspective and an observer's perspective. Thus, there is a natural mapping between the (actor, observer) pair to the (student work, intended solution) pair. In the language of actor-oriented theory, "taking an observer's point of view entails predetermining the particular strategy, principle, or heuristic that learners need to demonstrate in order for their work on a novel task to count" (Lobato 2012, p. 245). In contrast, from an actor's point of view, the researcher investigates how the student's prior experiences shaped their activity in the novel situation, even if the result is non-normative or incorrect performance (Lobato 2012). In summary, "solutions which might be viewed as erroneous from a disciplinary perspective, are treated instead as the learner's interpretation" of the task (Danish et al. 2017). In this way, the operational definition for intended solution becomes a "predetermined particular strategy, principle, or heuristic" and the focus of the present study is on how the participants interpret the task situation. Under actor-oriented theory, the *authenticity* of a task is determined by the extent to which the task context aligns with, and is amenable to, the participants' lived experiences. Thus, modelling problems are those that permit students to bring their knowledge to bear in defining their own variables and introducing their own assumptions.

The actor-oriented theory of transfer can be applied to modelling because it acknowledges that knowing and representation are products of how the student interprets the task situation and that the selection of ideas need not be intentional (Jornet et al. 2016; Lobato 2012). Within modelling, structuring refers to imposing mathematical structure on a real-world situation. This is accomplished through introducing variables and parameters which measure attributes of entities in the real world

Fig. 6.1 Analytic framework to examine students' decisions while structuring the problem situation to be mathematized

(Thompson 2011). Real-world conditions and assumptions are also identified. The variables, parameters, conditions, and assumptions are then put in relation to one another, using mathematical objects, their properties and structures, and relations and operations to join them. As discussed above, each of these activities depends on the individual modeller's current interpretations and prior experiences. The idea that structuring is an active process carried out by the modeller, rather than a passive process where an inherent structure is present in a situation and then discovered and extracted, is also emphasized in the actor-oriented theory perspective.

To study how individuals' models may diverge from intended solutions, an analytic framework capable of capturing student decision making while tracing the intended solution was needed. The framework needed to allow me to document how the participants defined a mathematical problem from a nonmathematical one. The process is not straightforward and there are many cognitive obstacles within it (Galbraith and Stillman 2006). Since the process includes anticipating the mathematical structures and procedures that could be used and then implementing that anticipation, the framework needed to include identifying, prioritizing, and mathematizing appropriate variables, conditions, and assumptions (Czocher and Fagan 2016; Niss 2010; Stillman and Brown 2014). The analytic framework, summarized in Fig. 6.1, zooms in on the *simplifying/structuring* phase of modelling (see Blum and Leiß 2007). The framework is appropriate because each successive step is a site where the modeller's choices may diverge from the intended solution. Therefore, the framework allows for divergence to be documented as described below in the methods section and allows for the research questions to be addressed.

6.3 Methods

I conducted a laboratory-based study of how student thinking diverged from intended solutions on tasks with intended curricular aims.

6.3.1 Data Collection

Data were generated via a set of one-on-one task-based interviews with twelve students enrolled in high schools (8) and universities (4) from different states in the

Table 6.1 Task and participant details in task-based interview

Problem (source)	Statement	Number of participants and mathematical level
Letter Carrier (Swetz and Hartzler 1991)	A letter carrier needs to deliver mail to both sides of the street. She can go to all the boxes on one side, cross the street, and deliver to all the boxes on the other side. Or she can deliver to one box, cross the street, deliver to two boxes, cross and deliver to two boxes and so on until all the mail has been delivered. Which is the best route?	4 post algebra, 3 algebra
The Cell Problem (Schoenfeld 1982b)	Estimate how many cells might be in an average-sized adult human body.	3 advanced, 2 post algebra
Water Lilies/Yeast (Czocher 2016)	Water lilies on a certain lake double in area every twenty-four hours. From the time the first water lily blooms until the lake is completely covered takes sixty days. On what day is half the lake covered?	2 advanced, 2 algebra, 2 post algebra
Empire State Building Problem (Ärlebäck 2009)	Devise a method to predict how long it would take to ascend the Empire State Building.	4 advanced, 2 post algebra

United States. There were four participants from each of the following levels: high school algebra, post-algebra (high school geometry and calculus), and undergraduate differential equations. The purpose of including mathematically and geographically diverse students in the sample was to explicitly seek similarities in their ways of approaching the problems, not to treat them as comparison groups.

This study examines student work on the four tasks presented in Table 6.1. As shown, the tasks were drawn from prior research and research-based educational materials. Tasks were appropriate to each student's mathematical level and each had a clear curricular objective, that is, mathematics content that would be brought out if the student carried out the task writer's intended solution. However, the tasks were presented in a way that allowed the participants to generate their own variables and assumptions. In this way, each task would allow me to trace the cognitive pathways learners might take which would reveal the tensions between student thinking and the intended solution.

At the start of each session, the participant was presented with a task and asked to read it aloud. I assured participants that they would not be graded as I was interested only in their thinking. Participants worked for as much time as needed to come to a conclusion (usually within 30 min). Follow up questions focused on understanding how important the students' choices for variables and assumptions were to them. In this way, the interviews elicited the students' mathematical thinking as they engaged in the modelling tasks, not on guiding the student to an intended solution.

6.3.2 Data Analysis

The participants generated 24 sessions, which were transcribed. Analysis focused on how students defined a mathematical problem to solve by decomposing student work according to the analytic framework (Fig. 6.1) and comparing their work to the intended solution for each task. Each student's work on each task was analysed for whether they engaged in mathematical modelling, what variables and assumptions were identified (mentioned explicitly), whether they were prioritized (designated as being important to the model), and whether or not they were mathematized (represented mathematically). "Variables" designated independent and dependent variables, parameters, or constants that referred to measurable attributes of a physical entity (see Thompson 2011). "Assumptions" were defined as constraints of the real-world situation that participants identified explicitly or implicitly as impacting the values of, or relationships, among variables of interest.

To understand how student-generated models diverged from the solutions envisioned by task writers, I examined the extent to which student-generated variables and assumptions differed from those in the task writers' intended solution. In the intended solutions, I classified a variable or assumption as identified under two conditions: (1) if it was mathematized or (2) if the intended mathematisation necessitated that a variable or assumption be ignored. An outline of the intended solutions, along with intended variables and assumptions, and curricular objectives (aligned with CCSSM 2010) follows:

Letter Carrier: Assume a straight road with length l and width w. Assume that the street has n evenly spaced mailboxes on each side of the street, that the mailboxes are directly across from one another, and that they are at the centre of each lot. Let d_A and d_B be the distance travelled along the first and second paths, respectively. Then $d_A = 2l + w - l/n$ and $d_B = nw + l$. We find that $d_A = d_B$ when $l/n = w$ or when the width of the road is equal to the width of each lot. As long as $w < l/n$, the second path will be shorter. *Curricular objective*: linear equations, working with variables and parameters. *Assumptions*: the road is straight, mailboxes are equally spaced, mailboxes are directly across from one another, mailboxes are at the centre of each lot, there are an equal number of mailboxes on each side, the "best" route has the shortest distance. *Variables*: number of mailboxes, length of the street, width of the street, total distance travelled.

Cell Problem: Assume cells are cubes whose dimensions are approximately 1/5000 of an inch on a side. Assume a human is a box with dimensions $6' \times 6'' \times 18''$. *Curricular objective*: proportions, rates, estimation. *Assumptions*: cells are shaped like cubes, humans are shaped like boxes, cells are packed inside of humans. *Variables*: cell side length, human height, width, depth, number of cells.

Water Lilies: Since the number of water lilies doubles every day, on day $N - 1$ there are half as many lilies as on day N. Therefore, on day 29 there are half as many lilies as on day 30. Since the lake is covered on day 30, the lake was half covered on day 29. *Curricular objective*: exponential growth. *Assumptions*: each lily produces one new lily during the growth period. *Variables*: growth period, growth rate, final time.

Empire State Building: For an object moving at a constant rate, distance is speed multiplied by time: $d = r \times t$. Estimate the height of the Empire State Building, the speed of the elevator, and solve for t. In order to use this model, one must implicitly assume that the elevator makes no stops and that its speed is constant. The latter is reasonable if r is taken to be the average velocity over the duration of the ascent. *Curricular objective*: rates, linear equations. *Assumptions*: elevator makes no stops, moves at constant speed. *Variables*: height of building, rate of elevator, time elapsed.

To understand how students handled the variables and assumptions they generated, I listed all variables and assumptions referenced by each participant on each task. The result was the set of variables and the set of assumptions identified on each task collectively by all participants. Note that some participants generated more than one model on a given task. I then tabulated the frequency that each variable and assumption was referenced across all participants: how many times a variable or assumption was identified, how many times it was prioritized for inclusion in a mathematical representation, and the number of times it appeared in a mathematical representation. If a variable or assumption was mathematized, it was assumed to have been prioritized. For example, if a participant included "height of the Empire State Building" in her representation but never stated it verbally, it was assumed to have been both identified and prioritized. Next, I calculated the following percentages from the analytic framework:

$$\%\ identified = \frac{\#\,\text{times identified}}{\#\,\text{participants who worked on the task}}$$

$$\%\ \text{Prioritized} = \frac{\#\,\text{times prioritized}}{\#\,\text{times identified}}$$

$$\%\ \text{Mathematized} = \frac{\#\,\text{times mathematized}}{\#\,\text{times prioritized}}$$

6.4 Results

Data analysis led to two interrelated findings: (a) participants' choices did not lead to the intended solutions and (b) their selection of relevant variables and assumptions reflected their desire to represent complexity (rather than to simplify). In elaborating these findings below I show how their episodic and encyclopaedic knowledge influenced their mathematical choices via examples of how their work diverged from the intended solutions.

No participants produced the intended solution for the *Cell Problem* or the *Letter Carrier Problem*. None produced the intended solution for the *Water Lilies Problem* on their first attempt. All identified important variables and assumptions for the *Empire State Building Problem*, but offered additional variables and assumptions which led to unintended solutions. Since the participants' work did not match the intended solutions, there are two straightforward interpretations I will refute. First, the problems were too hard for the participants. Second, the participants failed to transfer their mathematical knowledge to a novel, real-world problem. Closer inspection of the data revealed that neither interpretation is accurate. Indeed, in every case the participants used well-known standard mathematical structure (e.g., linear equations, proportions, etc. Niss et al. 2007) even though their work was idiosyncratic and sometimes ad hoc. Thus the participants' models were, in most cases, completely reasonable given the variables and assumptions they identified and prioritized.

Table 6.2 displays the number of assumptions and variables identified by the participants on each task and compares it to the number of assumptions and variables in the intended solution. On all problems, participants collectively identified more variables and assumptions (except for *Letter Carrier*) than were in the intended solutions. This fact (i) implies that the participants knew enough about the task situations to gain entry to the problems (ii) demonstrates that participants had little difficulty in this stage of modelling (iii) suggests that they were engaged in the problems, and (iv) relied on real-world knowledge to help them analyse the situations. These four inferences together refute the interpretation that the problems were too hard.

Table 6.2 Number of variables and assumptions collectively identified by participants, compared to number intended

Problem	Assumptions identified		Variables identified	
	Intended solution	Participant work	Intended solution	Participant work
Letter Carrier	6	4	4	11
Cell Problem	3	10	5	12
Water Lilies	1	4	3	13
Empire State Building	4	4	3	11

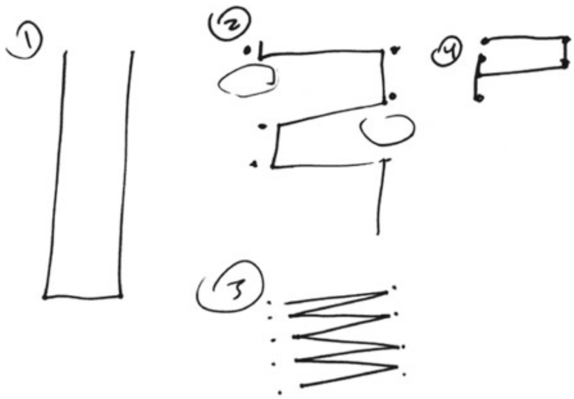

Fig. 6.2 Different potential paths for the letter carrier sketched by an algebra student (Czocher and Moss 2017, p. 657)

On the contrary, participants tried to include all variables and assumptions that could impact the selected dependent variable. For example, on the *Letter Carrier Problem*, 4/7 participants discussed various street layouts and mailbox arrangement and how each would impact the letter carrier's path (see Fig. 6.2). One participant explicitly assumed that the letter carrier did not skip any houses (otherwise, to her, the second path would not make any sense at all). Another noted that mailboxes could be arranged directly across from one another (as in the intended solution) or they could be grouped together in a common area where all residents could retrieve their mail.

On the *Empire State Building Problem*, all participants identified the intended variables: speed of the elevator, height of the building, and time elapsed. However, they also identified additional factors affecting the time of ascent: acceleration, weight, the number of stops made, how long it takes for people to load and unload. Along with these went a variety of assumptions and observations such as whether or not floors below the observation deck were open to tourists or whether the rate of ascent would be constant. In this way, the students treated the tasks authentically, based on their episodic knowledge of streets and mailboxes and elevators.

On the *Cell Problem*, only one participant (an undergraduate) gave a ballpark estimate. Instead, participants were concerned about cell shapes and sizes varying over the body, rather than the shape or size of the body. They noted that bones and various organs were made up of different kinds of cells. Some mentioned that nerve cells could be a metre long whereas reproductive cells were much smaller. These concerns signal an unease in accepting that a set of measurements which vary can be replaced by the average of those measurements, which the intended solution expects.

These observations both support and challenge the finding of Schoenfeld (1982a, b) that his participants were concerned with finding "more legitimate" solutions. The participants in this study identified sources of variation based on the function of the cells (rather than on location) and desired their models to reflect those sources of variation. This does not necessarily constitute a "wild goose chase" or metacognitive failure. Rather than assume some dimension of variation could be eliminated (or rendered irrelevant altogether in order to simplify the problem), the participants were driven by a desire for the model to accurately and precisely capture their real-world knowledge of the task situations.

Similarly to other reports, participants identified variables and assumptions based on their episodic knowledge, their encyclopaedic knowledge (Stillman 2000), or through immediate relevancy to their lives (Manouchehri and Lewis 2017). This was evidenced by responses containing value statements or clarifying questions. For instance, responses included: (1) The letter carrier should take the simplest path. (2) Is there traffic? If so, the letter carrier should take the first path which is safer. (3) Does the letter carrier need to return to her vehicle? (4) Does the letter carrier have to visit every house? (5) What shape is the street? (6) What is the purpose of estimating all of the cells in the human body? It would make more sense to count T-Cells or heart muscle cells after a heart attack. (7) Should I count the non-human cells? (8) It makes more sense to time the elevator. (9) It depends on how big the lilies (lake) are.

Whereas considerations like: Does the size of the lake matter? get right to the heart of the curricular objectives of a modelling problem that uses exponential growth, the others might be interpreted as attempts to avoid developing a model altogether. Others have suggested writing tasks that avoid this tendency (see, for example, Lesh et al. 2000). However, I offer an alternative interpretation: the participants were not necessarily "avoiding" the problem, but offering a logical, well-reasoned response based on their personal knowledge of the world and the heuristic "what would this situation actually look like?" Individuals develop heuristics for quickly handling decision-making in real-life situations (Gigerenzer 2008), which may support them in identifying important variables and assumptions for mathematical modelling. The participants' responses clarify the "rules" of the real-world situations described in the task statement and led sometimes to simplifying the situation to make it amenable to mathematical representation or at other times complicated it.

The majority of variables identified on a task were also mathematized in at least one participant's representation (9/11 on the *Letter Carrier Problem*, 11/12 on the *Cell Problem*, 11/13 on the *Water Lilies Problem*, 9/11 on the *Empire State Building Problem*. This suggests participants' difficulties lay in selecting the most important variables and assumptions in order to fit them to known mathematical concepts. For example, all participants who worked on the *Cell Problem* observed that the density or arrangement of cells varied over body parts, all of these acknowledged that the observation was important, but no one was able to mathematize the assumption. One undergraduate progressed so far as describing something like a weighted average for the different organs in the body, but abandoned this strategy before producing a mathematical representation. Similarly, on the *Letter Carrier Problem*, participants intended to include the shape of the street and variation of mailbox placement because both of these variables impact distance travelled. As a consequence, only 2/7 (29%) of the students were able to mathematize distance.

The majority of identified and prioritized variables did appear in at least one mathematical representation but this representation did not use the mathematics of the intended solution. For example, on the *Letter Carrier Problem*, two students focused on the arrangement of the mailboxes along the street because it would impact the total distance the letter carrier would travel. Their images of the street led to drawing a zig-zag path for the letter carrier to follow. Both created paths that would minimize distance between mailboxes, leading to mathematisation via the Pythagorean Theorem. These choices led to quadratic equations in two variables rather than the intended linear ones.

In the *Empire State Building Problem*, one undergraduate participant gave the following mathematical model:

$$t = 2T(p) + h/v; \quad T(p) = \text{enter/exit rate} \times p + \text{doors}$$

where t was the total time, *T(p)* was the length of time it takes for people to enter (or exit), h was the height of the building, and v was the velocity of the elevator. He computed the length of time for people to exit as some per person rate times the number of people plus the length of time for the doors to open and close. The student transformed a problem about rate into a pair of affine linear equations depending on the number of people riding the elevator.

In the intended solutions, many of the variables and assumptions identified, prioritized, and mathematized by the participants were assumed to be unimportant, leading to simper models. However it is not necessary or even necessarily natural for students to seek these simpler models. At the very least, the participants' choices led to mathematical concepts that were not the same as the curricular objectives of the tasks. And in these cases, the participants' models might be seen as "incorrect" when compared to the intended solutions.

6.5 Interpretation and Discussion

In this study, participants tended to prioritize variables and assumptions in order to authentically reflect the complexity they perceived in the situations. They did so regardless of whether the variable or assumption could be mathematized, regardless of the magnitude of its impact, or even whether the resulting mathematical problem could be analysed with their on-hand mathematical tools. Thus, as in other studies (e.g. Czocher 2013; Ikeda and Stephens 1998; Manouchehri and Lewis 2017) participants struggled to prioritize those variables and assumptions that could be mathematized using their current mathematics knowledge, over those that could not.

Even though each participant had difficulty prioritizing variables he or she identified, most variables and assumptions identified appeared in at least one participant's representation. Taken together, these observations refute the idea that participants were unable to transfer mathematical knowledge to a novel problem situation. Instead, the evidence highlights how students' knowledge contributes to the contrast between student work and intended solutions in ways that parallel tensions which arise for those who wish to teach with modelling in the classroom.

In particular, participants prioritized variables and assumptions that would *preserve precision*. Each participant prioritized different variables and assumptions, which were amenable to different mathematical content or representations. It was therefore uncommon that two participants starting from differing sets of initial variables and assumptions produced the same model, let alone the model of the intended solution which was tied to curriculum content goals. On the surface it would seem that letting students freely and authentically engage in modelling, even on routine or simple word problems, is incompatible with meeting the curricular goals a teacher might use these tasks for. Moreover, managing a classroom full of distinct solutions seems daunting, a tension that has been reported before (e.g., Chan 2013; Tan and Ang 2013).

Concluding that student models are incorrect because they do not match the intended solutions or use curricular mathematics implicitly assumes that the intended solution is correct. It assumes that many of the variables and assumptions important to the participants should be neglected or assumed constant. But such assumptions cannot always be justified. For example, in real life the (average) speed of the Letter Carrier will be slower if she chose the second option or stops at more mailboxes. The many crossings require her to change direction more often and also to check whether she can safely cross the road. If there is a lot of traffic, she may not cross the road at all until she arrives at a pedestrian crossing. In the *Empire State Building Problem*, the door opening and closing speeds could be conceptualized as a constant that affects time to ascend the building but would not vary from trip to trip, unless there were more or fewer people entering and exiting. Yet the idea that only the potential distance travelled by the mail carrier or the elevator or the number of hours worked should be considered and that "all other things are equal" is an implicit assumption. These assumptions reveal the mathematical structures aligned with curricular content and representations and were not adopted unproblematically for the participants exactly because of their lived experiences, for example, waiting to cross the road safely.

Such choices simplify the problem situation in order to make it fit the target mathematics. However, student success in using mathematics to model real world situations is tied to their ability to see a correspondence between the behaviour of the system to be modelled and its potential mathematisation (Camacho-Machín and Guerrero-Ortiz 2015). Evidence presented here supports the claim that students desire that the mathematical model accurately reflects their lived experiences and empirical observations. This desire can create tension with the conventional simplifications suggested by the intended solutions. Conventional simplifying choices may seem arbitrary to students and contradict what they know to be true about the world. However, the preference for conventional assumptions that target curricular mathematics amounts to just that: preference. Thus intended solutions are correct insofar as they are privileged above other models.

Part of the tension that arises when using modelling as a vehicle to foster students' engagement with mathematics content (Julie and Mudaly 2007) is between the intended solution and students' ideas. When student work does not align with the intended solution, it is natural to interpret the student's work as "incorrect." Another common response is to disregard curricular tasks as avenues developing modelling skills. An actor-oriented perspective offers a middle ground. First, students do transfer mathematical and real-world knowledge to the novel situation described in the task situation (Lobato 2006). Second, the intended solution is not *the correct model*, it may simply be a convenient, conventional, or curricular one. From this perspective, it is possible to predict what variables and assumptions students might suggest when allowing them time and space to work authentically on such problems. The participants in this study selected variables and assumptions that would increase their models' *precision* relative to their lived experiences with the task situations.

This interpretation shifts the locus of support to helping students *prioritize* those which can be modelled using either the mathematics they know or the intended mathematics. To meet the latter goal, it would be necessary to connect the student variables and assumptions to those in the intended solution. For example, making explicit that certain quantities adhere to conventions (e.g. assuming that elevator speed is constant), not because it is the "correct" assumption but because the assumption makes the problem amenable to a particular mathematical analysis which, in turn, provides insight into the problem. Other examples include variables like number of doublings in the *Water Lilies Problem*, which can be seen as a *proxy* for the intended variable time elapsed. Variables such as *number of mailboxes*, *distance between mailboxes*, *variation in mailbox placemen*t, and *number of times the street was crossed* can all be seen as *proxies* for *length of the street*.

6.6 Limitations, Future Directions and Recommendations

Greer (1997) asserted that "doing mathematics should be relatable to the experiential worlds of the pupils, and consistent with a sense-making disposition" (p. 306). The actor-oriented perspective offers a path toward Greer's ideals by illuminating the rationality of the participants' choices. The interview methodology allowed for close examination of participants' responses but the small sample size and laboratory setting of this study raise questions about the situativity not only of the participants' knowledge but also its analysis. That is, the findings were observable exactly because participants were free to identify, prioritize, and mathematize their own variables and assumptions without the imposition of the intended mathematics. Furthermore, the theory and methodology privilege student work and questions about what facilitator competencies might be necessary to bridge intended solutions to student thinking remain unanswered. Hypotheses are found already in the literature. For example, supporting student modelling processes will draw on skill sets like listening (Doerr 2006; Manouchehri and Lewis 2017) scaffolding (Schukajlow et al. 2015; Stender and Kaiser 2015), and attending to student validating and metacognition (Czocher 2014; Goos et al. 2002; Stillman and Galbraith 1998). The actor-oriented theory of transfer, and by extension a *transactional* view, applied to modelling, would be a useful perspective for exploring the viability of these conjectures because it views transfer as distributed across experiences, situations, and discourses among people (Danish et al. 2017; Jornet et al. 2016; Lobato 2012).

In conclusion, the issue is not that students fail to transfer (or suppress) their real world or mathematical knowledge or that tasks with intended solutions are too inauthentic to foster modelling skills. Students *do* engage sensibly in these problems and their willingness to engage in curricular tasks needs to be nurtured rather than discouraged. The path forward is to find ways to lead students to mathematics content that allows them to model the world as they see it, rather than constraining them to see the world as curricular mathematics allows. Part of learning modelling as a practice is learning the conventions about which variables or conditions can acceptably be ignored and under what conditions; but that is only part. Being explicit about the conventions and connecting the conventional decisions to the students' natural ways of thinking may help the facilitator and the student develop a shared understanding of the real model, how it was chosen, and why. It might not be enough to show students *that* some considerations can be ignored (or variables replaced with constants) but rather there is a need to explore justifications for *why this is so*.

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Chapter 7
Teachers as Learners: Engaging Communities of Learners in Mathematical Modelling Through Professional Development

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Abstract Mathematical modelling is a cyclic process in which a modeller evaluates a real-life scenario using mathematics. It is rarely included in the curriculum for pupils prior to secondary school in the United States and is thus unfamiliar to most elementary teachers. In this study, we begin by describing our perspectives and stance on professional development for elementary school teachers in mathematical modelling from both content and pedagogical aspects. We then describe how engaging teachers and students in mathematical modelling promoted mathematical communities of practice through classroom values of relevance, access, and engagement. Findings from a narrative analysis of field notes and transcripts from teacher study groups suggest that when teachers create modelling tasks with these values in mind, modelling provides opportunities for all students to use mathematics to solve problems that matter to them in a way that fosters and benefits community.

Keywords Elementary teachers · Communities of practice · Professional development · Mathematical modelling

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7.1 Introduction

The idea that mathematical modelling provides powerful learning opportunities for students is not new. In Principles and Standards for School Mathematics (2000), the National Council of Teachers of Mathematics (NCTM) asserted, "One of the powerful uses of mathematics is the mathematical modelling of phenomena. Students at all levels should have opportunities to model a wide variety of phenomena mathematically in ways that are appropriate to their level" (p. 39). Yet models and modelling perspectives on teaching and learning mathematics have yet to take hold in U.S. classrooms, especially at the elementary school level. There is, however, reason to believe this trend will change. A growing body of research highlights the affordances of mathematical modelling across grades K–12 (e.g. Brown 2013; Doerr and English 2006; Doerr and Lesh 2011; English 2004, 2006; Lesh and Doerr 2003). The Common Core State Standards for School Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010), wherein modelling is included as both a high school content strand and a Standard for Mathematical Practice, has led to a renewed focus on teaching and learning mathematical modelling in the USA.

A critical next step in developing and promoting high-quality modelling experiences for children and youth is to understand the learning opportunities teachers need in order to facilitate such experiences. Modelling certainly will not become an integral part of students' mathematical learning if their teachers are not prepared to provide classroom leadership in this area. This chapter describes our efforts to engage a community of elementary school teachers in professional development focused on mathematical modelling and our own discoveries as we followed the teachers into their classrooms to implement what they had learnt.

We envisioned mathematical modelling as a tool that elementary teachers could use to help students see mathematics, early in their academic careers, as open, rich, creative, and purposeful. As we enacted modelling tasks with teachers and as they engaged students in mathematical modelling, we found that modelling promoted mathematical communities of practice (Wenger 1998) in which modellers developed a shared purpose, relied on shared knowledge resources to determine solutions, and respected each other's mathematical contributions. We know that developing meaningful mathematical tasks and facilitating mathematical discussions where all students are heard is not easy (Smith and Stein 2001) and we investigated what it was about modelling that promoted these practices.

In this chapter, we describe why we believe modelling helps to foster communities of practice. We describe our research journey by providing rich description of professional development provided to our teachers. Next, we describe our theoretical framework, communities of practice (Wenger 1998) in relation to three teacher-developed modelling tasks to understand what attributes of the task may have led to community building. Lastly, we look across the teacher-developed modelling tasks to consider implications for our work and for future professional development.

7.2 Perspectives and Stance on Modelling Professional Development

Teaching mathematical modelling is demanding work. Teachers must draw on multiple dimensions of knowledge (Koellner-Clark and Lesh 2003) including, but not limited to pedagogical content knowledge (Blum 2011, 2015; Doerr and Lesh 2011), knowledge of the modelling process, knowledge of students' backgrounds and experiences, and knowledge of instructional practices that facilitate individual and group learning (Zawojewski et al. 2003). Moreover, the open nature of many modelling activities means that teachers must allow for, and respond to, student conceptions as they emerge.

Explicit attention to models and modelling is not typical in U.S. teacher preparation programs (Doerr 2007). Even experienced and skilled teachers may not automatically transfer their knowledge of teaching mathematics to teaching mathematical modelling (Niss et al. 2007). Thus, teachers need a variety of experiences and support in order to effectively engage students in mathematical modelling.

Professional development is most effective when it is sustained, intensive, and integrated into teachers' daily work (Garet et al. 2001). It should also be "learner centered" (Hawley and Valli 1999, p. 137), accounting for teachers' existing knowledge, experiences, and beliefs. We adopted a two-phase format for our professional development. First, teachers engaged in a week-long intensive summer institute focused on mathematical modelling. Then, during the school year, teachers were organized into study groups that worked together to develop and implement tasks in their own classrooms.

For our project, situating teachers' initial learning experiences outside their regular classrooms was important. We wanted our participants to begin to think about mathematics from new and different perspectives, perspectives that might conflict with, or call into question, their current classroom routines and environments. Putnam and Borko (2000) argue that such goals may require teacher learning to be removed from teachers' own classrooms, at least initially, because "the classroom is a powerful environment for shaping and constraining how teachers act" (p. 6). We did not want teachers' existing classroom routines, assumptions, and curricula to inhibit their initial experiences with mathematical modelling.

At the same time, our understanding of mathematical modelling as an open, exploratory, and dynamic process meant that learning to engage children in mathematical modelling had to include opportunities for teachers to learn from their own practice. All classroom teaching is relational work (Lampert 2010), and modelling, which involves ongoing negotiations around the meaning and importance of contexts, assumptions, representations, and mathematical strategies, intensifies the relational work between and among the teacher and the students. Thus, it was essential to engage teachers in the process of reflecting on modelling from two perspectives: that of a student who is learning to model and also as a teacher teaching others to model.

To support both of these needs, we adopted a situated (e.g., Greeno 1997), social (Wenger 1998) perspective on teacher learning in which the teacher took on roles as both modeller and teacher. When viewing learning as situated, educators acknowledge the "interactive systems that are composed of groups of individuals together with the material and representational resources they use" (Cobb and Bowers 1999). Individuals within such systems learn by "engaging in and contributing to the practices of their communities" (Wenger 1998, p. 7). From these perspectives, designing professional development is less about finding ways to convey information, strategies, and practices to teachers and more about examining the contexts, activities, and interactions that may facilitate shifts in teachers' perspectives, understandings, and classroom practices.

We established the following content and pedagogical goal for our professional development:

- Teachers will understand what mathematical modelling is and the processes involved.
- Teachers will experience modelling as learners of mathematics by investigating tasks relevant to their own experiences.
- Teachers will experience different types of models and will understand that different models emphasize different values.
- Teachers will relate mathematical modelling to other types of mathematical problems to analyse cognitive demand and attributes of the task.
- Teachers will learn about and experience classrooms routines and structures they can use to facilitate the modelling process.
- Teachers will develop modelling tasks that are mathematically appropriate for their students.

As highlighted in these goals, we wanted teachers to understand and experience the modelling process as learners and then use their understanding of modelling, coupled with pedagogical supports, to design and enact mathematically appropriate modelling tasks for their students.

7.2.1 Preparing Teachers as Modellers

In designing modelling tasks for teachers, we began with the phases in the modelling process (Borromeo Ferri 2006). To transition between these phases the following steps could be involved (1) Examining the situation and setting up a problem to be solved, (2) Identifying variables in the situation and selecting those that are essential, (3) Creating a model that best describes the relationships among the variables using geometric, graphical, tabular, algebraic, or statistical representations, (4) Formulating conclusions, (5) Interpreting the results for accuracy and relevance, (6) Refining the model through validating its potential to account for all relevant variables, (7) Testing model generalizability to other situations. However, we did not want teachers to see the modelling process as a pedagogical checklist where students simply needed to complete steps 1 through 7 to have 'done' mathematical modelling. We also wanted to give teachers opportunities to practice and experience broader features of mathematical modelling that stretch across the modelling cycle. We then identified four features of modelling practice that could be developed and used by novice, as well as experienced, modellers:

- Wrestling with openness in modelling
- Posing mathematical problems to address real world situations
- Making choices creatively whilst modelling
- Revisiting ideas and revising solutions during the modelling process

We believed that these features were unlikely to be a part of most teachers as learners' experiences in teaching and learning mathematics.

Many of the mathematical tasks used in elementary schools are word problems—applications in which either the real world does not affect the problem or there is a clear solution strategy (Tran and Dougherty 2014; Zbiek and Conner 2006). In mathematical modelling, problems can be open at the beginning of the investigation, allowing modellers to ask different mathematical questions about a scenario, open in the middle as modellers investigate different solution strategies, and open at the end as modellers consider ways the models do, or do not, apply to other situations. Wrestling with openness in modelling is a feature that conveys the idea that real-world situations do not always have a single, clear-cut beginning, approach, or solution.

Posing mathematical problems to address real world situations involves determining if, and how, mathematics can be used to investigate issues originating from lived experiences. In school mathematics, students are usually asked to solve problems, but rarely asked to determine and articulate the range of mathematical problems they could pose. Problem posing is a central feature not only of mathematical modelling, but of mathematical activity in general and "can occur before, during, or after the solution of a problem" (Silver 1994). Although often neglected in school mathematics, problem posing is as important as problem solving (Cai et al. 2015; Kilpatrick 1987) and warranted explicit attention in professional development focused on modelling.

Many students believe that mathematics consists of fragmented bits of information transmitted from teachers or textbooks, and that students are not capable of "constructing mathematics knowledge and solving problems on their own" (Muis 2004, p. 329). To be successful modellers, students need experiences that challenge these perspectives and reveal mathematics as a creative enterprise, wherein problem solvers are empowered to make choices as they pursue solutions. Making choices creatively whilst modelling focuses on the ability to determine what mathematics the modeller will use or develop to make progress on a task. It empowers students as mathematical thinkers whose perspectives, ideas, and decisions matter, both to the ways problems are formulated and to the solutions pursued.

Finally, revisiting ideas and revising solutions during the modelling process involves stepping back and considering whether the solution (either in progress or complete) makes sense in light of the initial problem. In the context of teaching and learning, revisiting ideas and revising solutions also involves returning to both the contexts and mathematical content that played a critical role in previous modelling tasks and applying them to new situations.

Keeping our four features in mind, we designed five modelling tasks for teachers during professional development. When determining what tasks to pose, we considered that modelling is a challenging process that requires persistence and time. In designing tasks, we considered four pedagogical features:

- Attributes of Modelling: What modelling practices (wrestling with openness in modelling, posing mathematical problems to address real world situations, making choices creatively whilst modelling, revisiting ideas and revising solutions during the modelling process) are made visible through this task?
- Variation: Does this task highlight a particular type of model?
- Access: Do the modellers have the appropriate mathematical tools to approach and solve this task?
- Relevance: Will the modeller care about this problem or situation?

To highlight each of these pedagogical features, we will describe them through the *Water Usage Task* adapted from Hoffman (2014). The water usage task asks problem solvers to quantify the amount of water used to grow, process, and distribute the ingredients needed to make one slice of cheese pizza. We asked our participants, "How much water is needed to make pizza?"

Hoffman (2014) suggests a specific solution based on critical assumptions and decisions about the meaning of the situation and the purpose in asking the question. Our purpose in using the *Water Usage Task* was to introduce teachers to wrestling with openness in modelling. We wanted teachers to consider the notion that a problem could be approached from multiple perspectives and reinforce the notion that the perspective and knowledge of the modeller matters (English and Watters 2005). We also wanted teachers to see that perspective can cause variation in the types of models that are produced and that situations and questions exist in which more than one solution could make sense. We anticipated teachers could use a variety of mathematical tools, at different levels of complexity, to make sense of the situation and knew all of the teachers had experiences making and eating pizza.

7.2.2 Preparing Teachers to Teach Modelling

Using mathematical modelling to solve a problem is markedly different from supporting students as they work through a modelling task. Elsewhere, we described the work teachers must engage in as they develop and enact modelling tasks (Carlson et al. 2016). This work involves three teaching phases: developing the task and anticipating student strategies, enacting the task alongside students, and revisiting the task as opportunities arise. As the phases suggest, enacting modelling tasks with students involves many of the demanding teaching practices that have been associated with tasks that have high cognitive demand (Stein et al. 2000). Teachers must anticipate the mathematics students might use as they approach the task and anticipate mathematics students might find confusing or challenging. If the teachers plan for students to work in small groups, they must manage group work. During task enactment, teachers facilitate classroom discourse, selecting and sequencing student work that will be shared in a whole-group discussion. In addition, teachers must consider what contexts will be interesting and engaging for students and try to predict how students' cultural and community-based "funds of knowledge" (e.g. González et al. 2001) give students access to, and agency within, the problem. In order to respond to the pedagogical demands of teaching mathematical modelling, we set aside time each day to focus on and develop teachers' capacity in areas like differentiating classroom instruction, facilitating classroom discourse, and managing group work.

7.3 Theoretical Framework: Mathematical Modelling as a Community of Practice

As teachers translated their experiences in professional development into their own classrooms, we wondered how attributes of modelling and associated pedagogical practices might influence their classroom instruction. Our qualitative analyses suggested that modelling fostered outreach and empowerment across the classroom, school, and local communities. We looked to the communities of practice literature as a way to make sense of the communities that formed when teachers enacted modelling tasks as well as the attributes of modelling that seemed to promote community knowledge and ownership.

Wenger (1998) described a community of practice as a "simple social system" (p. 1). Participants in a community of practice exhibit certain competencies, including:

- Understanding what matters, what the enterprise of the community is, and how it gives rise to a perspective on the world.
- Being able (and allowed) to engage productively with others in the community.
- Using appropriately the repertoire of resources that the community has accumulated through its history of learning. (p. 2)

Wenger went on to explain that communities of practice exist across broader systems that involve other communities.

In this chapter, we look across three teacher-developed modelling tasks to consider what attributes the teachers transitioned into practice that fostered mathematical modelling as a community practice. We address the following research questions:

1. In what ways does mathematical modelling promote mathematical communities of practice in the elementary classroom?
2. How do attributes of modelling and pedagogical practices help foster communities of practice as teachers engage elementary school students in mathematical modelling?

7.4 Setting and Method

This investigation was part of a National Science Foundation-funded study aimed at examining mathematical modelling in the elementary and middle grades. Working alongside two other universities and three school districts, we engaged in a three-year, multi-state project. Our overarching goal was to provide and study the effects of professional development in mathematical modelling for grades Kindergarten to grade 8 (K–8) teachers. We developed and implemented a week-long, intensive summer professional development course, described above, as well as facilitated semester-long teacher study groups for 28 elementary grades teachers at a U.S. university in the Rocky Mountain West. We set up study groups of grade level teams and each teacher team implemented at least one modelling task. Each of the authors as well as two teacher leaders facilitated a teacher study group of 4–6 teachers during implementation of the modelling tasks. In this investigation, we draw on data collected from classrooms during teacher study groups.

7.4.1 Data Collection

The 28 teachers worked in their study groups with a facilitator to discuss, debrief, and modify modelling tasks designed during the summer professional development. These study groups met six times through the autumn school semester. We audio-recorded teacher-study group sessions and took notes describing the nature of the discussions. In addition, we visited classrooms as the tasks were enacted and wrote field notes following each day of implementation. Our main data sources were video and audio of classroom tasks and teacher study groups as well as first-hand observations of, and field notes taken during, implementation of the modelling tasks.

7.4.2 Data Analysis

We used narrative analysis (Clandinin and Connelly 2000; Riessman 2008) to examine the data, primarily by examining the modelling tasks themselves. We used communities of practice (Wenger 1998) as a guiding framework. Narrative analysis uses artefacts such as field notes and conversations to understand the ways in which meaning is created. Using our notes, paired with transcripts from the teacher study groups, we first sought to describe each modelling task, the rationale, and the teachers' mathematical goals for the task. We found evidence that a majority of teachers used mathematical modelling tasks in their classrooms as a way to highlight community—classroom community, school community, or the civic community. Next, we considered the ways in which students worked through the task and decisions teachers made as students worked. We looked for ways, both in the design and in the enactment of the task, that modelling fostered community across the tasks and how this occurred. From this analysis, we constructed rich descriptions of the modelling tasks and teacher accounts.

As we analysed each of the modelling tasks, we looked for these three features of communities of practice:

- *Relevance*: Understanding what matters, what the enterprise of the community is, and how it gives rise to a perspective on the world (Wenger 1998, p. 2). We looked for ways that the modelling task was meaningful to the modeller.
- *Engagement*: Being able (and allowed) to engage productively with others in the community (Wenger 1998, p. 2). We looked for ways that multiple modellers' perspectives were heard, considered, and valued.
- *Access*: Using appropriately the repertoire of resources that the community has accumulated through its history of learning (Wenger 1998, p. 2). We looked for ways that all modellers in the class could make contributions to the creation of a model.

To help answer the second research question, we also analysed across modelling tasks considering how these attributes fostered modelling as a community of mathematical practice.

7.5 Results

In this section, we chronicle three different modelling tasks designed by teacher teams selected on the basis that they demonstrate fostering of classroom community (*Lunch Planning Task*), school community (*Pizza Party Task*), and the civic community (*City Park Ice Rink Design Task*) and discuss how the themes of relevance, engagement, and access emerged in each lesson and how these themes fostered modelling as a community endeavour.

7.5.1 The Lunch Planning Task

Third and fourth-grade teachers collaborated on the *Lunch Planning Task*. The purpose of the task was for students to design a lunch, within budget, for several classes to enjoy and use as a team-building experience. Students discussed that they needed to determine what to serve, how much food to order, and what the cost of the lunch would be. In addition, since the focus was on team building, several of the classes also researched how to spend their time during the lunch and what activities they could organize that would help to build community within the classroom.

The teachers worked with students on this task over the span of four weeks, addressing the task a few times each week. As the students worked on the task, the teachers allowed students to choose which portions of the task they wanted to help with. They described that all parts of the task were accessible to someone in the classroom. Initially, the students worked on determining what should be served at lunch and the quantity of food they should order. Students primarily used surveys, multiplication, counting, and measurement as mathematical tools to aid them in making decisions. Once students determined what should be served and how much they would need to order, they needed to determine where the food would come from and if the meal was in budget. The teachers helped by providing grocery store and restaurant advertisements, and students used multiplication and repeated addition in determining the total cost.

In planning the activity, the teachers discussed that the community lunch planning happens every year, but the teachers usually take on the responsibility of designing and planning the lunch. Since students usually enjoy the lunch and make suggestions on what should happen, they determined that the students would find the task *relevant*. As students worked on the task, the teachers commented that it promoted excitement, motivation, and perseverance because students felt they were given ownership of an important decision. In describing relevance, one teacher stated:

> I would also really encourage them [other teachers] to think about not just problems outside of school, like building a house or something but problems that are more real to the students. For fourth-grade, that has been really successful, you know? Our lunch was really successful and really motivating. There is no work I have to do, you know, no encouragement. We just started the process…and they were really excited. They knew what was going on. [It's important] to attack problem that they have some sort of connection to.

Engagement emerged quickly in this modelling task. When determining what lunch to serve, students had opinions on which meal was their favourite and why, however, they found that their opinions varied greatly. The students realized that 50 different meals was not a cost-effective model. Teachers encouraged students to consider how to best hear and acknowledge everyone's perspective, which eventually resulted in learning about surveying and implementing surveys in order to hear from all students. Even after a decision was made, according to the survey, students asked if it was fair to serve a meal that a few people did not like or could not eat. The teacher encouraged them to go back and consider these perspectives as they were making choices in the modelling process. For example, after pizza was chosen as the most popular meal, students decided they needed to find a solution for several lactose-intolerant students.

In describing the *Lunch Planning Task*, one teacher described that it was one of the few times where all students could feel included and *access* the mathematics together. She stated:

> The most amazing thing to me is that everybody is able, no matter who you are, can enter the process where you need to enter it. I just, my entire life, as a person, I have always had a hard time not including everyone and not having everyone feel like they are valued or important. And, I've, when I decided to become a teacher, as much as we like to think public education is inclusive, it's not. We have groups, pullouts and things because we need to service everybody. I totally understand, but it has always made me a little uncomfortable because I see the dynamics because of that. Roles are created…It's just reality. This was the first time that I had that "aha" moment in the class this summer when we were reading those

> articles. I was like ohhh, if this is how math could be in my classroom where everyone was doing math and didn't have a status role so to speak as the really smart math kid or the not so smart kid. If we all just had a part in this, that was totally mind-blowing for me. I got so excited.

The *Lunch Planning Task* promoted a community of practice through relevance, engagement, and access. Students found the task relevant because they were allowed to assume ownership of a classroom activity usually reserved for the teacher to design and enact on her own. Engagement was evident as the students wanted to make sure the model satisfied their wants and needs as a classroom. All students were able to access the task and share solutions because the teachers carefully thought about the mathematics involved in the task in relation to their grade level.

7.5.2 The Pizza Party Task

The *Pizza Party Task* involved first-grade and fifth-grade students. Traditionally, first-grade students (age 6–7) at the school have a pizza party on the last day of school and this year they wanted to invite their fifth-grade (age 10–11) "buddies" to join.

Similar to the *Lunch Planning Task*, this task was *relevant* to both ages of students and the teachers discussed the importance of relevance in motivating students. They also discussed how many mathematics problems they typically work on that are set in the "real-world" but the choices students make in solving the problem do not actually mean anything in the context of the problem. One teacher addressed relevance this way:

> The real worldliness of what we were doing was key. Because a lot of math that I teach on a daily basis I feel like has no connection to the real world. I mean maybe you can stretch it where we are talking about candy or in a story problem dividing it up, but it kind of loses something because it's not connected to a real-world thing that means something to the kids… The fact that they were actually going to get food at the end was important. It was engaging.

Engagement emerged in several different ways. First, the teachers did not reserve all of the mathematics for the older students. Instead, they allowed both age groups to be experts in their own right and describe solutions to parts of the problem to each other. Second, echoing what we found in the *Lunch Planning Task*, the students had to understand the wants and needs of one another as they ordered pizza.

Concerning *access*, the teachers were very thoughtful about considering how a modelling task could have multiple questions that could be approached in an appropriate way by students at varying grade levels. For example, the fifth-grade class determined where to buy the pizza by considering the area of the pizzas in relation to their size and price. They presented their findings to the first-graders, identifying the place that they had determined was the best, showing them the size of the pizza slices, and explaining the topping choices for a particular cost. The first-graders addressed a different question by determining how many of each pizza to buy and what kind. The first-graders also constructed surveys to gather data about how many pieces of each type of pizza each student wanted and determined how many pizzas they should order for their class. Using repeated addition, they extrapolated from their class to all of the classes who would be at the party. They concluded the task by presenting how many pizzas and what type should be ordered. Although different ages, each class was able to contribute to the overarching question in a way that was appropriate for their mathematical understanding.

The *Pizza Party Task* promoted a mathematical community of practice across grade levels through relevance, access, and engagement. In this case, students found the task relevant because the task was real and the choices they made had consequences that mattered to them. In terms of access, the teachers divided the task so that students, who varied in age by four years, could answer a question at their appropriate level of mathematical understanding. Lastly, engagement emerged as the students, across grade-levels, were able to take ownership and expertise of the task and share their solutions to parts of the problem.

7.5.3 City Park Ice Rink Design Task

The *City Park Ice Rink Design Task* emerged from fifth-grade students' discussion about the use of an ice rink at City Park. The students felt that the area currently designated for the skating rink was not being used appropriately. From their perspective, the hockey space was too small, causing hockey players to enter the free-skate area. Students who enjoyed free-skating felt unsafe because of the hockey equipment, and students who enjoyed hockey felt they did not have enough space to play. The teacher commented that she heard the students complain about the use of the park on a daily basis and felt it would have *relevance* for her students to explore the use of the park. She stated:

> We have a local park that they all use and they all talk about how the use of it isn't always appropriate, or appropriate as they see it as ten-year olds. So my thought was that we could explore the uses and see why the uses aren't equitable….I think we can use it [Google Earth]. That's part of the discovery that I really want them to be thinking about. How are you going to figure out how big the park is? How are you going to measure and do that?

Unlike the other two tasks we chronicled, the teacher described that for some students, lack of relevance impacted students' perseverance and interest in the problem. Several of the students were engaged in the task and solving the problem but a few seemed to lose interest because they did not use the park nor did they like ice-skating.

The teacher worked carefully to consider how students would *access* the task, examining what mathematics would be addressed in the activity and how it was connected to grade-level standards. Because students were studying area measurement, she wanted them to investigate the area of the skating rink and how they might fairly partition it for different types of skaters. During the task, the students found the area of the current skating space using tools like Google Earth and then researched appropriate areas for skating activities.

Engagement was represented in a different way in this task. The *City Park Ice Rink Design Task* intersects with the broader community. When students proposed changes, the teachers and students could not actually enact them. The teacher proposed that they first present and listen to each other's solutions to determine which model might be the best overall. After determining which solution might be best, the students were asked to voice their opinion to the town hall through a letter. The students were asked to describe the problem and the mathematical process they went through in determining a solution. Through this process, the students were given a voice in a larger community space. The teacher stated:

> They use that park all the time. It is very much part of their daily lives… Even if the city doesn't do anything, they have been able to voice their opinion in a constructive way with evidence, which is so important.

The *City Park Ice Rink Design Task* promoted a mathematical community of practice across grade levels through relevance, access, and engagement. In this case, many students found the task relevant because they cared about the city park and ice rink usage. The task also highlights that if the modelling task is not relevant to some students then they may not fully participate in the task. In terms of access, the teacher thought carefully about the mathematics involved in the task and how it aligned with grade level standards. Lastly, engagement emerged as the students were able to describe their solutions both inside their classroom and in the broader community.

7.5.4 Looking Across Tasks

Looking across the tasks, we found that the elements of communities of practice emerge in different ways. *Relevance* is evident where a community of practice has a shared vision or purpose and understands its role in relation to the greater community. Also, when problems had relevance to the students, this made the problem important and motivating for students to work to solve the task. Relevance was apparent in these modelling tasks when students would take on a coveted responsibility, take action toward an important problem, or work toward a goal that everyone valued.

Access is apparent when a community of practice draws on their shared experience and history of learning. In each of these tasks, the teachers thought carefully about the mathematics the students might use in the task to make sure everyone could contribute in a way that made sense to them. The *Pizza Party Task* highlights that the same modelling task could be approached using different questions and content knowledge to address the task in an appropriate way.

Engagement is perceptible when all members of a community of practice have the right and ability to be heard. Teachers fostered this in several ways as they implemented modelling tasks. First, the openness of the task and the variety of tasks embedded in one project allowed for students to solve the *Lunch Planning Task* in multiple ways. In the *Pizza Party Task*, teachers designated tasks to different grade levels to provide engagement by, and expertise to, all. Communities of practice acknowledge that multiple communities and spaces exist together at once and, within the *City Park Ice Rink Design Task*, the teacher helped students to transition their engagement to a community that encompassed the local town.

7.6 Discussion and Implications

In this research, we asked in what ways mathematical modelling promotes mathematical communities of practice in the elementary classroom and how attributes of modelling help foster communities of practice as teachers engage elementary school students in mathematical modelling. Our narrative analysis of the tasks designed and taught by teachers provides a chronicle of how teachers use the relevance of a task to build communities and how the nature of modelling allows for multiple access points and student engagement.

We chose tasks for teachers to engage in during professional development that were intended to help teachers experience and reflect on learning opportunities made available through mathematical modelling: (1) wrestling with openness in modelling; (2) posing mathematical problems to address real world situations; (3) making choices creatively whilst modelling; and (4) revisiting ideas and revising solutions during the modelling process. As a stance, the professional development reinforced the nature of mathematical modelling as a tool to solve problems important to the community and enacted by a community of learners. In the teacher study groups, teachers worked together to create tasks for their students that would exhibit these features. We found that the tasks teachers enacted in their classrooms promoted classroom, school, and civic community through mathematical modelling. In addition, we know that teaching mathematical modelling is demanding work and that teachers must coordinate knowledge of mathematical and pedagogical content knowledge to facilitate a modelling task (Blum 2011, 2015; Doerr and Lesh 2011; Zawojewski et al. 2003). Teachers saw opportunities to motivate mathematical thinking and perseverance by engaging students in solving problems that made a difference for themselves and for others.

In our professional development, we created opportunities for teachers to experience being modellers, to engage in relevant tasks, to work together as a community of learners, and to be thoughtful practitioners in teaching mathematical modelling. In this combination of goals in professional development, we framed how teachers saw community as an important component of modelling, which they then extended to their own tasks. We are interested to notice that the teachers and their students made the most of these opportunities to make social connections and to use modelling to find ways to make decisions that were for the good of the group. In the *Lunch Planning Task*, students tackled a task they understood and felt connected to. Their teachers, in that case, recognized the power of students using mathematics in ways that directly impact their own classrooms. In the *Pizza Party Task*, students investigated and analysed a situation that fostered connections between themselves and other students. In the *City Park Ice Rink Design Task*, students considered the role mathematics can play not only in solving problems, but in developing convincing arguments that can be shared with civic leaders. One teacher gave insight about why this might be when discussing a community-based task in her classroom of second-graders by saying, "[students] are really interested in these ideas and problem solving these big things. They don't feel incapable because they are young. I want them to understand that their work, and their work through math, has an impact."

7.7 Conclusion

The teachers responded to our conception of modelling as a tool used to solve relevant problems. They created tasks for their students that would engage students as modellers in using mathematics to solve problems that mattered to them.

Our approach to the mathematical modelling professional development was one in which we relied on a community of practice (Wenger 1998). This message about modelling, and how best to accomplish the teaching of modelling among those unfamiliar with it, may have permeated teachers' notions about how to teach modelling. It might also be that in their search for contexts with meaningful problems that lend themselves to classroom modelling, teachers found that school students were most interested in pursuing problems that affect themselves.

This narrative analysis leaves us with several lines of inquiry open to investigation. Why do teachers gravitate towards opportunities to model problems directly affecting community? Is there something inherent in the way we defined mathematical modelling in classrooms—as solving a problem that is based in authentic, lived experiences—that leads to tasks based in community improvement?

The way we approached modelling as an activity that takes place within a community of practice involved classroom practices of discourse, including making compromises and active listening. Does modelling in classrooms amplify this sense of negotiation and agreement because problems begin and remain open? Critical to modelling in elementary grades is the teacher facilitating community agreements about modelling decisions so that the class can move forward together in productive work (Carlson et al. 2016). It is not practical to think that in elementary grades teachers can let students pursue individual solutions to modelling problems. This means modelling in an elementary classroom requires the teacher to facilitate community agreement and progress in a way that is not inherent to modelling outside of classrooms.

Finally, we remain interested in the affordances and limitations of mathematical modelling in elementary grades, especially viewed in light of an already demanding elementary school curriculum. Our selection of modelling tasks in the professional development was a careful balance of mathematical topics we wanted to engage teachers in and modelling practices and values we wanted to expose them to. What kinds of mathematical decisions do teachers make when pursuing modelling opportunities for students? Do they hold high standards for mathematical work? There can be much classroom activity when students are engaged in solving problems with modelling, but how can teachers keep the activity focused on mathematical learning? Does inserting modelling in elementary curriculum sacrifice curricular coherence?

Mathematical modelling in elementary grade classrooms in the USA is a new arena—new to practice and new to research. We continue to pursue an understanding of how to best support teachers in their enactment of mathematical modelling.

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Chapter 8
Assessing Sub-competencies of Mathematical Modelling—Development of a New Test Instrument

Corinna Hankeln, Catharina Adamek and Gilbert Greefrath

Abstract The distinction between different phases of a modelling process and thus of different sub-competencies for carrying out these processes is widespread in the modelling literature. In this chapter, we present our research on the assessment of these modelling sub-competencies. Based on a conceptual clarification of sub-competencies, we consider various ways of operationalising them into test items and present examples. With the help of psychometric models, we show that the sub-competencies of modelling, simplifying, mathematising, interpreting and validating, can be treated as separate dimensions, rather than being subsumed in a two-dimensional model, in which *simplifying* and *mathematising,* as well as *interpreting* and *validating,* have been combined.

Keywords Interpreting · Mathematising · Simplifying · Sub-competencies · Test instrument · Validating

8.1 Theoretical Background

8.1.1 Mathematical Modelling Competency

In 2003, the German ministers of education in the various federated states stipulated mathematical modelling as a mandatory part of each school's mathematics curriculum (see KMK 2003). The German national standards require students to possess

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abilities to translate real situations into mathematical problems and vice versa. As is widely accepted in the modelling debate, these processes can be represented in an idealized manner as a modelling cycle (e.g. Blum and Leiß 2006; Maaß 2006). Being competent in mathematical modelling is being able to autonomously and insightfully carry out all aspects of a mathematical modelling process in a certain context (Blomhøj and Kjeldsen 2006; Niss 2004). Accordingly, the German standards require students to be able to translate a situation in mathematical terms, structures and relations, to work within the respective mathematical model as well as to interpret and check results in relation to the corresponding situation (KMK 2003).

In the research literature, there is a broad debate on how mathematical modelling competency can be defined (see Kaiser and Brand 2015). First of all, two different perspectives (holistic and analytical) can be identified, which are also evident through the use of certain terms. Firstly, from a holistic perspective, the term *modelling competence* is used and interpreted in relation to experiencing an entire modelling of a situation. Some authors who adopt this perspective, propose competence models that incorporate different levels. Greer and Verschaffel (2007), for example, distinguish between three levels of mathematical modelling: implicit (in which the student is essentially modelling without being aware of it), explicit (in which attention is drawn to the modelling process), and critical modelling (whereby the roles of modelling within mathematics and science, and within society, are critically examined). Blomhøj and Jensen (2003) also take up the distinction of different competency dimensions formulated by Niss and Højgaard (2011). They distinguish the *degree of coverage*, which relates to the part of the modelling process with which the students' work and the level of their reflection, the *technical level*, which refers to the kind of mathematics students use, and the *radius of action*, which describes the domain of situations in which students are able to perform modelling activities (see Kaiser and Brand 2015).

Secondly, in addition to these holistic approaches to mathematical modelling competence, other authors adopt an analytical perspective and refer to a *modelling competency* that can be subdivided into different elements or sub-competencies. This analytic view on competencies thus focuses on identifying different elementary competencies that are part of a more general modelling competency. Therefore, researchers who follow this perspective formulate models that focus more on the competency structure and not so much on its levels. Within this perspective, "competencies should be defined by the range of situations and tasks which have to be mastered" (Klieme et al. 2008, p. 9). The distinction between different sub-competencies, according to the different phases of the modelling cycle, is an example of this view within the modelling debate. Several authors (e.g. Kaiser 2007; Maaß 2006) formulate definitions of sub-competencies that are necessary for performing a single step in the modelling cycle.

Maaß (2006), for example, distinguishes between the following five sub-competencies: The competencies needed to understand the real problem and to build a model based on reality are referred to as *Simplifying*. This sub-competency includes the competency to make assumptions, identify relevant quantities and key variables, construct relationships between these variables and to find available information.

Mathematising refers to "competencies to set up a mathematical model from the real model" (Maaß 2006, p. 116). This includes competencies to translate relevant quantities and their relationships into mathematical language by choosing appropriate mathematical notations or by representing situations graphically. *Working mathematically* describes competencies for solving mathematical questions within the mathematical model by using mathematical knowledge or heuristic strategies. Again following Maaß (2006, p. 116), *Interpreting* can further be seen as the "competencies to interpret mathematical results in a real situation". This includes being able to relate results back to the specified extra-mathematical situation. Finally, competencies for verifying the solution and for critically reflecting on the solution, the assumptions made or the model used, are subsumed under the term *Validating*.

Even though these sub-competencies form the indispensable basis for a more general modelling competency, their mere existence is not sufficient. As research has shown, several additional factors such as metacognition or social competencies might be necessary for solving a complete modelling problem and carrying through a whole modelling cycle (see e.g. Blomhøj and Jensen 2003; Maaß 2006). Research has additionally shown that modelling competency is different from a technical mathematical competence and can also be empirically distinct (Harks et al. 2014).

8.1.2 Assessment of Modelling Competencies

The assessment of competencies generally depends on the underlying concept of a competency. Since we base our research on the functional concept of competencies as used, for example, in the Program for International Student Assessment (PISA), we assume that "modelling competencies include, in contrast to modelling abilities, not only the ability but also the willingness to work out problems, with mathematical aspects taken from reality, through mathematical modelling" (Kaiser 2007, p. 110). Therefore, "assessment might be done by confronting the student with a sample of … (eventually simulated) situations" (Klieme et al. 2008, p. 9). This confrontation can either be done with a written test, or with the help of observations or interviews (Dunne and Galbraith 2003; Maaß 2007). Written forms of assessment however, have the advantage that they can easily be applied to a huge number of students at the same time, that they are often more objective than interviews or observations (Smith et al. 2005) and that they can be confidential and anonymous.

Written tests do not necessarily have to be limited to solving tasks on paper, as Vos (2007) shows. In her hands-on tests, students even experimented with tangible material such as rubber bands, and afterwards responded to open-ended tasks. However, such tests require specific testing situations in which such activities are possible. Furthermore, coding of students' responses might pose difficulties as well (Smith et al. 2005). A more common way is to employ test items that can be solved on paper.

One of the most important distinctions for test items or tasks is the difference between holistic and atomistic tasks (Blomhøj and Jensen 2003). While in holistic tasks, students have to proceed through a complete modelling cycle to solve a problem, atomistic tasks pre-structure a modelling problem and focus on one or two sub-processes. Both forms of tasks can be used in written assessment, either of which has its own benefits and disadvantages.

If the aim is to assess students' ability to complete a modelling process (which is often called the *general modelling competency*), it is preferable to use holistic tasks. In atomistic tasks students only have to deal with problems that require a limited range of modelling competencies, so these tasks cannot be used to obtain information about whether a person is generally capable of completing a modelling process. Holistic items have been used by several researchers to measure students' modelling competency (e.g. Kreckler 2015, 2017; Rellensmann et al. 2017; Schukajlow et al. 2015).

The disadvantage in using holistic items lies in the interdependence of the modelling steps. If, for example, a person is weak in simplifying a problem, he or she might not reach the point of interpreting a mathematical result. Thus, this person would not be regarded as having a high modelling competency, despite being capable of conducting the modelling process once the problem has been simplified. To avoid this problem, some authors have employed atomistic tasks to assess different sub-competencies of mathematical modelling and interpreted the sum of the measured sub-competencies as a general modelling competency (Haines et al. 2001; Kaiser 2007; Maaß 2004), even though the sub-competencies are not sufficient for general modelling competence (as stated above). Therefore, some researchers have tried to combine both forms of task and evaluated their data with the aid of Item Response Theory (Brand 2014; Zöttl 2010; Zöttl et al. 2011).

However, if the aim is not to assess general modelling competency, but rather several modelling sub-competencies, it is preferable to use atomistic tasks in a test. Since the different steps of the modelling cycle are intertwined and based on one another, it is almost impossible to rate the different sub-competencies separately in holistic tasks. If, for example, a person fails to simplify a situation adequately, he or she might not even reach the point of validating a solution, since none was found. Therefore, it is then impossible to judge that person's competencies in validating a result.

Even though some researchers have focused on assessing sub-competencies of mathematical modelling (Brand 2014; Haines et al. 2001; Zöttl 2010), there is no sound empirical evidence that the theoretically assumed division into different sub-competencies adequately describes the structure of mathematical modelling competency. The two authors who assessed different sub-competencies of mathematical modelling, namely Brand (2014) and Zöttl (2010), summarized different sub-competencies. They subsumed the sub-competencies of simplifying and mathematising, as one dimension of modelling competency, and interpreting and validating as another. Additionally, they examined working mathematically and general modelling competency. Even though both authors use the same structure of combining sub-competencies they do not give any reason other than time economy.

We therefore wanted to determine whether it is possible to measure the sub-competencies of simplifying, mathematising, interpreting and validating as separate dimensions of modelling competency. If this proves not to be the case and the demands made in the different phases of the modelling cycle are very similar to each other, is the aggregated view of Brand (2014) and Zöttl (2010) the more suitable to depict the structure of mathematical modelling competency?

Based on the theoretical work concerning the sub-competencies of mathematical modelling, we expected it to be possible to assess these sub-competencies separately and hoped to create a test instrument that could be used, for example, to evaluate experimental interventions at the level of sub-competencies.

8.2 Methods

8.2.1 Item Construction

Based on the theoretical considerations, as well as existing test items, we began to construct atomistic test items that were intended to assess each sub-competency of mathematical modelling separately. As the basis for operationalisation, we used the familiar definitions of the sub-competencies as explained above (Kaiser et al. 2015, an English translation can, for example, be found in Maaß 2006).

As we had in mind using the new test instrument in further studies, we aligned our work with the requirements of these studies. For example, we focused on geometric modelling problems and chose grade 9 students (15–16 years old) to be our target group. There were no content-related reasons for these choices concerning the research questions formulated above, and we expect the results of our study to be transferable, to a certain extent, to other mathematical domains. However, as Blum (2011) states, learning is always dependent on the specific context, and hence, a simple transfer from one situation to another cannot be assumed. He emphasises that this applies to the learning of mathematical modelling in particular, so that modelling has to be learnt specifically. Thus, if a student is a good modeller in the field of geometry, he or she is not necessarily a good modeller in the field of functions. Of course, the restriction to geometric modelling problems limits the generalizability of our results, but allows us at the same time to gain more reliable and meaningful findings regarding the chosen topic.

Next, we present an example of a test-item for each of the four sub-competencies we measured, and explain, to what extent this item actually measures the sub-competency. To provide some evidence for the quality of the items, the solution frequency and the item-total-correlation as an indicator for its selectivity are given, as found in an implementation of the test in a large sample (3300 completed tests).

During their summer vacation, Marcus and Irina are standing on top of a lighthouse and enjoying the view. "How far is it to the horizon?" Irina asks.

Mark all of the following information that you consider to be important to calculating the distance to the horizon.

https://upload.wikimedia.org/wikipedia/commons/b/bf/Louisbourg_Lighthouse.jpg

☐	Between the lighthouse and the ocean, there are 25 m of sandy beach.	☐	The two are standing on the Atlantic coast in France.
☐	There are no clouds in the sky.	☐	The radius of the earth measures 6370 km.
☐	The lighthouse is 83 m high.	☐	The lighthouse's light shines as far as 10 km.

Fig. 8.1 The *Lighthouse Task*: multiple-choice item that measures competencies in simplifying a problem (translated task)

8.2.1.1 Simplifying: *Lighthouse* Item

An example of a test item that was used to measure the sub-competency of *simplifying* is the *Lighthouse Task* (see Fig. 8.1, translation). It is a modification of the well-known lighthouse question (Kaiser et al. 2015), which requires the use of a geometrical model and is suitable for grade nine students. The given situation is depicted by a picture of a lighthouse. The students' task is to select all the information that is relevant to calculate the distance to the horizon. Thus, the item measures competencies for identifying relevant quantities and key variables, which are part of the definition of the sub-competency simplifying.

The fact that more than one answer has to be selected, namely the radius of the earth and the height of the lighthouse, reduces the probability of selecting the correct answer by guessing. The alternative answers represent misconceptions, for example the answer "There are no clouds in the sky" reflects confusing the distance to the horizon with the visibility. The first two alternatives show different misconceptions of the dependence on location of the lighthouse, and the last alternative represents a misunderstanding of the question, or rather the misconception that the distance to the horizon depends on the range of the light.

The distractors were developed with the help of experts in the field of modelling. We collected various items of information we thought students might select as relevant, even though they are not. In our pilot studies, as well as in the implementation of the test with a large sample, we checked these distractors and found that all of them were chosen by at least some students. The two most common mistakes were to select the distractor: *Between the lighthouse and the ocean, there are* 25 m *of sandy beach* (25.2% of wrong answers) and not to select the second correct option: *The radius of the earth measures* 6370 km (13.5% of wrong answers).

A farmer has stacked up straw bales like in the photo on the right.
You can assume that all straw bales have the same size and are exactly round. You can further assume that all straw bales are 1.50 m in diameter and that they always sink 20 cm into the layer of straw bales below them.

Make a labeled drawing and set up a formula that you can use to calculate the height of the stack. You <u>do not</u> need to calculate the height!

Fig. 8.2 The *Straw Bale Task*: short answer-item that measures competencies in setting up a mathematical model (translated task)

The item was used in a study with a large sample which led to 1473 responses to this item. A total of 45.35% of the students was able to answer this question correctly and received 2 points. Students who selected one additional distractor or forgot to select the second correct answer without selecting one of the distractors still received 1 point. This was the case with 27.70% of the students. Even though it is thus a relatively easy item, its Item-Total-Correlation of r = 0.43 yields a satisfactory selectivity of this item in this sample.

8.2.1.2 Mathematising: *Straw Bale* Item

The item in Fig. 8.2 was used to assess the competencies for setting up a mathematical model from a simplified real situation (i.e. *mathematising*). This item is inspired by the *Straw Bale Task* in Borromeo Ferri (2011, pp. 84–85), which confronts students with a real-life situation of a stack of straw bales in a field. The idealizing assumptions that all straw bales are the same size and that they are evenly and exactly round are given in the text. So are the diameter of 1.50 m and the depth that the straw bales sink into the layer below them. The student's task is to convert this situation into a mathematical representation, both graphically in a labelled drawing and symbolically as a formula with the aim of calculating the stack's height. The item thus measures the competencies required for choosing appropriate mathematical notations or by representing situations graphically.

A correct answer must include the stack's diameter, the depth of sinking in and, as the unknown quantity, the height of the stack. Answers using the specific sizes and those using abstract variables to denote these quantities were acceptable. Students could achieve a maximum of two points for this item, one for the correct drawing and one for a correct formula.

Use in a large sample produced 1143 responses to this item, which was correctly solved by 24.58% of the students, 36.05% scored one point and 39.37% gave a completely incorrect answer. With an item-total-correlation of $r = 0.50$, its selectivity is also within a satisfactory range. Since this item is in a short-answer format, approx-

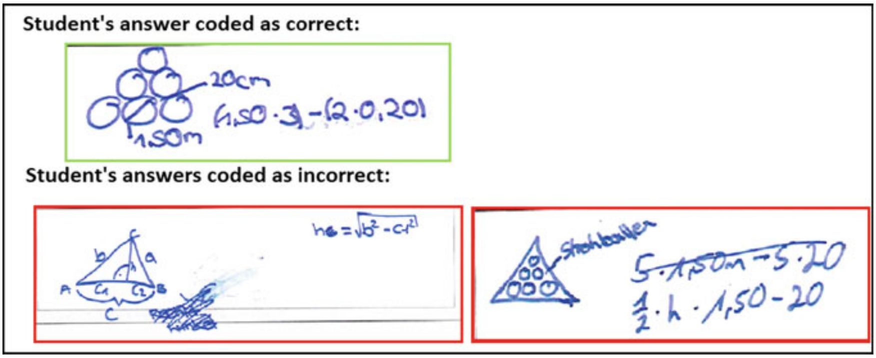

Fig. 8.3 Student responses to the *Straw Bale Task*

imately 40% of students' answers were rated by two independent raters according to a coding manual. The interrater-reliability Cohen's Kappa was $\kappa = 0.86$ and thus very good.

Figure 8.3 gives an example of how this item was coded. The first solution shows a correct solution given by a student. He or she was able to use the given relevant information to build a graphical and a symbolic mathematical model. The answers below show incorrect responses. The answer on the left shows that the student tried to apply Pythagoras' theorem and was not able to transform the given data into a mathematical model, with which it would have been possible to solve the problem. The response on the right shows a graphical representation of the situation where the straw bales are still shown (which is written next to the drawing). The formula used is an attempt to incorporate the given data, but on one hand does not pay attention to the units, and on the other hand, employs the formula for the area of a triangle. This mathematical model thus cannot be used for solving the task and was therefore coded zero.

8.2.1.3 Interpreting: *Dresden* Item

The sub-competency of *interpreting* a mathematical result and relating it back to the extra-mathematical context was measured with items such as Fig. 8.4. In this item, students are confronted with an extra-mathematical situation, which has been simplified and converted into a mathematical model. In the Dresden item in Fig. 8.4, a boy takes a look at a photograph, where he identifies his father standing in front of a giant arch at a Christmas fair. He mathematises the situation by measuring the height of his father and of the arch in the photo, and by setting up a mathematical term that combines all given numbers and yields the numerical result 3.8. In other words, the modelling cycle has already been carried out up to the point where the mathematical result has to be related back to the context. The student's task is to explain what the result 3.8 means in relation to the specified situation. Since the mathematical term

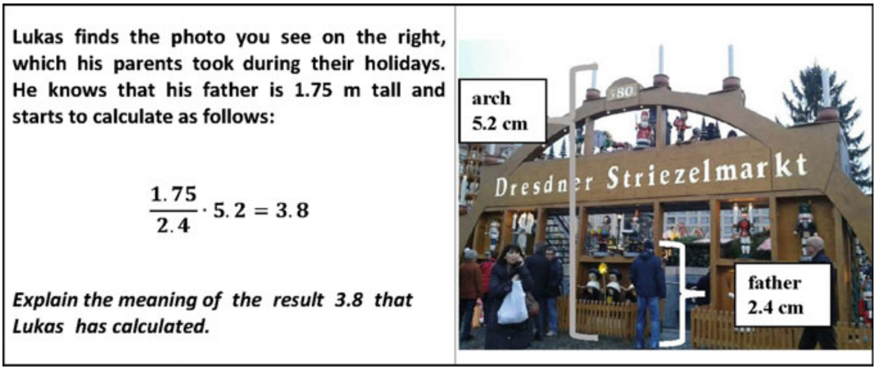

Fig. 8.4 The *Dresden Task*: short answer item that measures competencies relating a mathematical result back to reality (translated task)

represents the father's height in reality, divided by his size on the photo, multiplied by the size of the arch, the correct answer, which was rewarded one point, is that the arch is in reality 3.8 m high.

In our study, 56.05% of the students gave a correct answer. The most common incorrect response was that 3.8 represents the difference between the size of the father and the arch. This is probably due to the fact that the numbers given in the picture have a difference of 2.8. Students who do not pay attention to the 'borrowing' in the subtraction thus confuse the given result with the difference. These students clearly display a deficit in their competencies for interpreting a mathematical result, and subsequently did not receive a point for their answer. The selectivity for this item was satisfactory with a value of $r = 0.48$. The interrater-reliability (Cohen's Kappa) was very good with a value of $\kappa = 0.95$.

8.2.1.4 Validating: *Rock* Item

The sub-competency *validating* was perhaps the most difficult to assess. As the definition of this sub-competency shows, it consists of different facets, namely to critically check solutions, reflect on the choice of assumptions or of the mathematical model and also to search for alternative ways to solve the problem. To measure this sub-competency, we therefore employed a broader variety of items, which means that the items measuring the sub-competency validating were not as similar to each other as the items in the other sub-competencies. Figure 8.5 gives an example of an item that assessed the competencies for critically reflecting a result. In this item, students are confronted with a photo of a girl standing beside a rock. Without presenting a mathematical model, students are given the result of a calculation, namely the assertion that the rock is 8 m tall. They are asked to explain whether or not this result is plausible. To solve this task, students must use the photo and compare the size of the girl with that of the rock. As the rock is approximately three times as high as

Emilia has calculated that the rock she visited in her vacations is 8 m high. With the help of the photo on the right decide whether her result is plausible.

Fig. 8.5 The *Rock Task*: short answer item that measures competencies for critically checking a solution (translated task)

the girl, she would have to be more than two metres tall if the result was correct. A student's response, which clearly stated that the assertion is wrong and justified this answer by comparing the size of the girl and the rock, and additionally identified a maximum size for the girl was coded with two points. Answers like "No, since the rock is just approximately three times as big as the girl" which did not give a maximum size for the girl were still awarded one point. Answers that were coded as wrong mostly either were not justified at all or the result was found to be plausible.

Approximately half of the students (51.05%) acquired one point in this item, 27.56% were given two points. The selectivity was $r = 0.40$ and the interrater-reliability was again very good with $\kappa = 0.88$. Other items that assessed this sub-competency did not focus so strongly on checking a result, but confronted students with the choice of a mathematical model and asked them to decide whether the mathematical model would fit the given extra-mathematical situation. Additionally, there were items that assessed student abilities to find objects that help in determining the plausibility of a result. For example, students were given a photo of a dog and the claim that this dog is 28 cm high. They were asked to name one object that is approximately 28 cm high with which they could mentally compare the dog. In contrast to the Rock item in Fig. 8.5, students in this item were not asked to actually check the given result. This item assessed whether students were able to fall back on supporting knowledge (in German "Stützpunktwissen") as a basis for checking their results. We therefore had a broad variety of difficulty levels of items that assessed the different facets of the sub-competency of validating.

8.2.2 Testing of Items

Before constructing test booklets, we had a phase of intensive item testing. We first presented the items to experts in the field of modelling and asked them to comment on the tasks and to indicate what they thought the items would assess. All experts classified the items as we expected, but there were some that tended to assess more than just the one sub-competency. We reworked those items and related them more closely to the definitions of the respective sub-competency. Special attention was paid to the multiple-choice items and the choice of distractors. We asked the experts to comment on all answers that were part of the items and to add an answer to the item if they thought an answer or a typical mistake would be missing.

Subsequently, we gave the items to 36 students in a class, observed their working processes and asked them afterwards in groups what problems they had solving the tasks. Most of their answers referred to the poor quality of a photo which was then changed. In this phase, we identified formulations that were too complicated and made items too difficult to understand. With the help of students' comments, we simplified the language and made clear references for students who would subsequently be expected to use a picture as in the Rock item in Fig. 8.5. Students found some of the items "easy and interesting to solve, since they are different from conventional maths exercises", but "had to think intensively" about some of the items. These comments, as well as the analysis of their answers to the exercises, revealed a wide range of item difficulties, with a large number of items having a medium solution frequency, but also with a substantial number of items with a high as well as with a low solution frequency. No item remained unsolved, but no item was solved by all participants either. The qualitative analysis of students' answers made it possible to identify possible difficulties in coding the answers, which led to small changes in formulation. It was also the basis of a first draft of a coding manual for the test instrument.

Afterwards, we conducted a second pilot study with the aim of acquiring quantitative data to check the test's quality and to generate solution frequencies of the various items. In this study, no item was solved by all, or by none, of the 189 students. The answers the students gave additionally helped us to improve the coding manual.

8.2.3 Combining Items into a Test

One of the most difficult challenges in constructing a test that can be used in an experimental design is to ensure the comparability of pre- and post-tests. This challenge of creating parallel tests becomes redundant if one uses psychometric models and interprets responses to items as manifest indicators of one or several latent variables. The central idea is that the more distinct a person's latent variable is, the greater his or her probability of solving an item. Thus, in the simplest model, only the difficulty of the items and the person's ability are taken into account. The great advantage of

Pre-Test A	1	2	3	4				
Pre-Test B			3	4	5	6		
Post-Test A					5	6	7	8
Post-Test B	1	2					7	8

Fig. 8.6 Multi-matrix design of pre- and post-test: light grey boxes show the linkage between the pre- and post-test, dark grey boxes show the linkage between booklets at one point of measurement

this model is that the person's ability can even be determined if not all items are presented, which makes it possible to use a multi-matrix-design.

Figure 8.6 illustrates the test structure. Firstly, we constructed eight item blocks consisting of one item per sub-competency, a total of four items per block. No items were in more than one block. Secondly, we combined the item blocks into four test booklets, two for each point of measurement, so that each test booklet consisted of 16 items. We thereby paid attention to a similar average difficulty of the test booklets so as to avoid motivational problems for some groups of students. The fourteen multiple choice items were also equally distributed over the different booklets, so that all test booklets contained both item formats.

The two booklets we used at the first point of measurement were linked to each other via two blocks (blocks 3 and 4 in Fig. 8.6). Additionally, booklet A contained items that were not part of booklet B and vice versa. The same linking method was used for the post-test, where new items (blocks 7 and 8 in Fig. 8.6) were used to link the booklets. A person who answered test-booklet A in the pre-test also received post-test A, and the same for booklet B. By so doing, no student answered the same items twice. Nevertheless, since the item blocks 1, 2, 5 and 6 were used at both points of measurement, it was possible to link the two points of measurement. We determined the item difficulties using the data of all points of measurement, and then calculated the person's abilities for each point of measurement separately.

8.2.4 Methods of Data Collection

We implemented the test in 44 classes of grade 9 students who completed the test instrument three times each. This led to a total of 3300 completed tests which was the basis for the evaluation of the test instrument presented in this chapter.

Each testing lesson had a duration of 45 min, and since each student had to answer a set of just 16 items, no time pressure was observed. The testing was performed by the teachers strictly following a written test-manual, in which all details for conducting the testing process, as well as instructions to be read out, were recorded. In this way, it was possible to have a standardized execution in each of the participating classes. The correct implementation was controlled at random.

The completed test sheets were coded according to the coding manual. Some items were coded dichotomously and some had a Partial Credit scoring, receiving two points for a completely correct solution and one point for a partially correct solution. A sample (40%) of the completed test sheets were rated by two independent coders. The interrater-reliability for the open tasks was within a range of $0.81 \leq \kappa \leq 0.96$ (Cohen's Kappa) which reflects very good agreement.

The data were scaled using a one-parameter Rasch model with the help of the software ConQuest (Wu et al. 2007). For the estimation of item- and person-parameters, weighted likelihood estimations were used. To determine item parameters and to evaluate the test instrument, all three points of measurement in the main study were treated as if they were independent observations of different people, even though the same person could appear in up to three different rows in the data matrix. This approach is called 'using virtual persons' (Rost 2004) and is used in PISA (OECD 2012) and TIMSS (Martin et al. 2016), since it is unproblematic for the estimation of item parameters. These item parameters are the basis for evaluating the test instrument reported in this chapter.

8.2.5 Statistical Analyses to Answer the Research Questions

To be able to use the outcome of a probabilistic model for empirical data, it is necessary to check whether the a priori chosen model fits the data. Since the model that fits the data best is regarded as the best reproduction of the structure of the latent variable, this check of model fit can be used to gain more information about the competence structure itself. We therefore calculated various different models and compared the respective model fits. As we were interested to know whether it is possible to measure the different sub-competencies separately, we compared three models shown in Fig. 8.7. The first Model is a four-dimensional one in which each sub-competency is measured as a separate dimension. The second Model reflects the aggregation of sub-competencies as Brand (2014) and Zöttl (2010) chose for their research. The third Model is one-dimensional. If this was found to be the best fitting model for the empirical data this would mean the abilities students need to solve the different types of items, as presented in Sect. 8.2.1, were so similar that it would not be appropriate to model them as different dimensions.

When scaling empirical data with the help of item response theory (IRT), there are different ways to check how well a model fits the data. In the case of estimating item- and person-parameters, the algorithm used, iterates until the likelihood of observed responses reaches its maximum under the constraints of the given model. Therefore, the fit of two models can be compared by analysing their likelihood (L). After estimating the parameters, the programme ConQuest displays the final deviance (D) of the estimation, which derives from the likelihood by $D = -2ln(L)$. The smaller the final deviance, the greater the likelihood and the better the model fits the data. This measure does not take into account the sample size and the number of

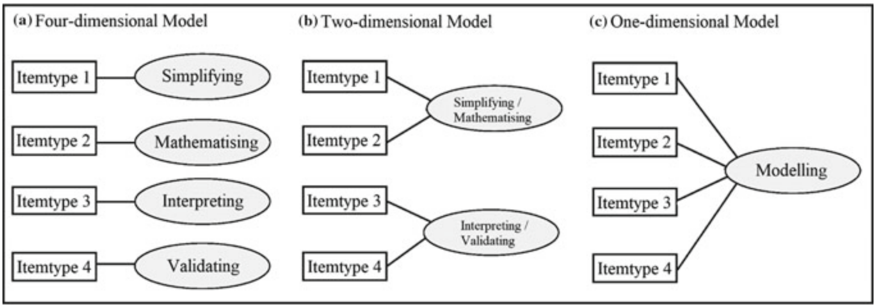

Fig. 8.7 Models used and compared to gain information about the competence structure

Table 8.1 Information criteria to compare models

	Model		
	Four-dimensional	Two-dimensional	One-dimensional
Sample size	3300	3300	3300
No. of estimated parameters	61	54	52
Final deviance	85,471.21	85,821.33	85,838.86
AIC	85,593.21	85,929.33	85,942.86
BIC	85,965.41	86,258.82	86,260.15

Note Lower values indicate a better fit for the model

estimated parameters. Therefore, the AIC and BIC are also reported.[1] AIC tends to prefer models that are too large whereas BIC prefers smaller models. If both criteria prefer the same model, this is likely to be the best of the candidate models (Kuha 2004, p. 223).

8.3 Results

Table 8.1 shows the final deviance, AIC and BIC for the three different models in Fig. 8.7. The four-dimensional model, for which each of the different types of items, which were designed to measure the sub-competencies separately, loads on one dimension each, fits the data best. All three measures are lower than in the two-dimensional model, in which *Simplifying* and *Mathematising,* as well as *Interpreting* and *Validating*, have been combined, and lower than in the one-dimensional model, where all types of items load on just one factor.

[1] AIC (Akaike Information Criterion) defined by $AIC = -2\ln(L) + 2n_p$ and BIC (Bayes Information Criterion) by $BIC = -2\ln(L) + \ln(N) \cdot n_p$ where n_p is the number of estimated parameters and N the sample size. A smaller value indicates a better fit.

The results indicate that scaling the test items used (which aimed to measure different sub-competencies) with a one-dimensional model, is the poorest of the tested options. The two-dimensional model fits the data slightly better than the one-dimensional model, as all measures have a lower value. That the four-dimensional model fits the data best, with a considerable margin concerning the information criteria, provides quantitative empirical evidence supporting the theoretically assumed, and qualitatively observed, sub-competencies of mathematical modelling.

Another possibility for checking the fit of the model is to look not only at the overall model fit, but to check the fit of the different items separately. There are also different measures that indicate the fit of items from which the weighted mean square fit (WMNSQ) is the most frequently reported value. This fit index reflects how much the empirically determined responses to an item differ from the solution probabilities that the model predicted (Wilson 2008). Since those whose abilities lay near the item parameters provide more information than persons with a more extreme ability value, it is sensible to weight their residua more strongly (Bond and Fox 2007). The WMNSQ can be z-standardised and tested statistically for significance. Following Bond and Fox (2007, p. 243) and PISA (OECD 2012), WMNSQ should be within a range of 0.8–1.2 for high stakes tests, and 0.7–1.3 for "run of the mill" tests. For the four-dimensional model, the WMNSQ was within a range of 0.93–1.11 and thus quite near to the generating value of 1, which reflects a perfect model fit.

As mentioned above, there are two parameters calculated in the models we used: those that reflect the *item difficulty* and the parameters that indicate the *degree of competency*. The item parameters within this approach were determined with an (item-separation)-reliability of 0.996, which is an excellent result, even though such high values are not unusual for large samples (Wu et al. 2007). The EAP/PV-reliabilities for the ability parameters lie within a range of 0.66 and 0.80 for the different sub-competencies at different points of measurement. Since the EAP/PV-reliabilities can be compared to Cronbach's Alpha, these values are within a satisfactory to good range, and certainly sufficient to compare groups, as planned for further studies.

8.4 Summary and Discussion

This chapter focused on the assessment of modelling competencies, taking up the notion of different sub-processes in a modelling process that requires different competencies. Previous research has already shown that modelling requires different competencies than purely technical mathematical competencies (Harks et al. 2014). Concerning the sub-competencies of simplifying, mathematising, interpreting and validating, it was still unknown whether it is possible to assess them separately. Those test instruments that assessed sub-competencies of mathematical modelling subsumed different sub-competencies, instead of treating them as independent dimensions of a more general modelling competency. We therefore constructed a new test instrument with specific test items for each of the four chosen sub-competencies. This chapter presented an exemplary item for each type. We explained to what extent these items are able to measure the sub-competencies of mathematical modelling. It was clear from the beginning that the new test instrument was not designed to measure a more global modelling competency, but to enable making statements concerning one or several sub-competencies. This is especially fruitful for empirical research, for example if the effects of certain interventions have to be evaluated. A test that objectively, reliably, and validly measures the sub-competencies can be used to compare student achievements before and after having experienced a certain treatment. If the treatment being examined is expected to have effects that differ from one sub-competency to another, it is preferable to evaluate the intervention at the level of sub-competencies, instead of building average scores.

We have shown that the test we presented can be used to do so. The psychometric model with four separate dimensions fits the data best. Thus, we took up the idea of Haines et al. (2001) to assess sub-competencies separately, continued the work done by Brand (2014) and Zöttl (2010), and succeeded for the first time in measuring the sub-competencies empirically as independent latent variables.

These results underline the different demands that a modelling process imposes on students and further confirms the empirically assumed division of a modelling process into different steps. However, our study of course has its limitations. Firstly, we limited our research to the field of geometric modelling. This was due to other studies for which the test was constructed and not grounded in reasons with regard to content. Further studies should expand the content areas, as well as test further age groups. Secondly, even though there are already some tests that assess sub-competencies of modelling, we have not yet had the opportunity to check the correlation of those tests with our new test, so as to control the validity statistically. This is also valid for discriminant validity.

Thirdly, we did not aim to assess general modelling competency and hence cannot make assumptions on the interplay between a general competency and the sub-competencies. The question arises as to how much we know about a person's general modelling competency, if we know his or her strengths and weaknesses in the sub-competencies. By answering this question, it could be possible to substantiate the assumptions, for example concerning meta-knowledge, that have been derived from qualitative studies with quantitative data.

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Chapter 9
The Influence of Technology on the Mathematical Modelling of Physical Phenomena

Miriam Ortega, Luis Puig and Lluís Albarracín

Abstract A study is presented in which students are asked to model two physical phenomena using applications on electronic tablets: a bounce of a ball and the extension of a spring. The analysis focusses on (a) the influence of characteristics of the applications on the tablets on the decisions that groups of 16-year-old students made during the modelling phases in which reality and mathematics are related, (b) mathematisation of the phenomena and (c) interpretation of the models. The phenomena were recorded using an app that requests users establish a set of references during the mathematisation process, which makes students focus on the way the references have been set to interpret the model properly. Our findings indicated inconsistences between student decisions made during mathematisation and their considerations during interpretation of the model. To conclude we suggest reasons students experience problems in working without a pre-defined reference system.

Keywords Mathematisation · Physical phenomena · Real data · iPad · Interpretation · Bounce of a ball · Lengthening of spring

9.1 Introduction

Usually, content in most upper secondary mathematics lessons is worked on in a formal and abstract way and students find difficulties in relating mathematical content

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to real-life situations (Henn 2007; Oliveira and Barbosa 2013). Many investigators (e.g. Greefrath and Siller 2017; Sala et al. 2017) have found that modelling tasks are a useful resource to show students the capability of mathematics to describe reality. In addition, the use of technology in classrooms can contribute positively to students' understanding of the phenomena in the world around us (Henn 2007; Stillman 2007). However, as Grigoraş et al. (2011) point out, "deeper analyses of the modelling situations are necessary to understand the interplay of tasks, techniques, technology and theories in modelling situations" (p. 94).

We designed teaching material for the concept of function in which two physical phenomena have to be modelled by using technological tools with the purpose of finding possible solutions to help students overcome the difficulties that may appear in relating the characteristics of functions to real phenomena. In this chapter we present a study where the influence of a technological tool on the students' decisions is analysed during the translations between the real world and mathematics.

9.2 Theoretical Framework

As Freudenthal (1961) already mentioned decades ago, there are many different ways of understanding the term, modelling, depending on the perspective from which we want to formalize or organize a real situation or phenomenon. From the didactics of experimental scientific disciplines and specifically from the didactical framework of scientific practice (Osborne 2014), modelling is conceived as a way of organizing physical phenomena taking into account the values of the physical magnitudes in order to shape scientific knowledge and scientific practices (Duschl and Grandy 2012). However, from a mathematical point of view, when modelling a phenomenon mathematically is concerned, we focus our interest on the mathematical content and processes that allow us to organize this phenomenon (Freudenthal 1983), which is the perspective that we adopt in our study. From this point of view, every mathematical modelling task requires translations between reality and mathematics, where reality is taken to be the rest of the world, other than the mathematical domain.

Parts of the modelling process can be described by the so-called modelling cycle, which can be guided by the questions posed in the task or promoted by the students themselves. There are several versions and interpretations of the modelling cycle (Perrenet and Zwaneveld 2012) but all start from the division proposed by Pollak (1979) in which modelling processes are studied by differentiating reality from the mathematical domain, even if we conceived of mathematics as part of the real world because reality is understood as the rest of the real world, that is the extra-mathematical aspects. In this way, under any of the theoretical perspectives adopted to study the modelling processes, several phases can be distinguished, among these, the transition from reality to the constructed mathematical model, that is, *mathematisation*, and the *interpretation of the results*, obtained in the mathematical domain, from the point of view of the real phenomenon studied.

Studies involving modelling tasks with the use of technology have shown that technology related activity takes place during all phases of the modelling cycle (Geiger et al. 2010; Greefrath and Rieß 2013). Research in the frame of the instrumental genesis approach shows that there is an interaction between conceptual understanding and the use of ICT tools, considered as mediators of human behaviour and learning (Artigue 2002; Morera et al. 2012). Specifically, technological mediation is at the heart of the relation between the tool or artefact and the user and it may happen that certain features or uses of the tool transform user understanding. This is the process whereby the artefact becomes an instrument for the user and the process is called instrumental genesis (Guin and Trouche 1999). Therefore, students have to perceive affordances (Gibson 1966) of technological environments and take advantage of them to be able to understand the task properly and act subsequently (Brown 2015).

Moreover, different types of technological supports impact different phases of the modelling cycle (Greefrath 2011). Despite this, those that have received more attention in the field of mathematics education are those that allow the treatment of the data obtained (Geiger 2011) instead of those in which data are captured through the use of technology, which will be the focus in our study.

9.3 The Research Study

In the present study, we adopted the theoretical and methodological framework for research in mathematics education of Local Theoretical Models (LTMs), developed by Filloy et al. (2008). In this framework, competence, cognitive, teaching and communication models (p. 34) are developed to make sense of the phenomena that occur in the teaching and learning processes of specific mathematical content with a particular group of students in a specific setting. The model is thus local and serves to give an account of what is observed on the basis that "if things were as characterized by the model then the phenomena would be as observed" (Puig 2010, p. 3). The LTM is descriptive, explanatory and predictive in nature but does not rule out other descriptions, explanations or predictions of the same observed phenomena (Puig 2010) but is adequate in the particular setting.

In our case, teaching material was designed to study linear and quadratic functions through the use of electronic tablets and the mathematical modelling of two different physical phenomena. The study presented here is part of a more general one in which students' performances are analysed from different points of view when they are working with this teaching material. In particular, what we show here is the effect of using a specific technological tool during the mathematisation of the phenomena on students' performances. Specifically, our research aim will be to answer the following question: *What is the effect of the decisions that students make during the mathematisation of the phenomena conditioned by Video Physics® app on the interpretation of the characteristics of the models?*

Another part of the general study that complements the research presented here can be found in Ortega and Puig (2017). In that chapter, the influence of a previous study of the qualitative properties of phenomena and families of functions on students' performances is analysed.

9.3.1 Participants and Teaching Methodology

We conducted our study at two different moments in our materials design trajectory, thus we implemented the teaching material with, and collected data from, two different groups of students. These were Year 11 upper secondary school students (16-years-old) with similar characteristics. They were intact class groups of 16 students each: 10 girls and 6 boys in the group in which the first material to study the quadratic function was implemented and 7 girls and 9 boys in the group in which material to study the linear function was implemented. Students of both groups did not have any previous experience in solving modelling problems or studying physical phenomena as families of functions from a mathematical perspective (although students of both groups had prior knowledge of Physical laws that describe the kind of relation between the variables studied). However, both groups had studied linear and quadratic functions before but not by using electronic tablets.

The teaching experiments had two different parts: classroom lessons and interviews. Both classroom lesson implementations were for two sessions of 55 min each with experiments with actual phenomena and the collection of data using electronic tablet apps occurring during this time. Students also completed worksheets. At the end of both experiments, 2 groups of 2 students were selected based on the results obtained after analysing their answers to participate in an interview. The interview had a dual purpose. The first was to identify students' decisions made during mathematisation influenced by the features of the Video Physics® app in the interpretation of the model. The second purpose was to guide the students to become aware of their misconceptions related to their performances during these phases and to overcome them. For classroom lessons, the teacher kept a diary and all sessions were video- and audio-recorded.

The students worked in groups of two (designated A1–A8 for Experiment A and B1–B8 for Experiment B) during both teaching experiments, except when they had to carry out and record the experiments. The purpose of working in pairs was to encourage the verbalization of what they were doing, thinking or wanting to do (Schoenfeld 1985) because, although this does not allow us to know the cognitive processes of the students when they are dealing with a task, at least it is possible for us to obtain the data of the explanations that they are forced to give in a collaborative environment. The teacher was the same during classroom lessons and interviews, not the usual teacher of the groups but one of the researchers. She provided minimal help during the lessons in order to allow the students to work at their own pace and avoid not being influenced by other students' ideas. However, during the interviews the teacher had an essential role: to guide the students through the questions and to provide suggestions for them to overcome the difficulties observed.

9.3.2 Data Analysis and Research Method

Due to the characteristics of the Local Theoretical Models, the methodology used for the analysis of the data will be those commonly used in this framework. This methodology will be described distinguishing between the data from student performances during the classroom lessons and the interviews.

The analysis of the data from classroom sessions started by extracting the data from worksheets, the teacher's diary, iPads and video cameras and relating them in a coherent way by describing students' performances. Next, similar types of students' performances were searched according to the noteworthy aspects of each item, grouped and described in detail. Finally, cognitive tendencies in students' performances with regard to the research aim were identified and described.

For the analysis of the interviews, firstly a written protocol was elaborated. Transcriptions of the interviews were included in the protocol as well as description of the students' gestures and performances when using specific apps on iPads, accompanied by images of written answers and screenshots of videos and tablet screens. Secondly, comments to give meaning to students' performances were added to the written protocol. Afterwards, these comments were organised by undertaking a rational reconstruction (Puig 1996), that consists of a narration, based on the students' performances, with the purpose of making sense of the teaching-learning situation. In this narration, it is assumed that all students' performances have a rational and consistent base that justifies them. Next, cognitive tendencies regarding the research aim were identified. Finally, results from the study of the group during classroom lessons and the case study were related in order to answer the research question.

Due to the characterization of model in the framework of the Local Theoretical Model, description of students' performances in both classroom lessons and interviews was based not only in describing what they did during the teaching experiments but also on making explanatory hypotheses. These explanatory hypotheses are descriptions of possible explanations for students' performances, not with the intention of asserting that current cognitive processes of students are those hypothesized but, if the cognitive processes were the ones described, the performances would be the ones we have observed. In addition, other results obtained from the data analysis process are not included here as they are not part of the study presented in this chapter.

We will now present characteristics of the design of the teaching experiments, the organization and implementations in classrooms before proceeding to the reporting of results followed by a discussion of these.

9.4 Design of the Teaching Experiments

Considering the distinction between modelling as content in its own right and as a vehicle for teaching mathematics in educational settings (Julie and Mudaly 2007; Gravemeijer and Doorman 1999), we designed two tasks considering modelling as a vehicle for the development of particular mathematical content: the characteristics of the families of linear and quadratic functions.

The two tasks studied were phenomena widely studied by classical physics and concerned the use of real data, which would be directly taken into the classroom by the students using iPads. In the first teaching experiment (A), the phenomenon studied was the relation between the time and the height of a ball dropped from a certain height restricted to the first rebound and the subsequent drop of the ball, that is, from the moment that the ball touches the ground for the first time until it touches it again. Since this phenomenon is a uniformly accelerated rectilinear motion, it can be described by one of the kinematics laws of Newton, $y(t) = y_0 + v_0 \cdot (t - t_0) + ½ \cdot g \cdot (t - t_0)^2$ where y is the distance travelled by the ball, t is the time in each instant and y_0, v_0 (null in our case), g and t_0 are constants. Therefore, it can be represented by a quadratic function where t is the independent variable and y the dependent one. In the second teaching experiment (B), the physical phenomenon was the relation between the lengthening of a spring (in the form of a slinky) and the number of marbles introduced into a glass that hangs from it. Since the phenomenon is asked to be studied only in the elastic zone of the spring, we can consider that the function that approximates this relation is a linear function. Specifically, as the force of the spring will be equivalent to the force of the weight of the marbles and considering Hooke's law, we will obtain that $k \cdot y = N \cdot m_i \cdot g$, where k is the elastic constant, y the spring elongation, N the number of marbles, m_i the mass of a marble and g the acceleration of gravity. Therefore, now the relation that has to be studied is the one between y and N, isolating the dependent variable we obtain that the function that describes the phenomenon in the studied area is $y(N) = (m_i \cdot g)/k \cdot N$. Although the theoretical reasons that prove the type of function that better describes the phenomena comes from Physics laws, during the teaching experiments this is not mentioned to the students.

In general terms, the questions of the experiments can be divided into three types of sets of questions that would be implemented in the order presented here. The first question type was designed to make an initial analysis of the qualitative properties of the phenomena and the families of functions, following the idea first developed in Puig and Monzó (2013) and subsequently applied in Ortega and Puig (2017) that the incorporation of this type of question can help students manage and control the modelling process. The second type of questions, which were called instructional questions, was intended to give students guidelines on how to conduct the experiments and how to obtain the data. Finally, the third question type was intended to guide the students to find a function that fits the data and, to work on specific aspects of the function obtained and to interpret some of its characteristics in relation to the experiment to verify suitability and validity. In Ortega and Puig (2015), further details of the design of the teaching material and the specific questions for the experiment of the bounce of a ball are given.

However, although in general the characteristics of both designs are similar, they differ in three main aspects. First, questions formulated in each teaching experiment were slightly different. This was because teaching experiment B was conducted after A, after considering how the characteristics of the design of teaching material used in A could be improved. Second, the way of obtaining the variables to study in each experiment was also different. Whereas in experiment A both variables were given directly by the app, in experiment B students had to construct one variable (number of marbles) and transform the values provided by the app for the other (to obtain a positive lengthening that initially is zero). Third, the way to obtain the functions, and consequently the apps used in each experiment, differ as well. While in experiment A students only had to choose the formula of the family of functions so that the Data Analysis® app provided the specific function, in experiment B students had to find the function by handling the values for the parameters in the formula and noticing the graphic transformations related to them in Desmos® app.

9.5 Implementation of Teaching Experiments

Although the teaching material was implemented at different times and with different groups of students, both teaching experiments took place in the students' usual classroom and were carried out during two sessions of 55 min each, not including the interviews.

In the first session of each experiment, the students had to answer the first set of questions and to obtain the data from real experiments following the second set of questions. In particular, they were asked to analyse the qualitative properties of the phenomenon and the families of functions sketching the graph that they expected to represent the phenomenon and choosing the algebraic expression of the family of functions that they thought would describe better the phenomenon from a given list. Next, they had to perform the experiments in the classroom recording with a video following the instructions given in the second set of questions in order to obtain the coordinates of the points that showed the relation studied in each case using the app Video Physics®. For the first experiment, the students had to stand on a chair and drop a ball. In the second experiment, the students from the other group had to place each end of a wooden strip over a table and then hang a spring from it with a plastic cup on the end of the spring, where up to 8 marbles would be introduced gently. Once they had video-recorded the experiments with an iPad, they had to send the video to the rest of the devices so that the students could continue working in pairs. Following this, they had to obtain the values for the variables on the trajectory of the moving object using the following actions in the app: (a) setting the referential axes in one of the photographs (Fig. 9.1a, b) locating a referential measure by marking a segment in one of the photographs and introducing its real value into the app (Fig. 9.1b, c) marking a point on the object that is changing its trajectory in each photograph of

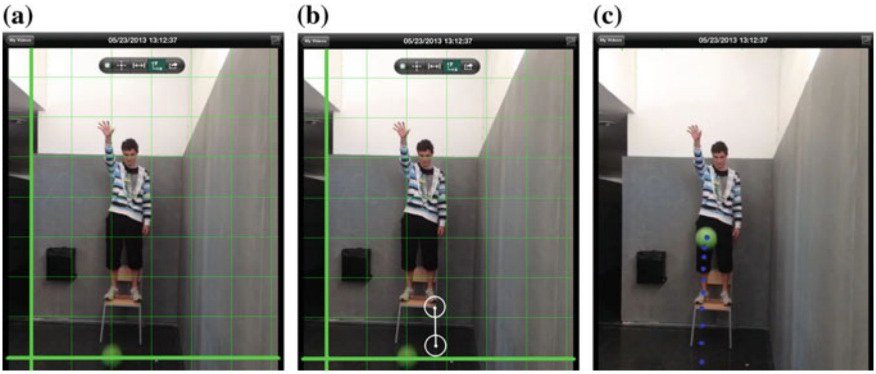


Fig. 9.1 Sequence of screenshots **a**–**c** of the process of obtaining data in the first experiment

the video, that is, on the ball in experiment A and on the end of the spring or on the plastic cup in experiment B (Fig. 9.1c). It should be noted that what the app does next is to show different graphs relating the variables time, distance of the points to the x-axis and distance of the points to the y-axis on the grid overlaying the video. Then, obtaining the values of these variables separated by columns in an Excel file has to be requested of the app, so the students can construct the coordinates of the points that describe the relation studied in each case. However, whereas in experiment A the coordinates can be obtained directly because both the time and the height of the ball are variables provided by the app, this does not happen in experiment B. In B, every point has to have as its first coordinate the number of marbles introduced into the plastic cup, which can be determined easily, and as the second coordinate the lengthening of the spring, for which values will have to be calculated adding, subtracting or multiplying by -1 one of the variables provided by the app, depending on where the students set the x-axis and the points.

Hence, the mathematisation of the phenomenon is conditioned by the characteristics of the app and the possibilities that it offers to the students. Moreover, the way in which students take references will affect the values that variables will take and, consequently, the functions. Therefore, students will have to be conscious of how they take references and how the app works, that is, the technological tool should become an instrument for the students (Guin and Trouche 1999).

During the second session, the students had to answer the last set of questions. In particular, they had to find the algebraic expression of the function that better described the phenomenon recorded. For that purpose, in experiment A, the students had to introduce the coordinates of the points obtained with Video Physics® into the app Data Analysis®. Then, they had to choose a formula of the family of functions that fitted better to the points from a list of options given by the app, which provided them with the concrete formula and the numeric values for the parameters. However, in experiment B the students were asked to copy the points to the app Desmos® and try to fit a graph to them handling the values for the parameters of the formula of the family of functions that they thought represented the experiment. For this, we would require the students not only to obtain the formula by themselves, but also to give meaning to the parameters of the family of functions $y = mx + n$ where m and n take real values for the particular experiment.

Following the experiments, students had to answer some questions to guide them through the interpretation of the model in relationship to the real phenomenon and the model validation. Firstly, the students were asked to (a) calculate some images of the function and (b) explain if doing this calculation made sense, where they were pushed to interpret its meaning in the real world. They had to realize that, although the domains of both functions are all the real numbers, the functions only make sense in the interval of values for which the phenomena studied are defined: just during the first vertical rebound and fall, or until the eighth marble is introduced into the plastic cup. For that reason, some of the points, for which images had to be calculated, were out of the interval and students were asked, among other questions, to explain why the answers did not show what really happened. Finally, students had to answer further questions by interpreting the peculiarities of the function (using the graph, the concrete algebraic expression or the coordinates of the points) comparing it to concrete characteristics of the experiment conducted in the classroom. In particular, in experiment A they were asked to calculate the values of time when the ball touched the ground and the values of time when it reached the maximum height. In experiment B, they had to calculate the lengthening of the spring when 4 marbles had been introduced into the plastic cup, the minimum length of the spring during the experiment, and the distance between the plastic cup and the ground. Finally, they had to explain how the function would change if they had to study the distance between the cup and the ground instead of the lengthening of the spring. To answer all these questions students were allowed to use the app Free Graphing Calculator®, which made it possible to represent functions graphically and let students calculate images of the function, amongst other possibilities.

After these classroom lessons, 2 groups of 2 students from each experiment were selected to participate in an interview, conducted by the teacher-researcher in one 55 min session per group. In the interview, students' reasons for choosing the references were identified and also the way they conceived of the variables during the interpretation of the phenomena in relation to the experiment, as will be shown in Sect. 9.6.1 and 9.6.2. In addition, they were guided to realize their misconceptions through teacher questions and suggestions.

9.6 Results

In the following subsections, the results of the analysis of the data will be presented in detail. Firstly, we will show the way in which the students took the references in the Video Physics® app during the mathematisation of the phenomena and what they conceived as the variables for each experiment. Secondly, we will describe how they interpreted the characteristics of the model: basing their answers in their prior

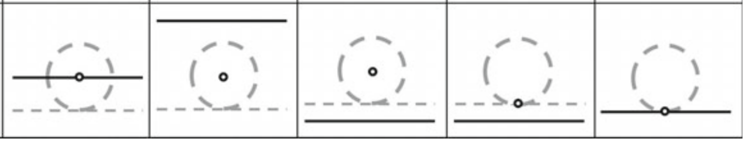
Case	1	2	3	4	5
Group	A1	A2	A3, A4	A5, A7, A8	A6
Diagram					

Fig. 9.2 Different schemes for establishing references in the ball experiment

conceptions instead of the way in which they had taken the references in Video Physics®.

9.6.1 Choosing References in Video Physics®

As noted previously, the characteristics of the technological tool used, the Video Physics® app, required students to choose a reference during the mathematisation, which is given by the combination of two factors: the position in which the axes have been set on the image and the position of the points marked. Therefore, the variables affected in each experiment by the way in which references are taken will be "height" (experiment A) and "lengthening" (experiment B). In particular, the significant references in both cases are the x-axis and the points marked.

For that reason, in experiment A, we focused on analysing the photograph of the video in which the ball touches the ground for the first time, where it is possible to observe both the axes and the first point marked. When categorizing the responses, we observed that students set the x-axis in different positions with respect to the ball: below it, above it and at the same height. On the other hand, they marked the points that showed the trajectory of the ball in three different spots: at the centre, at the top or at the bottom. Avoiding duplicates and reducing equivalent options we obtained 5 different cases as shown in Fig. 9.2, where the large dashed circle represents the ball and the dashed line the ground, the small circle represents the first point marked and the black line the x-axis. The high number of different cases is due to the application not providing any limitations when setting references.

It is important to note that some features of the diagrams of the different responses have been simplified as they are two-dimensional drawings which represent photographs of a three-dimensional space. In these diagrams, "the ground" has been represented as the horizontal line placed at the lowest part of the ball, which has been done taking into account the students' conceptions observed both in their answers and during the interviews. Considering this is important in understanding the students' responses during the interpretation of the model.

Moreover, most of the students placed the referential axis taking into account "the ground" during the selection of references, since it is usually taken as the reference. Two categories can be distinguished: the students who placed the x-axis at the ground

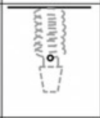
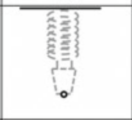
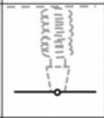
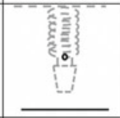
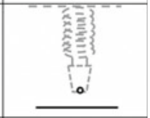
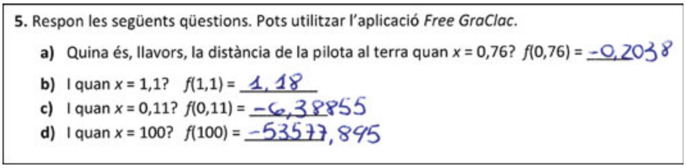
Case	1	2	3	4	5
Group	B1	B2	B3, B4	B5, B6, B8	B7
Diagram					
Kind of variable used	$L + l_s$	$-1 \cdot (L + l_c + l_s)$	$-1 \cdot L$	$d_{p,x} - L$	$d_{p,x} - L$

Fig. 9.3 Different schemes for establishing references in the spring experiment

or close to it (groups A3, A4 and A5) and the ones who placed it exactly at the chair seat where the student was standing and from which he/she dropped the ball (group A2). The students who performed as in case 1 did not make reference to the ground and they explained during the interviews that they set the referential axis in that position (see Fig. 9.2) because it was at the same height as the first point marked, therefore "the height will be zero", focusing only on the point and ignoring the rest of the elements. Therefore, it is already possible to see a certain tendency to view "the ground" as reference for variable height during the mathematisation phase because they want the height of the ball at ground level to be zero and positive above it, although only students of case 5 actually did this.

On the other hand, with regard to experiment B, we analysed the photograph of the video in which the students marked the first point, that is when no marbles were introduced as yet (or, in the case of the students who did not mark a point when the cup was empty, the photograph where just one marble was introduced). In this case, they marked the points that showed how the length of the spring would change in two different places: in the lower part of the spring and in the bottom of the cup. They set the x-axis in different positions: just in the wooden strip, in the lower part of the spring, in the bottom of the cup, just in the last point marked and on the ground (at the end of the table legs). A classification of how students took the references is shown in Fig. 9.3. In the diagrams, the spring and the plastic cup are represented with dashed grey lines, the x-axis by a black straight line and the first point marked by a small black circle.

After observing the way the students took the references in the different photographs, we analysed how they obtained the coordinates of the points. In relation to the number of marbles introduced into the cup, all students were consistent in their answers: if they marked the first point when the cup was empty, they wrote 0 as the first coordinate of the first point and if they did it when the cup contained one marble, they wrote 1. For the lengthening of the spring, most of the students considered the variable that gave the distance between the points marked and the x-axis, which was provided directly by the app itself, and they did not make any kind of calculation to ensure it was positive and initially zero. Only the students of group B1 changed the sign of the values that such a variable takes to obtain positive values for the lengthening of the spring, although they did not take into account that doing so


5. Respon les següents qüestions. Pots utilitzar l'aplicació *Free GraClac.*

a) Quina és, llavors, la distància de la pilota al terra quan x = 0,76? f(0,76) = -0,2038

b) I quan x = 1,1? f(1,1) = 1,18

c) I quan x = 0,11? f(0,11) = -6,38855

d) I quan x = 100? f(100) = -53577,895

Fig. 9.4 Students' answers (group A2) for values of the function at various values of x [Translation: 5. Answer the following questions. Free GraCalc app can be used. (a) What is the distance of the ball to the ground when $x = 0.76$? $f(0.76) =$; (b) And when $x = 1.1$? $f(1.1) =$; (c) And when $x =$ 0.11? f(0.11) =; (d) And when $x = 100$? $f(100) =$.]

was not enough to obtain the lengthening of the spring because, this way, they were also considering its initial length. The kind of variables considered for the second coordinate of the points in each case is also shown in Fig. 9.3. In the formulas, L represents the lengthening of the spring, l_s the initial spring length, l_c the length of the plastic cup and $d_{p,x}$ the distance between the first point marked and the x-axis. Therefore, no student actually took the lengthening as a variable.

9.6.2 *Interpretation of the Models*

During interpretation of the model in experiment A, all students considered that the height of the ball on the ground is exactly zero and it took positive values above it, regardless of how they had taken references with the app. In particular, students who took the references as in case 2 (see Fig. 9.2) should have considered the possibility of finding negative values for height. However, they did not conceive it this way. For students of group A2 the ball touched the ground at 0.718 and at 1.88 s when the height of the ball was −0.46 m because they set the x axis at the chair seat. So, when they calculated the height of the ball at 0.76 s, they obtained a negative value (see Fig. 9.4). Nevertheless, in question 6 they explained that none of the negative values made sense because obtaining negative values meant "there is no ground and the ball continues falling down", as they confirmed during the interview. Hence, they attributed the obtaining of negative values to the ball being below the level of the ground, which implies that they were considering the ground as the reference. Therefore, the negative values of the images caught the students' attention more than the fact that there were points out of the interval where the function was defined in relation to the phenomenon, even when one of the points ($f(0.76) = -0.2038$) would have made sense for their experiment. This fact explains how strong their conception was about the ground being the reference and how unfamiliar they were with interpreting mathematical results in real terms.

Otherwise, students who took the references as in cases 3 and 4 (Fig. 9.2) obtained the height of the ball at the ground is not exactly zero. However, all students considered the ground as the reference during the interpretation of their model. In particular,

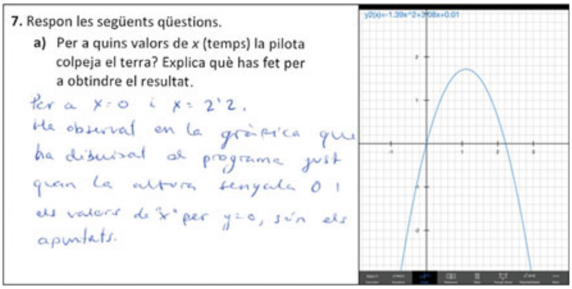


Fig. 9.5 Students' answers (group A5) at question 7 (left) and screenshot of graph (right) [Translation: 5. Answer the following questions. (a) At which values of x (time) does the ball touch the ground? Explain how you got the solution. "For $x = 0$ and $x = 2.2$. We have observed the graph drawn by the program just when the height is 0 and values of x for $y = 0$ are the ones we have written."]

when they were asked to calculate the time when the ball was on the ground, groups A5 and A8 made $y = 0$ and solved the resulting equation and groups A3, A4 and A7 looked at the values of x where the graph cut the x-axis, all considering the height of the ball at ground level as zero.

Finally, although the values for the height of the ball would always be positive and zero on the ground for the students of cases 1 and 5, only case 5 students took the floor as reference (see Fig. 9.2). In case 1, the students of group A1 set the x-axis in the first point marked "[…] as if the ball was half above and half below the ground", as they confirmed after becoming aware of it during the interview. Therefore, the height of the ball at ground level would actually be negative. In relation to the students of case 5, despite having taken the ground as a reference, group A5 obtained that the height of the ball on the ground was not exactly zero due to the lack of precision when taking references in Video Physics®. However, when the students were asked to calculate the values of time when the ball touched the ground in question 7, they answered that it is "when $x = 0$ and $y = 2.2$" because these are "the values of x when $y = 0$" (see Fig. 9.5) considering the height to be exactly zero, not using the data of their own experiment.

In relation to the interpretation of the characteristics of the function in experiment B, it was observed that students considered lengthening as a variable that always takes positive values and it was zero initially when no marbles were in the cup, independent of the kind of variable used for the lengthening and how the references were taken.

Specifically, students from cases 2 and 3 should have considered the possibility of obtaining negative values for the lengthening of the spring according to the way they defined the variable. Nevertheless, no student saw it that way. Students from group B3 calculated some images of the function and explained that $f(4) = -0.023$ made no sense because "lengthening cannot be negative". Students of groups B2 and B4 also indicated that obtaining negative values made no sense, although they did not explain why. However, when they were asked to explain what the calculated function

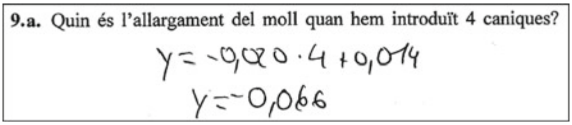

9.a. Quin és l'allargament del moll quan hem introduït 4 caniques?
y = -0,020 · 4 + 0,014
y = -0,066

Fig. 9.6 Students' answer (group B5) for the lengthening when 4 marbles are introduced [Translation: 9.a. What is the lengthening of the spring when we have introduced 4 marbles into the cup? "y = −0.020 · 4 + 0.014", "y = −0.066".]

represented, they explicitly indicated that the function showed the relation between number of marbles and lengthening of a spring. Therefore, they were considering that it was not possible to obtain negative values because lengthening cannot be negative.

We also noticed that most students assumed that initial lengthening was zero, despite taking references in a way that did not lead to this. Only those in case 3 could consider that the lengthening of the spring was zero at the beginning but also students in cases 1, 2, 4 and 5 conceived it in that way. For example, students from group B1 (case 1) considered an initial length for the lengthening because they set the axis just in the wooden strip and marked the points in the lower part of the spring. However, they said that the function showed the relation between the number of marbles and the lengthening of the spring, which "is zero because there aren't marbles in the glass" as they indicated during the interview.

Finally, despite no student actually taking the lengthening of the spring as a variable, understood as having a positive value that was zero initially, we thought that maybe they could have considered "lengthening" as another variable and still be coherent with it. In particular, during the mathematisation students from cases 1 and 2 considered the lengthening plus a certain length, students from case 3 a negative lengthening and students from cases 4 and 5 the distance to the ground or the last point marked (see Fig. 9.3). However, no student was coherent with these considerations during the interpretation. For instance, students of group B5 used their function to find the lengthening of the spring for 4 marbles (see Fig. 9.6), which actually gave the distance between the 4th point marked and the x-axis. Nevertheless, they conceived of lengthening as a positive magnitude that was 0 at the beginning because when we asked them during interview to explain why they had used a linear function to fit the data, they said that it was because it "gives the lengthening of the spring because the more marbles the larger numbers and the slope is positive". That is, they conceived that their function provided the "lengthening" per se, not the variable that they had considered.

9.7 Discussion and Conclusions

As was noted previously, more studies in which the interplay of technology is analysed in modelling situations are needed (Grigoraş et al. 2011), especially studies in which data are captured through the use of technology. In this study we have analysed how the way in which students take and process data during the mathematisation of phenomena using the Video Physics® app influenced students' performances. In particular, Video Physics® app gives students the liberty of establishing a reference system during the mathematisation of the phenomena, which makes them have to understand the task in a certain way (Artigue 2002) to be coherent with their decisions during the whole activity and, in particular, during the interpretation of the model.

However, we have found that students interpret the function obtained in relation to their prior conceptions about the values involved in the variables of each function, not basing their interpretation on how they have taken the references. In experiment A, we observed that students had a deep-rooted idea that they should use the ground as reference, which comes from most previous school activities, and the use of the technological tool did not change this. In fact, although most of the students in our study chose references different from the ground when using the Video Physics® app, all ended up considering the ground as a reference during the interpretation phase of the model. In particular, their interpretations were based on the height of the ball not taking negative values and height being exactly zero on the ground, so they were not consistent with interpreting what they did during the mathematisation. In experiment B, students with similar educational backgrounds and age did not perform coherently either. Specifically, during the mathematisation phase no student conceived the variable "lengthening of the spring" as it is defined in Physics: as the magnitude that describes the difference between the length of an elastic object at a given moment and its length when it is not stretched, so it always will take positive or zero values. Nevertheless, all referred to the "lengthening" during the interpretation phase. In addition, they could have considered "lengthening as another variable" and still be coherent, which would be correct from the perspective we took in these tasks in which phenomena have to be organised by mathematical content and processes (Freudenthal 1983). However, they used lengthening in the sense of a positive variable which is zero initially.

Therefore, it seems that in these cases students have not perceived what Gibson (1966) called affordances in general terms in Video Physics® app related to the freedom of taking references and these have become an added difficulty for students rather than a positive aspect. They do not seem to be aware that the way in which they took references influenced the kind of data they obtained because they did not pay enough attention when taking references during mathematisation, nor the way in which the reference taken influenced their data. In addition, this lack of coherence within the students' performances in either circumstances can probably be attributed to being a consequence of the students not being used to being asked previously in their educational experiences to set references nor to interpret mathematical results in real terms.

Hence, technological tools can be useful to require students to pay attention to aspects that normally are set for them and to make decisions that they are not used to doing. Moreover, students need experience in doing this for themselves and they should not be relieved of facing this kind of problematic by teachers doing this for them in task instructions or technological tools being set so they would have no access to aspects like this. However, it is necessary to emphasize the importance of the role of the teacher in the classroom to prompt students to notice this type of situation because, as we have seen in this study, at least the first few times they tackle this kind of task, they might not be aware that they have to pay attention in certain aspects, which will have decisive consequences during the whole task.

Consequently, mathematics lessons should include tasks in which the use of technological tools ensure students reflect on their performances and the decisions they make, especially when they are working on tasks that require relating the real world and mathematics, with which they are not used to dealing. Reflection on actions with tools and the consequences for decision making are critical to developing modelling competency in real-world tasks with technological tools where such tools are used in a meaningful manner.

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Chapter 10
Adopting the Modelling Cycle for Representing Prospective and Practising Teachers' Interpretations of Students' Modelling Activities

Juhaina Awawdeh Shahbari and Michal Tabach

Abstract Teachers play an important role in determining how students work on modelling activities. In the current study, practising and prospective teachers engaged with modelling activities to develop their ability to identify students' modelling process. The study sought to answer the research question as to how the teachers' participation in modelling affects their interpretation of students' modelling activity. Data included two sets of participants' reports on their observations of a video-recorded modelling activity carried out by a group of five sixth-grade students, pre and post participation in four modelling activities. The findings indicate that prior to engaging with modelling activities, most participants described the students' modelling as linear, noting only the final mathematical model and mathematical results. After participating in the activities, most of the practising teachers' reports and a third of the prospective teachers's reports identified cyclical processes in the modelling.

Keywords Modelling · Modelling process · Modelling cycle · Practising mathematics teachers · Prospective mathematics teachers

10.1 Introduction

Teachers play a pivotal role in fostering modelling among their students (Cai et al. 2014). Teachers' intervention may help students adopt strategies that facilitate the construction of situation models while engaging in modelling activities (Leiß et al. 2010). Effective intervention is related to teachers' subject matter knowledge and

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pedagogical content knowledge during modelling (Doerr and English 2006). For example, the way teachers think about modelling potentially influences the suggestions and ideas raised by their students while engaging with modelling activities (Borromeo Ferri and Blum 2010). Indeed, acquaintance with the modelling cycle can guide teachers towards effective intervention while their students engage with modelling activities (Blum and Leiß 2005). In addition, teachers' knowledge about the modelling process can contribute to their awareness of students following different modelling paths (Borromeo Ferri 2010). As well, this can assist teachers in identifying student blockages in transitioning between stages in the modelling process (Stillman et al. 2007).

Despite the importance of the teacher's role while students engage in modelling activities, studies (e.g. Borromeo Ferri and Blum 2010) indicate that teachers' interventions are mostly intuitive and not constructive. In addition, teachers' professional knowledge about modelling is limited (e.g. Frejd 2012; Lamb and Visnovska 2013). The importance of the role of teachers and their limited knowledge about modelling have led various researchers to suggest that teachers should take courses in modelling (for practising teachers: Maaß and Gurlitt 2011; Mischo and Maaß 2013; for prospective teachers: Anhalt and Cortez 2016; Bukova-Güzel 2011). Some researchers (e.g. Altay et al. 2014; Çetinkaya et al. 2016; Kang and Noh 2012; Shahbari 2018; Tan and Ang 2016) have suggested that practising and prospective teachers should engage in modelling activities as learners in order to become qualified in modelling.

Subsequently, the study proposed is an intervention in which practising and prospective teachers engage with modelling activities as learners. Our aim is to examine whether such an intervention has an impact on the abilities of practising and prospective teachers to identify their students' modelling process. Specifically, we seek to enhance participants' abilities to identify the actions between the phases in the modelling process and to recognize the cyclic nature of their students' modelling processes. We focus on this identification because each transition between phases in the modelling process is considered to be a possible source of blockage (Galbraith and Stillman 2006). Hence, such knowledge is critical for developing strategies to overcome student difficulties while transitioning from one step to the next (Çetinkaya et al. 2016).

We examine teachers' identification of students' modelling processes by adopting the modelling cycle proposed by Blum and Leiß (2005) (elaborated later). This modelling cycle enabled us to use a visual means to describe the cognitive analysis of a modeller's modelling process.

10.2 Theoretical Background

10.2.1 Modelling

Modelling is defined as the two-way process of translating between the real world and mathematics (Blum and Borromeo Ferri 2009). Modelling activities involve the partial, ambiguous or undefined information about a situation (English and Fox 2005) that learners working in small groups need to mathematize in ways that are meaningful to them (Doerr and English 2003). While traditional word problems require only one interpretation of a problem and hence, demand limited mathematical thinking (English 2003), engagement in modelling activities involves iterative cycles of translation, description, explanation and prediction of data outcomes and solution paths (Lesh and Doerr 2003).

The literature describes modelling cycles in various ways (e.g. Borromeo-Ferri 2006). In the current study, we use an adaptation of the modelling cycle proposed by Blum and Leiß (2005), who used a visual means to describe the cognitive analysis of modelling activity. Accordingly, cognitive activity is divided into phases and actions. The phases in our adaptation[1] include a situation model, a realistic model, a mathematical model, mathematical results, and realistic results. [The real situation phase is not included in the analysis for this study]. The actions consist of understanding the problem and simplifying a situation model (combining two actions from Blum and Leiß 2005 following our simplification of the stages); mathematizing, which leads to constructing a mathematical model; applying mathematical procedures that yield mathematical results (i.e. working mathematically); interpreting these mathematical results with respect to the real-world situation; and validating these results with reference to the original situation. If the results are unacceptable, the cycle begins again.

10.2.2 Teachers' Knowledge About Modelling

Teachers consider modelling to be difficult (Blum and Borromeo Ferri 2009). Several types of difficulties with modelling activities encountered by teachers have been identified: difficulty in discussing features of different models (Lamb and Visnovska 2013); lack of adequate knowledge about modelling (Çetinkaya et al. 2016; Chan 2013); and limited experience with the notion of mathematical modelling in mathematics education (Frejd 2012). Conversely, teachers' subject matter and pedagogical content knowledge influence how they interact with their students while participating

[1]Editors' note: This simplification reduces the representation of the Blum and Leiß (2005) cycle to that of Maaß (2006, p. 115) if Maaß's "real world problem" is taken to be equivalent to "situation" as used in the diagrams that follow in this chapter and "simplifying" including understanding. Kaiser and Stender's representation of the modelling cycle (2013, p. 279) is an extension of Maaß (2006) where these equivalences have been represented explicitly.

in modelling activities (Doerr and English 2006). Teacher awareness of modelling processes is naturally very important to the support they provide their students in developing modelling competencies (Blomhøj and Kjeldsen 2006). This awareness is also crucial for effectively mediating students' group work during modelling activities (Blum and Leiß 2005). The importance of teachers' understanding of modelling in implementing a modelling approach in their own classroom cannot be underestimated (Kaiser and Maaß 2007).

Teachers' knowledge about modelling activities can be developed through their active engagement in modelling activities (Kang and Noh 2012). Hence, several researchers (e.g. Bukova-Güzel 2011; Mischo and Maaß 2013) have proposed interventions in which practising and prospective teachers engage in modelling activities as learners. Besides having an impact on teachers' development of modelling competencies (e.g. Kaiser 2007), these interventions affect practising and prospective teachers in other ways as well, including teachers' pedagogical knowledge. The teachers develop an understanding of the nature of mathematical modelling, of the relationship between mathematical modelling and meaningful understanding, and of the nature of mathematical modelling tasks (Çetinkaya et al. 2016). Other studies have noted that an intervention made teachers aware of their changing roles in interacting with their students, including a focus on listening and observing, and on asking questions for understanding and clarification (e.g. Doerr and English 2006). Furthermore, Kaiser and Schwarz (2006) reported on affective aspects, such as changes in teachers' beliefs about mathematics.

The current research seeks to examine the effect of engaging in a sequence of modelling activities on the development of pedagogical knowledge among practising and prospective teachers. The study attempts to answer the following research questions:

1. How do practising and prospective mathematics teachers with no prior modelling experience interpret students' modelling activities?
2. Fom a cognitive perspective, does participation in a sequence of modelling activities change the ways in which practising and prospective mathematics teachers interpret students' modelling activities? If yes, to what extent? Is the effect the same both for prospective and for practising teachers?

10.3 Method

10.3.1 Participants and Procedure

Thirty-four practising mathematics teachers and 49 prospective mathematics teachers participated in the current study. The practising teachers taught grades 1–6 at primary schools and were studying towards a master's degree at a college of education. The prospective teachers were studying at a college of education to become primary school mathematics education teachers and were in their second of four years of study. All participants reported no prior experience with mathematical modelling. The participants were enrolled in a 'problem solving' course taught by the first author. Each group (practising and prospective teachers) took the course separately. The courses differed in structure and were similar only in the four lessons during which participants worked on the modelling activities that are the focus of the current study.

The participants were shown a 70-min video recording that documented five sixth-grade students working on a modelling activity known as the *Sneaker activity* from Doerr and English (2003). After watching the video and reading the transcript, participants were asked to write a report describing the work of the five sixth-graders on the *Sneaker activity* from the beginning to the end of their engagement with the task. Participants were instructed to emphasize the cognitive aspects only and to avoid other aspects, such as emotional ones. This first report (R1) was completed and submitted to the lecturer before the participants themselves engaged in a sequence of four modelling activities. After submitting R1, over the course of four lessons the participants worked on four modelling activities. Participants remained in the same group for each activity. Work on each of the four modelling activities took place once a week for approximately 90 min per session. The participants were then asked to watch the same *Sneaker activity* video and write a second report (R2) describing the students' work on the activity. In total, two sets of 46 reports were collected: 20 reports submitted by practising teachers and 26 reports by prospective teachers.

10.3.2 Modelling Activities During the Intervention

As mentioned, in the time between submitting R1 and R2 the participants worked on a sequence of four activities designed by the researchers. Because these activities are not the focus of the current study, we describe them only briefly.

The first activity was the *Summer Camp activity* (Shahbari and Tabach 2017). Participants were asked to choose the summer camp they deemed most suitable, to suggest a means of choosing suitable camps for the coming years, and to write a letter explaining their decision. The summer camp activity was presented via four tables providing information about six camps, with each table referring to several components, such as dates, transportation, food, and cost of each camp; types and number of entertainment activities at each camp; and parents' evaluations and ranking of the camps for the previous year.

The second activity was the *Flower activity*, in which an art teacher planned to recreate a picture of a flower (the image of the flower was given) through the participation of all 524 students at the school. In the re-creation, the students were required to wear yellow, green, or brown clothing in accordance with their location in the original picture. The prospective and practising teachers were requested to write a letter to the art teacher explaining how to enlarge the picture so that all the students could participate, how to place the students in the schoolyard, and how many students should be wearing clothing in each color (yellow, green, and brown).

The third activity was the *Toothpaste activity* (Shahbari and Tabach 2016). The participants were told that the opening of their toothpaste tube had been enlarged, and they were asked to write a letter describing how their toothpaste consumption from this tube might have changed compared to the original tube.

The fourth activity, the *Good Teacher activity* (Shahbari and Tabach 2017), also comprised four tables describing ten teacher candidates by providing different information, such as age, performance in practicum work, evaluation during an interview, and more. The participants were asked to choose the most suitable candidates for a teaching position, suggest a means of choosing suitable candidates for the coming years, and write a letter explaining their decision.

10.3.3 Sneaker Activity

The students who engaged in the *Sneaker activity* (Doerr and English 2003) were in the sixth grade. Their mathematical achievements, as reported by their mathematics teacher, differed. They were selected on a voluntary basis. The students were required to work as a group and answer the questions posed in the activity.

The *Sneaker activity* began with the opening question: 'What factors are important to you for when you buy a pair of sneakers?' After students discussed the factors they deemed important, they were given a list of ten factors (brand, size, comfort, fashion, purpose, grip, colour, quality, style, and cost) that are considered important in buying a pair of sneakers. The students were then asked to determine how to order these factors in deciding which pair of sneakers to purchase. Solving this assignment involved generating an ordered list of factors. For the current study, the students were given two additional lists of the same factors that were ranked differently than their list. They were told that these lists belonged to two other groups. In the original activity, each group in the class has its own list, so the number of lists is equal to the number of groups. In the current study we prepared two other lists such that all three lists were different. The final assignment was for the students to build a single list representing the priorities of all three lists.

10.3.4 Analyses of Students' Modelling Activity

The solution path of the group of five sixth-graders in the *Sneaker activity* was analysed using our adaptation of the modelling cycle of Blum and Leiß (2005) and our analyses described in a visual diagram (see Fig. 10.1). The numbers in Fig. 10.1 indicate the modelling actions: (1) *understanding and simplifying*; (2) *mathematising*; (3) *applying (i.e. working mathematically)*; (4) *interpreting the mathematical results*; (5) *validating*. The letters indicate the modelling phases: (A) *real model*; (B) *mathematical model*; (C) *mathematical results*; (D) *realistic results*. There were three mathematical models (B1–B3), three different sets of mathematical results (C1–C3),

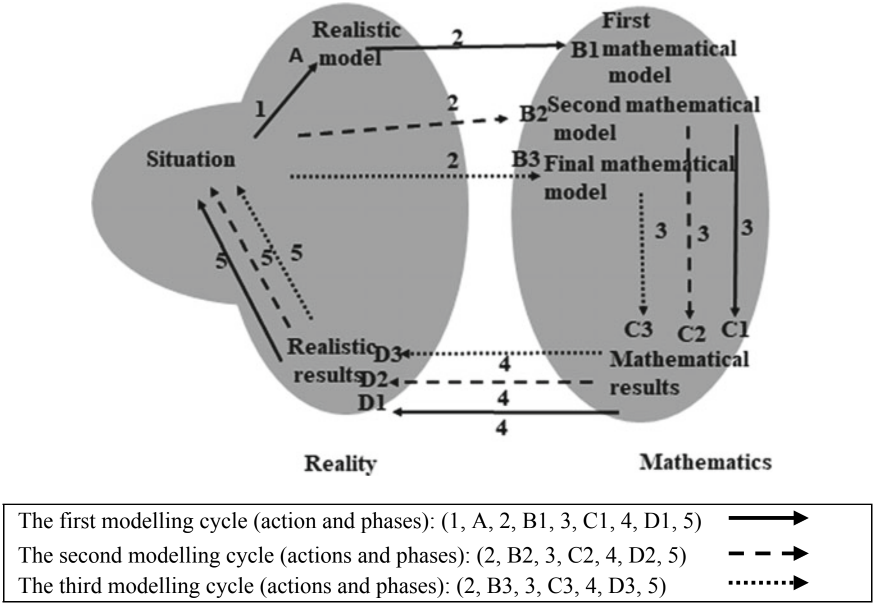


Fig. 10.1 Researchers' analyses of the solution path of the sixth-graders' activity using an adaptation of the modelling cycle of Blum and Leiß (2005)

and three sets of realistic results (D1–D3). The combination of letters and numbers (e.g. B1) in Fig. 10.1 indicates the number of the modelling cycle during each phase.

Analysis of the *Sneaker activity* indicates that the sixth-graders' solution included three modelling cycles, as indicated by the three types of arrows (see Fig. 10.1). The first modelling cycle included (1) interpreting the situation; (2) mathematising—quantifying the factors and trying to connect between the three lists, eliciting the first mathematical model (B1), dividing the factors into two groups of five factors each that contained the most important and the least important factors, and then ranking each group by using frequencies; (3) applying the frequency model to the situation to yield mathematical results (C1); (4) interpreting the mathematical results to yield realistic results (D1); and (5) validating the realistic results obtained by applying the frequency model. This is represented in Fig. 10.1 using solid arrows (→). The validating process revealed the following problem: Some factors were among the first five factors on some lists and among the last five factors on other lists. This problem led the students to begin a second modelling cycle.

The steps in the second modelling cycle were: (1) returning to the main situation; (2) mathematising—dealing with the number of factors and eliciting the second mathematical model (B2) using the average; (3) applying the average model to elicit mathematical results (C2); (4) interpreting the mathematical results in the situation, yielding the realistic results (D2) of applying the average model; and (5) validating the realistic results of applying the average model. This is represented in Fig. 10.1

Table 10.1 Sample analysis of participant's report

Participant's report	Researchers' analyses
"First the students read the problem and try to discover what is required of them… the students understand what the problem is about"	The participant pointed to the students' simplification and understanding of the action. (Action 1)
"the students understand the need for searching and organizing one list that represent the three lists … we must integrate the three lists"	The participant's description identified the real model of the situation. (Phase A)
"they applied several mathematical strategies, they give each factor a value"	The participant reported on the mathematisation process in the third modelling cycle. (Action 2)
"after a long discussion they agreed to use the average as a factor in the three lists… the average of the value that each factor gets in the three lists … and in the case of equal averages they decided to use estimation, … average and estimation was the strategy they used"	The participant reported on the final mathematical model (B3)—the use of average and estimation
"the students get answers by using the strategy, they get a value for each factor…, the size is ten, the colour is eight… the average of size is 9.66"	The participant reported on applying the mathematical model and the mathematical results (C3): "the biggest average is 9.66"
"they ranked the factors … the first and most important is size"	The participant reported on the interpretation of the mathematical results and the realistic results (D3): "the most important is size"

using dashed arrows (‒ ➔). This cycle revealed that some factors have the same average, leading the students to return to the situation for a third modelling cycle.

The third modelling cycle included the following steps: (2) mathematizing—dealing with the number of factors and eliciting the final mathematical model (B3) through the use of average and estimation; (3) applying the final model to elicit mathematical results (C3); (4) interpreting the mathematical results in the situation to elicit realistic results (D3); and (5) validating the realistic results obtained from applying the final model. This is represented in Fig. 10.1 using dotted arrows (····►).

10.3.5 Data Analysis of the First and Second Reports

The reports of the practising and prospective teachers were analysed by the authors. All reports were examined to see whether the participants referred to each of the modelling phases (A–D) and modelling actions (1–5) and were then compared to the researchers' analyses, as shown in Fig. 10.1. Table 10.1 provides examples from the participants' reports (left column, translated into English by the researchers) and the researchers' analyses (right column).

After coding each report, a visual representation was created for each report to facilitate identification of changes between the first report set (R1) and the second report set (R2) and to characterize the changes in the group. Finally, we assigned each report to one of three levels. Reports that included only one modelling cycle were assigned to the first (lowest) level; reports that included two modelling cycles were assigned to the second (middle) level; and reports that included three modelling cycles assigned to the third level. Each level was divided into different sub-levels according to the identified phases and actions.

10.4 Findings

First, we visually present the participants' descriptions of the students' modelling process in the two sets of reports. We arranged our findings in the order of their complexity, from less to more complex. We then discuss the modelling phases and actions in all of the reports.

10.4.1 Participants' Descriptions of Students' Modelling Process in R1 and R2

Our analysis of the practising and prospective teachers' reports yielded nine different categories of analysis of the solution path (Fig. 10.2a–i) describing the students' solution paths. The visual representation of one was presented in Fig. 10.1 and the other 8 are presented in Fig. 10.2a–h.

At the first and lowest level of complexity, the students referred to only one modelling cycle. For this level, we identified three sub-levels (a–c) in the participants' reports. Sub-level (a) represents the simplest description in which the participants referred to presentation of the situation and the final mathematical model (B3) while ignoring all the other modelling phases. At the next sub-level (b), the participants' description in the report considers the situation interpretation, describes the real model (A) of combining the three lists into one list, considers the mathematical work of quantifying the factors, and discusses the relations between them. In the report, the participants also consider the final mathematical model (B3)—the use of average and estimation—and describe the mathematical results (C3) elicited by applying the final mathematical model. In the third sub-level (c), the participants' reports considered all the phases and actions of the third modelling cycle from sub-level (b), and in addition considered the realistic results (D3) obtained from interpreting the mathematical results to reality.

Table 10.2 provides visual descriptions of the three sub-levels of the first level, in which the students referred to one modelling cycle only. In addition, it details the numbers of teachers producing a report of the students' modelling process that

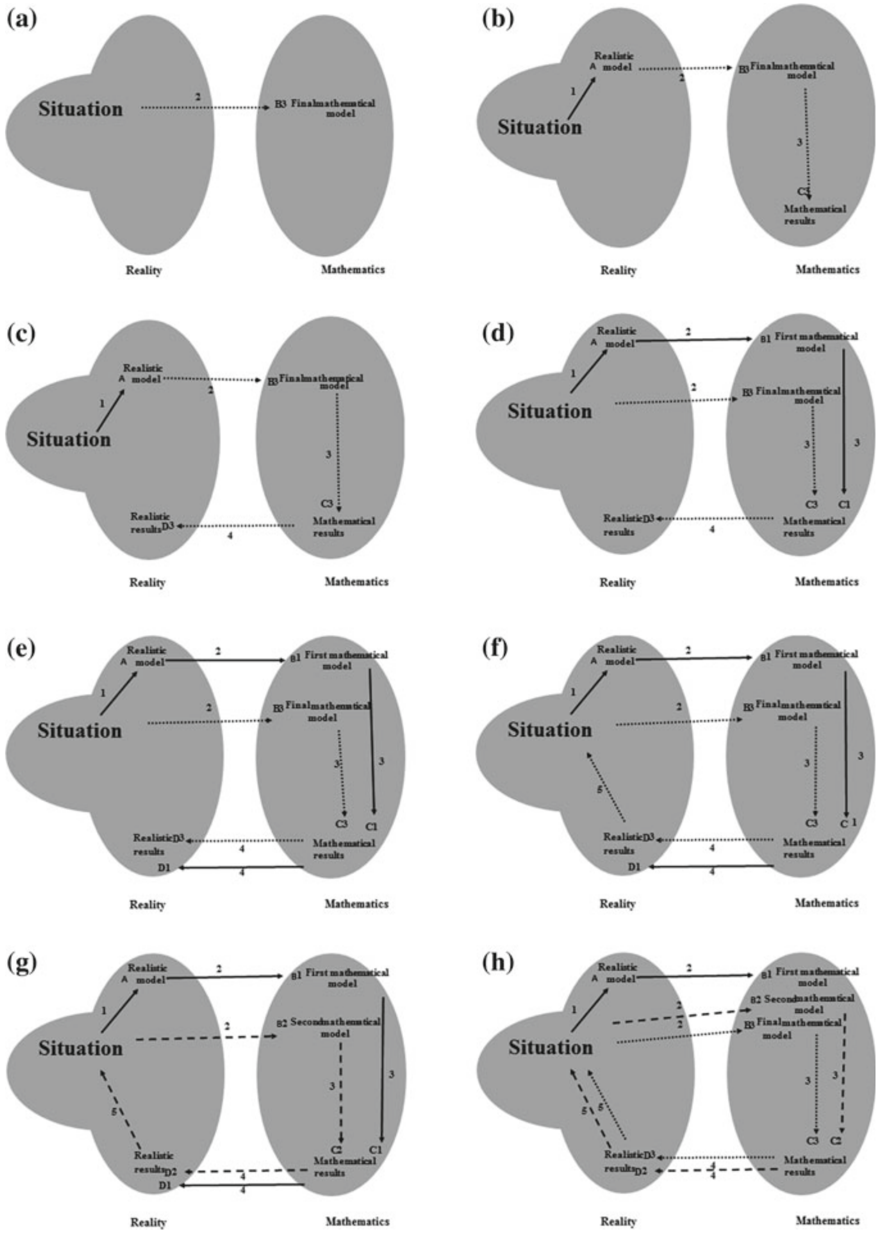
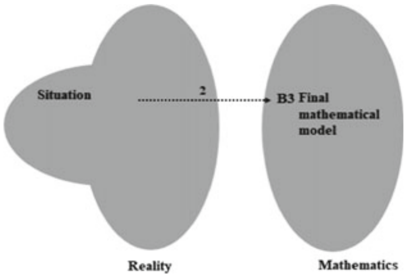
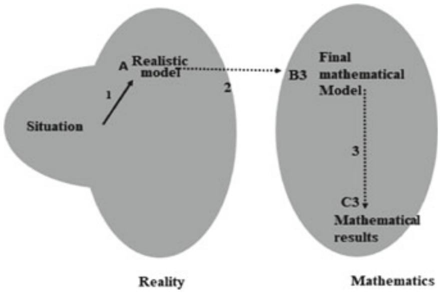
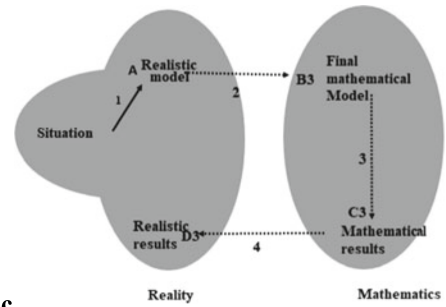


Fig. 10.2 Eight visual representations of the teacher analyses

Table 10.2 Distribution of the analyses in the reports of practising teachers (N = 20) and prospective teachers (N = 26) for the first level: sub-levels (a–c)

<table>
<tr><th>Groups</th><th>R1</th><th>R2</th><th>Visual representation</th></tr>
<tr><td>Practising teachers</td><td>8</td><td>1</td><td rowspan="2">

a</td></tr>
<tr><td>Prospective teachers</td><td>10</td><td>3</td></tr>
<tr><td>Practising teachers</td><td>7</td><td>–</td><td rowspan="2">

b</td></tr>
<tr><td>Prospective teachers</td><td>12</td><td>7</td></tr>
<tr><td>Practising teachers</td><td>2</td><td>–</td><td rowspan="2">

c</td></tr>
<tr><td>Prospective teachers</td><td>3</td><td>8</td></tr>
</table>

was categorised in this way for both R1 and R2. For example, the reports of eight practising teachers and 10 prospective teachers were classified at the lowest sub-level (a) for their initial report. After the intervention this number reduced to one and three repectively.

At the second level of complexity, the participants' reports included descriptions of two modelling cycles. Three of these descriptions considered the first and the third modelling cycles without describing the second modelling cycle, while one description considered the first and the second modelling cycles without reporting the third modelling cycle. This second level was separated into four different sub-levels (d–g). Sub-level (d) represents participants' reports that considered the first modelling cycle, which included working mathematically and deriving the first mathematical model by dividing the factors into two groups of five factors each—the most important and the least important; and considered the third modelling cycle [as in sub-level (c)], including the mathematical work, the final mathematical model (B3), the mathematical results (C3) obtained from applying the final model and the realistic results (D3). At sub-level (e) the reports indicate that the participants recognise that the task solvers worked through the first and third modelling cycles, but do not recognise any validating processes. At sub-level (f), the reports considered the first and third modelling cycles but did not consider the validating process of the first modelling cycle. At the sub-level (g), the reports point to recognition of the first and second modelling cycles but do not consider the validating process for the first modelling cycle. The visual descriptions of the four sub-levels of the second level are given in Table 10.3, and the data for the participants whose reports were classified this way.

At the third and highest level of complexity, the participants' reports included descriptions of the three modelling cycles. This level included two sub-levels: (h) and (i). At level (h), the reports show that the participants identified the second and third modelling cycles, considered all the phases and actions, but identified the mathematical work only in the first modelling cycle. Finally, sub-level (i) is the expert model, similar to the researchers' description in Fig. 10.1. At this sub-level, the reports included description of all the phases and actions in the three modelling cycles. In other words, the participants' reports considered the whole modelling process as elaborated in our explanation of the students' modelling activity in the method section. Table 10.4 shows the visual descriptions of the sub-levels of the third level, and the data for the participants whose reports were classified this way.

The data in each of Tables 10.2, 10.3 and 10.4 show the changes in the descriptions of practising and prospective teachers from the first to the second reports. Table 10.5 summarizes the distribution of the three levels of description between the first and the second reports.

The results in Table 10.5 indicate that by the second reports, the two groups were able to observe more modelling cycles. In addition, in the second reports a higher proportion of practising teachers observed more modelling cycles than the prospective teachers.

The analyses of the participants' first reports (R1) according to the specific phases and actions indicate that their attention was focused mainly on the third modelling cycle, as all the first reports (R1) included descriptions of the final mathematical model. In addition, three-fifths of the practising teachers' first reports (R1) considered the mathematical results of applying the final model (i.e. visual representations b, c, e, and h) and three-fifths of the reports considered the realistic model (i.e., visual representations b, c, e, and h). More than three-fifths of the prospective teachers' first reports considered the mathematical results of applying the final model (i.e., visual representations b, c and e). Little attention, however, was devoted to the modelling phases or to actions related to the first two modelling cycles. In contrast, the descriptions in the second reports (R2) pay attention to the three modelling cycles relative

Table 10.3 Distribution of the analyses in the reports of practising teachers (N = 20) and prospective teachers (N = 26) for the second level: sub-levels (d–g)

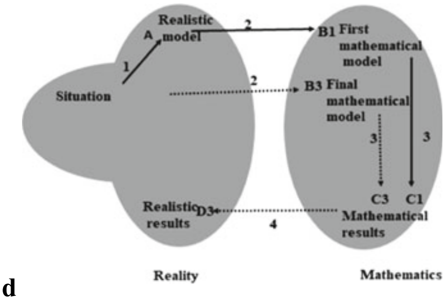
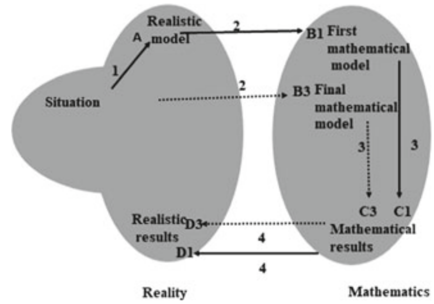
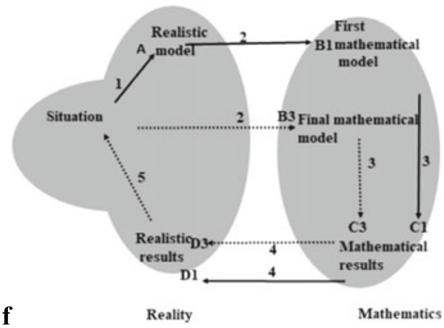
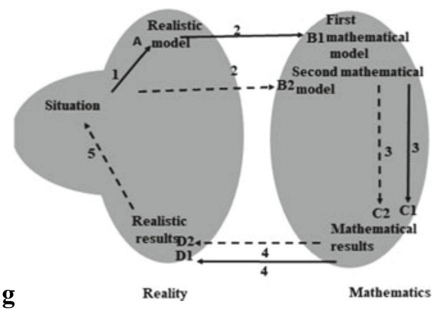
Groups	R1	R2	Visual representation
Practising teachers	–	4	 **d**
Prospective teachers	–	–	
Practising teachers	2	–	 **e**
Prospective teachers	1	–	
Practising teachers	–	4	 **f**
Prospective teachers	–	–	
Practising teachers	–	1	 **g**
Prospective teachers	–	–	

Table 10.4 Distribution of the analyses in the reports of practising teachers (N = 20) and prospective teachers (N = 26) for the third level: sub-levels (h) and (i)

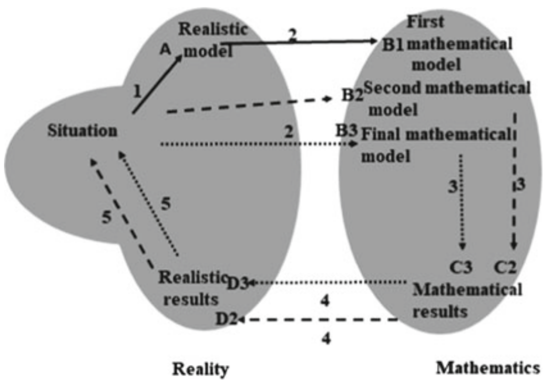
Groups	R1	R2	Visual representation
Practising teachers	1	3	**h** 
Prospective teachers	–	3	
Practising teachers	–	7	**i** Expert model: similar to researchers' analysis (see Fig. 10.1)
Prospective teachers	–	5	

Table 10.5 Summary of classification by level

Level	Practising teachers (N = 20)		Prospective teachers (N = 26)	
	R1	R2	R1	R2
First level	17 (85%)	1 (5%)	25 (96%)	18 (69%)
Second level	2 (10%	9 (45%)	1 (4%)	–
Third level	1 (5%)	10 (50%)	–	8 (31%)

to the modelling phases and actions in them. More attention was devoted to the first and third mathematical modelling cycles than to the second modelling cycle. The results also show that compared to other actions, the least amount of attention was directed toward validating processes in the three modelling cycles. The findings obtained from the analyses of the two reports among the two groups indicate that the practising teachers identified more phases and actions in both the first and the second sets of reports.

10.5 Discussion and Conclusion

The current study examined how participation in four modelling activities affected how practising and prospective teachers interpret students' modelling activities. The main findings indicate that the modelling cycle adapted from Blum and Leiß (2005) for this study made it possible to closely monitor the modelling phases and actions emerging from practising and prospective teachers' interpretations of students modelling activity. The use of this adapted modelling cycle to visualize the interpretations of practising and prospective teachers enabled us to monitor which specific phases and actions were noted or ignored and to identify the changes in their interpretation.

The findings indicate that before participating in the modelling activities, the practising and prospective teachers were unable to identify and document the students' entire modelling process. Most of the descriptions in the first reports considered the final mathematical model and the mathematical results of applying this model, while overlooking the realistic results and the validating process. In addition, most of the first reports disregarded the second and third mathematical modelling cycles and the modelling phases and actions related to these cycles. The practising and prospective teachers emphasized the final model without considering the first and second modelling cycles, indicating that they considered the solution path to be linear (typical of school problems). In other words, both the practising and the prospective teachers expected to see a specific computational solution rather than a more general strategy, as discussed by Doerr and English (2006). Furthermore, the practising and prospective teachers may have emphasized the final mathematical model because they were expecting it to be the result of the students' work. This observation is in line with the findings of Blum and Borromeo Ferri (2009), who reported that teachers impose their preferred solution through their intervention while students engage in modelling activities.

The active participation of practising and prospective teachers in modelling activities contributed to their awareness of the modelling phases and actions and of the processes by which mathematical models progress. Findings from our analysis of the second reports indicate that after their participation, more practising and prospective teachers considered the three modelling cycles and the cyclic process of the mathematical models' progress. These results are in line with those of Tan and Ang (2013), who reported that experience with modelling activities enhanced teachers' knowledge of different elements in modelling process phases. Our findings show that the validating process was the least recognised action described by the practising and prospective teachers in both sets of reports. It is important to note that this process is considered to be the most difficult for students when undertaking modelling activities, because in a 'regular' classroom activity it is the teacher who is responsible for the correctness of the solutions (Blum and Borromeo Ferri 2009).

Finally, the findings indicate that practising and prospective teachers who participate in modelling activities can develop modelling lenses for themselves that find expression in the way they consider their students' modelling. These modelling lenses can help teachers observe and monitor the modelling process more precisely. In addition, the findings indicate that the effect of engaging in a sequence of modelling activities was greater among the practising teachers. The difference between the two groups may be attributed to the fact that practising teachers have more comprehensive knowledge about students' problem-solving processes. They are likely to have gained some of this knowledge from their general experiences teaching mathematics. Another explanation for the differences can be attributed to the practising teachers being participants in higher degree studies, namely, a master's degree.

In light of the study findings, we recommend the use of the modelling cycle of Blum and Leiß (2005), or our adaptation of this,[2] to describe from a cognitive perspective how practising and prospective teachers interpret students' modelling activity. The use of the modelling cycle enabled us to monitor the modelling phases and actions that emerged when the practising and prospective teachers interpreted the students' modelling activity and to observe their professional development. In addition, we recommend integrating courses about mathematical modelling into professional teachers' programs in which teachers actively engage in modelling activities as participants. Based on the results of this study for prospective teachers, we recommend that dealing with interpretation of students' modelling activity be postponed until they have gained experience as practising teachers.

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[2]Editors' note: Please note other cycles such as Kaiser and Stender (2013, p. 279) capture this modification and so would also be appropriate.

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Chapter 11
Heuristic Strategies as a Toolbox in Complex Modelling Problems

Peter Stender

Abstract The support of students who are working on realistic modelling issues is a complex process, especially if it is intended, that the students work as autonomously as possible. In the underlying research project, actions of tutors were analysed as they were fostering students who were working on complex, realistic, authentic modelling problems over three days. The tutors were prepared previously in seminars. The whole process was videotaped and analysed afterwards, looking for examples of successful teacher interventions especially interventions based on the idea of strategic assistance (Zech in Grundkurs Mathematikdidaktik. Beltz, Weinheim, 1996). The findings in the research led to the insight that heuristic strategies developed within problem solving could be identified in the modelling process and are an appropriate concept to formulate strategic interventions. This is shown here by examples based on the analysis of observed student solutions and a standard solution.

Keywords Heuristic strategies · Modelling problem · Modelling activities · Teacher interventions

11.1 Theoretical Framework

In this chapter, modelling is understood as follows: A problem from outside of mathematics, in the "Rest of the World" (Pollak 1979), occurs, is simplified and then translated into a mathematical problem that is worked on. The solution found is translated back into the real world and it is validated whether the result answers the primary problem adequately. If this is not the case, the modelling cycle is run through again until a satisfactory solution is produced. The modelling cycle shown in Fig. 11.1 allows use with students in the classroom as well as being complex enough to illustrate the *two* steps from real world situation into the mathematics that occur, if the modelling problem itself is complex. The modelling cycle itself is a model of

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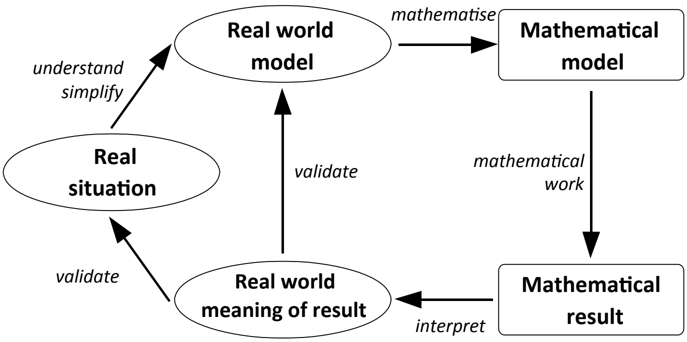


Fig. 11.1 Modelling cycle (Kaiser and Stender 2013, p. 279)

the modelling process so in reality modellers do not follow this cycle strictly but go back and forth as necessary as shown in Borromeo Ferri (2011).

11.1.1 Teacher Activities to Promote Independent Student Action

In the German discussion (e.g. Zech 1996; Leiss 2007), the concept of an *adaptive intervention* is used for the support of students in modelling and problem-solving situations with the aim of independent student activity: "Adaptive teacher interventions are defined as that assistance of the teacher to the student, which supports the individual learning and problem-solving process of students minimally, so that students can work at a maximum independent level" Leiss (2007, p. 65, own translation). As guidelines for the teacher supporting students who need support in their work, Zech (1996) proposed a step by step approach on five different levels. In this approach, the teacher should in the first two levels only motivate the students (first motivate: "You will make it"; second feedback "Go on like this!"). Only if this is not sufficient to enable the students to go on in the work on the task, should the next steps be done. In these steps, *strategic help* is given first, and then increasingly more assistance is given that relies more and more on the content of the task (e.g. calculations needed are explicitly shown). Strategic help provides students with support that relies on mathematical or other methods and strategies that regulate the work, not on the steps to fulfil these strategies. "Formulate an equation and then solve it!" is an example of strategic help, as long as the teacher does not explicitly show which kind of equation is appropriate or which are the single steps to build the equation or to solve it. While working on modelling problems, a reference to the modelling cycle can be used as a strategic help: "Simplify the situation!" "Try to bring this into a formula!" "What does the mathematical result mean in the real world?" "Does the result answer the real-world situation meaningfully?" Strategic help that is provided only when necessary supports the independent work of students in the best way, as the students are only supported to find a way to go on, but the solution itself must still be developed by the students themselves.

If the strategic help does not enable the students to go on in the solution of the problem*, content related strategic help* should be given. This means that additional content related information around the strategic help is provided. This could be a hint about what kind of equation would be appropriate or what kind of formulas could be used. Only if this content related strategic help is not successful is more content related help provided, for example direct support formulating an equation or in the mathematical process. Between the phases of intervention, the students need time to think about what they could do and to try different approaches, so they have the chance to solve the problem or the next step as independently as possible.

11.1.2 Heuristic Strategies

During the research described below hints occurred, that heuristic strategies which are a well-known concept in the problem-solving theory (e.g. Dörner 1976; Pólya 1973; Schoenfeld 1985), are also used when solving modelling problems and are a strong concept to formulate strategic help, which was already mentioned by Zech (1996).

From a theoretical point of view, a *heuristic strategy* is a possible approach to solve a problem (Dörner 1976; Schoenfeld 1985). To clarify this definition, one has to define what is meant by the word *problem*. Following Dörner (1976, p. 10, own translation), "A problem is a situation where you achieve a goal, but you don't know how to achieve the goal." Dörner points out that there is a barrier between the actual situation α and the goal ω. In this approach, a heuristic strategy is an approach to overcome the barrier between situation α and ω. To discriminate the concept from other situations, Dörner uses the concept of a *task* as a situation where no barrier exists. If you work on a task you know what to do, even if it may be difficult: if someone knows the Gaussian elimination, for example, it is a task to solve a 10 by 10 system even if it is a lot of work: you always know what to do as all steps are given by the Gaussian elimination procedure. Whether a situation is a problem or a task depends on the knowledge and the experience of the person, but if something is a problem there is no explicit answer for this person of how to solve it. That means heuristic strategies are always ideas you can try but maybe they are not successful, and you have to try another one. If you have no more ideas what to do, you could try a strategy out of a list of heuristic strategies. Therefore, the following list of heuristic strategies was gathered based on the problem-solving theory. The classification of these strategies was formulated to keep a better overview of the single heuristic strategies as there are too many to keep them all in mind (Stender 2018).

- *organise your material/understand the problem*: change the representation of the situation if useful, trial and error, use simulations with or without computers, discretize situations,
- *use the working memory effectively*: combine complex items to supersigns, which represent the concept of ‘chunks’, use symmetry, break down your problem into sub-problems,
- *think big*: do not think inside dispensable borders, generalise the situation,
- *use what you know*: use analogies from other problems, trace back new problems to familiar ones, combine particular cases to solve the general case, use algorithms where possible,
- *functional aspects*: analyse special cases or extreme cases, in order to optimise you have to vary the input quantity,
- *organise the work*: work backwards and forwards, keep your approach—change your approach—both at the right moment.

The use of these heuristic strategies is shown below within a complex realistic modelling problem. Using this example, the single strategies used in the specific modelling problem are explained in more detailed.

In addition, examples are displayed for using heuristic strategies to formulate strategic interventions. The connection is obvious: if you want to provide strategic interventions, you need to know the appropriate strategy in the specific situation and these are often heuristic strategies.

11.2 The Study

The aim of the research project is to find appropriate strategic interventions in situations where teachers are tutoring students who are solving complex, realistic, authentic modelling problems. As the research environment, “modelling days” were established. The empirical research led to two assumptions: heuristic strategies are used while working on complex modelling problems and they can be used to create strategic help for students. To proof the first assumption, descriptions of the modelling process were analysed due to the underlying use of heuristic strategies. In order to see whether it is possible to create strategic help with these strategies corresponding teacher interventions were formulated. Whether these are effective had not yet been an object of the empirical research but is an assumption based on Zech (1996). A summary of the whole research project is shown here and a detailed analysis of one modelling process according to the use of heuristic strategies is presented.

11.2.1 Modelling Days

"Modelling days" is a learning environment (see also Kaiser et al. 2013 for other examples), where students of grade 9 (15 years old) work for three full days in a school on only one single modelling problem. The modelling days were held several times, and are still held, at a school for higher achieving students in Hamburg (Germany). For the students from grade 9, three modelling problems were presented from which each student chose one (see Stender 2018, for use of heuristic strategies in The Bus Stop Problem). Then groups of four to six students were formed, so that in each group students worked together on the same problem, supervised by tutors.

The tutors were future teacher students studying for their master's degree. They were prepared in a university seminar on modelling. In the seminar, they worked on the three modelling problems that were the choice for the modelling days, they learnt about the theory of mathematical modelling and the theory of teacher interventions and scaffolding. Heuristic strategies were also content of the seminar.

11.2.2 Modelling: Roundabout Versus Traffic Light

In the research project presented here, students worked on the problem: "At which kind of intersection (a roundabout or a intersection with traffic lights) can more cars pass a crossing?"

In Germany, as in many other countries, many intersections have been reconstructed as roundabouts for several reasons. In other countries, a similar question to compare a traffic light and a four-way stop might be more appropriate. One aspect of the discussion is whether a roundabout can manage more traffic than an intersection with a traffic light. A sketched idea of a solution is presented here for a better understanding of the problem and the appearing complexity. Ideas that are more detailed are shown below connected to heuristic strategies.

In a first approach, it makes sense to assume the maximum possible symmetry in the situation: from all directions the same number of cars should come, and the drivers want to go in all directions with equal probability, with velocities and accelerations also being the same for all cars. The crossings are a simple four-road intersection, where the traffic light is green only for one direction at a time. The restrictive assumptions can be reduced during the modelling process, to arrive at a more sophisticated solution, but within the time available this did not occur in the modelling days. For the students, it took mostly a longer work process to arrive at these assumptions.

Once these real models (one for the roundabout, one for the traffic light) were formulated, the students had to calculate the time a whole line of cars needs to start, when the traffic light switches to green. This way one finds the number of cars that can enter the intersection during one green phase. In this calculation, one has to deal with constant and accelerated movements and the time a car has to wait until the

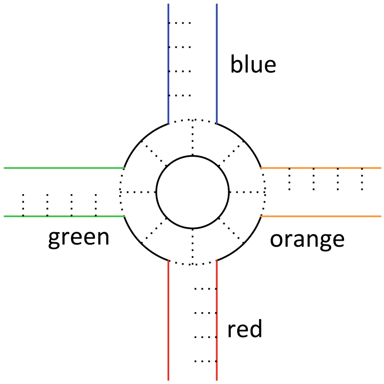


Fig. 11.2 Material for simulation of the roundabout

necessary distance to the car before occurs. One also needs to consider that the cars at the end of the line drive with constant speed according to the speed limit after a phase of acceleration.

The processes in the roundabout are more complex than those occurring at the traffic light, since the probability to enter the roundabout depends on the situation in the roundabout itself. If the roundabout is completely full of cars, a new car can only enter the roundabout if a car from inside the roundabout left the roundabout previously.

This process can be simulated with the help of a game, as shown in Fig. 11.2. A simulation with a computer is also possible, but that was usually beyond the capabilities of the students involved. The access roads shown in the figure are drawn in different colours (blue, green, orange, red). Now pieces of paper in the same colours are distributed representing cars on the streets, in a way that from each direction, say 21 "cars", approach in random order. The drivers of blue cars have the goal to drive in the direction of the blue street. In the line in the blue street there are obviously normally no blue cars. One turn of the simulation consists of the following steps: At first all cars in the roundabout that are at the right exit leave the roundabout. In the second step, all other cars in the roundabout drive one step ahead, which leads to free places at the entrance to the streets, where a car left the roundabout previously. Now cars from the waiting lines drive into the roundabout.

This simulation leads to a deeper understanding of the roundabout-process and to the probability of 50% for a car to enter the roundabout in one turn of the simulation. Afterwards it has to be calculated how long a single turn in the simulation lasts in reality. This leads to calculations similar to traffic light ones.

The results for the capacity of the two designs of the intersection depend on the values for velocities and accelerations used. A clear answer to the initial question cannot be given without clearing the parameters depending on the size of the intersection. It is therefore useful for the students to visit an intersection and do some measurements while working on the problem. These measurements could also be considered for the evaluation of the results of calculations. The results will always be based on the corresponding dimensions of intersections, but can be generalized by further calculations.

11.2.3 Empirical Survey

During the modelling days ten groups of students were videotaped in five rooms, with one camera for each group. The video-recordings include six hours of modelling activities for two days and a few hours on the third day. The phases during which the tutor communicated with the students and some minutes before and after every such communication was transcribed. In total, 238 contacts between teacher and individual groups were examined. The transcribed text passages were analysed and coded using qualitative content analysis (Mayring 2010). Three types of variables were used relying on the time before the intervention, during the intervention itself, and on the time after the intervention. Thus, the success of the interventions could be determined based on the coding. Findings were presented previously in Stender and Kaiser (2015) and Stender (2016).

While analysing single interventions that were not successful or delivered too much content related help, I tried to formulate alternative strategic interventions for further projects. In doing so, formulations using heuristic strategies seemed appropriate. This led to the next step in the research project to find out whether it is possible to provide evidence of the use of heuristic strategies in the modelling process and whether strategic help can be created based on the heuristic strategies that were used.

11.3 Results

11.3.1 Using Heuristic Strategies in Modelling Problems

In this section the process of modelling the problem “Roundabout versus Traffic Light” is analysed. Two materials are used: The first solution is that of the author, formulated while developing the modelling problem to make sure that there was a chance for the students to find a meaningful solution and to prepare the tutors for the modelling days. The second solution is a reconstruction of students’ solution. This reconstruction is based on the presentation of the students at the end of the modelling days and on videotapes of several groups from the modelling days. For a single group only parts of the modelling process and the approaches of the students are visible on the videos as there were always phases in which they worked without visible communication or documentation. Therefore, the visible parts of the modelling process of different groups were connected to one students’ solution.

Both solutions were examined step by step, analysing whether the heuristic strategies mentioned above occurred and how they were realized. From these heuristic strategies, strategic interventions were formulated. The results are shown under the headlines of single heuristic strategies and not in the strict timeline of the modelling process, but the first strategies described occurred earlier in this modelling process

Break down your problem into sub-problems: Pólya (1961, p. 129), citing Descartes, states: “Divide each problem that you examine into as many parts as you can and as you need to solve them more easily”. At the same place, Pólya also cites Leibnitz underlining the core problem connected with this strategy: “This rule of Descartes is of little use as long as the art of dividing … remains unexplained. … By dividing his problem into unsuitable parts, the unexperienced problem-solver may increase his difficulty” (p. 129). In the modelling process this strategy occurs as a basic approach to the process of looping through the modelling cycle several times (Pollak 1979, p. 20). In each loop, one single part of the modelling problem is (partly) solved and is the groundwork for the following steps. Besides this example, there are many situations in problem solving and modelling that have to be divided into sub-problems in order to access a solution.

During the modelling days students were faced with the challenge to convert one unit of velocity (km/h) into another (m/s). Some of the students knew the conversion number to be 3.6, but did not know how this was to be applied. Longer work phases on this problem with constantly decreasing motivation were observed without the students getting closer to the answer, as they always searched for a single step operation to perform the calculation. In the end, the tutor showed the students how the conversion is calculated step by step (which is not a strategic help!), after several interventions that gave only general information. The statement: “There are two units involved, so convert only one of them in the first run!” could have helped the students to develop the calculation on their own. This can also be formulated more concretely: “Convert only the km in the first step”, or more generally “It won’t work with one step, you have to make at least two steps!”

This observation led to the insight, for the author of this chapter, of the high relevance of heuristic strategies in the modelling process: if a student deals with a problem in the modelling process and there is a barrier (a “red flag situation” according to Goos 1998) that the student cannot overcome on his/her own, the teacher or the tutor has to analyse the next steps he or she would do himself/herself and then identify the underlying strategy of the own solving process. This strategy identified by metacognition, leads to the strategic intervention that could help the student. This process can then be a general method to formulate strategic help.

Build the real model as symmetric as possible: The most often used strategy in this modelling problem is the use of symmetry, as already mentioned above. “Try to treat symmetrically what is symmetrical, and do not destroy wantonly any natural symmetry” (Pólya 1973, p. 200). Pólya emphasized that symmetry is not only meant in the usual geometric meaning but also in a general, logical meaning: “Symmetry, in a general sense, is important for our subject. If a problem is symmetric in some ways we may derive some profit from noticing its interchangeable parts and it often pays to treat those parts which play the same role in the same fashion” (p. 199). The concept of symmetry is very broad. “In a more general acceptance of the word, a whole is termed symmetric if it has interchangeable parts. There are many kinds of symmetry; they differ in the number of interchangeable parts, and in the operations which exchange the parts" (Pólya 1973, p. 199).

In the analysed modelling problem, the symmetry is not necessarily there from the beginning but has to be created by the modeller: a four-street crossing is examined, and most students picked an example nearby the school in the first run of the modelling process. Mostly a bigger road meets a smaller one in their examples. According to the modelling cycle, the first approach to a real model should simplify the situation as much as possible. This simplification implies to model the situation as symmetric as possible: all four streets meeting at the crossing should "play the same role" and thus should be "treated in the same fashion". That means, in detail, that from all four streets the same number of cars arrive per hour and from each street one third of the cars is going to turn right, one third turns left and one third goes straight ahead.

A further aspect that should be treated symmetrically are the cars. Usually cars have different sizes, different accelerations and drive with different speeds and the drivers keep different distances to the car in front. All these quantities should be identical in the real model and the mathematical model. For example, the cars are all 5 m long, accelerate with 2 m/s^2 after starting at the green light and waiting in the line all drivers keep a distance of 1 m to the car in front.

The strategic help for the students can be formulated as: "Form the situation as symmetrically as possible!" or more concrete if necessary: "Treat all the streets and the cars in the situation in the same way in the first approach!"

In the modelling cycle using symmetry is one possible way to simplify the situation. A less specific strategic intervention, "Simplify as much as possible!" may help students with particular experience in modelling but others may need the idea, that creating the situation symmetrically is the appropriate approach to realise the simplification.

"*Here is a problem related to yours and solved before. This is good news; a problem for which the solution is known and which is connected with our present problem, is certainly welcome*" Pólya (1973, p. 110) with the related questions in the list: "Could you use it? Could you use its result? Could you use its method?" Everywhere in mathematics, it is an often-used method to apply the solution of solved problems in the form of proved theorems. In a modelling process, it also occurs that results from one-step of the modelling process can be used in further steps.

In the modelling problem analysed here, for both kinds of crossings there are cars waiting, then accelerating and driving through the crossing. When the traffic light switches to green, the first car accelerates, after a short time the second car starts and so on. To calculate, how many cars can pass the light in one green phase one has to model this process and thus to deal with the formulas $s = \frac{1}{2}at^2$ and $v = at$. Parts of the calculation for the first car can be used for the second car, one only has to add a delay as the second car starts later and drives a longer distance to pass the crossing. The calculation is almost completely similar for the following cars. Therefore, in the ongoing modelling process students can use what they have done before. Once they deal with the roundabout the calculations from the traffic light can be used again with slight changes. "You nearly had the same calculation before—adopt it to this situation!" "Work similar to your foregoing calculation!" Pólya (1973, p. 37) emphasizes, "Analogy is a sort of similarity. Similar objects agree with each other in some respect, analogous objects agree in certain relations of their respective parts." According to this, a teacher could also formulate: "Work in analogy to your foregoing calculation!"

Generalization is described in detail by Pólya (1973, p. 108 ff.). Generalization means to leave certain restrictions of the problem and thus come to a more universal problem. Even though this new problem covers many more different cases than the original one, it might be easier to solve because it has fewer restrictions than a special problem. In the modelling process, the modeller is free to build a more or less general model, depending upon which gives access to a solution and if this occurs in the beginning of the modelling process or in an advanced state. A more general real model is mostly also a more abstract one, which is less complex but, due to the abstraction, often less accessible to some students. In this research study, it turned out that the students clung to more concrete real models that were often too complex for them to work on so the students were stuck.

The acceleration process mentioned above can be done by calculating everything for each single car with concrete numbers. This is of course the first approach of a modeller but in the long run, it pays off to calculate with variables instead of numbers. This way one calculates the acceleration process for a group of cars with one single calculation. Pólya (1973, p. 110) underlines: "Such a generalization may be very useful. Passing from a problem 'in numbers' to another one 'in letters' we gain access to new procedures; we can vary the data, and, doing so, we may check our results in various ways." In this modelling situation, generalization helps to realize how the calculation for one car can be transferred to the other cars and to the similar situation at the roundabout as mentioned above. So, in this situation two heuristic strategies come together. "After realizing the calculation with numbers try to use letters instead of numbers so you can easier transfer your results to the other cars!"

Extreme cases are particularly instructive (Pólya 1973, p. 192): "The allegedly general statement is concerned with a certain set of objects; in order to refute the statement, we specialize, we pick out from the set an object that does not comply with it. … If, however, we find that the general statement is verified even in the extreme case, the inductive evidence derived from this verification will be strong."

In our modelling problem, the question is: At which kind of intersection (a roundabout or an intersection with traffic lights) can more cars pass a crossing? To answer this question, one has to find out the *maximum possible number* of cars that can pass the crossing in a certain time. So, at the traffic light there have to be always enough cars waiting so that during the green phase the maximum possible number of cars can drive through the crossing (same for the roundabout). This is not obvious for the students in the beginning of the modelling process nor for more experienced people. In an experimental comparison between the roundabout and the American four-way-stop (Mythbusters 2013), the cars driving through a roundabout were counted but one could clearly see, that the number found in the experiment is too small as there were not enough cars involved. Students dealing with this modelling problem often started with assumptions like "from each direction there come 100 cars per hour" and worked with this for a while. If they continue to use this assumption, the following advice is necessary for the students: "You have to calculate a situation where as many cars as possible go through the crossing!"

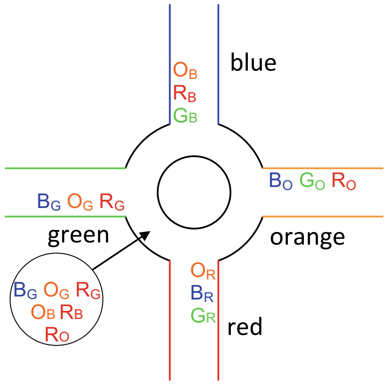
Use a simulation! Computers are mostly available while modelling nowadays and it is often mentioned that simulations with computers can be very helpful during the modelling process (Greefrath 2011). In many situations, the computer skills of the students are not sufficient to implement an appropriate computer simulation on their own. Thus, for the roundabout problem a paper simulation was used to examine the roundabout using material shown in Fig. 11.2. This material was prepared before the modelling days and was already used in the tutor-training. The simulation material, as described above, was essential for the understanding of the situation and led to deeper understanding of the roundabout traffic. The simulation was introduced to the students by handing out the material to them without further instructions. Once they were using the material, hints for its usage were provided like: "Let all blue cars drive in the direction of the blue street, and do so with the other colours." This simulation material already involved other heuristic strategies.

Discretize the situation! Discretization is a core method in mathematics. In applied mathematics, continuous situations must be transformed into discrete ones for example while solving a differential equation using computers or in school using Cavalieri's principle. In pure mathematics, discretization occurs too but at the end of a proof the discrete situation is transformed back into a continuous situation using limits, for example in the definition of the Riemann integral. In the modelling process a discretization can be used to build the real model or while transferring the real model into the mathematical model.

In the reality of the roundabout, the cars drive with constant speed through a roundabout but in the simulation, they move in steps like on a board game. As the material for the simulation, as shown in Fig. 11.2, was handed over to the students, the discretization was already settled by the material. The students acted with this discretization without any problems due to the similarity to a board game. Problems occurred later while the students tried to interpret the simulation results. One result was, that 21 cars from each road can drive through the roundabout in 42 turns of the simulation. The problem for the students was to connect this to a particular time in reality. To get back to a continuous process and connect each turn of the simulation to a definite time was a barrier, maybe due to the fact, that they had not discretized the situation by themselves. So, they needed the help: "Now think again how cars drive through the roundabout in reality. A car needs five turns of the game to drive through the roundabout. How can you calculate how long this is in seconds in reality?"

Use an appropriate representation! In the simulation, there are four colours used for the different streets and the same colours for the cars. Similar to the discretization, this representation was delivered to the students with the material and helped to execute the simulation. In general, it is very helpful for solving a problem and equally a modelling problem to select a good representation of the situation. One aspect of this idea was described by Pólya (1973, p. 103) discussing the use of figures. Using figures is a very important representation dealing with modelling problems. So "Draw a figure!" is a very important heuristic help. The simulation led to a well-educated

Fig. 11.3 Calculating the probability



guess that on average a car from one certain street can enter the roundabout in every second turn of the simulation. To prove this supposition another representation of the roundabout is needed.

In this representation (Fig. 11.3), B_R means cars that drive from the red street (R) to the blue street (B). So obviously in the red street there are cars O_R (red to orange), B_R (red to blue) and G_R (red to green) and similarly in the other streets. The cars drive through the roundabout against the clockwise direction. So which sort of cars appear at the point marked by the arrow? Usually there should be no cars coming from the red street because that only happens if the driver missed the exit which should be the orange, blue or green street. Cars heading for the blue street should only come from the green street as cars from the red or orange street would have already passed the exit to the blue street at this point. With the same argument, cars heading for the orange street may come from the green or the blue street while cars that want to go into the red street may come from all three possible directions. Putting this information together, in the circle one can directly see that half of the cars passing the position marked by the arrow will drive in the red street. Now we have to switch back to Fig. 11.2 and realize that in the simulation the last statement means that in 50% of the turns of the game a car from the red road can enter the roundabout. The representation with two letters as shown (B_R etc.) leads to the insight, why, on average, in every second turn of the simulation a car can enter the roundabout. This works together with the use of colours and the drawing of the roundabout. The appropriate representation is the key to this result. Furthermore, the use of the symmetry, as mentioned above, is essential for this result and, in addition, allows the transfer of the result for the red street to the other streets. If students know a kind of representation that is helpful in the situation, a tutor can just mention it, for example "Draw a figure!" If the representation is new for the students, the tutor has to give a little bit more help: "Use O_R for cars driving from the red street to the orange one and examine, which sorts of cars pass a certain point of the roundabout!"

Supersigns are a concept that was introduced in the problem-solving discussion by Kießwetter (1983) based on the concept of chunking described by Miller (1956).

Using supersigns means to chunk different items together to form a new idea (the supersign) in order to use the working memory more efficiently. The word "supersign" was used relying on information theory and it means a sign that represents several signs. Thus, the name of a mathematical set is a supersign but also a vector, a matrix, a function, an equivalence class and so on. Supersigns are used for structuring the situation in order to organise the material. In natural language, supersigns occur too, for example, to think about "a queue of cars" makes it possible to talk about 30 cars, say, without referring to each single car in the working memory.

In the roundabout simulation, each turn of the simulation includes many concrete steps. So, "one turn" is a supersign for which several times, in reality, are to be calculated. Several calculations that were done in the modelling process "dividing the problem into sub-problems" had to be combined to single ideas to work with them, which means to rebuild the supersign.

In order to gain insights from the problem-solving theory, building supersigns especially when using abstract patterns, is very challenging for students and it can only be expected from very gifted students to do this on their own. All others need support to formulate the supersign even if it is obvious to the tutor. So the tutor has to be conscious about the uses of supersigns. A teacher support using a supersign might appear when students focus on single steps and should combine them to a bigger pattern: "One turn in the simulation consists of several single steps. Build one number that describes all these steps together (for example the total time of one step in reality)."

11.3.2 Results Referring to the Modelling Cycle and Observations in the Empirical Research

Beside the strategic help relying on heuristic strategies, strategic interventions based on the modelling cycle (Fig. 11.1) were formulated in the tutor training and observed during the modelling days. The request to describe the situation precisely and then simplify it to build a real model, to transfer the real model into mathematics, to deal with the mathematics and then interpret the results in terms of the real model as with the request to validate the result with regard to the real-world situation or the real model were appropriate strategic interventions as they demand to do certain steps in the modelling process without giving specific support how these steps should be realised.

One very important strategic intervention observed during the modelling days was the request of the tutor to explain the work already done. This strategic help arose as a very powerful instrument as it has several advantages.

- For the tutors, it was very easy to apply this intervention that was part of the preparation seminar after the first observation. Even if the tutor did not know everything about the modelling problem and the solving process, he or she could ask this question.

- While the students answered the question, the tutor had time to ascertain the situation of the students in the modelling process and was able to analyse barriers or misconceptions. In other words, there was time for a good diagnosis for further interventions.
- The students are encouraged to reflect and structure their ideas. While answering, the students looked back on their own work and often realised thereby, what they have done and what went well or not. In the first approach, sometimes their arguments are poorly structured but asked to explain it again, because it is hard to understand, they rearranged the ideas and themselves gained more insight into their own results. There were situations, where the tutor only asked a group to explain their work when a group was stuck, and they started to explain and then shifted into a debate on their own work that enabled them to go on—while the tutor left the group without any other word.

Overall, strategic interventions turned out to be an appropriate approach to support students' work during complex modelling situations, but a tutor needed a deep insight into the modelling process, the modelling problem and possible solutions (see also Stender and Kaiser 2015).

So, tutoring students that are working on complex modelling problems needed a good preparation for the tutors. In the seminar for the tutors, they had to work on each modelling problem in groups. The process was accelerated a little bit compared with the setting in the modelling days, but the tutors overall worked three hours on each problem with phases of metacognition being the focus in the seminar in-between. This metacognition reflected the phases of the modelling cycle, possible assumptions and those made, and simplifications and expected problems in the modelling days and appropriate interventions. The three modelling problems were not solved in a row but there were seminar sessions that dealt with theory of modelling, strategic help and heuristic strategies. This way the metacognition of the tutors' own modelling processes included more and more theoretical aspects over the time.

11.4 Summary and Conclusions

Tutoring students who are working on complex modelling problems is a very complex challenge for the tutors. Essential for this work is good preparation of the tutors according to the special modelling problems and according to helpful theoretical background.

Heuristic strategies might be very helpful supporting the students but to apply them in concrete situations is not easy. One has to realize the barrier that prevents the students from undertaking the next steps, solve the problem to overcome this barrier and then find out the heuristic strategy one uses via metacognition. The last step needs a lot of experience in analysing solutions of modelling problems regarding the use of heuristic strategies and it needs time in the situation. This means that it is meaningful to prepare this kind of teacher intervention beforehand: tutors who are going to supervise students while modelling should model the problem on their own. While doing this, tutors should analyse their own work via metacognition and identify possible barriers and strategies used in the way shown above. Then strategic interventions can be pre-formulated. Teacher trainings for modelling activities could use this approach.

Here only the Roundabout versus Traffic Lights Problem was analysed but in Stender and Kaiser (2016) and Stender (2018) the use of heuristic strategies in The Bus Stop Problem are shown too, so this approach is not limited to a single modelling process of one modelling problem.

In further research, this approach should be examined in more detail as up to now there is no empirical evidence that heuristic strategies really improve the work of students while working on complex modelling problems. This research should include the teacher training and an appropriate modelling environment. In other areas of mathematical work there is (unpublished) evidence that strategic interventions based on heuristic strategies are successful: this approach was used supporting mathematics teacher students in the first semester doing high level mathematics with a very good outcome. This indicates that there is a good chance that using heuristic strategies, as a generalized toolbox to describe students' work via metacognition and support students' work via strategic help, is a very promising approach in all parts of mathematics.

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Chapter 12
Modelling Tasks and Students with Mathematical Difficulties

Ibtisam Abedelhalek Zubi, Irit Peled and Marva Yarden

Abstract This study is a part of a bigger study of 23 fifth graders observed as they worked in heterogeneous groups on a sequence of 12 modelling tasks for eight months. This chapter focuses on nine students identified as having difficulties in mathematics. Our research goal was to identify the nature of the changes that occurred as they worked on these tasks and is exemplified by one case, that of Sami. The findings show how Sami's mathematical knowledge and modelling competencies developed and how, simultaneously, his group's attitude towards his contributions was affected. At the beginning of the process he did not understand the task situation, and even when he gave relevant realistic considerations his peers ignored him. Later he became more active not only in offering realistic considerations but also in suggesting mathematical ideas, and eventually Sami became dominant and effective in the group and was well aware of this change.

Keywords Mathematical difficulties · Modelling competencies · Modelling tasks · Students with learning difficulties

12.1 Theoretical Background

Researchers claim that teachers emphasize high thinking processes in good classrooms, while in classrooms of students with learning difficulties, they use methods of instruction that require only low order thinking (Shepard 1991; Raudenbush et al. 1993; Zohar et al. 2001). This tendency exists also when working in a heterogeneous class (Yair 1997). Page (1991) adds that teachers expect good students to deal

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with complex and challenging learning materials, while students with difficulties are expected to learn basic skills only. On the other hand, Haylock (1991) emphasizes the need to provide meaningful and relevant activities for students with difficulties, as well as the importance of working in small groups in coping with these tasks.

Characteristics of students with difficulties include a variety of problems such as: memory difficulties, difficulties in generalizing and transferring acquired knowledge to new and unfamiliar tasks (Kroesbergen and Van Luit 2003). Haylock (1991) also mentions difficulties in: language, reading, spatial perception and anxiety in mathematics. In addition, Peled (1997) and Bachor and Crealock (1986) emphasize the passive response to instruction as one of the main characteristics of these students.

In looking at the source of these difficulties, Abel (1983) claims that environmental factors are more significant than congenital factors. Ginsburg (1997) analyses many possible factors that might contribute to students' difficulties. He suggests that often instructional methods are to blame rather than some cognitive deficit. Thus, it can be concluded that low performance and achievement in mathematics can also result from inadequate instruction and lack of motivation (Barnes 2005; Ginsburg 1997; Karsenty and Arcavi 2003; Reusser 2000).

Peled (1997) claims that students who are passive and do not participate in class, often because of lack of confidence, experience less teacher reaction and miss the chance to improve their knowledge thus falling into a vicious cycle. She suggests that mathematics programs should encourage students with difficulties to participate more in class, receive more feedback and, thus, have a better chance of learning and progressing. In this study, our assumption is that, modelling activities can be appropriate and effective for students with difficulties in mathematics.

Mathematical modelling has been defined by Blum and Borromeo Ferri (2009) as a bi-directional translation process between the real world and mathematics. Peled (2007) defines modelling activities as a process of organization, analysis and observation of situations and phenomena through models and mathematical tools. Blum and Leiβ (2007) describe the expected stages in a modelling solution as a cyclical process. The ideal modelling cycle starts in the "situation (usually realistic) world" with analysis, simplification, organization, and structuring of the given situation. This stage is followed by mathematisation of the structured situation by choosing and using mathematical models and representations, and by performance of some mathematical processes. The next stage involves interpretation of the results in terms of the situation, validation of the results and deliberation on whether new (realistic or mathematical) considerations are needed and another cycle is called for. The analysis of modelling competencies by Maaβ (2006) and Maaβ and Mischo (2011) is closely related to the modelling cycle because the problem solver should be able to perform each of the cycle's stages in order to follow this cycle. Niss et al. (2007) define competency as the ability to perform certain appropriate actions in a problem situation and mathematical modelling competency as the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution.

In this chapter, we have chosen to focus on the main competencies. This includes an awareness of the need to understand, analyse and simplify the situation and the ability to do so; an ability to choose mathematical representations and use mathematical concepts in building mathematical models for describing the simplified situation; and a need to make sense of the results by reasoning and using argumentation. In addition, the whole process involves an organizational competency that includes the ability to take part in group work and keep track of it through clear documentation.

Maaβ (2005) claims that, the close connection between reality and modelling activities makes mathematics more useful, interesting and understandable, especially for students with difficulties. Moreover, the work on modelling tasks is usually done in small groups (Zawojewski et al. 2003). This setting might result in having the better students dominate the discussion while the weaker students stay passive and are ignored when they try to contribute to the group discussion (Peled 1997). While this latter drawback exists, there is also some support for potential benefits of group work for these students. A report of an advisory committee in the USA (Steedly et al. 2008) suggests that low achievers and students with learning difficulties in mathematics might benefit from working with other students and listening to their peers' mathematical discussions.

In our own experience in several studies where we implemented a sequence of modelling tasks we noticed positive changes in the work of students identified by their teacher as having difficulties in mathematics (e.g. Filo and Peled 2012). Following these informal observations, we designed this study with the purpose of focusing specifically on these low-performing students and observing changes in their *modelling competencies*, their *mathematical knowledge* and how these changes are exhibited through the *nature of their participation*.

12.2 Method

The study was conducted in an Arabic primary school of a low-class population. The studied group was a fifth-grade class of 23 students, nine of whom were identified by their teacher as having difficulties in learning mathematics. Her definition matched that of the school and the Ministry of Education, which relies on the results of standard tests. The mathematics instruction method in this class was a whole class method, with the teacher being the source of knowledge and authority. The main mathematical topics in fifth grade are: fractions and decimals, geometry focusing on triangles, quadrilaterals, areas and circumference of polygons and deepening of the four arithmetic operations in natural numbers.

The study was conducted in school during regular school hours. It involved one lesson (of 45–60 min) each week for eight months during one of the lessons assigned as mathematics lessons. The task sequence consisted of 12 tasks, detailed in Table 12.1, including individual pre-test and post-test tasks. The study included all students in the class; they worked in heterogeneous groups of 4–5 students each group, two of them were students with difficulties. However, the observations focused on the students with mathematical difficulties in each of the groups. We followed their learning process in order to examine two questions:

Table 12.1 Task sequence

Task	Context
1. Pre-test: *Fun day*	Planning a fun day schedule by choosing a sequence of activities
2. *Class party*	Ordering refreshments for a class party given items and prices
3. *Pizza*	Ordering pizza given two pizzerias, different items and prices in two places
4. *Cookie bakery*	Providing orders of cookies packaged in given package sizes
5. *Tangram*	Pricing different tangram parts given the total price of the game
6. *Ads-task A*	Preparing advertisement for a store sale of different products including toys, clothes, etc.
7. *Ads-task B*	A similar task as in part A with a more limited number of products
8. *School time*	Verifying the claim "most of the year is spent at school" (Maaβ and Mischo 2011)
9. *Tents*	Deciding what tents in terms of size, number, and manufacturer the school should purchase
10. *Body relations*	Checking whether there is a constant relation between the head and body of a person
11. *Volleyball*	Choosing players based on a table of quantitative and qualitative data (Zawojewski et al. 2003)
12. Post-test:	Planning a schedule by choosing a sequence of activities
a. *Fun day*	Similar to the pre-test task 1 with different activities
b. *Farm visit*	Similar to the Fun day task with a different context and more complicated data

1. (a) Can modelling competences be developed among students with difficulties in mathematics? (b) Which modelling competences will develop through the implementation of a modelling task sequence?
2. Does this implementation of modelling tasks also have an impact on their mathematical knowledge?

The study was conducted using the Design Experiment approach developed by Cobb et al. (2003). This approach can be defined as a research method aimed at developing theories and materials, based on 'how learning works'. It involves iterative task design where tasks are evaluated and redesigned following observations of students' learning. Tasks were designed using contexts to motivate the students, and involved situations taken from their daily life. Further on in the study, some tasks were redesigned based on students' suggestions or needs.

The final task sequence (Table 12.1) consisted of twelve modelling tasks, some designed in advance and some constructed during the implementation of the sequence, as will be detailed further. All the tasks were designed using principles for constructing Model Eliciting Activities (Lesh et al. 2000), keeping in mind the goals of the study. The main features of these tasks involve the use of context in a way that will elicit and encourage an analysis of the given situation, organization of the situation, and making choices about representing and mathematising it. These tasks are expected to elicit the development of modelling competencies leading to a solution process that is depicted by the Blum and Leiβ (2007) modelling cycle.

Table 12.1 details the twelve tasks with a short description of their context. The first task and the last task were similar; the difference between them was in the type of activities in order to suit the fun day at the end of the year, serving as individual pre-test and post-test for examining the change in modelling competencies in addition to the evidence collected from students' work through the whole sequence.

Our analysis of the progress of Sami and the rest of the students was conducted using a variety of data collection instruments: (a) individual pre-test and post-tests for examining modelling development competencies, (b) individual pre- and post-tests for examining mathematical knowledge (testing the concepts of multiplication and fractions), (c) observation notes and student work during implementation of a sequence of modelling tasks in which the students worked in heterogeneous groups and (d) individual interviews following the implementation of the modelling tasks and post testing. Pre-test and post-test data on the *Fun Day* and *Farm Visit* tasks were analysed for evidence of modelling competencies: awareness of the need to understand, analyse and simplify the situation and the ability to do so; an ability to choose mathematical representations and use mathematical concepts in building mathematical models for describing the simplified situation; and a need to make sense of the results by reasoning and using argumentation, and an organizational competency that includes the ability to take part in group work and keep track of it through clear documentation. In our observations through their implementation of the tasks sequence we focused on modelling competencies development through changes in the nature of their participation, chronicling any development from completely passive behaviour to becoming involved in the working and organisation of the group and constructing and documenting mathematical models.

The study was conducted as a part of the regular school day, one session per week for eight months with the exception of holidays. In addition to working on modelling tasks, the students continued with their mathematics classes according to the regular curriculum. As will be mentioned further on, the effect of the 'other' regular mathematics classes was controlled by data from another class, which did not participate in the study and served as a control group.

12.3 Findings

The information on the development of modelling competencies and mathematical knowledge was obtained by analysing data from the pre-tests and post-tests and from the observations during the task sequence. The comparison of the pre-test and the post-test for every student showed that seven of the nine students with mathematical difficulties had developed all competencies and showed progress in their mathematical knowledge.

We chose to describe in detail the follow-up on Sami's development in order to give sense and understanding to how the changes occurred. Sami was one of the seven students for whom we found a significant development in modelling competencies and mathematical knowledge. Sami's development was similar to the development of the other six students and hence serves as a representative example. The results of the pre-test for examining of mathematical knowledge were consistent with the assessment of Sami's performance in the standard school tests on these subjects.

In addition, the mathematics teacher noted at the beginning of the study that Sami did not show interest in mathematics lessons and did not participate in the class discussions despite his high verbal abilities expressed in social fields.

12.3.1 Sami's Pre-test in Modelling Competencies

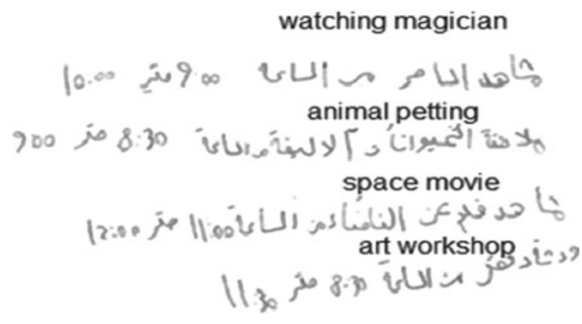
The pre-test *Fun day* (Appendix 1) examined several modelling competencies: simplifying and understanding the situation, setting up a mathematical model, reasoning and documentation. In this task, the students were requested to plan their own sequence of shows or activities from a given list with a starting time and duration of the shows.

Sami wrote down his favourite choices in a short and partial list without giving any explanation for his choices (Fig. 12.1). In addition, it would have been impossible to carry out his choice because of time overlaps between activities. For example, the art workshop overlapped all the other activities. Although he wrote the durations of the activities, he did not take them into account. He was thus demonstrating little to no modelling competence.

12.3.2 Sami's Performance and Role During the Task Sequence

As mentioned earlier, Sami worked in a heterogeneous group of five students. Two of the students, Sami and Noor were students with difficulties in mathematics. In the follow-up on Sami's work, we observed a gradual development of Sami's modelling competencies, which we describe below in four stages:

Fig. 12.1 Sami's work in the modelling competencies pre-test



1. Passive.
2. Active in realistic considerations only.
3. Active in organization and partly active in setting up the mathematical model.
4. Dominant in all stages.

12.3.2.1 Passive

The first task (after the pre-test) in the group sequence, *Class party*, was a relatively simple decision-making task with limited data. It was supposed to create a bridge between traditional school word problems and modelling tasks. The students were requested to choose items from a given list of refreshments under the constraint that each student had to pay 6 New Israeli Shekels (NIS).

At the beginning of the group's work, Sami asked Noor, the other weaker student, not to interfere with the calculations: "you and I shouldn't do the calculation." This act expressed his concern and awareness of his own weakness in doing mathematics. Seeing the task as a kind of competition between the groups, he wanted his group to have the best mathematical solution, and therefore thought it would be better if the other group mates do the work.

As the group continued working, Sami did not stay completely passive. He *pointed out some realistic considerations* suggesting not to choose too many sweet drinks and snacks because they are not healthy. Unfortunately, his group mates totally ignored his ideas.

When the groups presented their choices, it was evident that Sami's group and all other groups provided similar solutions. They constructed some arithmetic expressions that would reach an amount equal to the number of students in class multiplied by the amount paid by each student, but their solutions did not take into account any realistic considerations.

The final stage in working on each of the modelling tasks involved a presentation of the groups' solutions to the whole class. This was supposed to give the different groups a chance to comment and discuss each other's ideas. However, as it turned out, the groups' presentations of this task did not trigger any discussion or argumentation of the presented solutions.

While the students did not make any comments or ask any questions, possibly not accustomed to doing so, the mathematics teacher made a move that could be considered an intervention. In, what seems to be, an effort to hint that the (new) rules (or didactical contract) allow discussing problem solutions, she raised a question about the validity of the solution: "Are the quantities of items you chose realistic?" As a result of the teacher's question, Sami felt confident and said with excitement, "I was thinking that too many sweet drinks and snacks is unhealthy! I even told that to my group mates, but they did not listen!" At the end of this meeting all the groups asked to solve this task again.

12.3.2.2 Active in Realistic Considerations Only

Following their request, the groups were given the opportunity to work again on the *Class party* task. While they started to work, Sami suggested a criterion for choosing the products "Pita is most important, let's order 24 pitas so that there will be an extra one for those who will still be hungry". Then he added, "Two bags of snacks will be enough if you put them on plates so everyone can take, and we do not need a lot of juice, three bottles will be enough for the whole class". This time his group mates listened to him and took his suggestions into account.

The next task, *Pizza*, was similar in text complexity to the *Class party* task but included more complex data and more choices. Again, Sami had an important role in *making decisions related to realistic considerations*. For example, he asked his group mates for their favourite extras and he looked in the table to find out which company supplied these. Although he *did not deal directly with calculations*, he showed interest in the written exercises and their solutions. In the presentation in front of the class, Sami listened to other groups, he criticized the solution of one of the groups that did not make (realistic) sense.

Up to this point, as just described, although Sami had become active in his group and in class discussion, his participation only involved realistic considerations. He was still avoiding taking any part in setting up the mathematical model.

12.3.2.3 Active in Organization and Partly Active in Setting up the Mathematical Model

The next task, *Cookie bakery* (Appendix 2), was complex and required quite an extensive situation analysis and data organization. The students' task was to provide orders of different amounts of cookies. They were expected to role play cookie factory workers, make decisions, plan, and prepare efficient delivery of cookies that come packaged in given package sizes.

In the first meeting Sami took upon himself the role of *group work organizer*. At his initiative, he distributed empty pages to the group members. He asked each student to check 20 different numbers. Sami chose to check the small numbers, 1–20. He identified quantities that could be provided including all multiples of four and all multiples of six (4, 6, 8, 12, 16, 18, 20). However, he did not notice that 10 and 14, combinations of the two package sizes, could also be provided.

Sami took his self-chosen role as an organizer seriously. At the end of this meeting he asked the researcher to keep the pages of all the members of his group in order to continue their work in the next meeting. As we will see, he continued taking upon himself this responsibility in the following lesson.

In the second meeting Sami was the first to ask the researcher for the documentation pages from the previous meeting. He took a blank page and began to collect the data from all the group mates. He suggested "I think we should draw three columns: one for the number of cookies, one to specify whether it is possible to provide or not and the third column to represent how to provide the cookies (number and type of packages)". His group mates agreed with his suggestion.

Sami presented the possible quantities that could be provided from the set of numbers he had examined. He said that it was possible to provide: 4, 8, 12, 16, 18 and 20 cookies. A mathematically strong student in the group told Sami that it was also possible to provide 10 cookies in one package of six and 1 package of four and that 14 is also possible using 1 package of six and 2 packages of 4. Sami acknowledged, "Oh, I did not think about that!".

While checking the rest of the numbers, another mathematically strong student said that 26 was not possible to provide because it is not a multiple of four or six. Sami told her that he thought it was possible, "I found that the 20 is possible, so we will add another one package of 6. 26 could be provided with 5 packages of four and one package of six". Sami listened carefully to the explanation of Noor, another mathematically weaker student, why 39 could not be provided: "38 cookies is possible: 5 packages of six and 2 packages of four. You cannot provide 39 because you can't add one cookie. You know that! If it's odd—you cannot provide it". Sami wasn't sure of Noor's conclusion and began checking some odd numbers and said, "I think Noor is right", and he suggested writing this conclusion in their group solution: "We cannot provide odd numbers of cookies". He also said, "We need to check only the even numbers".

It is interesting to note that Sami related to a solution that was better than his own as something exciting to be learned from. He did not see it as something that undermined his own discoveries, but rather as a group effort to progress together. Similarly, he listened carefully to his group mates and was able to identify, point out, and put on record good ideas.

The third meeting involved group solution presentations. When Sami's group mates introduced the group's solution, they asked the class to become active and participate in a simulation of ordering and providing cookies. The students were expected to make orders and the group would figure out "live" how to provide these orders, and if it was possible to provide a certain order at all. Sami actively participated in the role of the seller and was proud of himself for using his generalization about odd numbers and giving an immediate "not possible to deliver" answer when an odd number came up.

Fig. 12.2 Sami's pricing suggestion

2 NIS	small triangle
2 NIS	small triangle
2 NIS	square
10 NIS	big triangle
10 NIS	big triangle
2 NIS	parallelogram
2 NIS	medium triangle
30 NIS	

During working on this task, Sami was very *active in organization and documentation* and he was able to learn and apply the knowledge acquired from his group and class mates.

The next task, *Tangram*, required pricing each part of the Tangram game-set under the constraint that the price of a complete set would be 30 New Israeli Shekels (NIS). Saying "I do not like shapes", Sami resisted and withdrew when he saw the geometric shapes which were associated with the task.

Nevertheless, while the group started working Sami recommended pricing the parts by categorizing them into two sizes: large and small. The idea was to price each large part (the two large triangles) at 10 NIS ($2 \times 10 = 20$) and each other part 2 NIS ($5 \times 2 = 10$). The members of his group accepted his idea and priced them accordingly (see Fig. 12.2).

Following this first solution one of the mathematically stronger students in the group noticed that there were 3 different sizes and suggested another pricing solution: square 5, medium triangle 3, large triangle 7, parallelogram 3, small triangle 2.5. Sami liked this solution and said, "That is right, I noticed that two small triangles cover one square".

In the class presentation, another group priced the shapes according to the exact ratio between the shapes areas. Their solution was: big triangle 8 NIS, parallelogram 4 NIS, medium triangle 4 NIS, square 4 NIS and small triangle 2 NIS. The total price of the various tangram parts, in this solution, amounted to 32 NIS. Sami noticed from their solution that the three shapes (parallelogram, medium triangle and square) are the same size. He said to his team members, "We did not do right". A mathematically weaker student in this group, justified the fact that the total price of the seven pieces was higher than the price of the complete set (30 NIS), because the sale was in individual parts. Sami agreed with her and said, "It is always like that, when you buy in parts it is more expensive than buying the same parts as a set".

Despite his initial dislike of the geometric shapes, Sami was an active partner and even dared to offer a pricing model for the group. His ability to learn from other students was evident during all the task stages. The task and the group work helped Sami overcome his dislike of geometry, as he noted at the end of this task, "At the beginning of the task I had a fear of the shapes but now I'm feeling that I even do like a little bit geometry".

12.3.2.4 Dominant in All Stages

All the tasks that were given after the *Tangram* task were more complex in terms of numbers and data types. In all these tasks, Sami was dominant in organizing the work of the group. He contributed a lot to the mathematical and realistic considerations and to setting up the mathematical model. He also played a significant role in documenting and presenting the product to the entire class. It should be noted that he worked out of interest and learned from his class mates and applied the new knowledge. For example, in the *School time* task (Maaβ and Mischo 2011) no data were given and the students asked to verify the claim, "Most of the year is spent at school". Sami was dominant in analysing the situation, finding out relevant data and in defining concepts such as: most of the year, school time, home time, etcetera. He was also dominant in setting up the mathematical model and in the documentation.

Another example that reflects his dominance at all stages of the solution is the *Volleyball* task. This task was based on a task of Lesh and colleagues (Zawojewski et al. 2003) and it was given after Sami and other students' request to have a task that dealt with sport. The task is a complex task requiring decisions on relevant factors and how to weight them taking into account quantitative and qualitative data. In this task, the students were requested to divide 15 players into three groups.

As in the previous tasks, Sami took on the role of the organizer. He suggested that first of all each one of them look individually at the data and divide the players into three groups and then they decide together on the final solution.

Sami looked at the overall players' points, chose the three players with high scores and listed each in a separate group, same for the three weakest players with low scores. The rest of the players, he divided among the three groups taking into account the opinion of the coach. For this dividing, he used mapping using three different marks, 1-2-3 to distinguish three categories, as can be seen in Fig. 12.3.

At the end of the individual work they moved on to set up a group model. At this stage Sami noticed that one of the students made a division according to the order of players in the given table without taking into account any of the players' data. Sami explained to this student that strong and weak players must be mixed because otherwise, there will be a weak team that will lose all the time and a strong team that will win all the time. Sami suggested that he would show to his group his solution and they would make changes if it was necessary. As a result of a dispute between the members of the group about the dividing of the players, he drew a new table and wrote down the names of the players after he obtained the consent of all members of the group.

In the presentation in front of the class, he noticed that other groups calculated the sum of the points for each player and used it as a tool to compare between the players. Sami was excited and told his group mates, "This way is easier than ours, we can know easily which player is strong and which is weak, the dividing of the players in this way will be surely fair".

As described above in this task, which was the last one in the sequence, Sami had a significant role in *analysing the situation*, *setting up a mathematical model*,

معطيات عن اللاعبين الذين سيشتركون في مخيم كرة الطائرة.

توصية وتقييم المدرب	أفضل إنجاز بالقفز إلى الأعلى بالسنتيمترات	عدد النقاط التي جمعها كل لاعب خلال آخر 5 ألعاب					اسم اللاعب	
		لعبة 5	لعبة 4	لعبة 3	لعبة 2	لعبة 1		
مثابر وجدي	26 سم	2	1	4	5	3	رامي	1
سريع	28 سم	2	3	4	3	2	سامي	3
منضبط	26 سم	0	4	3	2	1	يوسف	1
سريع	30 سم	5	5	6	5	6	جدي	1
خفيف الحركة	24 سم	2	1	1	2	3	سليم	2
مثابر وجدي	29 سم	5	4	5	7	6	راوي	3
سريع	26 سم	2	0	5	4	3	أسير	2
ملتزم	30 سم	6	6	6	5	4	نسيم	1
سريع جدا	30 سم	7	5	6	6	5	وسام	2
خفيف الحركة	28 سم	3	4	5	4	3	وليد	1
خفيف الحركة	27 سم	6	5	4	2	0	موسى	2
مثابر وجدي	29 سم	6	5	4	5	6	راني	3
سريع	25 سم	3	4	2	1	2	حماد	3
سريع وملتزم	30 سم	7	6	7	6	5	يونس	3
مثابر وجدي	27 سم	5	0	8	1	5	بلال	2

Fig. 12.3 Sami's mapping during dividing the players into 3 groups with the annotations on the right indicating the assigned group number of the player in that row

reasoning, *validatio*n and *documentation,* as well, his ability to learn from his group mates stood out in this task.

To check if the development of the modelling competencies in the post-test were not affected by the identity between the post-test and the pre-test, a similar task *Farm visit* with complicated data and different activities was given to the students. Sami's results for this task (see Fig. 12.4) were identical to the results of the post-test.

12.3.3 Sami's Progress in Mathematical Knowledge

Beside Sami's development of modelling competencies there was also a development in his mathematical knowledge as can be seen in the results of his post-tests for mathematical knowledge in multiplication and fractions: Fig. 12.5 presents Sami's pre-test answer to a question that diagnoses student conception of an array by asking the students to determine the number of small rectangles in the figure. Figure 12.6 presents his post-test result for the same question. It is important to note that in the pre-test Sami used a counting strategy to find the number of rectangles in the figure.

Activity	Starting time	End	Decision
Soccer	8:30	9:00	Because I want to play another game
water games	9:00	10:00	I like water games
Inflatable games	10:00	10:30	Inflatable games are fun
Rope games	10:30	11:00	I play with friends
Basketball	11:00	11:30	I like basketball
Running race	11:30	12:30	I am fast

Fig. 12.4 Sami's work on the modelling competencies post-test

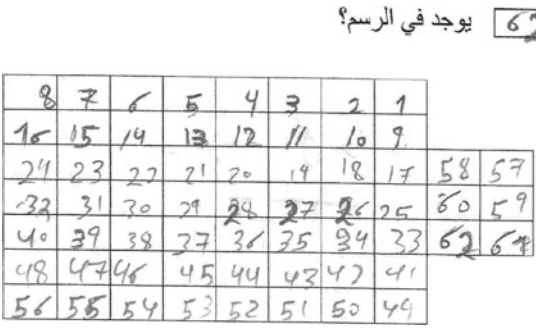


Fig. 12.5 Sami's answer in the multiplication pre-test

In the post-test, he used his multiplication knowledge to figure out the number of rectangles.

Another example from the Fraction pre- and post-tests is given in Figs. 12.7 and 12.8. In these figures, we can see Sami's answer to the question: "Given 2 identical containers ½ of the first and $^2/_3$ of the second are filled with oil. Is it possible to transfer all the oil from the first to the second?"

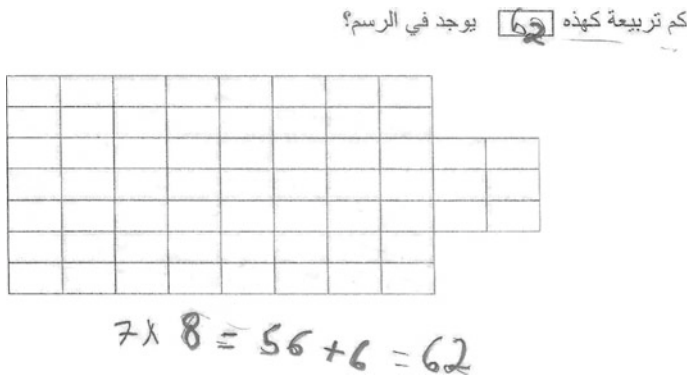


Fig. 12.6 Sami’s answer in the multiplication post-test

Fig. 12.7 Sami’s answer in the fraction pre-test

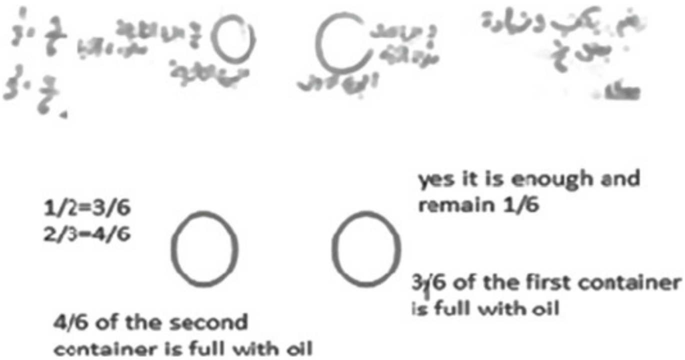


Fig. 12.8 Sami’s answer in the fraction post-test

In the pre-test, Sami wrote a multiplication exercise with the two numbers without understanding and analysing the situation. He did not give any reasoning for his solution and did not attribute any realistic significance to its result. On the other hand, in the post-test, he demonstrated a deeper understanding of the situation, which helped him to set up a realistic mathematical model.

Since the class continued learning according to the regular curriculum, the improvement in Sami's mathematical knowledge could be a result of the regular work rather than an effect of the task sequence. Therefore, the performance of Sami and the other eight students with mathematical difficulties was compared with a similar fifth grade taught by the same teacher. This class served as a control group and included 10 students with learning difficulties in mathematics. While the performance of the nine students in the modelling group increased in both concepts, the 10 students in the control group exhibited minimal knowledge development of both concepts.

12.4 Discussion

Our research examined the effect of a modelling task sequence on the development of modelling competencies and mathematical knowledge among students with difficulties in mathematics. In this chapter, we focused on one student, on Sami's development. Sami was one of the seven students for whom we found a significant development in modelling competencies and in mathematical knowledge. His mathematics teacher described him, before the beginning of the research, as a passive and unmotivated student in mathematics lessons. The teacher's description was compatible with the characteristics of students with difficulties as described by Peled (1997), Ginsburg (1997) and Bachor and Crealock (1986).

At the beginning of the process, Sami was passive and his role in the group work was insignificant. Soon afterwards, Sami started to use his daily life experiences in making realistic considerations suggesting factors that might be taken into account in a given situation. As the work progressed, he became active in organization of the group work and also started making mathematical suggestions with regard to possible representations or calculations. Close to the middle of the process, Sami became an active participant in working with the group at all stages of the modelling process, from the *analysis of the situation* through its *mathematisation* and *validation*.

In addition, Sami's experience with modelling tasks changed his work habits. He changed his own norms with regard to the time one is expected to devote to solving a problem, and especially the time spent on simplifying and understanding the situation, a stage that is crucial for building a sound mathematical model (Schoenfeld 1992). It is interesting to note that the development of the competency of *simplifying and understanding the situation* is also reflected in extending the time devoted for reading and analysing mathematical problems given in the mathematical knowledge post-test. This change might have been one of the factors affecting his mathematical knowledge development. He also began attributing importance to the reasons behind his own decisions as is seen in both the modelling competencies post-test (Fig. 12.4) and mathematical knowledge post-test (Fig. 12.8).

Not less important was the effect of these changes on Sami's self-image, he saw himself as an active partner, initiator and decision maker, as he expressed in the interviews at the end of the process:

"I was an effective member in the group, in deciding what to choose and how to calculate".
"I had an important role in the group, they listened to my ideas".
"I enjoyed the activities because it was much more than just solving exercises, I could express my opinions and help the group making decisions".
"I started to like geometry"
"I learned useful things so I can manage a store when I grow up".
"I am the king of math".

What caused and triggered all these changes in Sami's knowledge and work habits?

Modelling tasks seem to have raised Sami's interest and encouraged him to become an active participant in the group's work. The nature of the modelling tasks made the situation accessible, but their effect went much beyond that. The problem situation also facilitated the understanding of the mathematical structures that were associated with it.

What this means is that the development of Sami's modelling competencies and mathematical knowledge as well as his shift in motivation and participation were actually a result of the difference in the instructional approach. If Sami managed to undergo these changes following the introduction of a new type of task together with new problem-solving norms, it means that the source of his difficulties, to begin with, was not some cognitive deficit.

12.5 Conclusion

Sami's case, supports the more general claim that a significant part of the students who experience failure in mathematics should not be labelled as mathematically disabled. Their low achievements in mathematics might be attributed to instruction that is not adequate for them (Ginsburg 1997; Reusser 2000). Like Sami and his peers, they might be helped by making a curricular and instructional change that involves instruction that is more appropriate and meaningful for them.

Thus, despite the intuitive tendency to avoid giving students with difficulties complex problems, this experience with modelling tasks with these particular students seems to have opened powerful learning opportunities for students weaker in mathematics. It facilitated the development of modelling competencies and the development of mathematical knowledge, thus helping them develop the ability to cope with situations they might encounter in their life.

Appendix 1

Task: *Fun day*

The school management organized a fun day for the students.
The fun day starts at 8:30 and ends at 12:30 pm and includes activities and shows.
Please plan your sequence of shows/activities.
Comments:

1. There is more than one start time for every activity.
2. It is permissible to take a break up to half an hour only during the fun day.
3. You do not have to participate in all the activities.

Attached below, names, starting time and duration of activities.

Appendix 2

Cookie bakery

The "Magic Bakery" sells chocolate chip cookies in two types of packages: packages of 4 cookies, and packages of 6 cookies.
Imagine that you work at this bakery and people come to buy a quantity of cookies (up to100 cookies), you have to give them their exact order as soon as possible, if it is possible. Also, you should know how to serve it: number and type of packages.
Try to find a way to help you to provide any order efficiently.

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Chapter 13
Conclusions and Future Lines of Inquiry in Mathematical Modelling Research in Education

Jill P. Brown and Toshikazu Ikeda

Abstract This final chapter overviews the 12 contributions to the monograph, organising this along the lines of inquiry suggested by Stillman. Contributors share understanding of mathematical modelling as solving real-world problems. The value and purposes of implementing modelling varies, in part due to local curricula. Theoretical underpinnings of the research include prescriptive modelling, modelling cycles, and modelling competencies. The challenges of engaging in modelling see empirical foci on modellers, teachers, and tasks whilst acknowledging interactions between these. Other important areas of the field, where researchers need to focus in the future include research with experienced student modellers, research on experienced teachers of modelling, and successful mathematisation by modellers.

Keywords Modelling tasks · Teachers of modelling · Prescriptive modelling · Affordances

13.1 Mathematical Modelling: What Lines of Inquiry?

Defining the bounds of research reported in this monograph is important. This includes the shared understanding by authors as to what mathematical modelling is. Modelling occurs when teachers, students, mathematicians, and others attempt to describe some aspect of the real-world in mathematical terms in order to understand something better or take or recommend actions (e.g. Blum 2015; Blum et al. 2007). All authors in this volume view the real-world as important throughout engagement

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in mathematical modelling, as the task solver is aware that any solution that does not make sense in the real-world is no solution at all. The real-world may take a back seat to the mathematical world at times, however, it is never entirely absent (e.g. Julie and Mudaly 2007).

Understanding how chapter authors define mathematical modelling in this book is clear. However, this is not always the case in research reporting, as articulated by Brown in her chapter discussing the multiple varied meanings of context, task context, and real-world task context—all critical to reading and doing research about mathematical modelling and applications. Brown's study was of research reported in general mathematics education research, however, even within the modelling community, we should not assume others have the same understanding or definitions as ourselves, or even assume there is only one interpretation of these. In other words, the mathematical modelling research community must be scholarly in both our work and the reporting of this.

The need for a shared understanding of modelling by authors extends to other key terms but it is impacted by the value placed on modelling and its place in various curricula around the world. The value of mathematical modelling and applications should be clear to all. An explicit articulation of this can be found in the statement by Blum et al. (2007) that "nearly all questions and problems in mathematics education, that is questions and problems concerning human learning and the teaching of mathematics, influence and are influenced by relations between mathematics and some aspects of the *real world*" [emphasis added] (p. xii). However, there is still much variation as to whether this importance is recognised by curriculum writers and included in school curricula, and, where included—if this is implemented by teachers.

In Germany, Maaß (2016) reports that mathematical modelling is part of the national standards of mathematics education. Hankeln, Adamek and Greefrath (this volume) note that the German national standards, include the expectation that students translate real situations to mathematical problems, solve the mathematical problem, and interpret and check the results in terms of the real-world situation. This has meant professional learning on various aspects of mathematical modelling is available and textbooks include some modelling tasks. Maaß notes that many teachers, but not necessarily the majority of teachers, include mathematical modelling as part of their teaching repertoire.

In contrast, the framework of the Japanese mathematics curriculum is based on pure mathematics. Mathematical modelling has been given some emphasis and it is more emphasized in the next curriculum to be introduced from 2020. Some teachers implement aspects of mathematical modelling in their daily classroom teaching. However, there are difficulties related to incorporating mathematical modelling into a curriculum based on pure mathematics (Ikeda 2015). Namely, it is not explicitly described in the national curriculum at which grade and with what content teachers might introduce mathematical modelling.

Blomhøj reports that in the Danish secondary school systems, modelling, whilst included in the curriculum, is not "really integrated, in the curriculum" in practice. Fulton et al. report that modelling is rarely part of the US primary school curriculum even though it has come into the high school curriculum. Caron's chapter considers the feasibility of introducing modelling into the school curriculum in Canada, so we can infer it is currently absent, or scant, in some Canadian states. Similarly, grade 11 students in the study by Ortega, Puig and Albarracín, had no previous experience with modelling, so we infer, modelling is either absent from the Spanish curriculum, or present but not implemented. Similarly, the study in Israel by Zubi, Peled, and Yarden was introducing primary students to modelling tasks, so we infer this is not the norm.

One reason for the limited focus on mathematical modelling may be due to its high cognitive demand (Stillman et al. 2009). This is the nature of mathematical modelling as students make sense of the messy real-world and simplify this in order to bring it into the mathematical world in a way that can be managed and solved. In addition, the complex nature of modelling often sees students working in groups. Collaborative group work can enhance opportunities for successful solution of a given task, however, students need to learn how to work in groups, and do so collaboratively. If this is not a normal classroom practice, then an additional challenge exists as students learn to work collaboratively during modelling.

On the one hand, there are issues related to the value of mathematical modelling and its place in curricula documents, and the challenges for students in working together to solve such tasks. On the other hand, there exist issues related to teachers, their belief that modelling is an important part of mathematics, and being prepared to implement modelling tasks with students and face the challenges involved. As with students, the distance between the usual classroom practices implemented by the teacher and those required when modelling, increases the level of challenge for the teacher. Blum (2015, p. 83) gives insight into this distance when he laments,

> generally speaking, the well-known findings on quality mathematics teaching hold, of course, also for teaching mathematics in the context of relations to the real world. This seems self-evident but is ignored in classrooms around the world every day a million times.

Whilst this situation continues to be the case, the distance between the normal or usual classroom teaching and learning environment—for both teacher and students—and that necessitated by engagement in mathematical modelling increases the challenge of implementation by the teacher and successful solving by students.

13.1.1 Goal, or Purpose, of Mathematical Modelling

The goal of implementing modelling varies across the studies reported. Consistent with all chapters in this monograph, Ortega et al. take the stance that all mathematical modelling involves translating from reality to the mathematical world and back, "where reality is taken to be the rest of the world other than the mathematical domain" (Ortega et al. 2019, p. 162). Caron focusses on the need to live in the real-world as she argues strongly that if curricula were organised around *habits of mind* students, future professionals, and citizens in general would be better prepared for life in today's complex world. By habits of mind, she follows Cuoco et al.'s (1996) construct of "being comfortable with ill-posed and fuzzy problems … to look for and develop new ways of describing situations" (p. 373) and hence be more prepared to deal with decisions associated with problems in the world today.

Many have discussed the dual purposes of teaching "modelling as a vehicle" (Julie and Mudaly 2007, p. 503) to learn mathematics and *modelling as content* in its own right (e.g., Galbraith et al. 2010). Blomhøj argues that both are important and proposes ways to support teachers integrating modelling in secondary teaching practices. He noted teachers' difficulty was how to connect the students' modelling to understanding the mathematical knowledge in the curriculum. Czocher argues there has been an increased emphasis on mathematical modelling in curricula, and suggests the shift has been toward the modelling as content approach, although much emphasis is still on the former. Given that Julie (2002) noted "it is during the engagement with mathematical *modelling as content* that windows of opportunities are opened for dealing with relevance relevantly" (p. 8, emphasis added), this is a concern. Julie (2002) noted that teachers tend to prefer modelling as vehicle as the relevance to current teaching and learning mathematical content is more obvious. Sadly, development of learners as problem solvers and mathematical modellers seems less important. It appears teachers are still challenged in situations where different solution paths are followed by different students (Tan and Ang 2013). Teachers need to accept that real-world problems are likely to have multiple possible solutions and approaches to reaching these solutions (Blum 2015). Along these lines, Fulton et al. consider how communities of practice support teachers in being ready to respond to multiple student ideas.

Several authors focused on task development with Czocher noting that when this is part of a planned learning trajectory the task must have intended solutions which can be problematic when the intention is *modelling as content*. Ortega et al. took a *modelling as vehicle* approach with a teaching experiment focussed on learning about linear and quadratic functions. Whilst context was important, they found students tended to use prior knowledge when interpretation was required, rather than the functions they had found to mathematise the real phenomenon.

Caron describes multiple specific habits of mind students should be developing, including thinking of change analytically, thinking of systems in terms of flows, algorithmic and iterative ways of thinking, modelling interactions as inflows and outflows, and use of functions as building blocks for modelling. This is in contrast to modelling with functions via curve fitting and or regression which, she argues, along with Doerr et al. (2017) and Galbraith (2007), allow only a restricted understanding of the real-world situation being investigated. Brown (2015b) concurs noting "the enactment of multiple *Data Display-ability* simultaneously with multiple *Function View-ability* has the greatest potential in the model finding phase" (p. 437) and provides visual representations of both the data and model simultaneously, allowing modellers to keep the real and mathematical worlds at the forefront of their minds. The purpose of modelling, argues Caron, is not only to support students in understanding and integrating mathematical ideas but also "as a goal in itself of mathematics education" (Caron 2019, p. 83), that is, modelling as content.

Mathematical modelling plays an important role within social-critical research of mathematics education as a result of the relationship to the real world. The socio-critical perspective accounts for all participants' situations and backgrounds and aims to position learners as independent decision makers and critical users of information. Araújo's goal for mathematical modelling in her chapter is quite different. She presents initial steps toward a framework based on the notion of a mutually dependent dialectic relationship between practice and research. Relationships might be between practice and research, researcher and teacher, or student and research participant. Such a framework is a valuable contribution to socio-critical research specifically and modelling research more generally.

13.2 Theoretical Lines of Inquiry

In the opening chapter, Stillman (2019) discusses four theoretical lines of inquiry. The three local lines of inquiry, that is, those particular to mathematical modelling, are prescriptive modelling, modelling frameworks or modelling cycles, and modelling competence. The general line of inquiry discussed is anticipatory metacognition. Further research involving all three local lines of inquiry arose in this book and are discussed here. Anticipatory metacognition was not part of research reported and will be discussed along with other future lines of inquiry in the concluding section of this chapter.

13.2.1 Prescriptive Modelling

Meyer (1984) clearly defines mathematical modelling, models and mathematical models, with mathematical modelling being "an attempt to describe some part of the real world in mathematical terms…an endeavour as old as antiquity but as modern as tomorrow's newspaper" (p. 1). Meyer (1984) writes of different types of models, "a descriptive model, which tells how something works, and a prescriptive model, which tells the ideal way for it to work" (p. 60). He notes that prescriptive models are also known as optimisation or normative models. The difference is related to the purpose of use. A prescriptive model "is a tool for human decision making" (p. 61) whilst a descriptive model describes what is going on, and often "can be turned into a prescriptive one" (p. 61).

The example Meyer (1984) uses involves the manager of a retail store selling 20 soccer balls each day needing to know: How frequently, and what number of balls should be ordered from the supply factory. The descriptive model is presented in terms of an algebraic representation of a yearly cost function, $C(x)$, with variables, r, the rate the soccer balls are sold per day, s, the storage cost (of as yet unsold balls), k, the ordering cost (e.g. time of staff involved in ordering process), and x, the number of soccer balls per order (assuming each order is for an identical number of soccer

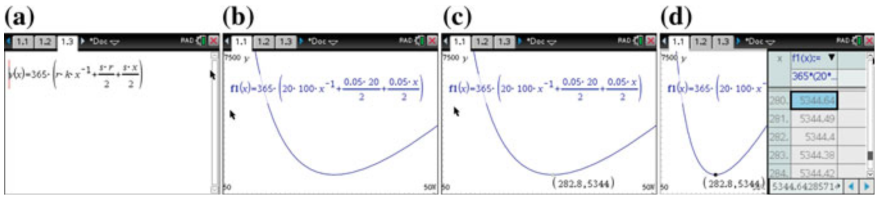

Fig. 13.1 **a** Descriptive model **b** specific model **c** mathematical 'best' **d** allowing actions

balls). The model can be used to find the cost per year for any combination of values of the variables. Thus, the model can be used to describe the yearly cost for any such situation (i.e. a descriptive model). In a given store, the first three variables may be assumed to be fixed hence a specific cost function can be found and represented, using several methods although a graphical representation should be the simplest. A visual inspection of the graph will show if the optimal value occurs at an endpoint of the domain or at a local turning point. The mathematical values then need to be interpreted as an integer number of balls and verified if that many can be both delivered and stored and thus is a solution to the real-world problem. The model is thus prescriptive as the task solver is expected to make recommendations as to what is best in terms of the number of soccer balls per order and frequency of orders. Figure 13.1 shows how digital technology can be used to (a) represent the general function or descriptive model, (b) find the algebraic and graphical representations of a situation given known parameter values, (c) find the mathematical best and (d) additionally use the graphical representation to begin to interpret what is best in the real world—this might include considering a range of possible values for the ball order, thus allowing other real-world considerations.

Davis (1991) argued we can distinguish at least three interrelated goals of applied mathematics, description, prediction, and prescription or "what *is*, what will *be*, what therefore to *do*" (p. 6). He elaborates with descriptions related to planetary motion and population predictions. Prescriptions, he argues require actions such as acting to stop smoking given statistical evidence or prescribing the fuel tank volume for a plane designed to fly non-stop from Copenhagen to Singapore. Davis is clear on the intertwining of these goals for modelling and our need to attend to all. Niss (2015) has reminded us of the need for an increased emphasis on prescriptive modelling (see Chap. 1).

Several examples of these types of modelling with their related purposes of use appear in the chapters. The *Yellowstone Game Task* (Caron), *Morning Shower and 100 metre Sprint (*Blomhøj) involved descriptive modelling as the intention was to describe the situation or context under investigation. Similarly, the *Water Usage Task* used by Fulton et al., aimed to describe how much water is used in making a pizza is descriptive. Czocher's *Letter Carrier* task could be described as prescriptive as the aim was to determine the best route for the mail deliverer to follow. One task used by Zubi et al. with grade five students involved prescriptive modelling as the task involves recommending the best three volleyball teams using 15 players. Stender's

Traffic Task required solvers to recommend whether to construct a roundabout or traffic lights at a given intersection and this is prescriptive.

Whilst no chapter authors referred specifically to descriptive versus prescriptive modelling, analysis of the tasks used and the purposes of their use in the research reported, shows both types were present and the majority, but not all, modelling tasks would be classified as involving descriptive models or descriptive modelling. This may be an artefact of the contexts investigated, the grade level of the modellers, the approach to modelling as vehicle more so than as content, limited previous modelling experience of the modellers, or a combination of these.

13.2.2 Modelling Frameworks and Modelling Cycles

Modelling cycles featured in the chapters of Blomhøj, Hankeln et al., Shahbari and Tabach, and Stender. These were used as an analytical tool by Shahbari and Tabach, and Blomhøj and as a structure for the research plan of Hankeln et al. and Stender. The modelling cycle acted as a structure for assessment tool development by Hankeln et al., in their research on modelling sub-competencies. Blomhøj reports that teachers use the modelling cycle as a tool for planning modelling activities. The modelling cycle was used as a structure for using heuristic strategies as strategic intervention in the study of Stender. Teachers in the study by Blomhøj used the modelling cycle as a means to analyse student work.

As an analytical tool, Shahbari and Tabach mapped pre- and post-intervention teacher observation reports of students engaged in modelling to the modelling cycle of Blum and Leiß (2005). The intervention saw the teachers work on four modelling tasks themselves. Post-intervention, teachers were more observant of modelling activity. There was however, a large number of future teachers still at the lowest of three classification levels, that is, failing to describe the majority of modelling activity undertaken by student modellers. More attention was given by future and in-service teachers to the final modelling cycle, perhaps, initially underestimating its critical role in students getting to the final solution.

13.2.3 Modelling Competence and Competencies

Building on work in the field (e.g. Kaiser and Brand 2015), Hankeln et al. focus on modelling competence, described by Blomhøj and Højgaard Jensen as "someone's insightful readiness to act in response to the challenges of a given situation (2007, p. 47). Hankeln et al. note that whilst their focus is on the sub-competencies, simplifying, mathematising, interpreting, and validating (Maaß 2006), "their mere existence is not sufficient" (Hankeln et al. 2019, p. 145). The focus of Hankeln et al. was to determine if these sub-competencies can be measured as separate dimensions or not.

Hankeln et al. present four sample tasks, one for each sub-competency, with a focus on geometric modelling. Analysis of each item is presented including selected incorrect responses and the success rate. Their research included 44 grade 9 classes and over 3000 completed tests. The thoroughness of this research is evident in the development and trialing of the items, compilation into test booklets, implementation and development of coding manuals for consistency of analysis. The findings by Hankeln et al. indicate that their statistical analysis shows it is possible to measure individual sub-competencies, at least with regards to geometric modelling situations relevant to grade 9 students.

13.3 Empirical Lines of Inquiry

In this section, following Stillman, lines of inquiry in the chapters that focus on the modeller, the task, and the teacher will be overviewed. Naturally, these are intertwined. A fourth and fifth line of inquiry, on the affordances of Technology-Rich Teaching and Learning Environments for modelling, and verification and validation concludes the section.

13.3.1 Focus on the Modeller

Fulton et al. note that in a country where modelling is rarely included in the primary school curriculum, teachers have an important role to play if modelling is to become integral to the mathematical learning of students. They recognise as challenging that primary mathematics teachers need support if modelling in primary schools is to become more widespread. Part of the support is of the *teacher as modeller* as Fulton at al. (and many others in the modelling community) opine that to teach modelling, one must first engage in modelling oneself.

13.3.1.1 Impact of Modelling on Learning

Zubi et al. focus on underachieving students in grade 5 in an Israeli school. They argue the role of the teacher, and expectations of students vary from classroom to classroom. In particular, in classes with perceived capable students, the expectation is on higher order thinking, but the converse is true in classrooms with low achieving students. Consequently Zubi et al. argue that low achievement is a result of the learning environment. Their study involved weekly modelling tasks, with the learning environment during the study contrasting with the norm. During the modelling sessions, students were expected to work in mixed ability groups on increasingly complex tasks. The chapter describes Sami, as typical of the low achieving students in this class, and his progress with regard to modelling competencies, mathematical knowledge, and participation in the group as they solved the tasks.

Teacher actions impacted on Sami's initial change of behaviour and motivation towards mathematics. Following the first task (where Sami decided to leave the mathematical work to more capable others), the teacher questioned the validity of the solutions. At this point, Sami, confidently and excitedly, shared that he had in fact proposed realistic considerations to his group but they ignored him. Subsequently all groups requested that they revisit their task solution. One can surmise, this was the first time Sami felt as if he had control over the direction of his own learning. Sami took an increasingly more active role in his group, from contributing realistic considerations, to actively organising his group, contributing mathematical ideas toward his group's model and finally to being dominant in organising his group, setting up a mathematical model and documenting the solution.

All low achieving students increased their mathematical knowledge of the content covered in the regular lessons during the study, whereas in a control class, taught by the same teacher, the low achieving students showed minimal development. The extended engagement of students with relevant real-world problems, that they were expected to solve in groups, and knowing that the solution was not predetermined by the teacher, allowed Sami and his peers, not only to develop modelling competencies and collaborative group work expertise, but also to view mathematics differently. It appears the expectations as to the role of the learner as modeller impacted on their engagement and motivation to learn in the regular classroom. Zubi et al. suggest that the introduction of, and student experience with, sustained modelling activity can also influence learning beyond modelling.

13.3.2 Focus on Teachers of Modelling

The teacher featured in the chapters by Caron, Czocher, Fulton et al., Ortega et al., Shahbari and Tabach, and Stender who all saw the teacher's role as critical. The focus included teacher knowledge about modelling (Fulton et al., Shahbari and Tabach), teacher knowledge of implementing modelling tasks (Fulton et al.), interactions between the teacher and modellers (Ortega et al., Stender) and expectations of teachers (Czocher).

Shahbari and Tabach recognised that teachers tend to lack knowledge about modelling, have limited experience in modelling, and teaching modelling. Similar to Zubi et al. and Fulton et al., the participants in the study of Shahbari and Tabach engaged in modelling activity themselves and undertook additional professional learning centred around watching and re-watching a video of a group of grade 6 students engaged in a modelling task. A second interaction with the video occurred after working on four modelling activities themselves. After each viewing of the video, the participants independently wrote a report of their observations. This activity enabled most teachers to become more alert to modelling occurring in student activity.

Ortega et al. note the important role the teacher plays in supporting student engagement in, and reflection on, decisions related to technology use and mathematical modelling and the interactions between these. Stender also focuses on the role of the (future) teacher, with the intention of providing minimal intervention, or adaptive intervention for students during modelling. These interventions aimed at providing minimal help to maximise student learning and problem solving. Intervention levels include motivational support, strategic help, and content related strategic help and should be used in this order—from least to most support. In addition, six heuristic problem-solving strategies that can be used, differentially, when solving a particular modelling task were also considered as appropriate strategic interventions.

Stender found the strategic intervention of asking student modellers to "explain the work already done" (Stender 2019, p. 209) was very powerful as it was easy to implement, provided opportunity to diagnose students' immediate needs, and provided an opportunity for student modellers to reflect on progress. Teacher interventions for substantive modelling tasks should be prepared, rather than be in-the-moment according to Stender. To do this, teachers need to solve the task prior to task implementation to identify potential barriers during the task as well as possible strategic interventions.

Fulton et al. had primary teachers engage in a week-long intensive professional learning program, participating in the process of mathematical modelling and then, reflecting from the perspectives of a student learning to model and as a teacher, teaching others to model, the first feeding forward to the second. Four features of modelling were in focus: the openness at all stages of the modelling cycle and grappling with this idea as a norm; posing problems, not just solving someone else's problem and understanding that modelling begins with the real-world context rather than the real-world problem; making choices about what mathematics to use in solving the problem as posed; and looking back at various stages of the solution to revisit ideas and consider revising the solution. Professional learning included teachers developing a modelling task, anticipating potential solution approaches; implementing the task with their students; and finally revisiting the task. Fulton et al. found that teachers engaging with modelling tasks themselves, resulted in the development of mathematical communities of practice which were supportive of subsequent collaborative task development and implementation. Caron also argued that collaboration between teachers could be productive, although she was suggesting cross-discipline collaboration, for example between mathematics and science teachers to design real-world problems.

Fulton et al. found the teachers in their study responded to the view of modelling as real-world problem solving and worked together in communities of practice to develop and implement mathematical modelling tasks addressing problems that mattered to students. Relevance, engagement, and access (i.e. tasks allowing all learners to participate in the task solution) were present in tasks designed and implemented by the teachers. The teacher participants clearly saw the power of mathematical modelling, and correctly believed their students could use mathematics to successfully solve relevant real-world problems. Moreover, modelling was seen as providing opportunities to promote mathematical thinking, encourage perseverance, and increase student engagement with mathematics. Most importantly, the solution path followed by the teacher must be bracketed when the task is implemented (see Blum 2015). Blum and Borromeo Ferri (2009) refer to the "teacher's own *favourite solution"* (p. 53) and note that, all too often teachers, consciously or not, direct students toward this solution.

Drawing on empirical findings, Blum (2015, p. 83) sees *individual solutions* as an element for teaching and learning mathematical modelling and applications. Unpacking what Blum means by this, it is not-as may appear on the surface-that students work individually, nor favour the teacher solutions. Rather, Blum is arguing for the need for the teacher to actively encourage multiple solutions to any given task and that this be considered the norm by students.

13.3.3 Focus on Modelling Task

Focusing on task development, Czocher noted that any solution depends on particular assumptions. Different assumptions may lead to a different focus and/or a different solution method which may not be what the teacher intended. A critical aspect of mathematical modelling is, that the modeller makes decisions, for example, considering some, but not all real-world aspects in one's initial solution, describing how to interpret terms such as 'best'. Such mathematical thinking naturally leads to diverse solutions, but the task must be presented in such a way as to allow this.

Blomhøj also focused on task design as he worked with teachers and teacher educators to support mathematical modelling in Danish secondary schools. His intention was to allow modelling as both vehicle and content and the contexts used were very familiar to the task solvers. For *the Yellowstone Game Task*, Canadian educators and mathematicians worked on modelling a real-life ecosystem (Caron 2019). Groups worked together to try to represent the situations and or solve the problem as to why a recent significant increase in the bear population had occurred. This task was seen as important as multiple paths and multiple solutions could be, and were, found.

Fulton et al. determined four features wrestling with openness in modelling, posing problems, making choices through a creative process, and revisiting ideas and solutions (Fulton et al. 2019), as critical to modelling activity and hence task design. The first of these relates to the reality of the messy real-world and the need for modellers to grapple with this in order to make sense of the situation and determine a possible way forward. Problem posing relates to the expectation that teachers and students should pose modelling problems. Making choices focused on the modeller needing to decide the solution path and what mathematics might be needed, noting all too often the mathematics expected to be used in primary school is explicitly conveyed to the learners, whereas the making choices focus was more likely to generate multiple paths and solutions. The final feature highlights that a first solution may not be a real solution to the problem, so modellers should not consider the first answer as meaning, problem solved.

Blomhøj presents two modelling situations suitable for secondary students. *The Morning Shower Task* sees students makes observations, collect data, and produce a poster communicating these. Digital technologies can be used to represent the data collected numerically and graphically to support communication of key ideas related to the mathematical function obtained and the real situation it represents. The emphasis of Blomhøj is certainly inclusive of the development of mathematical ideas (modelling as vehicle and as content). *The 100 m Sprint Task* saw students collecting data about a real-world situation and focused on understanding speed as the rate of change of distance over time. Digital technology use provided opportunities for multiple representations of the situation which could then be analysed and may lead to a deeper understanding of the mathematical concepts, or at least insight for students when these concepts are formally part of their mathematical learning.

Czocher used four tasks in her study of eight secondary school students and four university students, from several US states. Her tasks, the *Letter Carrier Problem,* the (human) *Cell Problem, the Water Lilies Problem,* and the *Empire State Building Problem,* ranged in degree of closeness to the real-world and level of complexity involved. Authenticity in Czocher's study, using actor-orientated theory, is based on the degree of alignment between task context and the task solver's lived experience. Consequently, task solvers are expected to use their own knowledge and make assumptions when solving a modelling task.

There was a strong emphasis on tasks allowing multiple pathways and solutions. Fulton et al. provided explicit criteria for task selection and design as appropriate to mathematical modelling. In contrast, Blomhøj saw any real-world context as providing opportunities to engage with both the real-world and the mathematical world, to develop understanding of both and the links between them. Czocher took an alternative approach as she tried to ascertain what solution was in the mind of the task setter and contrasted this with actual student solutions.

13.3.4 Affordances of Technology-Rich Teaching and Learning Environments

It is well known that digital technologies play an important role in mathematical modelling. Galbraith et al. (2007) described the "use of technology as central…and its integration with mathematics within the modelling process as creating essential challenges about which we need to know much more" (p. 130). This approach to digital technology use when modelling is taken up by Ortega et al. They acknowledge that use of digital technology to model provides opportunities to transform understanding. However, affordances of the environment including the technologies need to be perceived and acted upon (Brown 2015a) for this to occur.

Use of technology, its potential and ubiquitous nature should impact on the complexity of real-world contexts and modelling tasks explored by students of today. For several types of digital technology discussed, Caron highlights both the affordances and obstacles identified. By affordances she is following Gibson (1979) who made up the term to describe “the complementarity of the animal [i.e. the human] and the environment” (p. 127) and the definition used in Brown’s research (2015a) on affordances in technology-rich teaching and learning environments as “the opportunity for interactivity between the user (the actor) and the technology (the object or the artefact) for some specific purpose” (p. 113) (see also Brown and Stillman 2014; Frejd and Ärlebäck 2017).

In the study reported by Ortega et al. the focus turns to capturing data using technological devices and identifying how decisions made by grade 11 students during mathematisation affect interpretation. Students in this study used multiple digital technologies. One class investigated the phenomenon of a bouncing ball using an iPad and an application allowing the motion to be video recorded and graphed after the user set the scale and origin of a coordinate system. The data collected (i.e. points on the image) are specified by the user. The data were exported to a second app allowing coordinate pairs to be plotted, regression analysis undertaken, and the subsequent function model graphed simultaneously with the data plot. With regard to the regression model, students were able to test multiple function types and ascertain which model best fitted the data.

A second class investigated the phenomenon of a spring’s motion as marbles were added to a cup hanging from the spring. Again, video was captured using the iPad but then the data were exported to an app that did not perform regression analysis. In this class, students had to make additional decisions, regarding the type of function to use and subsequently to determine parameter values of that function. Post experiment, students in both classes used a graphing calculator app as desired to answer interpretation and validation questions. Galbraith (2007) would certainly see the use of technology in the second class as going “beyond the low hanging fruit” (p. 79) of “modelling as curve fitting” (p. 81) as students were expected to keep in mind the real world and its relationship to the mathematical model.

Students in the study by Ortega et al. did not perceive the affordances of the environment with regard to *reference-point set-ability* or understand the impact this has on the data collected, model determined and interpretation of that model and its outputs. However, given there is no evidence the students had previously used the technologies involved nor engaged in modelling, this is hardly surprising. Of course, the conception that a function’s parameters are explicitly related to the location of the origin and scale factor of the imposed axial system critical in modelling is also an important aspect of pure mathematics that upper secondary students should be aware of.

Caron explored uses of system dynamics software (e.g. *Stella*) allowing the development of experimenting, tinkering, and qualitative analysis of problem situations such as the spread of viruses and social policies. Depending on the level of mathematics of the student modellers, *Stella* can be perceived as black box technology if the software uses mathematical analysis techniques, such as integration, not yet understood by the students. However, affordances here include *experiment-ability* and *tinker-ability* as student modellers with technology have the opportunity to develop or engage in these ways of mathematical thinking about complex situations. Both could be seen as part of what a mathematician might do before model generation.

Caron also explored cellular automata, that is a digital array of cells where the behaviour of an individual cell is determined by the cells surrounding it and their state in the previous generation. A local focus at the individual cell level over time allows insight into the global system behaviour. Such discrete models of dynamic systems can model situations including wildfires and spread of infections. The technology could involve a spreadsheet or online simulator. The affordances here are *local behaviour predict-ability* and *global behaviour predict-ability* with recursion being a key mathematical idea that could be introduced and understood as student modellers explore the way such systems evolve. Greefrath and Siller (2017) recommend "the uses of simulations that naturally link modelling with the use of digital tools" (p. 537). See also Frejd and Ärlebäck (2017) who used simulation to investigate a pandemic.

Agent-based models (e.g. *NetLogo*) are described by Caron as allowing modellers to investigate behaviour of an individual in a system and of the system itself. The behaviour of a nesting pelican and the colony it is a part of would be typical examples of this phenomenon. Predator-prey models can be utilised to ask and answer questions related to each of the necessary elements of the system. The affordances of such an environment allow modellers to engage in reasoning at the agent-based (*individual-within-system behaviour reason-ability*) and aggregate level (*system behaviour reason-ability*). The latter focusses on the rate of change of the populations within the system (Jacobson and Wilensky 2006).

There is no doubt the use of technologies in mathematical modelling results in higher order thinking needing to be undertaken by students with multiple decisions being made. From a modelling perspective, the more decisions made by students, the more mathematical thinking they engage in, and the greater their connection is to the real-world situation they are using mathematics to explore. As Ortega et al. conclude, students need more experiences in decision making in mathematical modelling and technology use and critically the interactions between these, as discussed previously by Galbraith et al. (2007).

13.3.5 Verification and Validation

Verification and validation are an area of modelling that needs to receive further empirical inquiry and attention (Czocher et al. 2018). In the study by Zubi et al. it was the teacher who initially questioned the validity of the students' solutions. None of the grade 5 student groups had taken the real-world into account as they proposed solutions to their first of a series of modelling tasks. This questioning by the teacher, and subsequent class discussion was the catalyst for change. From that point on, students increasingly valued the real-world as they searched for more authentic solutions.

Model interpretation and validation were also a focus of Ortega et al. They found students had little experience in interpreting mathematics results in terms of reality (e.g. a negative height, or height of zero, predicted in the ball dropping experiment). In the spring experiment, students failed to notice, or account for the spring length changing. Ortega et al. found that rather than learning from observation of real data, students tended to revert to prior understandings even when these were inconsistent with their experimental activity (e.g. not setting the ground as a reference point but assuming this was the case when interpreting their model). In addition, students seemed not to be aware that the sign of a 'distance' is related to how and where it is measured. This resonates with the students in Czocher's study who, rather than make simplifying assumptions, maintained complexity of the situation under investigation. Both results suggest the need for increased student experiences with mathematical modelling and all the mathematical activity contained therein (e.g. simplifying, mathematising, interpreting, validating, verifying).

Caron also explored the idea that as solutions to real-world problems, and the mathematical techniques used to solve these, often involve approximations, both verification and validation are critical. Caron draws on the work of Roache (1998) to describe verification as ensuring "that the error has been controlled and that the equations have been solved correctly" (Caron 2019, p. 97) whereas validation relates to "external consistency, that is, ensuring the model and its associated solution adequately represent the situation". Caron argues that verification and validation are not typically part of mathematics teacher experience or expertise and this needs to be addressed.

Hankeln et al. in their research identify *Validating* as a modelling sub-competency. Their use of the term includes competencies for verifying a solution, critically reflecting on that solution and assumptions specified and the model selected. They argue this was the most difficult sub-competency to construct items to assess. The simplicity of these items belies the difficulty the researchers had in developing them but needs to be noted if using in teacher professional learning. Teachers in the study of Shahbari and Tabach gave the least amount of attention to validating processes when observing and interpreting students' modelling activity. These authors suggest this lack of attention to validating is, following Blum and Borromeo Ferri (2009), a result of this mathematical activity more typically being undertaken by the teacher in the normal classroom environment.

13.4 Future Lines of Inquiry

Future lines of inquiry for research in modelling, arising from the foregoing chapters, include more research with experienced modellers, the impact of teachers positioning themselves as modellers, strategic interventions by teachers during modelling, subcompetencies, anticipatory metacognition, and verification and validation.

Students in the study by Ortega et al. had no previous modelling experience, and other than in Germany (i.e. in the work by Hankeln et al. and Stender) this was typical. It is clear we need more research with students experienced in modelling as they engage with modelling tasks. Of course, this is not possible if curricula and or teachers do not value modelling and hence students are not engaged in mathematical modelling regularly.

One continuing line of inquiry should focus on the impact of teachers solving modelling tasks themselves before implementing these. Concurrent with undertaking this modelling, teachers must consider what blockages, difficulties, and challenges (Stillman et al. 2010) students might face when solving the task themselves. Interventions can be planned specific to the task, particularly following Stender, those intended to keep the student modellers doing all or most of the modelling and mathematical work. Following Vygotsky (1978), the interventions should relate to the Zone of Proximal Development of the task solver(s) and be strategic. In the first instance, if the intention is, as described by Blum and Borromeo Ferri (2009), to minimise teacher input and maximise student independence, teachers should consider strategic interventions "which give hints to students on a meta-level" (p. 52). This would include teacher responses such as: Can you imagine the situation? What is your aim? What else do you need to know? What does this (interim) result mean in terms of the real situation?

The work of Hankeln et al. regarding the possibilities of measuring individual sub-competencies should certainly be extended by themselves and other researchers to include non-geometric modelling situations and other grade levels of students. It would be beneficial to see studies of these same students engaged in complete modelling tasks as well.

Research is clearly needed with respect to teaching and learning approaches focused on issues related to students' mathematising successfully with both novice and experienced modellers. In 2010, Niss proposed the construct 'implemented anticipation' theorising as to the cognitive and metacognitive processes whereby student modellers foreshadow what might be useful mathematically in progressing a given problem, making decisions and implementing actions to bring what was anticipated to fruition. This is central to students being able to model (Stillman et al. 2015).

Subsequently, Stillman and Brown (2012) have found evidence of two aspects of anticipated implementation from classroom data. Furthermore, unsuccessful attempts at mathematisations were related to student's inability to use relevant mathematical knowledge in the modelling context rather than lack of mathematical knowledge, an application-oriented view of mathematics or persistence. In a following analysis modelling attempts when students were participating in an extra-curricular modelling event, Stillman and Brown (2014) found evidence of further aspects of implemented anticipation and that, again, unsuccessful modelling attempts could be explained using Niss' enablers of successful mathematisation. They suggest deliberately scaffolding the process of implemented anticipation as a "means of gaining a resolution of the long-standing issues of problem formulation and specification and their successful mathematisation" (Stillman et al. 2015). There is now a current research project investigating this further (see Geiger et al. 2018) with Year 10–11 students. This is most definitely a fruitful line of inquiry for others as well.

Finally, verification and validation should be considered as an important line of inquiry for future research. Clearly, both are important in modelling, but challenging to implement, teach, and research. They also need to be clearly defined (see Czocher et al. 2018). This is not currently the case, as the terms are often used without definition or interchangeably. Further work along this line of inquiry is recommended.

13.5 Conclusion

In conclusion, this book has indeed presented a broad spectrum of valuable research in the field of mathematical modelling and applications in education through extended contributions by a small selection of presenters at ICME-13 in Topic Study Group 21: Mathematical Applications and Modelling in the Teaching and Learning of Mathematics. Issues related to mathematical applications and modelling in the teaching and learning of mathematics have continued to grow in interest from previous International Congresses on Mathematical Education. This is a very broad field both in terms of educational level range, from elementary school to tertiary education, and from the perspective of mathematical content and processes involved. The Topic Study Group thus attracted and catered for a breadth of participants through the plenaries and individual talks which addressed several theoretical issues and/or reported on diverse empirical studies. To unify this diversity, 15 authors or groups of authors were selected by the editors of this book and invited to start with their presentation and extend into a chapter but to link to the overarching theme of *Lines of Inquiry in Mathematical Modelling Research in Education* as had been elaborated and exemplified in the opening plenary by Stillman. Twelve chapters remained after the extensive review process.

The chapters covered a wide variety of educational levels from elementary and primary school students (Fulton et al., Zubi et al.) to secondary (Blomhøj, Hankeln et al., Ortega et al.), and tertiary students (Araújo, Caron, Czocher) as well as pre-service and in-service development of their teachers (Blomhøj, Fulton et al., Shahbari and Tabach, Stender). Research on the teaching and learning of modelling provides a theoretical basis (e.g. conceptualisation of modelling competencies: Hankeln et al.) for the design and investigation of many different ways of implementing and organizing mathematical modelling in classrooms across these levels with the aim of developing student modelling competencies and, or, to support student learning of mathematics. The latter purpose is also informed by more general research in mathematics education (e.g. learning difficulties in conceptual and procedural development or learning trajectories for particular mathematical concepts).

The chapters in this book contribute to several lines of inquiry in researching or theorising with respect to teaching, learning, and assessing of modelling. Caron explored the approaches of mathematicians, mathematics teachers, and mathematics educators to solving complex dynamical systems (e.g. ecological systems) with a view to introducing such systems in school and university mathematics programs. Czocher analysed solutions of secondary and tertiary students to compare their solutions with that intended by the task setter. She found students were reluctant to simplify the situation as they saw this as creating a less authentic problem, not realising this is a critical component of task solving. She argues strongly that solutions other than the solution intended by the task setter in providing opportunities to address curriculum objectives, or those using mathematics not matching the curriculum being taught, must not be considered incorrect. Acknowledging the challenge of implementing modelling tasks at primary school, Fulton et al. investigated how this promoted meaningful task development by teachers and meaningful mathematical discourse by students. Hankeln et al. designed test items (multiple choice and short answer) for grade 9 geometric modelling ideas and showed these can be used to separately assess the modelling sub-competencies of simplifying, mathematising, interpreting, and validating. Ortega et al. explored how the available digital tools influenced grade 11 students' mathematisations when solving two functions-based tasks modelling physical phenomenon taking the stance of modelling as vehicle to enhance understanding about functions. Shahbari and Tabach investigated the impact of pre- and in-service teachers' engagement with modelling tasks themselves on their capacity to notice the complexity of modelling occurring when observing students engaged in modelling activity. Zubi et al. explored how a focus on development of modelling competencies by low achieving students led to improved mathematical understanding outside that focused on in the modelling tasks. This increase in mathematical understanding was, at least in part, a result of the changed expectations the young learners developed as active participants in the learning process.

In many cases researcher and pedagogical practice take place concurrently particularly if the researcher is also the teacher in the research study. Araújo's chapter addresses the dual role of researcher and educator in this setting from a socio-critical perspective in order to present her initial steps towards a framework for a dialectical relationship between pedagogical practice and research. Although clearly of application in mathematics education and educational research more generally, such a framework is particularly pertinent to mathematical modelling educational research given that mathematical modelling strongly depends on the situation of the learner which is not always the case in other parts of mathematics education. Other chapters addressed more general issues that inform the teaching and learning of mathematics through mathematical applications and modelling such as meanings of key terms such as context, task context, and real-world used by researchers in journal publications (Brown), and intervention strategies when managing modelling by others (Stender). Blomhøj argues that to ensure modelling and applications are fully integrated into secondary mathematics classrooms, modelling must be seen and understood as a didactical means for supporting students' learning of mathematics not just to develop students' modelling competency. In order to do this he makes the case that there is a need for the development of tools that allow teachers to make better use of theories of learning of mathematical concepts and develop the pedagogical foresight to view modelling activities in this way.

Most lines of inquiry explored in the research represented here need further research as has been discussed in this chapter. Other major areas, not the subject of chapters in the book, include metacognition and modelling, affect and modelling, and the relationship of mathematical literacy to modelling. Each of these should be the subject of future research lines of inquiry.

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Refereeing Process

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