János Pintz

Some new results on gaps between consecutive primes

Abstract: The paper presents a brief history of results about small gaps between consecutive primes and mentions some recent results of the author, in some cases only with a sketch of the main ideas, in some cases with detailed proofs.

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1 Approximations to the twin prime conjecture before 2000

In the present work we will give a survey about some new results concerning small gaps between consecutive primes. In the following let $\mathcal{P}$ denote the set of the primes, p_n the n^{th} prime and d_n the n^{th} difference between consecutive primes, that is,

$$d_n = p_{n+1} - p_n. \tag{1.1}$$

The famous Twin Prime Conjecture,

$$d_n = 2 \quad \text{infinitely often,} \tag{1.2}$$

would answer the most important problem about the smallest values of d_n which occurs infinitely often. However, we will investigate also various conjectures and results which do not follow from (1.2).

We will also discuss the well-known generalization of (1.1), formulated in 1849 by de Polignac [18]:

$$d_n = 2k \quad \text{infinitely often for any } k \in \mathbb{Z}^+. \tag{1.3}$$

We will call an even number $2k$ a Polignac number if $d_n = 2k$ infinitely often.

Polignac's conjecture (1.2) can be formulated with this definition as

Polignac Conjecture. *Every even number is a Polignac number.*

The most important weaker form of the Twin Prime Conjecture and the Polignac conjecture is the

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Bounded Gap Conjecture. *We have* $\liminf_{n\to\infty} d_n < \infty$, *or, equivalently, there is at least one Polignac number.*

If we are looking for small gaps between consecutive primes, then the starting point is that the Prime Number Theorem,

$$\pi(x) \sim \frac{x}{\log x} \sim \int_2^x \frac{dt}{\log t}, \tag{1.4}$$

implies (in fact, it is even equivalent to)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{d_n}{\log n} = 1. \tag{1.5}$$

Therefore we have

$$\Delta_1 := \liminf_{n\to\infty} \frac{d_n}{\log n} \le 1. \tag{1.6}$$

On the other hand, any estimate of type

$$\Delta_1 < 1 \tag{1.7}$$

is already non-trivial.

Equation (1.7) was proved first under the assumption of the Generalized Riemann Hypothesis (GRH) in an unpublished work of Hardy and Littlewood in 1926 (see [19]) providing the conditional estimate

$$\Delta_1 \le \frac{2}{3} \quad \text{under GRH.} \tag{1.8}$$

The first unconditional estimate was shown only in 1940 by Paul Erdős [5], who proved

$$\Delta_1 \le c_1 < 1, \tag{1.9}$$

with an effective but unspecified value $c_1 < 1$. After various refinements of (1.9), a large step was done by the unconditional estimate

$$\Delta_1 < 0.467 \tag{1.10}$$

of Bombieri and Davenport [2] in 1966 based on the large sieve of Bombieri [1]. After several smaller sharpenings, Helmut Maier improved (1.10) in 1988 to

$$\Delta_1 < 0.2486 \tag{1.11}$$

using his famous matrix method [16].

The mentioned results before 2000 left open the question whether $\Delta_1 = 0$, which can be formulated as the

Small Gap Conjecture. $\Delta_1 = 0$.

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2 Distribution of values of $d_n/\log n$

The Small Gap Conjecture is equivalent to the fact that the number zero belongs to the set of limit points of the sequence $\{d_n/\log n\}_{n=1}^{\infty}$. In the following, let J denote this set, i.e.

$$J = \left\{\frac{d_n}{\log n}\right\}'. \tag{2.1}$$

Erdős [6] and independently Ricci [20] proved that J has positive Lebesgue measure and Erdős formulated the conjecture that every nonnegative number is a limit point of the sequence $\{d_n/\log n\}$, that is,

$$J = [0, \infty). \tag{2.2}$$

However, the paradoxical situation was that no finite limit point of J was known. (The result $\infty \in J$ was already known by the result of Westzynthius [23].)

We would like to remark here that neither the Twin Prime Conjecture nor the Polignac Conjecture would imply any limit point of the sequence $d_n/\log n$ beyond the single number zero.

3 Results between 2000 and 2010

The Small Gap Conjecture, equivalently $0 \in J$, was shown finally in a joint work with D. A. Goldston and C. Y. Yıldırım in 2005.

Theorem A ([9], [10]). $\liminf\limits_{n\to\infty} \dfrac{d_n}{\log n} = 0.$

Somewhat later we showed the above in a much stronger quantitative form as

Theorem B ([11]). $\liminf\limits_{n\to\infty} \dfrac{d_n}{(\log n)^{1/2}(\log\log n)^2} < \infty.$

The method of proof of Theorem A provided a surprisingly strong estimate concerning small values of d_n, the truth of the already mentioned Bounded Gap Conjecture, if we suppose that primes are sufficiently uniformly distributed in arithmetic progressions. In order to formulate the result we need the following

Definition. Primes have an admissible distribution level ϑ if

$$\sum_{q \le X^{\vartheta-\varepsilon}} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{p \le X \\ p \equiv a (\mathrm{mod}\, q)}} \log p - \frac{X}{\varphi(q)} \right| \ll_{A,\varepsilon} \frac{X}{(\log X)^A} \tag{3.1}$$

for any constants $A > 0$, $\varepsilon > 0$ and any $X > 2$.

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Concerning known results about admissible distribution levels ϑ for the primes, the strongest result is due to Bombieri [1] and A. I. Vinogradov [22], the celebrated Bombieri–Vinogradov Theorem. This asserts that the value $\vartheta = 1/2$ is an admissible distribution level for the primes.

The strongest possible conjecture about the maximal distribution level of the primes was expressed by Elliott and Halberstam ([4]) right after the proof of the Bombieri–Vinogradov theorem.

Elliott–Halberstam Conjecture (EH). *$\vartheta = 1$ is an admissible distribution level for the primes.*

Our previously mentioned result can be formulated as follows:

Theorem C ([10]). *If there is an admissible level $\vartheta > 1/2$ of the distribution of primes, then*

$$\liminf_{n\to\infty} d_n \le C(\vartheta) < \infty. \tag{3.2}$$

In particular, if the Elliott–Halberstam conjecture is true, then

$$\liminf_{n\to\infty} d_n \le 16. \tag{3.3}$$

Theorem A implies for any $\varepsilon > 0$ an infinitude of prime gaps satisfying

$$d_n < \varepsilon \log n. \tag{3.4}$$

However, it left open the interesting problem whether (3.4) occurs with a positive frequency for any given $\varepsilon > 0$. Goldston and Yıldırım [13] showed that this is true if $\varepsilon > 1/4$. We have to mention that Maier's method [16], leading to a slightly smaller gap of size (1.11), did not yield small gaps with a positive frequency (unlike the result (1.10) of Bombieri–Davenport [2], which in fact, did).

4 Recent results

The aim of the present section is to give a survey about the most recent results in this area. The first result is a sharpening of the hitherto best estimate of Theorem B.

Theorem 4.1. $\liminf\limits_{n\to\infty} \dfrac{d_n}{(\log n)^{3/7}(\log\log n)^{4/7}} < \infty.$

The next one is a common generalization of Theorem C and the celebrated result of Green and Tao [14] according to which there are arbitrarily long (finite) arithmetic progressions in the sequence of primes. We need the same conditions as in Theorem C but the consequence is stronger.

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Theorem 4.2 ([17]). *For every $\vartheta \in (1/2, 1]$ there is an integer $C(\vartheta)$ and a positive even number $d \leq C(\vartheta)$ with the following property. If the primes have an admissible level ϑ of distribution, then for any positive integer k we have a k-term arithmetic progression of primes $\{p_i^*\}_{i=1}^k$ such that $p_i^* + d$ is the prime following p_i^* for all elements of the progression. If the Elliott–Halberstam conjecture is true, i.e. $\vartheta = 1$, or, at least $\vartheta > 0.971$, then the above is true with a $d \leq 16$.*

As we mentioned earlier, the existence of an admissible level $\vartheta > 1/2$ for the distribution of primes implies the Bounded Gap Conjecture (cf. Theorem C), consequently the existence of at least one Polignac number. However, we can prove more under the same condition, namely

Theorem 4.3 ([17]). *If $\vartheta > 1/2$ is an admissible level of distribution of the primes, then Polignac numbers have a positive lower density at least $c'(\vartheta)$.*

The same assumption $\vartheta > 1/2$ gives also further information about the limit points J of the sequence $\{d_n/\log n\}_{n=1}^{\infty}$. Thus, under the above deep hypothesis we obtain further progress in the problem of Erdős beyond the fact $0 \in J$ (following from Theorem A unconditionally).

Theorem 4.4. *If $\vartheta > 1/2$ is an admissible level of distribution of the primes, then we have a number $C_1(\vartheta)$ (depending in an* ineffective *way on ϑ) such that*

$$[0, C_1(\vartheta)] \subset J. \tag{4.1}$$

Concerning the frequency of small gaps, i.e. the gaps satisfying

$$d_n < \varepsilon \log p_n \tag{4.2}$$

for a small but fixed $\varepsilon > 0$, we could show in a joint work with D. Goldston and C. Yıldırım the following

Theorem 4.5 ([12]). *The sequence S_ε of indices satisfying* (4.2) *has a positive lower density for any $\varepsilon > 0$.*

5 Main ideas of the proofs of Theorems A, B, C

The first step in proving Theorems A–C is to investigate the following far-reaching generalization of the Twin Prime Conjecture. This was formulated more than a hundred years ago in a qualitative form by Dickson [3] and nearly two decades later (probably independently) in a quantitative form by Hardy and Littlewood [15]. In the formulation of the conjecture we shall call a k-tuple $\mathcal{H} = \{h_i\}_{i=1}^k$ of different nonnegative integers *admissible* if for any prime p $\mathcal{H}$ does not cover all residue classes mod p.

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Conjecture DHL. *If $\mathcal{H}$ is admissible, then there are infinitely many values n such that all $n + h_i$ $(i = 1, 2, \ldots, k)$ are simultaneously primes.*

The next step is the formulation of a weaker form of the above DHL Conjecture (which is weaker than Polignac's conjecture and it is equivalent to DHL in the simplest case $k = 2$).

Conjecture DHL $(k, 2)$. *If $\mathcal{H}$ is an admissible k-tuple, then there are $i, j \in \{1, 2, \ldots, k\}$ such that for infinitely many values n we have $n + h_i \in \mathcal{P}$, $n + h_j \in \mathcal{P}$.*

We remark that the truth of the conjecture DHL $(k, 2)$ for any particular k implies the Bounded Gap Conjecture.

The strategy is to attack the above Conjecture DHL $(k, 2)$ for some large values of k. We will miss it (thereby also the Bounded Gap Conjecture) just by a hairbreadth. However, this will help us to prove Theorem A and will immediately furnish a proof of the conditional Theorem C.

Let us consider the product

$$\mathcal{P}_{\mathcal{H}}(n) = \prod_{i=1}^{k}(n + h_i). \tag{5.1}$$

If we try to attack the original very deep conjecture DHL for any given $k \in \mathbb{Z}^+$ it is plausible to use the weights of Selberg's sieve,

$$a_n = \left\{\frac{1}{k!} \sum_{\substack{d \mid \mathcal{P}_{\mathcal{H}}(n) \\ d \le R}} \mu(d) \left(\log \frac{R}{d}\right)^k\right\}^2 \quad n \in [N, 2N) \tag{5.2}$$

with some $R \le N^{1/2}$.

In fact, we can evaluate the sum of these weights for $n \in [N, 2N)$ and obtain for $R \le N^{1/2}(\log N)^{-c(k)}$ the asymptotic

$$A_N := \sum_{n \in [N, 2N)} a_n \sim \frac{N}{k!}\mathfrak{S}(\mathcal{H})(\log R)^k, \tag{5.3}$$

where $\nu_p = \nu_p(\mathcal{H})$ denotes the number of different residue classes covered by $\mathcal{H}$ mod p and

$$\mathfrak{S}(\mathcal{H}) := \prod_p \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}. \tag{5.4}$$

We note that $\mathfrak{S}(\mathcal{H}) > 0$ if and only if $\mathcal{H}$ is admissible.

Let us denote by $\chi_{\mathcal{P}}(n)$ the characteristic function of primes, i.e.

$$\chi_{\mathcal{P}}(n) = 1 \text{ if } n \in \mathcal{P}, \quad \chi_{\mathcal{P}}(n) = 0 \text{ if } n \notin \mathcal{P}. \tag{5.5}$$

Then we can investigate the frequency of those $n \sim N$ for which $n + h_i \in \mathcal{P}$, i.e. $\chi_{\mathcal{P}}(n + h_i) = 1$ for any given $h_i \in \mathcal{H}$, or more generally of those n for which $n + h \in \mathcal{P}$ with some $h \notin \mathcal{H}$.

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It turns out that in order to be able to evaluate the sum

$$B_N(h) := \sum_{n\in[N,2N)} a_n \chi_{\mathcal{P}}(n+h) \tag{5.6}$$

we need a much stronger condition than $R \leq N^{1/2}(\log N)^{-c(k)}$, namely $R \leq N^{(\vartheta-\varepsilon)/2}$ (which is only in the hypothetical case $\vartheta = 1$, i.e. in case of the truth of the Elliott–Halberstam Conjecture near to the bound $N^{1/2}(\log N)^{-c(k)}$, which is the barrier for the evaluation of A_1 in (5.3)).

If the evaluation of the quantities A_N and $B_N(h)$ is successful, we can investigate the average weighted number of primes among $\{n+h_i\}_{i=1}^k$ if n runs through the interval $[N, 2N)$.

In case of $h_i \in \mathcal{H}$ we obtain

$$B_N(h_i) \sim \frac{2N}{(k+1)!}\mathfrak{S}(\mathcal{H})\frac{(\log R)^{k+1}}{\log N} \sim \frac{2A_N}{k+1}\cdot\frac{\log R}{\log N}. \tag{5.7}$$

Hence, if we choose the maximal possible R, namely, we set

$$R = N^{(\vartheta-\varepsilon)/2}, \tag{5.8}$$

then we obtain

$$\frac{B_N(h_i)}{A_N} \sim \frac{\vartheta-\varepsilon}{k+1}, \qquad \frac{\sum_{i=1}^{k} B_N(h_i)}{A_N} \sim \frac{k}{k+1}(\vartheta-\varepsilon) \tag{5.9}$$

which is unfortunately less than 1 even under the assumption of the Elliott–Halberstam conjecture and unconditionally it is even less than $1/2$.

However, if we consider the weaker conjecture DHL $(k, 2)$, we have no heuristic reason to consider only the k-dimensional Selberg's sieve, which is a truncated version of

$$\Lambda_k(\mathcal{P}_{\mathcal{H}}(n)) = \frac{1}{k!}\sum_{d\mid\mathcal{P}_{\mathcal{H}}(n)} \mu(d)\left(\log\frac{\mathcal{P}_{\mathcal{H}}(n)}{d}\right)^k \tag{5.10}$$

which approximates the situation

$$n+h_i \in \mathcal{P}, \qquad i = 1, 2, \dots, k. \tag{5.11}$$

This observation opens the door towards the introduction of other, higher dimensional sieves, or, equivalently, to the introduction of a larger class of possible weights depending on a new parameter $\ell \geq 0$. Thus, let us consider the new weights in the form

$$a(n, R, \mathcal{H}_k, \ell) := \left(\frac{1}{(k+\ell)!}\sum_{\substack{d\leq R\\ d\mid P_{\mathcal{H}}(n)}} \mu(d)\log^{k+\ell}\frac{R}{d}\right)^2 \tag{5.12}$$

and the analogous quantities $A_N(\ell)$, $B_N(h, \ell)$ instead of (5.3), (5.6), corresponding to the case $\ell = 0$.

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The new quantities can be evaluated too, if k and ℓ are fixed, $N \to \infty$ and $R = N^{(\vartheta-\varepsilon)/2}$. (It is sufficient to suppose $R \le N^{(\vartheta-\varepsilon)/2}$ but larger values of R yield better results.) However, surprisingly enough, we obtain nearly twice as many primes in a k-tuple $n + \mathcal{H}$ on average if n runs through the interval $[N, 2N)$ under the simple condition that ℓ and k/ℓ are both simultaneously large (for example $\ell = [\sqrt{k}]$ and k is large). We obtain, namely, in case of $h_i \in \mathcal{H}$ in place of (5.3), (5.7) and (5.9)

$$A_N(\ell) \sim \frac{N}{(k+2\ell)!}\binom{2\ell}{\ell}\mathfrak{S}(\mathcal{H}_k)(\log R)^{k+2\ell}, \tag{5.13}$$

$$B_N(h_i, \ell) \sim \frac{N}{(k+2\ell+1)!}\binom{2\ell+2}{\ell+1}\mathfrak{S}(\mathcal{H}_k)\frac{\log R}{\log N}(\log R)^{k+2\ell}. \tag{5.14}$$

Consequently we obtain for any fixed $\varepsilon > 0$ instead of (5.9) the better asymptotics

$$\frac{B_N(h_i, \ell)}{A_N(\ell)} \sim \frac{2(\vartheta-\varepsilon)}{k+2\ell+1}\left(1 - \frac{1}{2\ell+2}\right), \tag{5.15}$$

$$\frac{\sum_{i=1}^{k} B_N(h_i, \ell)}{A_N(\ell)} \sim 2(\vartheta-\varepsilon)\left(1 - \frac{2\ell+1}{k+2\ell+1}\right)\left(1 - \frac{1}{2\ell+2}\right) > 2(\vartheta - 2\varepsilon) \tag{5.16}$$

if $k > C\varepsilon^{-2}$, $\ell = [\sqrt{k/2}]$, for example.

The result (5.16) immediately yields that choosing the weighted average implied by the weights in (5.12) we obtain on average for $n \in [N, 2N)$ more than one prime among $\{n + h_i\}_{i=1}^{k}$ if we suppose the crucial condition

$$\vartheta > 1/2 \tag{5.17}$$

which implies Theorem C or more generally DHL $(k, 2)$ for $k > C_0(\vartheta)$.

However, in the unconditional case we have only $\vartheta = 1/2$ by the Bombieri–Vinogradov theorem. Hence we obtain, say, more than

$$1 - 4\varepsilon \tag{5.18}$$

primes on average for $n \in [N, 2N)$ among $\{n + h_i\}_{i=1}^{k}$, so we fail by a hairbreadth to prove the Bounded Gap Conjecture (or Conjecture DHL $(k, 2)$ for some large bounded values of k).

On the other hand, we have still the possibility to count the weighted number of primes in a short interval

$$h \in [1, H], \quad H = \eta \log N, \quad h \notin \mathcal{H}_i, \tag{5.19}$$

where η is a small positive constant. An argument, similar to that yielding (5.13)–(5.14) gives in this case for any admissible k-tuple $\mathcal{H}_k$

$$B_N(h, \ell) \sim \frac{A_N(\ell)}{\log N}\frac{\mathfrak{S}(\mathcal{H}_k \cup h)}{\mathfrak{S}(\mathcal{H}_k)}, \tag{5.20}$$

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consequently

$$\frac{\sum\limits_{h=1, h\notin\mathcal{H}_k}^{H} B_N(h,\ell)}{A_N(\ell)} \sim \left(\sum_{h=1, h\notin\mathcal{H}_k}^{H} \frac{\mathfrak{S}(\mathcal{H}_k \cup h)}{\mathfrak{S}(\mathcal{H}_k)}\right)\frac{1}{\log N}. \tag{5.21}$$

In the original work [10] a beautiful result of Gallagher [8]

$$\sum_{\mathcal{H}_k \subset [1,H], |\mathcal{H}_k| = k} \mathfrak{S}(\mathcal{H}_k) \sim \frac{H^k}{k!} \quad (k \text{ fix}, H \to \infty) \tag{5.22}$$

was used along with an averaging over all possible k-tuples $\mathcal{H}_k \in [1,H]$ to obtain on average over $n \in [N,2N)$ further η primes among $n+h$ with $h \in [1,H] \setminus \mathcal{H}_k$ which finally led to more than one prime in some interval

$$[n, n+\eta\log N] \quad \text{ if } \eta > 4\varepsilon \tag{5.23}$$

with a suitable $n \in [N,2N)$, $N > N_0(\varepsilon)$. This proved Theorem A, the relation $\Delta_1 = 0$. However, it might be interesting to note that Gallagher's estimate can be avoided and one can show (even in a simpler way) for any fixed admissible $\mathcal{H}_k$ either the relation

$$S(\mathcal{H}_k, H) := \sum_{h=1, h\notin\mathcal{H}_k}^{H} \frac{\mathfrak{S}(\mathcal{H}_k \cup h)}{\mathfrak{S}(\mathcal{H}_k)} \sim H,$$

or in a simpler way the weaker inequality

$$S(\mathcal{H}_k, H) > c^* H \text{ for } H > H_0(k) \tag{5.24}$$

with a suitable absolute constant $c^* > 0$, which is still enough to show Theorem A. We obtain a particularly simple proof if we show (5.24) just for some suitably chosen $\mathcal{H}_k$ (for any large k), which is again sufficient for the proof of Theorem A.

The proof of the quantitative Theorem B is based on the same ideas but it also requires many further ideas and is technically much more complicated. We just mention that in [11] we apply similar weights but the parameters k and ℓ tend to infinity with N as

$$k \asymp \frac{\sqrt{\log N}}{(\log\log N)^2}, \quad \ell \asymp \sqrt{k}. \tag{5.25}$$

Furthermore, we have to replace Gallagher's theorem (5.22) with an inequality (shown in a completely different way) which is still valid if

$$k < \varepsilon H / \log H \tag{5.26}$$

(for which range (5.22) is probably no longer true).

Additionally we have to replace Bombieri–Vinogradov's theorem by a more general one which gives a non-trivial estimate also in the case when the moduli run only through a sparse sequence, namely through the multiples of a given modulus

$$M \le X^{c/\log\log X}. \tag{5.27}$$

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6 Main ideas of the proofs of Theorems 4.1–4.5

As we mentioned in the last section, the proof of Theorem 4.2 needed a choice of $k, \ell \to \infty$ as $N \to \infty$ (in the way given in (5.25)). This represented the essentially optimal choice of the parameters k, ℓ if we work with weights of type (5.12). It turned out already in [10] that for concrete small values of k (like for $k = 6$ in case of (EH), i.e. $\vartheta = 1$) it might be more advantageous to substitute the expression $(\log x)^{k+\ell}$ in (5.12) by a linear combination of similar terms with different values of ℓ (like $\ell = 0$ and $\ell = 1$ in case of $k = 6$, $\vartheta = 1$). This idea is crucial in obtaining Theorem 4.1, where a suitably chosen polynomial

$$P(\log x) = \sum_{\ell=L_1}^{L_2} b_\ell (\log x)^{k+\ell}, \quad L_1, L_2 \asymp L \asymp k^{1/3} \tag{6.1}$$

is used in the definition of the weight corresponding to $a(n, R, \mathcal{H}_k, \ell)$ and k is chosen instead of (5.25) as

$$k \asymp (\log N)^{3/7} (\log\log N)^{4/7}. \tag{6.2}$$

At first sight it may seem paradoxical that a linear combination of weights performs better than the best term itself. However, the new weight a_n', induced by $P(\log x)$ in (6.1), is itself not a linear combination of the earlier weights $a_n(\ell)$. In fact, the relation between them is

$$a_n' = \left(\sum_{\ell=L_1}^{L_2} c_\ell \sqrt{a_n(\ell)} \right)^2, \quad c_\ell = b_\ell (k+\ell)!\,. \tag{6.3}$$

The proofs of the further Theorems 4.2–4.5 are naturally different and require several new ideas. However, they all need an important common basis. This base is that the weights a_n are concentrated on integers with the property that all prime factors of all 'coordinates' $n + h_i$ $(i = 1, \dots, k)$ are at least of size N^η with a small constant $\eta > 0$ (depending on k). This can be expressed by Lemma 4 of [17] which we formulate here as

Main Lemma. *Let* $N^{c_0} < R \le N^{1/(2+\eta)} (\log N)^{-c_1}$, $\eta > 0$. *Then we have for the weights in* (5.12) *for any* $\ell \le k$ *and any admissible* k*-tuple* $\mathcal{H}_k$

$$\sum_{\substack{n \in [N,2N) \\ \exists p < N^\eta, p \mid \mathcal{P}_{\mathcal{H}}(n)}} a(n, R, \mathcal{H}_k, \ell) \ll_k \eta \sum_{n \in [N,2N)} a(n, R, \mathcal{H}_k, \ell). \tag{6.4}$$

This result makes it possible to reduce the summation in all proofs with a bounded value of k to numbers n where all elements of $\{n + h_i\}_{i=1}^k$ are almost primes. This information provides a significant help in the proofs of Theorems 4.2–4.5. We have, for example, for these numbers automatically

$$\frac{a(n, R, \mathcal{H}_k, \ell)}{(\log^{k+\ell} R)^2} \ll_{k,\ell,\eta} 1, \tag{6.5}$$

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while all other n values having a small prime factor of $P_{\mathcal{H}}(n)$ can be neglected. This property is, in fact, crucial in the proofs of Theorems 4.2 and 4.5.

7 Proof of Theorem 4.1

The difference compared with the proof of Theorem B, that is Theorem 2 of [11], is that instead of a weight function of type

$$\left(\frac{\log \frac{R}{d}}{\log R}\right)^{K+\ell}, \quad K \asymp \frac{\sqrt{\log N}}{(\log\log N)^2}, \quad \ell \asymp \sqrt{K} \tag{7.1}$$

we use a linear combination of such type weights,

$$\widetilde{P}\left(\frac{\log \frac{R}{d}}{\log R}\right)^{K+\ell}, \quad \widetilde{P}(x) = \sum_{\ell=L_1}^{L_2} b_\ell x^{K+\ell} \tag{7.2}$$

where the parameters now satisfy

$$K = c_0\left(\frac{\log N}{\log_2 N}\right)^{3/7}, \quad L_1, L_2 \asymp M = \lceil C_1 K^{1/3}/6\rceil \tag{7.3}$$

where $\log_\nu x$ denotes the ν-fold iterated logarithm, c_0 is a small constant, C_1 is a large one.

Since the proof of Theorem 2 in [11] uses the conditions (cf. (4.1)–(4.2) of [11])

$$K \le \frac{c\sqrt{\log N}}{(\log_2 N)^2}, \quad \ell_1, \ell_2 \asymp \sqrt{K}, \tag{7.4}$$

we will first check the needed changes for the proofs of the modified Theorems 4 and 5 of [11] under the new condition:

$$\ell_1, \ell_2 \asymp M \asymp K^{1/3}. \tag{7.5}$$

This requires to check Sections 5–11 of [11] under the new condition (7.5). In contrast to Section 5 we will now choose

$$\delta_0 = \frac{M}{K \log_2 N}. \tag{7.6}$$

After a careful checking of Sections 5–10 we find that the results of Sections 6–9 and the most part of Section 10 remain valid and only (10.1), Lemma 9 and Section 11 need any revision. (We remark here two small errors in the original work [11]. First, in the 3$^{\text{rd}}$ line of (8.3) j has to be replaced by j' and in (8.9) $\log_3 N$ has to be replaced by $\log_2 N$, but the last inequalities of both formulae remain unchanged valid.)

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According to the new definition of δ_0 in (7.6) we obtain now by (7.12) of [11] instead of (10.1) with the contours $\mathcal{L}_0'$ and $\mathcal{L}'$ defined in (5.3) of [11]:

$$|D(s,-s)| \le e^{CM} D(0,0) \quad \text{for } s \in \mathcal{L}_0'. \tag{7.7}$$

Consequently, this implies together with Lemma 8 instead of the original Lemma 9 the corresponding inequality

$$|D(s,-s)| \le e^{CM} \max\{1,|t|\}^{-(a+b)/2} D(0,0) \quad \text{for } s \in \mathcal{L}'. \tag{7.8}$$

We can now turn to the necessary changes in Section 11. The new form of Lemma 9, the above (7.8), means that in (11.1) we have now

$$|D(s,-s)| \le e^{CM} D(0,0) \quad \text{for } s \in \mathcal{L}' \tag{7.9}$$

(instead of the old $|D(s,-s)| \le e^{C\sqrt{K}} D(0,0)$) and so we can change $e^{C\sqrt{K}}$ to e^{CM} in the 2nd and 3rd lines of (11.1) too. So we have now

$$B_{j'}(s,\mathcal{H}_1,\mathcal{H}_2) \ll e^{CM} \left(CK^2 \log_2 N\right)^{j'} \frac{\delta_0^{-(u+v)}}{|s|^2} D(0,0), \tag{7.10}$$

and consequently we obtain instead of (11.2) of [11] now

$$C_j'(\mathcal{H}_1,\mathcal{H}_2) \ll e^{CM} \left(CK^2 \log_2 N\right)^{j'} \delta_0^{-(u+v+1)} D(0,0). \tag{7.11}$$

Due to $u, v \asymp M$ and (7.6) we obtain in place of (11.3)–(11.4)

$$\begin{aligned} I_{2,1} &\ll \frac{e^{CM} D(0,0)\delta_0^{-(u+v+1)}(\log R)^{d-1}}{(d-1)!} \sum_{j'=0}^{d-1} \left(\frac{CK^2 \log_2 N}{\log R}\right)^{j'} \\ &\ll \frac{e^{C(u+v)} D(0,0)(\log R)^{d-1} \left(KM^{-1}\log_2 N\right)^{u+v+1}}{(d-1)!} \\ &\ll \frac{D(0,0)(\log R)^{d+u+v}}{(d+u+v)!} \left(\frac{CK^2M^{-1}\log_2 N}{\log R}\right)^{u+v+1} \\ &\ll \frac{D(0,0)(\log R)^{d+u+v}}{(d+u+v)!} (\log N)^{-M/50} \end{aligned} \tag{7.12}$$

and

$$I_2 \ll \frac{D(0,0)(\log R)^{d+u+v}}{(d+u+v)!} (\log N)^{-M/50} + e^{-c\sqrt{\log N}}. \tag{7.13}$$

The final few lines of Section 11, including the crucial formula (11.5) remain therefore unchanged valid. Consequently, the original Theorem 4 of [11] remains valid under the new choice (7.3) of the parameters. The same refers to Theorem 5 of [11] since the evaluation of the crucial integral there is obtained by the above modification of Sections 5–11 of [11] and no change is required in the formulation of the modified

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Bombieri–Vinogradov type theorem (Theorem 6 of [11] in Section 12 or in the arguments of Section 13), yielding the proof of Theorem 4.5.

Since Section 14 is independent of the others, we have to turn our attention to the last section, Section 15, which finishes the proof of Theorem 2 of [11].

The major change in Section 15 arises from the use of the weight function with the polynomial

$$\widetilde{P}(x) = P_K(x) = \sum_{\substack{\ell=M \\ 2\nmid \ell}}^{2M} g\left(\frac{\ell}{M}\right)\left(\frac{K}{2}\right)^{\ell} \frac{x^{K+\ell}}{(K+\ell)!}, \quad g(y) = (y-1)^4(2-y)^4 \tag{7.14}$$

which was analyzed in the last section of [7] where we proved that (see (34) of [7])

$$D_K\left(K + C_1 K^{1/3}\right) - 4E_K \geq 0 \tag{7.15}$$

with the notation

$$D_K = \sum_{\substack{\ell_1=M \\ 2\nmid \ell_1}}^{2M} \sum_{\substack{\ell_2=M \\ 2\nmid \ell_2}}^{2M} g\left(\frac{\ell_1}{M}\right) g\left(\frac{\ell_2}{M}\right) \frac{(K/2)^{\ell_1+\ell_2} K!}{(K+\ell_1+\ell_2+1)!} \binom{\ell_1+\ell_2+2}{\ell_1+1} \tag{7.16}$$

$$E_K = \sum_{\substack{\ell_1=M \\ 2\nmid \ell_1}}^{2M} \sum_{\substack{\ell_2=M \\ 2\nmid \ell_2}}^{2M} g\left(\frac{\ell_1}{M}\right) g\left(\frac{\ell_2}{M}\right) \frac{(K/2)^{\ell_1+\ell_2} K!}{(K+\ell_1+\ell_2)!} \binom{\ell_1+\ell_2}{\ell_1}, \tag{7.17}$$

which corresponds to (43)–(44) in [7].

We would like to mention that in the argument of Soundararajan [21] he works with a weight function of type (we defined $P_{\mathcal{H}}(n) = \prod_{i=1}^{K}(n+h_i)$ in (5.1))

$$a_{n,1} = \sum_{\substack{\mathcal{H}\subset[1,h] \\ |\mathcal{H}|=K}} \left(\sum_{d\mid P_{\mathcal{H}}(n), d\leq R} \mu(d) P_K\left(\frac{\log(R/d)}{\log R}\right)\right)^2. \tag{7.18}$$

In contrast to this, we used in our works [10], [11] more complicated weights of the form

$$a_{n,2} = \left(\sum_{\substack{\mathcal{H}\subset\mathcal{A} \\ |\mathcal{H}|=K}} \sum_{\substack{d\mid P_{\mathcal{H}}(n) \\ d\leq R}} \mu(d) P_K\left(\frac{\log(R/d)}{\log R}\right)\right)^2; \tag{7.19}$$

however, with the simple polynomial $P(x) = x^{K+\ell}$, $\ell \asymp \sqrt{K}$. ($\mathcal{A}$ is in our case an arbitrary set of integers $\leq N$ with H elements, so similar to but more general than (7.18).)

In the present work we will use weights of type (7.19) with the polynomial (7.14). The different structures of the weights (the order of squaring and summation over k-tuples $\mathcal{H}_k$) actually yield different optimization problems for the polynomial $P(x)$ but the difference is not significant in the most crucial case when we consider (after squaring in (7.19)) pairs $\mathcal{H}_1, \mathcal{H}_2$ when $(\mathcal{H}_1 \cup \mathcal{H}_2) \setminus (\mathcal{H}_1 \cap \mathcal{H}_2)$ is relatively small in size.

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This is the reason why despite the different optimization problems we can use here the polynomial from [7], which is quasi-optimal for the weights of type (7.18).

Additionally we remark that the question of optimization of weights of type (7.19) cannot be formulated in such a simple and nice way as in the case of the weights of type (7.18), analyzed in [21] and [7]. We also note that some nice functions can replace the polynomial $P_K(x)$ but it is hard to predict in advance what sort of properties we exactly need for the function in question so that the asymptotic evaluation of the quantities A_N and $B_N(h)$ in (5.3)–(5.6) should be possible. We just mention here that the case $K \asymp \sqrt{\log N}$, $\ell = K^\alpha$, $\alpha > 1/2$ would already cause serious problems in [11], even in the simple case of $P(x) = x^{K+\ell}$.

Finally, we remark that although the optimization problems are similar at weights of type (7.18) and (7.19) the final result for gaps between consecutive primes would be much weaker in case of the use of weights of type (7.18), namely $(\log N)^{3/5+\varepsilon}$ compared to our present $(\log N)^{3/7+\varepsilon}$. (In case of $P(x) = x^{K+\ell}$ the two methods yield gaps of size at most $(\log N)^{2/3+\varepsilon}$ in case of (7.18), whereas (7.19) led to gap size of at most $(\log N)^{1/2+\varepsilon}$ in [11], quoted as Theorem B here.)

The proof of our new result follows closely the arguments of Section 15 of [11], however the final result for the analogue of the crucial quantity S_R' in (15.9) (the analogue of S_R in (10.1) of [10]) is more complicated due to the use of the polynomial $\widetilde{P}(x) = P_K(x)$. Additionally, we have to change the value of most parameters. What makes the procedure more difficult is the fact that Section 15 relies on Section 10 of [10], so its proof is already not self-contained in [11] either. Below we will describe the necessary changes compared to Section 15 of [11] (all numbers of formulae beginning with 14 or 15 refer to [11]).

We do not change the values of R, θ, ξ and V given in (15.2) but in place of (15.3)–(15.5) we choose in accordance with (7.3) (with a sufficiently small constant c_0 and a large enough C_1) and the last section of [7]

$$K = c_0 \left(\frac{\log N}{\log_2 N} \right)^{3/7}, \quad M = \lceil C_1 K^{1/3}/6 \rceil, \quad \varphi = \frac{M}{K}, \tag{7.20}$$

$$x = \frac{1}{100\varphi^2} = \frac{K^2}{100M^2} = \frac{\log R}{H} \sim \frac{\log N}{4H}, \tag{7.21}$$

$$\widetilde{f}(r) = \binom{K}{r}^2 x^r, \quad r_0 = K(1-9\varphi) = K - 9M, \quad r_1 = K(1-7\varphi) = K - 7M. \tag{7.22}$$

Under this choice we have $\widetilde{f}(r+1) < \widetilde{f}(r)$ for $r > r_0$ and

$$\frac{\widetilde{f}(r_1)}{\widetilde{f}(r_0)} = \prod_{r_0 < r \le r_1} \left\{ x \left(\frac{K-r+1}{r} \right)^2 \right\} \le \left(\left(\frac{9\varphi}{1-7\varphi} \right)^2 x \right)^{2M} \le e^{-M/3} \tag{7.23}$$

analogously to (15.16); furthermore we have with $C_2 = \frac{25}{36} C_1^2 c_0^{-\frac{4}{3}}$ from (7.21)

$$H \sim \frac{25 \log N \cdot M^2}{K^2} \sim \frac{25}{36} C_1^2 K^{-\frac{4}{3}} \log N = C_2 (\log N)^{3/7} (\log_2 N)^{4/7}. \tag{7.24}$$

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This choice of parameters implies that $k\log_2 N/H$ will be smaller than any positive constant if c_0 is chosen sufficiently small and $k \le 2K$ arbitrary. Consequently, by Lemma 16 of [11] we obtain for the average of the singular series

$$S^*_{\mathcal{A}}(k) = \frac{1}{H^k}\sum_{\mathcal{H}\subset\mathcal{A},|\mathcal{H}|=k}\mathfrak{S}(\mathcal{H}) \tag{7.25}$$

(cf. (14.6)–(14.7) of [11])

$$S^*_{\mathcal{A}}(k+1) \ge e^{-1/40}S^*_{\mathcal{A}}(k), \tag{7.26}$$

hence in analogy with (15.18)

$$\frac{S^*_{\mathcal{A}}(2K-r_0)}{\max\limits_{K\ge r>r_1} S^*_{\mathcal{A}}(2K-r)} > e^{-9M/40}. \tag{7.27}$$

The formulae (7.23) and (7.27) together yield for any $K \ge r > r_1$

$$f(r)S^*_{\mathcal{A}}(2K-r) < f(r_1)S^*_{\mathcal{A}}(2K-r_0)e^{9M/40} < f(r_0)S^*_{\mathcal{A}}(2K-r_0)e^{-M/10}. \tag{7.28}$$

Using the notation $\theta(n) = \log n$ if $n \in \mathcal{P}$ and $\theta(n) = 0$ otherwise, we can evaluate the crucial quantity

$$\begin{aligned}\widetilde{S}_R = &\sum_{\substack{\mathcal{H}_1\subset\mathcal{A}\\|\mathcal{H}_1|=K}}\sum_{\substack{\mathcal{H}_2\subset\mathcal{A}\\|\mathcal{H}_2|=K}}\sum_{\substack{\ell_1=M\\2\nmid\ell_2}}^{2M}\sum_{\substack{\ell_2=M\\2\nmid\ell_2}}^{2M} g\left(\frac{\ell_1}{M}\right)g\left(\frac{\ell_2}{M}\right)\frac{\left(\frac{K}{2}\right)^{\ell_1+\ell_2}}{(K+\ell_1)!(K+\ell_2)!} \\ &\times\sum_{\substack{N<n\le 2N\\ n\in A(\mathcal{H}_1)\cap A(\mathcal{H}_2)}}\widetilde{\Lambda}_R(n;\mathcal{H}_1,\ell_1)\widetilde{\Lambda}_R(n;\mathcal{H}_2,\ell_2)\Big(\sum_{h\in\mathcal{A}}\theta(n+h)-\log(3N)\Big)\end{aligned} \tag{7.29}$$

by the aid of the modified Theorems 4 and 5 of [11] for all pairs $\mathcal{H}_1,\mathcal{H}_2 \subset \mathcal{A}$, $|\mathcal{H}_i| = k$, where now

$$\widetilde{\Lambda}_R(n;\mathcal{H},\ell) = \sum_{\substack{d|P_{\mathcal{H}}(n)\\ d\le R}}\mu(d)\left(\frac{\log\frac{R}{d}}{\log R}\right)^{K+\ell}; \tag{7.30}$$

the difference to $\Lambda_R(n;\mathcal{H},\ell)$ in (3.6) of [11] is just in the normalization by the division with $(\log R)^{K+\ell}$. Using the same argumentation as in Section 10 of [10] or, equivalently, that of Section 15 of [11] we obtain for the contribution of all pairs $\mathcal{H}_1,\mathcal{H}_2$ with $|\mathcal{H}_1\cap\mathcal{H}_2| = r$ instead of (15.9)–(15.10) of [11], taking into account also the different normalization

$$\widetilde{S}_R \ge \sum_{r=0}^{K} f(r)S^*_{\mathcal{A}}(2K-r)F_{K,r} \tag{7.31}$$

with

$$\begin{aligned}F_{K,r} = &\sum_{\substack{\ell_1=M\\2\nmid\ell_1}}^{2M}\sum_{\substack{\ell_2=M\\2\nmid\ell_2}}^{2M} g\left(\frac{\ell_1}{M}\right)g\left(\frac{\ell_2}{M}\right)\frac{(K/2)^{\ell_1+\ell_2}r!}{(r+\ell_1+\ell_2)!}\binom{\ell_1+\ell_2}{\ell_1} \\ &\times\left\{\frac{(\ell_1+\ell_2+1)(\ell_1+\ell_2+2)}{(\ell_1+1)(\ell_2+1)}\cdot\frac{K}{r+\ell_1+\ell_2+1}-4+O\left(\left(\frac{\log_2 N}{\log N}\right)^{5/14}\right)\right\}.\end{aligned} \tag{7.32}$$

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If $r < r_1 = K - 7M$ then we will show that $F_{K,r}$ is positive and consequently so is $f(r)S^*_{\mathcal{A}}(2K - r)F_{K,r}$. The connection between $F_{K,r}$, D_K and E_K (see (7.15)–(7.16)) is

$$F_{K,K} = KD_K - 4E_K\left(1 + O\left(\left(\frac{\log_2 N}{\log N}\right)^{5/14}\right)\right). \tag{7.33}$$

(This is why we asserted that the weights of type (7.18) and (7.19) lead to slightly different but similar optimization problems.) First we note that it is easy to see that

$$F_{K,r} > 0 \text{ for } r \le K\left(1 - \frac{3}{M}\right), \tag{7.34}$$

since in this case the quantity in the brackets is already positive. Using the notation $\ell_1 + 1 = u$, $\ell_2 + 1 = v \in (M, 2M]$ we have, namely, in this case

$$\begin{aligned}\frac{(u+v)(u+v-1)}{uv} \cdot \frac{K}{r+u+v-1} &\ge \left(4 - \left(\frac{1}{u} + \frac{1}{v}\right)\right) \cdot \frac{K}{K\left(1 - \frac{2}{M}\right)} \\ &\ge 4\left(1 - \frac{1}{2M}\right)\left(1 + \frac{2}{M}\right) \ge 4 + \frac{4}{M} > 4 + \frac{1}{(\log N)^{1/7}}.\end{aligned} \tag{7.35}$$

In the crucial case

$$K - \frac{3K}{M} < r < r_1 = K - 7M \tag{7.36}$$

the evaluation and estimation of $F_{K,r}$ is nearly exactly the same as that of D_K and E_K in the last section of [7] leading to (7.15). The only difference outside the brackets is the change

$$\prod_{i=1}^{\ell_1+\ell_2} \frac{1}{K+i} = j(K) \text{ to } \prod_{i=1}^{\ell_1+\ell_2} \frac{1}{r+i} = j(r) \tag{7.37}$$

but the two quantities are in case of (7.36) of the same order of magnitude, namely,

$$j(K) = j(r)\left(1 + O\left(\frac{1}{M}\right)\right)^{4M} \asymp j(r). \tag{7.38}$$

This means that the procedure in [7] yields for these values of r, similarly to (7.15) for any r with (7.36)

$$\begin{aligned}&\sum_{\substack{\ell_1=M \\ 2\nmid \ell_1}}^{2M} \sum_{\substack{\ell_2=M \\ 2\nmid \ell_2}}^{2M} g\left(\frac{\ell_1}{M}\right) g\left(\frac{\ell_2}{M}\right) \frac{\left(\frac{K}{2}\right)^{\ell_1+\ell_2} r!}{(r+\ell_1+\ell_2)!} \\ &\qquad \times \left\{\frac{(\ell_1+\ell_2+1)(\ell_1+\ell_2+2)}{(\ell_1+1)(\ell_2+1)} \cdot \frac{K + C_1K^{1/3}}{K+\ell_1+\ell_2+1} - 4\right\} \ge 0.\end{aligned} \tag{7.39}$$

If we show that for $r < r_1 = K - 7M$ the above quantity in the brackets is smaller than the one appearing in the definition of $F_{K,r}$ in (7.32) for any pair (ℓ_1, ℓ_2), then $F_{K,r} > 0$ holds, since the terms outside the brackets in (7.39) are all positive. Taking into account $M = \lceil C_1K^{1/3}/6 \rceil$, it is sufficient to show

$$\frac{K+6M}{K+\ell_1+\ell_2+1} < \frac{K}{r+\ell_1+\ell_2+1} + O\left(\frac{\log_2 N}{\log N}\right)^{5/14}. \tag{7.40}$$

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Let

$$r = K - \Delta, \quad \Delta > 7M, \quad m = \ell_1 + \ell_2 + 1 \in (2M, 4M). \tag{7.41}$$

Then we have

$$\begin{aligned} \frac{K+6M}{K+m} - \frac{K}{K-\Delta+m} &< \frac{6M(K-\Delta+m) - K\Delta}{(K+m)(K-\Delta+m)} \\ &< \frac{6MK - 7MK}{2K^2} = -\frac{M}{2K} < -\frac{1}{(\log N)^{2/7}}, \end{aligned} \tag{7.42}$$

which thereby proves (7.40) and so settles the case $r < r_1$.

The analysis of the last section of [7] teaches us that the maximal possible order of magnitude of $|F_{K,r}|$ does not change significantly for $r \in [r_1, K]$, so due to the strong decrease of the additional factor $f(r)$ we have in view of (7.28) in fact a similar inequality to (7.28) if we multiply all terms with the corresponding $F_{K,r}$, namely for $r > r_1$ we have, say

$$F_{K,r} f(r) S^*_{\mathcal{A}}(2K - r) < F_{K,r_0} f(r_0) S^*_{\mathcal{A}}(2K - r_0) e^{-M/20}. \tag{7.43}$$

This shows (cf. (7.29)–(7.32))

$$\widetilde{S}_R > 0. \tag{7.44}$$

This means that for some $n \in (N, 2N]$ we have

$$\sum_{h \in \mathcal{A}} \theta(n+h) - \log(3N) > 0 \tag{7.45}$$

and so we have $h_1, h_2 \in \mathcal{A}$ with $n + h_1, n + h_2 \in \mathcal{P}$, therefore two primes p', p'' with $p' - p'' = h_1 - h_2 \in \mathcal{A} - \mathcal{A}$. In the general formulation of Theorem 4.2, $\mathcal{A} \subseteq [1, N] \cap N$ was any sequence with H elements, where

$$H \geq H_0 = C(\log N)^{3/7}(\log_2 N)^{4/7}. \tag{7.46}$$

In particular we can take $\mathcal{A} = [1, H]$ yielding two primes p', p'' with $0 < p'' - p' < H_0$.

Added in proof for (06/30/2013 and 10/30/2013). Very recently Yitang Zhang (Bounded Gaps between Primes, *Ann. of Math.*, to appear) succeeded in showing by an essential refinement of the methods sketched in the present paper and by an additional use of the fact that only smooth numbers have a significant effect for the sieving process (Y. Motohashi and J. Pintz, A Smoothed GPY sieve, *Bull. London Math. Soc.* **40** (2008) no. 2, 298–310) that bounded gaps between primes occur infinitely often. The original bound 70 million is now diminished slightly below 5000 by the Polymath 8 project of T. Tao.

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