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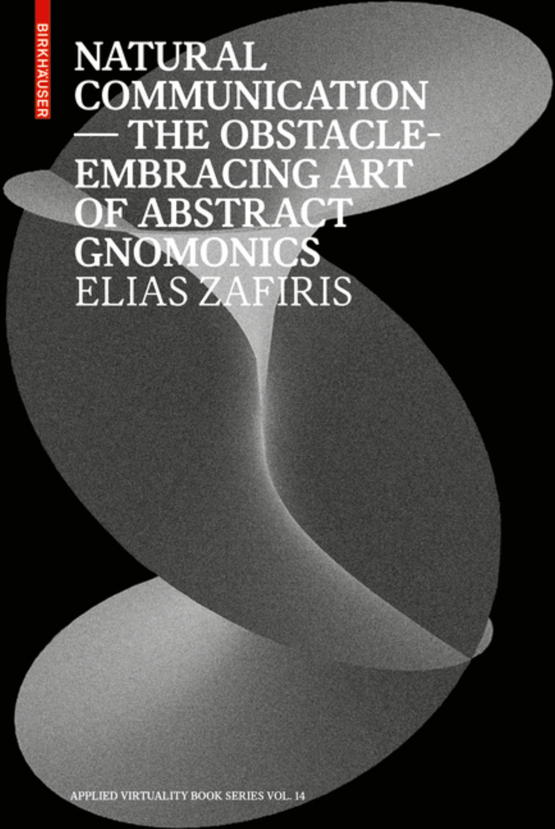
NATURAL COMMUNICATION — THE OBSTACLE-EMBRACING ART OF ABSTRACT GNOMONICS

ELIAS ZAFIRIS



APPLIED VIRTUALITY BOOK SERIES VOL. 14

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Basel

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LAYOUT AND COVER DESIGN: onlab, CH-Geneva, www.onlab.ch
TYPEFACE: Korpus, binnenland (www.binnenland.ch)
PRINTING AND BINDING: buch.one, D-Pliezhausen

Library of Congress Control Number: 2020943214

Bibliographic information published by the German National Library
The German National Library lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

ISSN 2196-3118
ISBN 978-3-0356-2075-7
e-ISBN (PDF) 978-3-0356-2080-1 Open Access

P.O. Box 44, 4009 Basel, Switzerland
Part of Walter de Gruyter GmbH, Berlin/Boston

The book is published open access at
www.degruyter.com

9 8 7 6 5 4 3 2 1

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TO THE FOND MEMORY OF ATHANASIOS AND ANASTASIOS

Οὐδέποτε ὠρέχθην τοῖς πολλοῖς ἀρέσκειν.
ἅ μὲν γὰρ ἐκείνοις ἤρεσκεν, οὐκ ἔμαθον.
ἅ δ᾽ ᾔδειν ἐγώ, μακρὰν ἦν τῆς ἐκείνων αἰσθήσεως.

φύσις κρύπτεσθαι φιλεῖ.
ἁρμονίη ἀφανὴς φανερῆς κρείττων.

ἓν τὸ σοφόν· ἐπίστασθαι γνώμην ὅκη
†κυβερνῆσαι† πάντα διὰ πάντων.

PROLOGUE

The main motivation behind setting up and articulating the theoretical model of *"Natural Communication"* in a book of this form proceeds from the intention of the author to criticize, and transcend the current, "target-oriented" paradigm of complexity science, by proposing and elaborating an alternative one, envisioning and implementing a fundamental architectonics of communication. The proposed model of "Natural Communication" encapsulates modern theoretical concepts from mathematics and physics, in particular category theory and quantum theory respectively, not for the sake of a technical formalization, but in order to abstract accurately basic notions that lead to a conceptual appreciation of this theory. Additionally, this makes it possible to re-consider and re-evaluate novel ways of thinking about complexity deeply-rooted in the past, which have been unfortunately, either oversimplified and distorted, or forgotten and left to oblivion. The author believes that only by looking to the past, does it become possible to establish a continuity and coherence in our current way of thinking, in particular regarding complexity, which is the pre-condition for any serious future development on these matters.

The fundamental realization underlying the generative reason of this treatise is that a certain architectonics of relations based on communication is ultimately necessitated in all cases, where direct accessibility to sharply distinguishable domains of objects and their behaviour is not feasible, due to obstacles or obstructions of any particular type. In these domains, objects are intrinsically shaped according to foamy or cloudy patterns, and they are characterized by topological plasticity, emergent properties and generically probabilistic attributes. The application of pre-specific, readily-tailored design ontologies to these domains, based on the reductive notion of some hypothetical sharply-distinguishable elementary constituency, not only distorts the architectonics of their non-trivial connectivity patterns, but limits and restricts, albeit inadvertently, the potential to unravel their rich computational capacities. In light of this, the notion of computation cannot be disentangled from the architectonics of relations based on communication. In the opposite case, complexity is reduced to a particular form of *complicatedness* due to mixing hypothetical elementary constituents, with the end result that information is treated exclusively in terms of statistical data attributes, effectively depriving these domains from the possibility of manifesting genuine novelty.

The basic idea proposed in this book is the following; to address and utilize the architectonic modelling of not directly accessible, or more generally, obstacle-laden domains. Instead of analysing them in terms of constituent set-based elements and their hypothetical absolute relations, the approach is to adjoin to them other adequately-understood or directly accessible domains, which can provide pointers and open up communication channels with the originals. The process of adjoining should not be ad-hoc and should not depend on artificial choices, meaning that it should be qualified as structurally-respecting, at least, locally or partially. This refers to the "naturality requirement" of the model, a term which is exemplified in the context of category theory. Technically, this adjunction process can be always abstracted properly in terms of a pair of adjoint functors between the categorical domains involved. The essence of adjoining in order to open up or potentiate communication channels is based on the idea of partial or local structural congruence, and implies a certain type of modularity in the treatment of obstacle-laden domains. It is deeply rooted in the old art of *"Gnomonics"*, i.e. in the masterful articulation of sundials, calendars, and atlases, to probe some domain inaccessible by direct means and obtain information in modular relation with respect to the *gnomon* adopted, by forcing or effecting a certain similarity or congruence relation.

In practice, the process of adjoining a controllable domain to another one, initially not directly accessible, or obstacle-laden categorical domain, amounts to viewing these domains as different categorical levels in a stratified universe of discourse, which are bi-directionally connected by means of oppositely-orientated bridges, to be thought of as communication channels. The architectonics of communication targets precisely the conception and explicit construction of these bridges, once suitable probing domains have been structurally delineated for adjunction to the directly inaccessible domain. The bridge directed to the controllable domain plays the role of an encoding bridge, whereas its inverse plays the role of a decoding bridge.

These connecting bridges effect the communication between the domains in question, in the sense that they naturally establish universal bidirectional communication channels through which a holonomic schema of *"metaphora"* is accomplished, based on the notion of an *"obstacle-encircling"* flow. In topological terms, the initially inaccessible categorical domain is resolved cyclically by a process of unfolding with respect to the probing domain that has been adjoined to it. In this manner, the invariants emerging by the process of unfolding depict the invariant characteristics of the reciprocal communication flow between these domains. Consequently, the complexity of the not directly accessible, or obstacle-laden, domain is not specified constitutively on the basis of a pre-assumed or axiomatic elementary ontology, but relationally, modularly, and, in the technical sense naturally, in terms of the invariants emerging in the bidirectional communication flow established with some appropriate probing domain. From an algebraic viewpoint, these invariants can be qualified in terms of structural group-type ciphers for the symbolic encryption/decryption of the induced flow.

The obstacle-embracing schema of "metaphora" giving rise to this cyclic flow, called the *"logical conjugation method"*, is always implemented on the basis of a legitimate logical manoeuvre through controllable or directly accessible domains, which are adjoined to an obstacle-laden domain as markers or pointers, providing eventually the means of specifying it indirectly through communication. The logical conjugation method should be thought of in terms of a *"motivic key"* that bears the capacity to unlock harmonically the complexity of the inaccessible domain depending on the nature and type of the obstacle encountered. The thing to be emphasized is that it always gives rise to a partition spectrum of the latter. Each cell of this spectrum is characterized completely by the pair of encoding/decoding bridges utilized for communicating an aspect of the directly inaccessible domain with respect to some probing domain.

The creative art consists in the innovation of genuine bidirectional encoding and decoding bridges between these two domains that make possible the instantiation of a cyclic communication flow capable of embracing the present obstacle. In this manner, the theoretical model of "Natural Communication" is deeply rooted conceptually to the old art of "Gnomonics", and it is meant as a continuation of this paradigm, articulating a schema of an "abstract gnomonics" in a modern theoretical scientific context.

For all practical purposes, the model of "Natural Communication" can be implemented briefly as follows: We consider a problem in the context of a domain whose objects and relations are inaccessible by direct means. It is instructive to think of this domain as a particular level in a broad universe of discourse, which can engulf other possible levels as well. First, we move out of the context of the problem, formulated at the level of the inaccessible domain, by adjoining to it another controllable domain, assuming existence within the same universe. In order to accomplish this, we have to set-up an encoding bridge from the level of the inaccessible domain to the level of the accessible domain, such that some certain form of congruence can be established between these domains. Once we have succeeded in setting up this bridge, we are able to mirror the initial problem at the level of the controllable domain, where we have the means to address it effectively. The process is completed by setting up an inverse decoding bridge from the level of the controllable domain to the level of the inaccessible one. In this way, the available means or knowledge pertaining to the controllable domain can be lifted at the initial context of the problem. Thus, the pertinent problem can be effectively resolved in the context of its initial formulation by the embracing of the obstacle it engulfs via the communication channels established with the other domain. The reciprocal encoding and decoding bridges constitute the means of a novel architectonics of communication. It has to be stressed that this procedure can be iterated by the adoption of more than one controllable domain complementarily adjoined to the inaccessible domain. The skeleton of this spectral resolution process remains invariant under the adjunction of deeper levels, and most important, it always gives rise to a partition spectrum of the inaccessible domain, amenable to possible refinement, whose cells are indexed by the respective pairs of encoding/decoding communication bridges.

The present treatise consists of eleven chapters. These chapters run in no linear order. Rather, the reader may discern a multiplicity of navigation paths through the exposed landscape depending on the type of questions, insights, topics, and depth, she or he wishes to be engaged with in this exploration. Faithful to its communicational roots this book does not intend to install any pre-assigned order for the mind of the reader. The hope is there will always be certain widths of spectral frequencies to resonate with readers coming from diverse backgrounds. Although the development is based on mathematical and physical ways of thinking and arguing, essential care has been taken for quite detailed conceptual articulations of the treated subjects, which may appeal to readers not wishing to delve into the more technical aspects of this work. The main novelty consists in the attempt to transcend the strict linear chronological order, imposed by a misconceived sense of historical coherence, and establish bonds among ideas, notions, and thinkers, which are seemingly unrelated if not viewed through a gnomonic perspective. In this manner, a new view of coherence emerges, which in the author's opinion characterizes the diachronic value and persistence of all these elements. Since the method of the book is mainly synthetic, traditional divisions among mathematical and physical concepts, as well as divisions within a discipline playing a mere organizational, less than organic role, are systematically avoided, or they are bridged together appropriately, focussing on the coherence in the functioning of a body of knowledge.

Undoubtedly, the most suitable mathematical framework to express the model of "natural communication" is the framework of category theory. The major problem, beyond the technicality of this framework, is that it is usually introduced in an ad hoc axiomatic fashion, as unifying together all the most important parts of mathematical thinking and knowledge. This approach to the framework of category theory dangerously predisposes to a view of its formalism and concepts merely as the means of generalization and unification, and not as the means of innovation in relation to the whole body of mathematical thinking. Rather, it is the deep roots of category theory in canonics and gnomonics, going back to the generative forces shaping mathematical thinking itself that underlies the value of this framework. Unfortunately these roots, which emerged only after the fundamental theory of adjoint functors shone through the whole endeavor, remained nevertheless suppressed creating the artificial distorted unifying generalization of an "abstract nonsense", a humoristic expression used more or less seriously by the insiders in the field. This bias can be remedied only by tracing back the roots of category theory in canonics and gnomonics, i.e. by viewing this framework as a continuation of the old schools on gnomonics that historically re-emerged as an emergency exit out of the set-theoretic paradise that threatened to absorb everything mathematical and physical, although never acknowledged as such. For this reason, category theory is introduced officially only after Chapter 8, and after the reader has acquired a quite thorough acquaintance with the notions of "abstract gnomonics", and "natural communication". Moreover, the presentation of the powerhouse of this framework follows an inverted order, in comparison to the usual account. The reason for this is that category theory shines through the crystallization of the notion of adjunctions, and in this sense, it is a higher level, structural continuation of the old art of gnomonics, based on the same conditions of metaphora and communicability between different structural species.

Besides the role of category theory in formalizing "natural communication" in modern structural algebraic and topological terms, the conceptual compass of the book is especially sensitive to the currents of thinking of our ancient predecessors who initiated the whole endeavor of *"Natural Philosophy"*, not as an internalized closed mental systematics of encompassing the workings of the whole world, but mainly as a means to qualify their dialogue with Physis, i.e. primarily as an open-ended means of experimentation and communication with what is directly inaccessible in order to unveil and remember it. The same stance is also necessary to re-examine and re-evaluate major modern advances in mathematics and the physical sciences in order to unravel the invisible threads of continuity and coherence through historical time.

Essentially all the mathematical and physical notions introduced in this work are associated with the names of the thinkers who first conceived of them, or played a major role in their establishment. Notwithstanding this, all these notions are scrutinized in light of their emergence from the gnomonic perspective permeating the whole volume, i.e. all these notions are recast in relation to their role in a corresponding process of communication. One of the artifacts of this association is that citation becomes particularly simple. Direct numerical citations in the text are avoided, and the reader may find bibliographical references in endnotes, directly after the name of the person involved appearing in the main text. In this manner, the credit goes to the innovators themselves, and not to textbooks, as is usually the case. As a general rule, the original works, although more difficult to read, and in many cases more fuzzy in comparison to latter formulations, expositions, simplifications, and appropriations, contain the real gems shaping a new field. It goes without saying that self-citations are intentionally completely suppressed in this work.

An aspect of this book that may disturb some readers is that it contains a significant amount of seeming neologisms in English. This is only apparent, since all the non-standard English terms used are actually rooted in the ancient Greek language. The difficult choice was to settle for a standard term, or to introduce a neologism at some point, as a viable way to emphasize some basic concept. The dilemma essentially originates through the linguistic loan of a term into English, and the subsequent appropriation of this term in the context of a specific discipline that, in many cases, delimits and even distorts the original meaning and functional applicability of this term. This is the case, especially, regarding the basic notions of "analogia", and "metaphora", employed in the text, since in English, the corresponding terms of "analogy", and "metaphor" have either a general meaning, or are used in specific linguistic or logical contexts unrelated to the etymological underpinnings of the original Greek terms. Additionally, in mathematical phraseology, the prefixes "homo" in algebra, and "homeo" in topology, have been adopted to characterize certain types of mappings associated with particular types of equivalence. Since the notion of an equivalence relation is formulated in set-theoretic terms, the abstract gnomonic type of these mappings is captured here by the term *"homeotics"*. As another example, the mathematical term "monodromy", is widely used in a sense opposite to the meaning of this term, i.e. "monodromy" is used instead of "polydromy", which would be the correct usage. Finally, in category theory the term "topos" is used, the plural of which is even conjugated in the absurd form "toposes", without the realization that this is a term originating from rhetoric. In other words, a topos is meaningful only through the lines of argument or communication permeating it, the so called "geometric morphisms" in the technical context. The same applies for the notion of an "object of truth values" characterizing a topos, which is absurd if not qualified through the term *"aletheia"*, encapsulating the energetic act of unveiling in Time and remembering.

I would wholeheartedly like to thank Vera Bühlmann and Ludger Hovestadt for their kind invitation to include this treatise in their Book Series, and most of all, for their genuine friendship, and their active interest and engagement, during long and exhausting series of discussions that took place in various spaces and times, on communication, computation, information, and gnomonics. The present

form of this book owes much to all these. It should also be mentioned that the term “gnomonics” itself has been suggested in the context of our conversations by Vera Bühlmann. A special thanks also goes to all my students, who patiently followed my long series of lectures on these subjects during the last four years in the context of the postgraduate teaching and research modules on “Gnomonics” and “Mathematical Thinking”, and attempted to obtain a deep conceptual and applicable comprehension of all the relevant subjects. Last, but not least, I would like to thank my family, and especially my wife Anna, who created the conditions for undisturbed work on this book, and who managed to reconcile with my peculiar sense of time.

Elias Zafiris

CIRCULATION: ENCODING-PARTITIONING-DECODING

1

The present treatise arose out of a curiosity to delve deeper into the nature of obstacles or obstructions that prevent a single, uniform, and linear approach to dealing with systems, entities or, more generally, beings, and their communication. The current motivation comes from a certain degree of dissatisfaction with the overflow of scientific production on what is called “science of complexity” and the qualification of what counts as “information of complex systems” based exclusively on specific “target-oriented approaches”.

The thread of this amphiboly starts from questioning what is considered to be “complex” in contradistinction to what is considered “simple”. Usually these terms are implicitly pre-loaded with an ontological meaning, which essentially identifies complexity with certain aggregations of elementary sharply distinguishable constituents developing emergent properties that are behaviourally observed under specific interactions or conditions. This approach is assumed to be valid irrespectively of scale, depending on what is axiomatically baptized as an elementary constituent, so that certain statistical patterns can be applied upon them targeting the simulation of their behaviour. A natural set of questions in this setting is the following: What makes an entity a constituent, how an entity can be characterized as elementary, and most important, how can an entity be sharply distinguishable?

The afterthought of these considerations is that the epithet “complex” arises from the opposition to what is called “simple”, where what is “simple” is identified with the “axiomatic elementary” in the context of the vast majority of these “target-oriented approaches”. Notwithstanding these scientific tendencies of the present, this is not actually the way that humankind came to terms with complexity. The targets were not predefined, but rather they always emerged out of a necessity to cope with obstacles in communication of every particular sort. In this respect, the meaning of complexity is altered dramatically if theorized from an “obstacle-oriented” standpoint instead of a “target-oriented” one. More precisely, the ability to locate obstacles or obstructions forcing a deviation from some standard condition of uniformity becomes the primary task. In contrast, “target-oriented” approaches are based on the shaky foundations of some pre-given axiomatic elementarity, usually identified with the foundations of set theory, on the top of which, pre-designed statistical methods claim to provide universal truths as the feeding source of policy-making on a larger scale. In this type of approach, where the form of the target is well-defined itself *ab initio*, precluding any chance for genuine novelty, even the standards of congruence effecting the condition of inertial variation in some domain are not determined with respect to the singularities of this domain, but on the basis of general classification methods applied in a more or less ad hoc manner.

In an "obstacle-oriented" approach the presence of an obstacle necessitates, firstly, the localization of this obstacle, and secondly, the unfolding of the obstacle into an appropriate partition spectrum that allows its indirect embracing by means of resonance with the marked frequencies of this spectrum, or reciprocally, but equivalently, synchronization with its corresponding temporal periods. By inversion, the first of the above, points to a viable understanding of what intelligence is about, whereas the second points to a viable understanding of the notion of an irreducible duration, together with its role in what memory refers to. Of course, the ability to obtain such a partition spectrum is always conditioned to the ability to identify a suitable *gnomon*, which actually provides the means to resonate or synchronize with the obstacles in question. This leads inevitably to a valuable association of *gnomonics* with the old art of *harmonics*. We will examine this association in detail in what follows, together with the intervening notion of *canonics*.

At present, a basic fact worth pondering at is that the localization and unfolding of an obstacle, requires a process of metaphora around this obstacle, i.e. a potential circular flow around it that allows its embracing. If the obstacle is considered as merely an obstacle in space, then the potential flow clearly takes place within the space where the obstacle is located. The drawback of this is that the notion of space is pre-assumed in relation to the localization of the obstacle, whereas the opposite should be the case; rather the notion of space should emerge out of the nature of the obstacle, or else, should be cohomologous to the nature of the obstacle with respect to the employed gnomon of measurement. Put simply, the labyrinth is not pre-existing in any sense, but it is the space opened up by all these meandering paths which serves to embrace its sacred center. This is at any rate the route we have to take in case the obstacle is an obstacle in time.

Actually this is always valid, even in the simplest case whereby we think of an obstacle as a hole in a single linear dimension, where the latter is considered as the line of inertial variation, i.e. the straight path of motion with constant velocity. The attendant need for an imaginary dimensionality, orthogonal to the initial real dimension, to bring into central focus the "unit circle" enunciated in potential or power terms via the exponential function, and culminating in the arithmetics and geometry of the complex numbers, should be properly thought of in temporal terms and not in spatial ones. In other words, the imaginary unit, or imaginary ring, supplying the complex structure, is a precise *symbolon* of metaphora through another level of *hypostasis* that allows the embracing of the obstacle localized at the zero point of the complex plane.

The notion of a symbolon bears a periodic temporal connotation, as opposed to its static reflective spatial manifestation as a symbol, which is further deteriorated to a mere formal sign. This connotation amounts to the instantiation of a rhythm that bears the potential to persist covariantly and resolve a moment of time, as manifested in the linear real dimension, into a vertical, orthogonally placed, spectrum of imaginary-valued frequencies. This resolution induces a *metabole* in both the qualitative and quantitative conceptions of time around the obstacle that spatially are manifested as relative phases on the unit circle, functioning in this sense as the homeostatic symbolic imaginary locus, i.e. the base sign-recording indexing of the metabole. Simply put, this is the subtle difference between an icon as a temporal and rhythmic symbolon and its spatial image as a semiotic indexing snapshot. From this viewpoint, an indexing real-valued sign should be never considered independently of its spectral depth in terms of the associated imaginary-valued frequencies. In all interesting cases, this frequency spectrum is actually quantized, i.e. it is displayed in terms of discrete quanta engulfing the integer multiples of some fundamental period, for instance, the period of circulation around the obstacle.

From a purely topological standpoint, the localization and unfolding of an obstacle may be equivalently thought of as a process which exhaustively encompasses and universally covers the obstacle. In the case of a single obstacle, manifested as a hole on the plane, the universal covering space is displayed as a helix, whose different layers correspond to the different frequencies comprising the universally covering or unfolding spectrum. The underlying topological idea is that an obstacle is the source of multiple-connectivity, enacted by all different types of paths embracing it, which are classified by their characteristic winding number. The universal covering space in this case may be thought of as a discretely fibred space, i.e. over any real-valued point on the base there is a discrete fiber above it consisting of the frequencies of the spectrum. A fiber is the carrier of the spectral depth of a base point in the sense that a loop based at this point and embracing the obstacle is uniquely lifted to the universal covering space, such that its starting and ending point belong to the same fiber above the base point and their relative spectral difference on this fiber qualifies the winding type of the base loop. All fibers of the universal covering space are not disconnected from each other, but cohere; there is a connectivity structure that binds them all together, giving rise to a helix. The important thing is that the helix is simply connected, i.e. the universal covering space is actually a geometric space, meaning that it bears a geometric form, and as such it resolves the multi-connectivity issue of the base induced by the obstacle. It is precisely in this spectral manner that the simply-connected obstacle-covering geometric form topologically unfolds the obstacle.

The topological unfolding process of universally covering an obstacle, giving rise to a simply-connected geometric spectral form, can be properly supplemented with the complex structure induced fiber-wise by the imaginary unit bearing the function of the symbolon of metaphora through another level of hypostasis. The setting here pertains to a multi-valued complex function deemed to be analytic; there exists a power series expansion of this function locally, and the purpose is to extend this local domain of definition by a method called analytic continuation, effectively furnishing a connection expressed locally by means of imaginary-valued gauge potentials. Singularities appear, for instance, in terms of poles where the function becomes infinite. Then, the unfolding procedure can be implemented in this context with the exception of these singular points of ramification, where branching behavior appears. Intuitively, the fiber over this point becomes degenerate collapsing to a single point, through which different branches collide together and reciprocally open up into well-defined multiplicities.

The significance of the complex structure in this setting, or equivalently, the conformal structure, properly lies in the domain of harmonics. The imaginary ring, i.e. the gnomon qualifying the conformal structure, gives rise to a type of resonator, where the unfolded branches, like bounded helical strings of a musical instrument vibrate under the action of the imaginary gauge potentials. In this way, a discrete series of harmonics is instantiated that have the capacity to resolve a moment of real-valued time, together with their consonances and dissonances. This rhythmic enunciation of the imaginary ring together with the harmonic series admits a choreographic interpretation in terms of oriented angle-preserving transfigurations, called conformal morphisms, looking like global metamorphoses from a geometric perspective that respect only the relative angles so as to maintain the rhythm. Another viable way to think of this choreography is in the terms of a helical standing wave and its associated harmonics, where this wave is not thought of as being in space in any sense, but rather, space arises out of its vibrations, periodicities, and resonances conformally in historical time. The latter is not comprised by the real-valued moments of events, but rather, it pertains to the coherent aggregation of all fibers bearing the harmonics over these moments, or equivalently, the quanta of resonance and synchronization. In a nutshell, complex harmonics is an expression of the economy of historical time in its entirety as it is unfolding through metaphora.

In this manner, and referring to the domain of complex harmonics, the imaginary ring is like the translocal imaginary umbilical cord, which as a symbolon of metaphora through another level of hypostasis, forces the translocal *homeosis* of the local with the global in terms of angle-preserving periodically repeating, and thus, homeostatic iconic tessellation of the whole by the part; In the two-dimensional case of an unfolded *epiphaneia*, this gives rise to only three distinct universal covering, simply-connected, geometric forms characterized by constant positive, zero, or negative curvature respectively. Simply put, the geometric form arises out of complex harmonics as a homeostatic crystalline epiphaneia of the choreography subordinate to the imaginary rhythm. It is a crucial fact that the domain of complex harmonics does not bear any distinctive ontology, rather its nature is akin to a magma. Notwithstanding this fact, emanating from the reciprocal and complementary relation between the chords and the potentials with respect to the imaginary resonator-symbolon in the acoustic articulation of the pure harmonics, a magma has *entelechy*, which is precisely manifested geometrically in one of three universal distinctive curvature forms.

The geometric, hence visual, form of the universal covering epiphaneia in the setting of complex harmonics, requires for its articulation a *canon* of *metalepsis*, effecting a transgression from the acoustic domain of pure harmonics to the visual domain of colours. This requires the detuning of the pure harmonics within equally-partitioned chromatic intervals. In this sense, the canonics from the acoustic domain to the visual domain amounts to a process of imaginary logarithmization with respect to equally-distanced angular intervals, so that any pitch in the chromatic domain is situated at equal distance from its nearest neighbours. The canonics of this *heteromorphism*, which essentially arises out of the adjunction of the chromatic to the harmonic domain, transfuses an affine character to the rhythm in its chromatic manifestation that allows its generation and progression infinitesimally and differentially in a continuous manner. The suppression of the pure harmonics determined by the type of the obstacle, i.e. the quotient of the induced chromatic spectrum by the module of pure discrete harmonics, leaves a trace for imprinting a memory element on the imaginary ring of metaphora revealing thereby its global role. More precisely, this trace is a global non-integrable relative phase factor, i.e. a global irreducible residual phase marking the *anholonomy* of the metaphora due to the embraced obstacle. As such, it incorporates both topological and geometric information. On the other side, the suppressed pure harmonics, as the implicit harmonic invariants, guide the extraction of the complex roots of unity on the imaginary ring. In a well-defined way, the complex roots of unity negate the punctual character of any real-valued moment, in the sense that they open up channels of potential resonance via the harmonics with other non-locally related moments, recorded as geodesic paths of connectivity in the corresponding universally unfolded curved geometric form.

In all cases concerning the embracing of an obstacle, the metaphora is effective if the cyclic flow is communicative, i.e. it establishes bidirectional bridges connecting any two hypostatic levels. It is in this sense that an "obstacle-oriented" approach is not a directly "problem-solving" approach, but an indirectly "problem-embracing" one. The different levels are not in any relation of *hypotaxis* of one to another, but only in a relation of *parataxis*, and what really matters is the facility in the passage from one level to another in a bidirectional way via the bridges. This facility requires that the bidirectional bridges should not be ad hoc, i.e. they should not be designed as referring to particular subjective choices, but they should apply under covariant variations on their respective inversely related hypostatic domains and codomains. This requirement is captured by the adjective *"natural"*, which pertains to a double articulation of an algebraic nature: First, the hypostatic domains and codomains should be though of as *categorical*, and second, the bridges between them should be *functorial*, i.e. not depending on particular choices of objects in the interrelated categorical domains and codomains of the bridges.

Generally speaking, algebra pertains to a structural enunciation of the "obstacle-embracing", communicative process of metaphora between any two hypostatic levels. This enunciation is formulated operationally in terms of symbolic algebraic structures like groups, rings, modules, and categories. The notion of an algebraic symbol does not bear, neither the temporal connotation of a symbolon, nor the spatial connotation of a sign. For this reason, an algebraic structure maintains an independence from both the harmonic, and the geometric connotation of its symbols, although it may properly mediate between them and abstract from both of them. The key idea is that an algebraic structure, in the context of the "obstacle-oriented approach", plays the role of a structure expressing both the invariant, and the covariant characteristics of the metaphora with respect to some specified notion of unity or equivalence displayed as an algebraic identity. For instance, in the case of a group structure it is the notion of a neutral element with respect to the implied operation that characterizes the algebraic identity. This becomes prominent in the case of structure-preserving morphisms, like homomorphisms and isomorphisms that transfer the group structure in some particular way. In this sense, the concept of a unit implicated by a gnomon, like a gnomon of discrete counting or rational measuring, is imprinted symbolically in the neutral element of the associated group structure. This is indispensable, since inversion of an operation is not feasible without the specification of the neutral element in any group-type structure. The ability to form inverses in achieving the closure requirement of a group-type structure is conditioned on the extension of elements into algebraic power domains, which elevates the exponentiation operation to a place of prominance, subsuming the operations of addition and multiplication. Its inversion, i.e. the logarithmization operation, cannot be performed unconditionally.

More precisely, the expression of the irrational numbers via the real logarithm requires a topological continuity condition, whereas the expression of the imaginary numbers via the complex logarithm requires a local topological simple-connectivity condition. In other words algebraic operational definability and topological continuity or connectivity become inextricably intertwined and inseparable in the transcendental realm. This is exactly the reason that a group-type of structure is, in principle, capable of expressing invariants of a topological nature.

The transcendental realm, as the cumulative realm of algebraic powers and their inversion or reciprocation, is particularly suited to express symbolically potential circular flows embracing obstacles of any type. This leads to the idea that there should exist a minimal, in the sense of the most economical, algebraic symbolic description of a metaphora between any two hypostatic levels that can be qualified as structural. In other words, any metaphora with respect to an obstacle should, in principle, be expressible in terms of a symbolic *"motivic key"* capable of unlocking operationally the communication capacity between the domains it is applied to. Thus, its function is doubly significant: First, since the key is symbolic, it always allows the structural encapsulation of the metaphora in terms of invariants, which can be qualified effectively by means of standard algebraic structures; Second, since the key is motivic, its economy is not subordinate to artificial choices pertaining to the bridging of the hypostatic domains, and thus, it should ne natural, fulfilling the requirement of covariance.

Regarding the issue of "naturality" in the specification of this symbolic motivic key, in the general case that the communicative domains-levels are heteronymous if hypostasized structurally, it is essential that the structural qualification follows from a deeper algebraic categorization that admits the possibility of heteronymous bridging via heteromorphisms. This is not imposed on the basis of some universal axiomatic system, but it arises out of necessity when operating in the heterogenous symbolic. The reason is that heteronymous domains cannot be bridged together directly, but require a certain canonics that makes them partially structurally adaptable to each other. We have already introduced this notion in relation to bridging together the harmonic with the geometric domain, or the acoustic with the visual domain. Here, algebraic canonics of metalepsis assumes a particular type of heteromorphisms between the concomitant categorical characterization of these domains, which can be internalized homonymously, i.e. expressed in terms of homomorphisms within each category, under the existence of initial or terminal objects via which the potential cyclic flow factors through. In categorical language, they are called *adjunctions*, expressed in terms of natural isomorphisms arising from the bidirectional bridging of the relevant categorical domains via adjoint functors. It is important to emphasize that this is how "natural communication" inverts the usual understanding of category theory, since it is the canonics between heteronymous domains that entails the category-theoretic characterization of those domains. In other words, the latter appears out of the *metaleptic economy* characterizing the "natural communication" between heteronymous domains in the terms of adjoint functors, and not as an appeal to any type of structuralist foundations.

Since the central issue is the notion of an obstacle and the metaphora required to embrace it via different hypostatic domains capable of entering into "natural communication", the notion of a foundation is totally misconceived. Instead, what is crucial always is the notion of an architectonic scaffolding that potentially is able to bridge together these domains so that the metaphora leading to the communication between these domains can be elucidated. The framework of adjoint functors in category theory pertaining to the symbolic and operational aspects of this metaphora can be appreciated if evaluated properly in its function as a sophisticated abstract architectonic scaffolding. The harmonic, topological, and geometric aspects of the metaphora, as explicated concisely above, also bear an indispensable role in the articulation of "natural communication" and the efficacy of abstract gnomonics in this fashion, which would be impossible by restricting only to the categorical and the symbolic realm. Put equivalently, the algebraic symbolic *"morphe"* should be consistently elaborated and ingrained by the harmonic symbolon of the metaphora, together with its topological schematism, and eventual geometric spectral form.

At a further stage, and as a result of this elaboration, the bridges between different hypostatic domains in the most economical articulation of "natural communication" in the terms of a motivic key can be thought of as *cobounding* the conditions of communication between these domains. Since cobounding is always of a local or even infinitesimal nature, these conditions transcribe the norms of harmonic congruence that can be eventually unfolded multi-periodically in some simply-connected geometric form. The important thing is that the norms of harmonic congruence allow the evaluation of any other possible metaphora between the domains involved with respect to this norm, i.e. *cohomologically*. This means that the qualification of information under a process of metaphora involving the communication between two domains is essentially cohomological, or else, entropy as a measurable magnitude of this information is cohomologically quantifiable.

The model of "natural communication" envisions an architectonics of communication as the most prominent conceptual stance in tackling the problem of complexity. The founding realization is that such an architectonics of structural relations, which are based on communication between appropriate correlated domains, poses itself as a necessity in all these cases, where obstacles and obstructions of any particular type prohibit or prevent the direct accessibility to hypothetical sharply distinguishable elements of complex domains. The shaping of objects in these domains takes place according to foamy patterns, characterized by topological plasticity, emergent properties and generically probabilistic attributes. The application of pre-specified, ready-tailored constitutional and elemental design ontologies to these complex domains, according to analytic methods designed for those ideal cases where sharp elemental distinguishability is feasible, not only distorts the architectonics of their intricate connectively weaving patterns, but limits and restricts, even inadvertently, their potential computational capacities.

The basic idea to address and utilize the architectonic modelling of these domains is the following: Instead of following the standard analytic method of dissecting ontologically complex objects, i.e. objects situated in non-directly accessible domains, in terms of the collections of their hypothetical set-based elements and their concomitant absolute relations, we adopt a synthetic method: We let them unveil themselves by adjoining to them other adequately-understood, or directly accessible domains, which can provide pointers and open up communication channels with the complex ones. The difficulty in this synthetic act rests on the fact that an adjoined domain should be capable of opening up such a communication channel, otherwise its adjunction is vacuous. The crucial condition that should be fulfilled for this approach is the viability of setting up a bidirectional bridging scaffolding of encoding/decoding relations between these domains.

These relations may be thought of as giving rise to a sieve through which the eventual unveiling of the connectivity patterns characterizing the objects of the complex domain becomes effective. In this sense, the study of the architectonics of communication is tantamount to the realization of the possibilities of unveiling a complex or obstacle-laden domain through an appropriate weaving sieve. A sieve with this function bears the capacity to reveal the intricate bonds beyond direct access, which give rise to the coherence of the complex domain. Therefore, complex objects are unveiled via percolation through a sieve of communication relations, an invariant process which is depicted precisely by the ancient Greek term *"aletheia"*. In this setting, the synthetic act of adjoining a domain for communication should not be ad-hoc, and should not depend on artificial choices, meaning that it should be designed to respect any structure encountered, at least, locally or partially. This refers to the "naturality requirement" of the model, a term which belongs to the register of category theory. Technically, as it will turn out as the argument develops, the adjunction process is concretely modelled in terms of a pair of adjoint functors between the involved categorical domains.

In practice, the process of adjoining a controllable or directly comprehensible domain to a complex, or obstacle-laden, categorical domain, is tantamount to considering these domains as different categorical levels in a broad stratified universe of discourse, which are amenable to bi-directional correlation by means of oppositely or reciprocally oriented bridges. The architectonics of communication targets precisely the conception, explicit construction, and manifestation of these bridges, once suitable probing domains have been structurally delineated, due to their partial or local congruence properties, to be adjoined to the directly inaccessible complex domain. The bridge directed from the latter to an adjoined controllable domain plays the role of an encoding bridge, whereas its inverse or reciprocal plays the role of a decoding bridge. These level-interconnecting bridges effect the communication between the involved domains, in the sense that they bear the capacity to open up and establish natural and universal bidirectional communication channels through which a schema of metaphora can be accomplished, which is based on the notion of an "obstacle-encircling" flow.

From a topological standpoint, the initially inaccessible categorical domain is being spectrally resolved in the fashion of a spiral, i.e. by a process of cyclic unfolding with respect to the various probing domains that have been adjoined to it, corresponding to different layers of spectral resolution. In this manner, the invariants emerging by the process of unfolding depict the substantive characteristics of the reciprocal communication flow between the domains involved. Consequently, the complexity of the non-directly accessible, or obstacle-laden, domain is not specified constitutionally on the basis of a pre-assumed or axiomatic elementary ontology, but relationally and functorially, in terms of the spectral invariants emerging in the bidirectional communication flow established with suitable probing domains. From an algebraic viewpoint, these invariants can be described in terms of structural group-type ciphers for the symbolic encryption/decryption of the induced flow.

The obstacle-embracing schema of metaphora giving rise to this transitory multi-levelled cyclic flow, called the "logical conjugation method", is always implemented on the basis of adjoining controllable or directly accessible domains to an obstacle-laden domain in their role as markers, or pointers, or more generally gnomons, providing eventually the means of unveiling it indirectly through communication. This constitutes a legitimate synthetic logical manoeuvre in the specification of a complex domain that has the capacity to unveil it by means of metaphora through the weaved sieve of encoding/decoding relations. The temporal percolation due to this sieve constitutes the –aletheia– of the complex domain, that is, what is unveiled and should not be forgotten about the coherence of this domain. The logical conjugation method effectuating the metaphora plays the functional role of a motivic key that bears the potential to harmonically unlock the complexity of the inaccessible domain, depending on the nature and type of the obstacles located. The creative art consists in the innovation of the reciprocal encoding/decoding bridges acting as the means of communication between an obstacle-laden domain and a conjugate accessible domain.

The function of the gnomon, enunciated through the encoding/decoding bridges, is instrumental in the eventual schematism of a partition spectrum pertaining to the obstacle-laden domain. Each partition block or cell of this spectrum is characterized completely by the equivalences induced by this pair of encoding/decoding bridges utilized for communicating an aspect of the non-directly accessible domain with respect to the probing conjugate domain. An important stipulation for the role of a probing domain, bearing a gnomon of partial congruence with respect to the obstacle-laden one, is that it is not to be considered, in any sense, as the foundational background of the latter, namely in terms of absolute elemental constitution, as would be the case in mathematical set theory. Rather, the functional role of a probing domain is to open up a communication channel with the complex domain, such that the partial congruence between them can be properly thought of as a resonance within certain intervals of frequencies comprising the induced partition spectrum. In other words, a probing domain, endowed with a gnomon enunciating the encoding/decoding capacity of the associated communication bridges with the complex domain, gives rise to an architectonic scaffolding for the qualification and quantification of the information gained by the embracing of an obstacle characteristic of the complex domain.

The simplest possible articulation of the proposed schema invites us to consider a problem in the context of a complex domain, i.e. a domain whose objects and relations are non-directly accessible due to various types of obstacles. First, we have to move out of the context of the initially posed problem, formulated at the level of this domain, since it is not directly accessible, due to obstacles preventing any possible type of sharp analysis. For this purpose, we act synthetically by adjoining to the complex domain another controllable probing domain at least locally. In order to accomplish this, we have to set-up an encoding bridge from the level of the inaccessible domain to the level of the accessible probing domain. Once, we have succeeded in setting up this bridge, we are able to transfer the initial problem, even locally, at the level of the controllable domain, where the means to resolve it effectively exist. This is tantamount to the construction of a resolving partition spectrum that covers the complex domain, and groups its communicable attributes with the probing domain into distinctive blocks, i.e. communication channels. The schema is completed by setting up an inverse decoding bridge from the level of the controllable domain to the level of the inaccessible one. In this way, the resolving spectral capacity of the controllable domain can be raised at the initial context of the problem, and thus, the problem is indirectly resolved through decoding in the context of its initial formulation. The reciprocal encoding and decoding bridges constitute the means of a novel architectonics of communication. It should be emphasized that the procedure implemented above can be iterated by the involvement of more than one controllable domains, which are adjoined respectively to the inaccessible domain. The skeleton of this algorithmic procedure of resolution remains invariant under the adjunction of deeper spectral levels. In all cases, there emerges a partition spectrum of the inaccessible domain, whose cells are indexed by the respective pairs of encoding/decoding bridges. The procedure of resolution conducted by synthetically adapting motivic keys to an inaccessible domain, according to the above schema, is technically called logical conjugation, whence the communicating domains, represented by levels inter-connected by the encoding/decoding bridges, are called conjugate domains. The adjective "logical" adopted for the conjugation method, is intrinsically related to the fact that the unveiling of the complex domain in this way, constitutes its -aletheia-, a notion that extends and enriches conceptually the standard bare logical notion of "truth".

1.3 THE "OBSTACLE-ORIENTED" APPROACH: INVARIANCE, ACTION, AND *GNOMONS*

First of all, it is worth focussing on two interrelated aspects of what we term the "obstacle-oriented" standpoint of enquiry. The first issue refers to the nature of metaphora around an obstacle. Simply stated, in what way is a cyclic flow initiated, giving rise to communication between two heterogeneous levels of hypostasis by connecting them through bidirectional bridges? The second issue refers to the specification of an obstacle as a source of invariance with respect to the specific context of its localization. These two issues are closely related by the temporal notion of action. In turn, it is the notion of action that qualifies the notion of connectivity independently of any spatial instantiation.

We start following the thread that identifies an obstacle as a source of invariance. The notion of invariance is not absolute, but it is modular, meaning that invariance is meaningful only within the specific context of localization of an obstacle. Invariance can be operationally characterized only through action directed initially away from the level or context of the obstacle. The effect of action is to initiate a stream flow that is capable of retracting the inaccessibility or obstruction imposed by the obstacle to some generic situation at another level through which a passage becomes viable, and then re-direct the flow black toward the initial level, so that the obstacle can be embraced. Successfully embracing an obstacle always leaves a residue, to be thought of in terms of "countable quanta of *metabole* by periodic action". These quanta are spectral quantities, denoting rhythmic arrangements within regular temporal cycles, to be thought respectively as frequencies. Most important, these quanta encode the invariance of the obstacle they refer to with respect to all possible embracing circular flows initiated by temporal actions.

The crucial issue is that a residue of a cyclic stream flow is something associated with a differential, i.e. it is the result of an integration procedure along a temporal cycle surrounding the obstacle. This has the following consequences:

i Temporal cycles can be distinguished only by the countable number of winding actions around the obstacle, and thus, the frequency spectrum is indexed or quantized by an integer number;

ii A differential is a quantity that does not assume any value at a point, meaning that it is viable only in germinal form with respect to a cloud or foam surrounding any point around the obstacle;

iii As a consequence of the above, an indistinguishability or ambiguity is induced in relation to the stream flow, expressed by the notion of the multiple-connectivity associated with an action;

iv In turn, this multiple-connectivity is annihilated by unfolding the domain of action continuously in successive branches, giving rise to a helicoidal staircase until uniformity or simple-connectivity prevails;

v The universal helicoidal staircase as the unfolded domain of all temporal actions to embrace the obstacle constitutes the quantized spectrum of these actions;

vi The nature of metaphora is explicated by the connectivity bridges between the level of the obstacle and any level of the helicoidal staircase together with the associated countable quanta of metabole;

vii In terms of metaphora, the invariance of the obstacle is expressed in a unequivocal way upon perpetual completion of all temporal action in terms of the anholonomy of metaphora;

viii The anholonomy of metaphora serves as the memory element of the process of encompassing the obstacle.

In context of the obstacle-oriented approach, there are two further intertwined issues that deserve special attention. The first has to do with the fact that the notion of quanta requires something that distinguishes among them, meaning a mark or a gauge or a boundary. The second has to do with the conditions of qualification of a quantum as a spectral information unit. These two issues open up the vast subject we address by the term *"gnomonics"*, i.e. creatively devising gnomons suited to the *physis* of obstacles that allow flow streams of communication, or else chains of connectivity, to bind together different levels of hypostasis.

The gnomonic enrichment of the obstacle-oriented approach is inseparable from the conception of obstacles as modular sources of invariance. This has been the case since the beginning of natural philosophy and natural science initiated by the magnitude measurement of the height of a pyramid by Thales. It was his innovation to use a vertically placed measuring stick as a gnomon, which to the theory of *homeothesis* connecting the level of actual objects with the level of their shadows by means of proportionality, or invariance of angle, under the temporal action of light fixed at the same time of the day. The notion of an appropriate gnomon in the context of any type of obstacle is always instrumental for obtaining a spectrum, consisting of distinct equivalence classes, or partition cells, or fibers, or finally, distinguishable orbits of a multiply-connected temporal action generating a stream flow around an obstacle.

The abstraction of the initial connotation of a gnomon from its association with the sun-dial of a stick emerged early during the flourishing period of the Alexandrian school of mathematics. A gnomon with respect to an obstacle becomes any suitable form with the following property: If the gnomon is adjoined to the obstacle-laden form it gives rise to a new form self-similar to the original one. This conception of the function of a gnomon in relation to an obstacle is not only ingenious, but it paves the way to a gnomonic derivation of the whole framework of category theory, which represents undoubtedly the most abstract part of modern mathematics. At this early stage, we mention for the sake of the curious and eager reader that the notion of adjoint functors forming an adjunction in category theory is nothing else than a technical elaboration of the way a specific category of frames serves as a gnomon with respect to another obstacle-laden and non-directly accessible one. For the time being, it is enough to highlight that the invention of a suitable gnomon with respect to an obstacle always gives rise to bidirectional bridges binding together the level where the obstacle is located with another level of hypostasis whose meaning derives only from relation to this gnomon. This constitutes the algebraic manifestation of the function of a gnomon. In cases where the bridges are exact inverses to each other, a group structure is generated, where the gnomon is enciphered as the neutral element of this group. In case that the bridges are only conceptually inverse, or else, adjoint to each other, what is generated is rather a categorical *monad* structure.

It is necessary now to examine in more detail the relation between these different conceptions of obstacles as modular sources of invariance and gnomonics. We consider the case that the algebraic manifestation of the function of a gnomon gives rise to a group of temporal actions. The notion of a group is associated with the conception of symmetry relations established by the action-elements of this group. Symmetry means common measure with respect to a standard of measurement, or a standard of demarcation, which is to say a gnomon. In other words, the idea of symmetry is subordinate to the function of a gnomon. Coming back to the context of our inquiry, symmetry of temporal actions can be established only with respect to a standard of partitioning for these actions into distinct classes of equivalence with respect to this standard. Given that these temporal actions initiate cyclic stream flows around the obstacle, such a standard of partitioning can be enunciated only in terms of boundaries for these flows. As such, the notion of a boundary serves as the cipher, or equivalently, the neutral element of the group structure partitioning symmetric temporal actions into distinct cells, and thus, forming a spectrum.

In more technical terms, we obtain for our purposes, a *metabasis* from *homothesis* to *homology* and *cohomology* by inventing a new gnomon bearing the property of topological deformation invariance with respect to the obstacle. This new gnomon is more potent in power than the homothetic one, since

a boundaries form an Abelian group structure, meaning that the cipher becomes of a structural type, and
b the gnomon detects and operates on germs of actions due to the deformation invariance property.

Due to these properties, the modular source of invariance forced by the obstacle is qualified in terms of symmetric, and thus, equivalence classes of germs of temporal actions with respect to the gnomon giving rise to a spectrum. Therefore, the pairing between homology and cohomology is spectrally induced, providing countable means of distinguishing among different

> quanta. Each quantum, namely each symmetric class of germs of temporal actions, specifies a concrete rhythmic arrangement within a regular temporal cycle surrounding the obstacle; it is of a certain harmonic nature.

The above brings us to the second fundamental issue of gnomonics in the context of the obstacle-oriented approach, the one pertaining to the conditions of qualification of a quantum as a spectral information unit. More precisely, we have seen how the function of this deformation invariant gnomon induces a spectrum, each cell of which is indexed harmonically in terms of "countable quanta of metabole by periodic action". The first crucial thing to stress is that the whole spectrum appears only through the simply-connected universal helicoidal covering of the obstacle. Since this universal covering is the global and uniform unfolded domain of all temporal actions embracing the obstacle, we conclude that a quantized spectrum becomes manifest only in a simply-connected domain.

In other words, the manifestation of a spectrum necessitates the gnomonic annihilation of multiple-connectivity associated with a temporal action. In turn, multiple-connectivity is of a foamy and ambiguous nature, that is, it prevents the distinction among streams of flows around the obstacle. Given that these streams amount to chains of connectivity, it is only through boundaries that this fog of objective probabilistic ambiguity may be gradually lifted in time. Note that this form of ambiguity is of an objective character, since it is not based on any type of subjective ignorance. This lifting is enacted by the perpetual completion of all germs of temporal action until simple-connectivity prevails gnomonically and the spectrum of distinguishable quanta becomes manifest.

The second crucial thing to stress is that a distinguishable quantum of the spectrum does not constitute a unit of spectral information yet. It becomes such if and only if it is actually distinguished. Put differently, the spectrum is only the pre-condition for conveying information in terms of quantum units, but it does not qualify any of its elements as pre-existing units of information without any act of actual distinction among them. This can be explained by the fact that the spectrum constitutes the articulation of the symmetry of the gnomon, and as such, the spectrum is precisely the bearer of the simple connectivity emancipated by the gnomon. On the other side, the objectification of a quantum of a certain frequency from the spectrum as the bearer of in-formation, requires an actual distinction that breaks its connectivity bridges with all other symmetrically connected ones with respect to the gnomon. We conclude that a spectral information unit amounts to breaking the symmetry in the connectivity pattern of the gnomon, the latter being the price for information. Note that, by the principle of gnomonic constitution of the spectrum, a spectral information unit pertains to germs of temporal actions, and thus, it is a modular unit of actual distinction between different cells. Henceforth, maximal symmetry amounts to minimal information, whereas minimal symmetry amounts to maximal information. Put differently, connective symmetry and spectral information constitute a *Galoisian* complementary pair with respect to the function of the gnomon in the context of the obstacle-oriented approach.

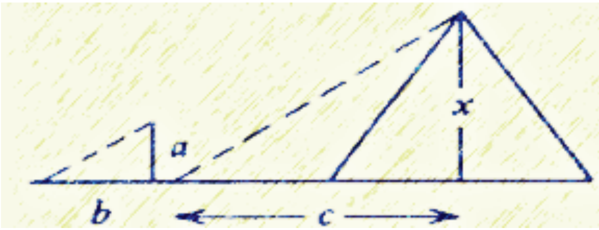
1.4 *HOMEOTICS* OF COMMUNICATION: OBSTACLE-EMBRACING *METAPHORA*

The notions of *analogia* and *metaphora* are considered in their broadest possible meaning, where a quantity, or an object, or a structure, or even a category, is amenable to a certain process of comparison with another of the above kinds. In this sense, analogia and metaphora are detached from the restricted cognitive and linguistic connotations that the widely used notions of analogy and metaphor carry with them. Notwithstanding this fact, the former notions are included as special cases of analogia and metaphora, which are endowed with a well-defined mathematical content, being at the basis of what we call "-natural communication-". The underlying basic idea is that the purpose of this process of comparison is to embrace the obstacles, which are associated in some specific manner with some domain occupied by any of the above kinds through transference to another conjugate domain. The transference is thought of bidirectionally, which means it involves both an encoding and a decoding bridge from some domain to its conjugate domain. The obstacle-embracing function of analogia and metaphora is articulated through congruence or similarity of relationship between instances and not as similarity between the instances themselves pertaining to any of the above kinds. This is a subtle difference that needs to be emphasized, since it discloses the basic characteristic of any analogia or metaphora underlying the power of this function and binding it with the notion of communication between conjugate domains.

Thinking in terms of instances in two different domains between which a congruence or similarity of relationship is established, one is generally not directly comprehensible or accessible, while the other is assumed to be better or more easily tractable. It is important to clarify that according to the above, an analogia pertains to a congruence relation not between two instances, but between the *relations* of these two instances. Thus, an analogia as a congruence relation between relations, involves (at least) two terms, each of which is itself a relation. This congruence relation between relations is called a *homeotic* relation.

As a simple first example, if we use the scaffolding of naive set theory, such that a simple relation is thought of as a binary relation between two sets, then an analogia requires four terms in order to be expressible. The four terms are distributed in two distinct levels or domains, where two of the four terms are placed on the same level so as to express a simple relation. Furthermore, three of the four terms are assumed to be known or directly measurable, or accessible, or more generally, determinable by some method, and the purpose is to determine the fourth.

The primary example of the notion of analogia emanates from Thales' theory of homeothesis or proportionality. It is important to emphasize that the purpose of Thales' theory of proportions had been the measurement of non-directly accessible magnitudes. More concretely, the objective of Thales was to find the directly inaccessible height x of a pyramid, given the length c of its accessible shadow, as well as the height a, and the shadow length b, of an accessible object placed vertically as a reference stick, which plays the role of a measurement rod in homeothesis. The analogia devised by Thales for the resolution of this problem is based on the idea that light coming from the sun induces a congruence relation between the level of heights and the level of shadows for each specific time recording the magnitudes of all the four variables involved. The analogia of homeothesis is expressed symbolically as follows:



(a to b) is as (x to c)

In algebraic terms, the above analogia is expressed by the simple equation $\frac{a}{b} = \frac{x}{c}$, from which the non-directly accessible magnitude x can be determined indirectly as $x = \frac{ac}{b}$. Note that the four terms of this proportion between magnitudes are arranged into two distinct levels according to some qualifying characteristic, i.e. a and x occupy one level as vertical heights, whereas b and c occupy the other level as horizontal shadows.

In point of fact, Thales provided a geometric solution to the problem addressed by homeothesis, since the set-up involving the algebraic equation of proportionality of magnitudes together with its simple algebraic solution presented above, was not available at that time.

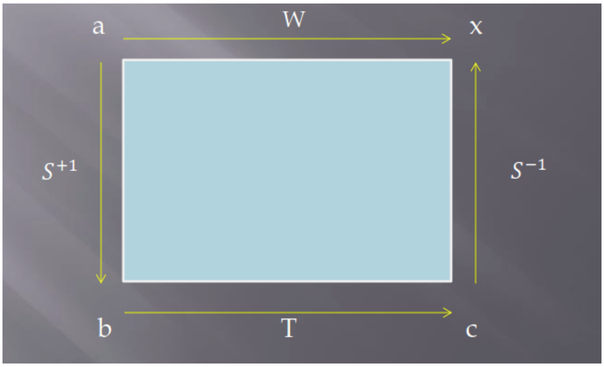
The important fact is that the solution of the proportionality equation involves the group-theoretic operations of multiplication and, inversely, division of positive integer magnitudes. Thus, from the viewpoint of natural communication, the geometric theory of proportions, i.e. the theory of homeothesis, contains all the seeds of abstraction leading to the conception of the modern algebraic structure of a multiplicative group.

In particular, given the multiplicative monoid structure of the positive integers, the solution of the Thalesian problem of proportionality or analogia of magnitudes requires the option to invert the multiplication operation, i.e. it requires the operation of division. In turn, since this is not possible in the context of positive integers, the operation of division entails their algebraic extension to the wider context of the rationals, culminating in the multiplicative group structure of the rationals.

Conclusively, the determination of an unknown magnitude in the Thalesian setting, by analogia, interpreted now algebraically, requires the introduction of the multiplicative group structure of the rational numbers in order to provide a solution to the associated proportionality of magnitudes equation expressing that analogia. In a suggestive manner, we can rewrite the solution of this equation as follows:

$$x = M_a c M_b^{-1}$$

meaning that to obtain the non-directly accessible magnitude x, "multiply by a" (denoted by M_a) the magnitude c, and then, divide by b (denoted by M_b^{-1}). Thus, the determination of inaccessible magnitudes by means of analogia, algebraically necessitates the introduction of the group-theoretic closure structure on magnitudes, equipped with the operation of multiplication and possessing an inverse, which is division.



By extrapolating, we may assume that the resolution of a more general problem, based on analogia (not restricted to the situation of proportionality of magnitudes) implicitly requires for its algebraic manipulation the following:

Firstly, the distribution of the four terms of an analogia into two distinct levels, two of the four on each level, where three of the four terms are assumed to be directly determinable, and the purpose is to determine the fourth.

Secondly, the introduction of an appropriate closed algebraic structure with respect to a process that bridges together the two distinct levels, playing a similar role to the operation of multiplication (between magnitudes at different levels). This multiplicative adjunctive process can be thought of as a directed bridge which connects the upper level with the lower one, where each level is occupied by things belonging to the same class, or domain, or universe of discourse.

Thirdly, the possible determination of the inverse to the multiplicative adjunctive process, called the division process. In many of the cases an exact inverse process (being suggestive of the global schematism of reversibility via another level) may not be attainable, and thus, partially or locally inverse processes should be employed, satisfying appropriate conditions.

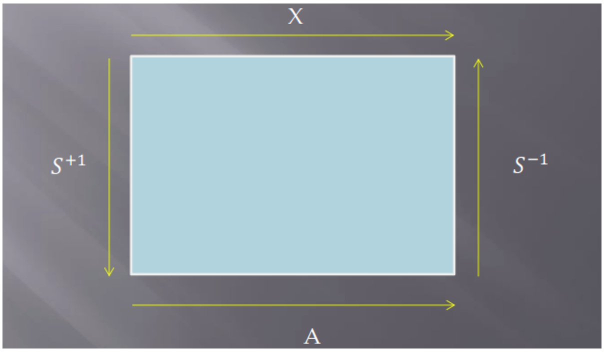
According to the above, in the case that an exact inversion process is available or globally constructible, facilitating an effective exact round-trip between two delineated levels, we call the analogia a metaphora. This conception has an *Aristotelian* origin, formulated in the statement in *Poetics*, according to which: "Metaphora is the substitution of the name of something else, and this may take place from genus to species, or from species to genus, or from species to species, or according to proportion."

Projecting this statement back to the general environment of analogical relations, we conclude that a general analogy between instances may be concerned with class membership or class characterization.

In a nutshell, an analogia, formulated as a relation among four terms distributed at two distinguished domains or levels follows a unifying conceptual thread: Starting from a term at some level the determination of an inaccessible term with respect to the first, at the same level, via a cyclical global round-trip process through another understandable level, involving three stages:

First, setting up an encoding multiplicative adjunctive bridge of correspondence of the initial term at the first level with another term conceived as an instance at the other level. Second, processing the required task at this other level. Third, devising a decoding bridge of correspondence, inverse to the multiplicative one, which facilitates the return at the initial level, and simultaneously resolves indirectly the problem of inaccessibility, or equivalently embraces the obstacle encountered at the initial level.

Subject to the above, characterizing the general thought pattern of an analogia or metaphora, as an attempt to extract the conceptual essence of the Thalesian theory of proportions of magnitudes, and then, abstract it algebraically, we express an analogia or metaphora in terms of the following symbolic relation:



$$X = SAS^{-1}$$

where, the unknown X, at the obstacle-laden level, may be specified by an ordered three-stage process, through some quite easily determinable A at another viable level mediated via the opposite pointing bridges S and S^{-1} connecting the two levels. In case where the bridges S and S^{-1} are exact inverses, and A is considered to be noise-free or *homeorhetic*, we say that the analogy is effective, characterized as a metaphor. In the general case, where the bridges S and S^{-1} are not exact inverses to each other, but only conceptually inverse, they are called adjoint.

The underlying idea in all these cases is that a communication is established between these two levels capable of embracing the obstacles encountered at the initial level. The characteristic feature of an analogia is that a problem, located at a domain or level, requires for its effective treatment to move away from the context of this domain, i.e. to transfer the problem into another non-obstacle-laden domain. This is possible by designing appropriate encoding and decoding bridges between these two domains and following the three-stage process indicated previously, in the specified order. The effect of this, is that, as a result of the established congruence between the domains involved, the return to the initial domain, carries within itself the indirect solution to the problem that has been actually derived at the non-obstacle laden domain. Of course, the designation of the encoding/decoding bridges, being either exactly inverse or adjoint, is of fundamental significance for the operational or computational manifestation of a round-trip between these levels, interpreted as a process of communication. Note that communication is possible if and only if a specific global, or local, or even partial, congruence becomes attainable between the involved domains. The requirement is that the bridges are somehow stable, i.e. they are not dependent on ad hoc choices or oversimplifying assumptions. This is expressed by the adjective natural, which obtains a well-defined meaning in mathematical category theory, as we will discuss later.

From an algebraic viewpoint, the symbolic relation $X = SAS^{-1}$ admits a dual interpretation, namely one in terms of substances and another in terms of operations. In the context of communication, the operational interpretation is preferable, since it places the emphasis on the process devised for overcoming the initial inaccessibility. In this sense, the *symbol* X (indirectly determinable by analogia), followed by the sign of equality, may be interpreted as signifying the total ordered series of the three actions needed for its effective determination via another level, connected to its own by two bridges in opposite directions. It is also instructive to notice that the meaning of the operational interpretation can be captured even from its dual substantive viewpoint, under the convention that the symbolic relation of analogia can be extended in the notational form:

$$X(l_1, l_2) = S(l_1)A(l_1, l_2)S^{-1}(l_2)$$

where, the symbols l_1 and l_2 denote some kind of base locality or base indexing parameter.

1.5 ALGEBRA OF *METAPHORA*: CONJUGATION BETWEEN HYPOSTATIC DOMAINS

In general mathematical terms, the presentation of an effective analogia, or metaphora, in the symbolic form

$$X = SAS^{-1}$$

defines X to be conjugate to A under S, where S^{-1} is considered to be the conceptual inverse of S. This is a useful observation because it associates the algebraic principle of conjugation with the functional role of a metaphora. Since a specific algebraic structure is not pre-supposed ab initio, we call the principle leading to the algebraic expression of a metaphora as the principle of logical conjugation operating between two hypostatic domains.

The algebraic expression $X = SAS^{-1}$ consists of two basic organic structural parts: The first part is delineated by the two conceptually inverse vertically displayed arrows S and S^{-1}, forming the outer part, or the boundary of the analogia or metaphora, interpreted abstractly as a bidirectional bridge of information encoding/decoding between two different levels entering into a communication with each other. The second part is constituted by the horizontally displayed arrow A, forming the inner part of the analogia, and interpreted as a directed process of *rhesis* or *stasis*, i.e. transfer or storage, within the level specified by the first vertical directed bridge. Rhesis or stasis always give rise to a partition spectrum at the obstacle-free level, which contains the blueprint for embracing the obstacle of the initial level under the action of the decoding bridge. Note that the functionality of an analogia or metaphora is always crucially dependent on the interpolation of some appropriate inner part A between the succession of the actions of the inversely pointing bridges. More precisely, if the inner part A is absent, then the outer part simply collapses since it cancels out. Based on this fact, we can formulate the basic properties of logical conjugation as pertaining to an effective analogia as follows:

1 Horizontal Extension of Metaphora in Length: This is equivalent to the juxtaposition of two metaphoras, i.e. two metaphoras sharing the same bridges can be combined horizontally simply by juxtaposing one with another as follows: if $X_1 = SA_1S^{-1}$ and $X_2 = SA_2S^{-1}$, then $X_1X_2 = SA_1A_2S^{-1}$;

2 Vertical Extension of Metaphora in Depth: This is tantamount to the stacking of two metaphoras arising from the substitution of the inner part of a metaphora by another metaphora, such that, the initial metaphora can be accomplished via a splitting into a deeper level of hypostasis, and so on, as follows: if $X = SAS^{-1}$ and $A = TBT^{-1}$, so that, $X = STBT^{-1}S^{-1}$, then $X = (ST)B(ST)^{-1}$;

3 Inversion of Metaphora: This means that if a process X is conjugate to a process A at another level under the action of a bridge S, then A is conjugate to X under S^{-1}, as follows: if $X = SAS^{-1}$, then $A = S^{-1}XS$.

An interesting type of logical conjugation arises in cases where a bridge S equals its own inverse, that is $S = S^{-1}$. An immediate consequence is that if the inter-level transformation S is repeated twice in succession, then it gives the identity, viz. $S^2 = 1$. In this case the bridge S is called an involution bridge. The most well known example of an involution bridge is provided by any device operating strictly between two states, represented by the simplest Boolean algebra containing two truth values (True and False, or 0 and 1).

Then, if the bridge S represents the transformation from the one state to the other (acting like a Boolean negation operator between the levels of truth and falsity), its repeated application for a second time brings us back to the original state. In logical terms, the negation of negation is equivalent to the identity, and therefore, an involution bridge functioning between two states distributed in two distinct levels is a picturesque way of expressing the law of excluded middle in Boolean logic.

Due to the properties listed above, and making temporary use of the scaffolding of naive set theory, an effective analogia expressed by means of logical conjugation can be presented in the form of an equivalence relation, namely as:

$$X \sim_S A$$

stating that X is conjugate to A under S. This is an equivalence relation because it is reflexive, transitive and symmetric: First, due to the property of metaphora extension in length if $X_1 \sim_S A_1$ and $X_2 \sim_S A_2$, then $X_1X_2 \sim_S A_1A_2$. Second, due to the property of metaphora extension in depth, the transitivity condition is established since, if $X \sim_S A$ and $A \sim_T B$, then, $X \sim_{ST} B$. Finally, due to the property of metaphora inversion, the symmetry condition is established since, if $X \sim_S A$, then $A \sim_{S^{-1}} X$.

1.6 PARTITION SPECTRUM AND CIPHERS: VIRTUAL AND ACTUAL DISTINCTIONS

The equivalence relation $W \sim_S T$ stating that the process W is conjugate to T under S, gives rise to a finite partition of all possible processes into blocks or cells constituting the observable spectrum of this partition. Each block of the partition is algebraically the equivalence class of W under some S, denoted by $E(W)$. In other words, $E(W)$ is the class of all processes V that are equivalent to W under S. In this sense, V is in the equivalence class of W if and only if $E(W) = E(V)$. Note that the cells of a partition are non-overlapping, and more precisely, they are mutually exclusive and jointly exhaustive. This is the crucial property that characterizes the notion of the induced observable spectrum, indexed or classified below in terms of different colours:

It is important to point out that a metaphora involving two hypostatic levels pre-supposes implicitly that these two levels can be in principle differentiated. Of course, this is only possible in terms of the inversely pointing bridges S and S^{-1}. Concomitantly, two processes W and T, which are equivalent under S, i.e. they belong to the same cell of the partition, can be differentiated within this cell by the intervention of S and S^{-1}, although they cannot be actually distinguished. The underlying reason is that a cell of the partition should be properly thought of as an intrinsically indistinguishable element by itself, which however, can be potentially differentiated internally (e.g. by enforcing the encoding and decoding bridges S and S^{-1}), and most important, can actually be resolved, refined, and distinguished externally (e.g. by external acts of distinction).

For this purpose, in the context of a finite partition, it is instructive to introduce the difference between virtual and actual distinctions. An actual distinction is characterized by pairs (W, W'), where W and W' belong to distinct cells of the partition. In contrast, a virtual distinction is a differentiation internally within some cell of the partition, which is invoked by the explicit enforcement of S and S^{-1}. The interesting question, which has far reaching consequences in relation to the notion of information, is when a virtual distinction becomes an actual distinction. This is only possible by means of refining the partition. Equivalently, partition refinement requires a decoupling of the encoding and decoding bridges, since for a pair (W, W') qualified as an actual distinction W and W' belong to distinct cells, which entails that there are no bridges between them in force. If we adopt the provisional definition that information emerges through actual distinctions of a partition in the present context, then the procedure of partition refinement amounts to decoupling bridges of metaphora, and in this way, obtaining new information.

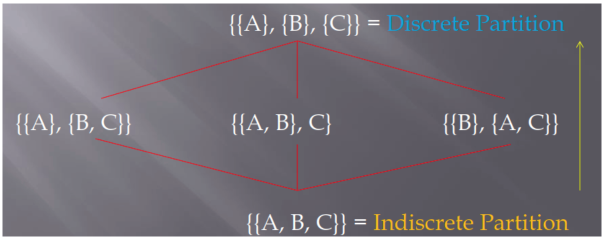
As a consequence, the notion of metaphora expressed through conjugation, incorporates three distinct types of structural ciphers. The first is the structural cipher of an algebraic group that expresses the notion of symmetry. In particular, an encoding bridge S together with the inverse decoding bridge S^{-1} differentiating a process W from a process T within a cell of the partition they enforce, make W and T symmetric to each other. The second is the structural cipher of a partial order that expresses the notion of distinguishability. In particular, external acts of refinement, i.e. acts of refining the grain of resolution in a spectrum to obtain actual distinctions -and thus- discern new information, induce a partial ordering relation among partitions. The third is the structural cipher of a category as a common abstraction between the notions of a group and a partial order. In particular, in our context the notion of a category is the algebraic cipher required to express the duality between sets and partitions, which emerges simply from qualifying the differentiation within a cell of a partition, given the bridges S and S^{-1}, as a set of virtual distinctions.

It is worth focussing, at this stage, on these structural ciphers in more detail. The structural cipher of an algebraic group is tantamount to the identification and classifications of the cells of a partition defined by metaphora, i.e. through conjugation. A group G is closed with respect to an operation, i.e. addition or multiplication, and it always contains an identity element, called the neutral element of the group. In this manner, the structural action of a group as a cipher is encoded in its neutral element. It is because of the neutral element that for any S in the group G, there exists a unique inverse S^{-1} such that their operational composition results in this element. A group is characterized via its action, i.e. it is the notion of a group action that structurally expresses the notion of equivalence in the cells of the partition. In this manner, the notion of a partition cell, under a group action that realizes it, is equivalent to the notion of an orbit or fiber of this group action.

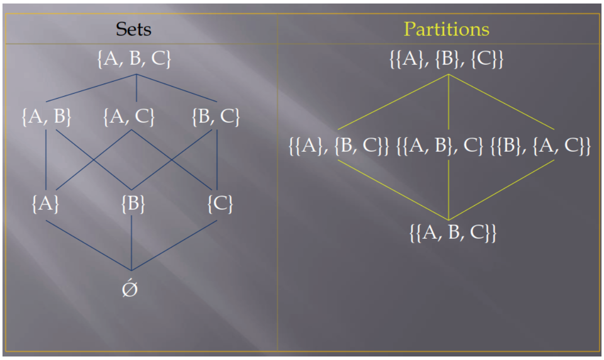
Next, we may consider the totally indiscrete partition that can be potentially differentiated under a group action, which essentially involves the enforcement of the encoding and decoding bridges in this partition. Since this partition is indiscrete by hypothesis, all virtual distinctions within it are symmetric to each other. In other words, we derive that the state of maximal symmetry equals a state of maximal indistinguishability. Equivalently, since there are not any actual distinctions in the indiscrete partition that constitutes a single block of the same colour, e.g. black colour, there is zero information that is extractable from it without any refinement.

The procedure of refinement of the indiscrete partition is possible by symmetry breaking; in other words, actual distinctions in a refinement of the indiscrete partition require the breaking of the maximal symmetry of that partition. Symmetry breaking makes information discernible in terms of the actual distinctions of the refined partition. On the other side, maximal refinement destroys all symmetry, i.e. it breaks all the bridges, and in this way is equivalent to maximal information, since everything becomes actually distinct in the discrete partition. Refinement is definable as a relation of partial order among partitions, where the indiscrete partition is the minimal element, and the discrete partition is the maximal element of this partial order.

To sum up, given a group action, we think of differentiation within a cell of the induced partition in terms of the set of virtual distinctions belonging to an orbit or fiber of this action. Moreover, we consider the partial order of refinement in terms of virtual and actual distinctions. Then, in the simple case comprising three elements amenable to actual distinctions, the order of refinement looks as follows:



The partial order of partition refinement is dual to the partial order of subsets of a set, and this fact renders the respective categories dual to each other. Thus, starting from metaphora and its properties as expressed through conjugation, we derive the categorical duality between partitions and sets. Note that the dual partner of an element of a subset is given by an actual distinction of a partition. Clearly the indiscrete partition does not have any actual distinctions, whereas all the possible actual distinctions are the actual distinctions of the discrete partition.



Moreover, since differentiation within a partition cell is qualified by the set of virtual distinctions that can be made within this cell, and due to the fact that a partition cell is identified with an orbit of a corresponding group action, based on the notion of symmetry induced by the inverse bridges of a metaphora, the duality between sets and partitions arises from the inverse correlation between symmetry and information. In particular, this inverse correlation boils down to the existence of two partial orderings that are in cadence with each other but the first is increasing whereas the other is decreasing. Note that both of these partial orders are bounded from above and below, such that the state of maximal symmetry in the first ordering corresponds to the state of null actual distinctions, and thus minimal information, in the second ordering, and so much is equally true of the converse. These two inversely correlated partial orders may be thought of as orthogonal to each other, if depicted jointly, since symmetry and information are complementary in the context of metaphora.

Now, suppose that $M \subseteq K \times K$ is the equivalence relation induced by logical conjugation on a set of processes or relations K. We may consider a category (K, M) in which K is the set of objects (standing for processes), M is the set of arrows, and the source and target maps $M \to K$ are given by the first and second projection. Then given X and A in K, there is precisely one arrow (X, A) if X and A are in the same equivalence class, viz. they are metaphorically related by conjugation, while there is none if they are not. Then transitivity assures us that we can compose arrows, while reflexivity tell us that over each process X in K there is a unique arrow (X, X), which is the identity. Finally symmetry tells us that any arrow (X, A) has an inverse (A, X). Thus, (K, M) is a *groupoid* (category in which all arrows are isomorphisms) such that, from a given object of this category (process) to another there is at most one arrow (if they are metaphorically related). Conversely, given a groupoid, such that from a given object to another there is at most one arrow, if we denote by K the set of objects and by M the set of arrows, the source and target maps induce an injective morphism $M \infty K \times K$, which gives an equivalence relation on K with the concomitant interpretation.

1.7 INTELLIGIBILITY OF THE COSMOS: STRUCTURAL EXTENSION OF ALGEBRAIC SCALARITY

Let us now examine the functionality of logical conjugation from a structural algebraic standpoint. We have already claimed previously that the resolution of the Thalesian problem of determination of an inaccessible magnitude by the method of proportions, implicitly contains the seeds of discovery of the multiplicative group structure of the (positive) rationals. More precisely, multiplication is an essential operation that can be performed on the integers endowing them with the closed structure of a multiplicative *monoid*. Division, the inverse operation to multiplication, is nevertheless not a total operation on integers, and thus, the determination of inaccessible magnitudes on the basis of proportion cannot be effectively performed within the reference domain of integers.

To achieve a total operation of division, to resolve the Thalesian problem, we are obliged to extend the initial domain into a new domain of numbers, where the required inverse operation can be always implemented. This means that the resolution of the problem requires an appropriate extension of the initial closed structure (integers) with respect to the operation of multiplication into a new structure (rationals), which is closed with respect to both multiplication and its inverse operation of division. This is a recurring theme in universal algebra and thus it deserves a closer analysis in order to explain the particulars of its implementation by means of the logical conjugation strategy.

For this purpose, let us state explicitly the ordered series of three processes that have to be performed, according to the general pattern characterizing metaphoras, for the construction of the field of rationals from the ring of integers. We recall that the rationals constitutes the set of all fractions a/b, a and b integers, $b \neq 0$ with the usual relation $a/b \equiv c/d$ if $ad = bc$, which makes invertible every non-zero element of the integers.

The basic ingredient for the construction of the field of fractions is the fact that the set of non-zero elements of the integers is multiplicatively closed. The structural metaphora characterizing completely this construction is technically called the process of localization of the commutative unital ring of the integers $\mathbb{Z}$ with respect to the multiplicative closed subset of the non-zero integers. The whole purpose of this structural metaphora by conjugation is to make every element of the multiplicative closed subset of non-zero integers invertible, such that the new structure of numbers obtained in this manner, fulfills the following objectives: First, it bears a structural similarity to the initial domain of numbers, viz. it is also a commutative unital ring with respect to addition and multiplication. Second, the operation of division (inverse to multiplication) can be performed by the existence of inverses of non-zero integers, which have been incorporated in the new extended closure domain of numbers. Third, as a consequence of the above, the initial domain of numbers together with their arithmetic can be embedded in the new one.

We consider the commutative unital ring of integers $\mathbb{Z}$ and let $S \subseteq \mathbb{Z}$ be the multiplicative closed subset of non-zero integers. The first step is to set up a directed bridge from the level of commutative unital rings to the level of sets, encoding the process of extending the underlying set-theoretic domain of integers $\mathbb{Z}$ into a new domain formed by the cartesian product of sets $\mathbb{Z} \times S$. Note that the ordered pairs of integers (a,s) with $s \neq 0$, are not supposed to have any a priori structure, since their existence is required at the level of sets by means of the encoding directed bridge connecting the structural levels involved. In this extended new set-theoretic domain the initial task can be facilitated by imposing the homological equivalence criterion, according to which the ordered pair of integers (va,vs) should be equivalent to (a,s) for any non-zero integer v. Technically this condition is described in the following way:

In the set $\mathbb{Z} \times S$ we define the following binary relation: (a,s) $\Diamond$ (b,t) if and only if there exists $v \in S$ such that: $v(at - bs) = 0$. The relation $\Diamond$ is an equivalence relation, partitioning the set $\mathbb{Z} \times S$ into equivalence classes. We will denote the quotient set by $\mathbb{Z}_S$, and the equivalence class of (a,s) by the fraction symbol a/s. Thus, the quotient set $\mathbb{Z}_S$ contains elements which can be interpreted as fractions, bearing the semantics of numbers allowing division by non-zero integers.

The structural metaphora is completed via logical conjugation by setting up an inversely directed decoding bridge from the level of sets to the level of commutative unital rings, effectuating the indirect round-trip as follows: We set $a/s + b/t := (ta + sb)/st$, $(a/s)(b/t) = (ab/st)$ for every a/s, $b/t \in \mathbb{Z}_S$. The operations are well defined and endow $\mathbb{Z}_S$ with the structure of a ring. The zero and unit elements are, respectively, $0/s$ and s/s, for every $s \in S$. Finally, we define the canonical morphism of rings $h:\mathbb{Z}\to\mathbb{Z}_S$, given by $h(a)=a/1$, for every $a \in \mathbb{Z}$. Note that for any $s \in S$ we have that $1/s$ is the inverse of $h(s)$ in $\mathbb{Z}_S$. Hence, $\mathbb{Z}_S$ is the smallest ring containing $\mathbb{Z}$, in which every element of the multiplicative closed subset of non-zero integers S is invertible.

Thus, the extension of scalars of the commutative unital ring of integers $\mathbb{Z}$ by means of algebraic localization, with respect to the multiplicative closed subset of non-zero integers, is understood as a structural algebraic metaphora implemented by logical conjugation. The structural effect of this metaphora by conjugation is the addition of multiplicative inverses to the elements of the multiplicative closed subset $S\subseteq\mathbb{Z}$, such that the extended ring $\mathbb{Z}_S$, consists of fractions a/s, where $a\in\mathbb{Z}$, $s\in S$. Moreover, the conceptualization of algebraic localization as a structural metaphor for the resolution of the general problem of making division a total operation by congruent extension of structure via the logical process of conjugation, permits its application in generalized structural environments as we shall see in the sequel.

1.8 STRUCTURAL *METAPHORA*: ADJOINING - PARTITIONING - QUOTIENTING

It is instructive to explicate in more detail the conjugation strategy related with the efficient functioning of the above structural metaphora. First, we observe that the encoding process of the underlying set-theoretic domain of $\mathbb{Z}$, utilized as an architectonic scaffolding, into the new domain formed by the cartesian product of sets $\mathbb{Z}\times S$ takes place by means of extending the scalars of $\mathbb{Z}$ with respect to the scalars of the multiplicative closed subset S of $\mathbb{Z}$. This means that the extension of scalars of the set-theoretic domain of $\mathbb{Z}$ is effectuated by adjoining to $\mathbb{Z}$ the scalars of a well-defined internal algebraic part S of $\mathbb{Z}$ distinguished by its anticipated operational role.

Second, the level of sets can be thought of as a temporary underlying scaffolding via which logical conjugation can be effectively applied. More precisely, at the level of sets the operational role of the distinguished part S of $\mathbb{Z}$ can be implemented by the imposition of an appropriate homeotic equivalence relation on the previously extended set-theoretic domain $\mathbb{Z}\times S$. The conceptual underpinning of this process is the identification of those elements of the extended domain $\mathbb{Z}\times S$, which exhibit a certain homeotics of behaviour, which we symbolize by the relation R. Any suitable criterion of homeotic indiscernibility must lead to a partition of $\mathbb{Z}\times S$ into disjoint classes of elements bearing the imposed homeotic relation R, and hence R must be an equivalence relation. Since, the imposition of such a relation R effectuates a classification of the elements of $\mathbb{Z}\times S$ into disjoint classes of equivalent elements, partitioning it in the particular way determined by R, the latter can be thought of as a homeotic perspective. It follows that an equivalence class modulo the homeotic perspective R, consists of all the elements of $\mathbb{Z}\times S$, indiscernible with respect to R, and thus homeotically identical.

More specifically, the homeotic perspective R imposed on $\mathbb{Z}\times S$, requires that the ordered pair of integers (va,vs) should be homeotically identical to (a,s) for any non-zero integer v, under the intended interpretation of the homeotic class of (a,s) by the fraction symbol a/s. Note that the homeotic classes (a,s) are metaphorically interpreted as elements a/s, being assigned a new name, viz. fractions, of a new set, namely of the quotient set $\mathbb{Z}_S$. It is important to notice that consequent to the transition from $\mathbb{Z}\times S$ to $\mathbb{Z}_S$ is the replacement of equivalence modulo R, viz. R-perspective homeotics, by equality (identity) of elements in the quotient $\mathbb{Z}_S$.

Third, the structural metaphora realizing the result of the applied logical conjugation is completed by means of the inversely directing bridge from the level of sets back to the initial level of commutative unital rings. The semantic aspect of this bridge amounts to a re-casting of the elements of the quotient set $\mathbb{Z}_S$, as elements of a new ring, viz. as elements of the same closed structural genus as the initial $\mathbb{Z}$. This is accomplished by modifying appropriately the addition and multiplication operations referring to these new elements (fractions). This modification takes place according to the principle that the new operations should incorporate and reproduce the effect of the old ones, when restricted to the old elements, being dressed in the new form imposed by the adopted homeotic perspective.

The important thing to notice is that the completion of the structural metaphora according to the logical conjugation strategy described above, accomplishes the task of making the operation of division total, and thus, resolves the geometric problem of homeothesis in a structural way. In this way, from the standpoint of the ring of integers, the structural metaphora permitting the unconstrained action of the division operation on magnitudes, belonging now to an extended closed partially congruent structure of the same algebraic genus (ring of rationals), accomplishes the interpretation of division as an emergent well-defined total operation. This is due to the fact that the operation of division acts properly on this new kind of species (fractions), which remains closed with respect to its action. The logical conjugation resolves the original Thalesian problem structurally because fractions are formed at the set-theoretic level, and then lifted at the ring-theoretic level by means of encoding/decoding bridges. In particular, fractions are formed by the inverse processes of extending the set-theoretic domain $\mathbb{Z}$ to the larger one $\mathbb{Z}\times S$ with respect to the part S, and then restricting this extended domain by collapsing it, viz. by partitioning it homeotically into disjoint classes, with respect to the imposed internal homeotic perspective subsumed.

In more general terms, the above algebraic localization structural metaphora is a particular application of the logical conjugation strategy designed for the resolution of a specific problem involving (at least) two delineated structural hypostatic levels, and based on the existence of a pair of inversely pointing bridges connecting these two levels, as follows: First, by means of an extension bridge, encoding the information of a structural domain into a new extended one assuming existence at a different level. Second, performing the required task at that level by realizing an appropriate equivalence relation, and subsequently forming the associated quotient structure. Finally, by means of a reciprocal bridge, decoding the acquired information in a structural form congruent to the form of the structural domain we started with, according to the specification of the initial level.

1.9 INDIRECT SELF-REFERENTIAL *METAPHORA* AND *HOMEOTIC* CRITERION

At a further stage of development of these ideas, we realize that the successful implementation of the conjugation strategy, concerning structural metaphoras, necessitates primarily the investigation of the meaning of an effective analogia within the same algebraic structural genus. This task is important, because it clarifies the nature of an indirect analogical self-referential relation taking place within a certain closed structural genus. From the general context of the preceding analysis, it has become clear that at least, referring to the set-theoretic level of magnification, a set can be related to a distinguished part of it by the imposition of an equivalence relation on their jointly formed cartesian product with respect to a homeotic perspective, which reciprocally required the delineation of that distinguished part, in the first place. The total process can be cast into the pattern assumed by a self-referential structural metaphora as follows:

Initially, we assume that a set of elements, considered as an individual object within the genus of sets (characterized by the membership relation), can relate to itself by separation of a well-defined part of it, viz. a subset bearing the functional role subsumed by a particular homeotic perspective. In turn, this homeotic perspective can be applied to the extended object obtained from the initial object by adjoining the distinguished part. Finally, using the quotient construction, we collapse the extended object into a new partitioned object belonging to the same genus. Of course, this is only possible if all of the following conditions can be fulfilled: First, if the initial object can split its substance between two internal levels or hypostases within the same genus, such that the latter, formed by extension with respect to a part, is also an object of the same genus encoding the former. Second, if the application of the homeotic perspective on the extended object partitions it into equivalence classes, forcing in this way a homeotic criterion of identity, or equivalently, an indiscernibility relation with respect to this homeotic perspective, at the same level. Thirdly, if the equivalence classes of the quotient can be re-interpreted as elements of a new object of the same genus, being formed at the initial level by identifying equivalent elements with respect to the homeotic perspective.

It is important to realize that an indirect self-referential relation, implicated by logical conjugation within the same genus, accomplishes precisely the satisfaction of the above conditions. This is possible by means of two inverse internal bridges connecting these two separate levels of hypostasis into a non-contradictory circular pattern as follows: the first bridge carries out the extension process of an object to another level of hypostasis, being formed by adjoining to it a distinguished part, delineated by the functional role subsumed under a homeotic perspective. At the new level, an appropriate equivalence relation on the extended object implements the functional role of the homeotic perspective, that is, implements a homeotic criterion of identity. As a result, we end up with a partitioning of the extended object into a set of equivalence classes constituted by indiscernible elements with respect the imposed criterion. Finally, an inverse bridge performs the transition back to the initial level, by collapsing the extended object with respect to the homeotic perspective, and thus, transforming the homeotic relation into an equality (identity) of elements in the quotient set, formed back at the initial level. Notice the crucial point that the quotient structure formed by returning to the initial level has to be again a set-theoretic object, that is, it must be congruent to the structural specification of the initial object we started with.

After this series of remarks, there arises the natural problem of applying the logical conjugation strategy realizing an indirect self-referential metaphora into the context of objects belonging to some algebraic structural genus, like groups, rings and algebras. This becomes possible, if we formalize the notion of a homeotic perspective as an equivalence kernel of a comparison morphism (homomorphism) between structures of the same algebraic genus. Note that the functional role subsumed by a homeotic perspective, elevates the relation of equivalence among elements belonging into the same equivalence class at the level reached by descending the first bridge, to a relation of equality (homeotic identity) at the initial level regained by ascending back through the inverse bridge. In turn, this constitutes the precise implementation of what we call a *homeotic criterion of identity*.

Set-theoretically speaking, this amounts to the implication that if two elements α and β of the extended set, at the new internal level of hypostasis, are equivalent with respect to a homeotic perspective R, viz. $\alpha R \beta$, then their images inside the quotient set, interpreted as new elements, at the initial level, are identical, viz. $[\alpha]_R = [\beta]_R$. Based on this argument, we can deduce the modeling of the notion of a homeotic perspective between structures of the same algebraic genus, by passing into some appropriately restricted type of equivalence relation by means of logical conjugation, depending on the algebraic genus considered.

1.10 EQUIVALENCE AND *HOMEOTIC* KERNELS OF ALGEBRAIC *GENUS*

In a general context, the minimum requirements for an algebraic system include the existence of a set S with an equality relation for which there is defined a binary law of composition, namely, a single-valued function of pairs α, β such that $\alpha\beta$ is in S for α, β in S. Adopting this as our starting point, we superimpose an equivalence relation R on S in order to investigate how a desired restricted type of equivalence relation arises. Namely, denoting by Σ the set of equivalence classes $C_\alpha \bmod R$, we raise the following question: Can an operation $\circ$ be defined in Σ based upon the composition operation in S?

We proceed along the lines of what might be a first attempt to investigate this question by defining:

$$C_\alpha \circ C_\beta = C_{\alpha\beta}$$

The above apparently makes the product dependent upon the choice of class representatives. This deficiency can be amended by requiring that, if $C_{\alpha'} = C_\alpha$ and $C_{\beta'} = C_\beta$, then $C_{\alpha'}C_{\beta'} = C_\alpha C_\beta$. This amounts to the assertion, if $\alpha' R\alpha$ and $\beta' R\beta$, then $(\alpha'\beta')R(\alpha\beta)$. Equivalently stated, we obtain the condition: $\alpha' R\alpha$ implies that $(\alpha' x)R(\alpha x)$ and $(x\alpha')R(x\alpha)$ for all x. We call regular those equivalence relations which satisfy the condition above. The latter constitutes a necessary and sufficient condition upon R in order that

$$C_\alpha \circ C_\beta = C_{\alpha\beta}$$

stands for a well-defined operation. Then, we can easily deduce that the correspondence φ of S onto Σ defined by: $\varphi(x) = C_\alpha$ if and only if $x \in C_\alpha$ is an algebraic homomorphism, called the natural homomorphism. Essentially, from a reciprocal standpoint, Σ should be a homomorphic image of S under a correspondence, mapping all elements of S belonging to an equivalence class onto an element of Σ. But the existence of such a homomorphism immediately implies the existence of one mapping the class containing α upon C_α, and the homomorphism property then requires that $C_\alpha \circ C_\beta = C_{\alpha\beta}$ holds true.

The central idea explained previously can be now easily applied to structures of some *algebraic genus*, for example, to groups. In this case, we consider a group S together with a regular equivalence relation R. Then, defining an operation in $\Sigma = \{C_\alpha, C_\beta, \ldots\}$ according to the composition rule $C_\alpha \circ C_\beta = C_{\alpha\beta}$, we obtain a homomorphic image of S. Since a homomorphic image of a group is necessarily a group, we deduce that Σ is actually a group whose identity element is C_e, where e is the identity element of the group S.

The above construction shows that the process of shrinking a group S with the aid of a regular equivalence R produces a homomorphic image Σ of S being also a group, thereby preserving the structural specification of its algebraic genus. Conversely, given a homomorphic image Σ of S, there is defined a partition, and therefore, an equivalence relation R on S. Moreover, the homomorphism property implies that R is a regular equivalence relation.

In a nutshell, we conclude that in the case of groups, the problem of finding all homomorphic images of S reduces to that of finding all regular equivalence relations over S. For this purpose, we make use of the coset decomposition of a group S with respect to a subgroup H. More precisely, we define, $\alpha R\beta$ if and only if $\alpha = h\beta$, where, $h \in H$.

We can easily show that R is actually an equivalence relation, such that the equivalence class $C_\alpha = H\alpha$, called the *right coset* of H. Moreover, since $\alpha' R\alpha$ implies that $(\alpha' x)R(\alpha x)$, the equivalence relation R is right regular.

But conversely, starting with a right regular equivalence R in S we find that C_e is a subgroup and $C_\alpha = C_e\alpha$, since $\beta R\alpha$ implies that $(\beta\alpha^{-1})Re$; Hence $\beta\alpha^{-1} \in C_e$, or, $\beta \in C_e\alpha$ and conversely.

Thus, the problem of finding the various right regular equivalence relations in S is reduced to the problem of determination of the right coset decompositions of S with respect to its subgroups.

Precisely analogous considerations establish that the various left regular equivalence relations in S are completely determined by the left coset decompositions of S with respect to its subgroups. Thus, we conclude that, if R is a regular equivalence relation, then, on the one side, it defines a left coset decomposition with respect to the subgroup H of all elements x such that xRe, and on the other side, it defines a right coset decomposition with respect to the same subgroup. Hence R stems from a subgroup for which the left cosets are identical with its right cosets. Such a subgroup N is called a normal subgroup of S, satisfying: $xN = Nx$ for all x in S.

Thus, a regular equivalence relation R in S stems from a normal subgroup N of S, viz., a subgroup remaining invariant under logical conjugation, meaning that $N = xNx^{-1}$ for all x in S. Conversely, a normal subgroup of S defines a regular equivalence relation on S. Now, if N is a normal subgroup of S, then its cosets $C_x = xN$ form a group with the following composition rule of closure:

$$\alpha N \circ \beta N = (\alpha\beta)N,$$

or equivalently, $C_\alpha \circ C_\beta = C_{\alpha\beta}$ holds. The resulting quotient $\Sigma = S / N$ is a group homomorphic to S and constitutes that group, which collapses the normal subgroup N of S to the identity element of Σ.

Conversely, every homomorphic image of S can be duplicated by, hence it becomes isomorphic to, such a quotient group.

The completely analogous analysis for the case of rings yields the corresponding homomorphism theorem with the same efficiency. Thus, we have deduced the modeling of the notion of a homeotic perspective between structures of the same algebraic genus, by the concept of regular equivalence relations. Consequently, the implementation of self-referential metaphoras within the context of objects belonging to some algebraic structural genus, becomes possible if we formalize the notion of a homeotic perspective precisely as a regular equivalence kernel of a comparison morphism (homomorphism) between structures of the same algebraic genus.

More concretely, in the case of groups, we have the following: Let S and T be groups and let ϕ be a group homomorphism from S to T. If e_T is the identity element of T, then the kernel of ϕ is the subset of S consisting of all those elements of S which are being mapped by ϕ to the element e_T:

$$Ker(\phi) = \{x \in S : \phi(x) = e_T\}$$

Since a group homomorphism preserves identity elements, the identity element e_S of S must belong to $Ker(\phi)$. By the preceding analysis, it turns out that $Ker(\phi)$ is actually a normal subgroup of S. Thus, we can form the quotient group $S / Ker(\phi)$, which is naturally isomorphic to $Im(\phi)$, viz. the image of ϕ (which is a subgroup of T).

Analogously, in the case of rings with a unit element we have the following: Let S and T be rings and let ϕ be a ring homomorphism from S to T. If 0_T is the zero element of T, then the kernel of ϕ is the subset of S consisting of all those elements of S which are being mapped by ϕ to the element 0_T:

$$Ker(\phi) = \{x \in S : \phi(x) = 0_T\}$$

Since a ring homomorphism preserves zero elements, the zero element 0_S of S must belong to the kernel. It turns out that, although $Ker(\phi)$ is generally not a subring of S, since it may not contain the multiplicative identity, it is nevertheless a two-sided ideal of S. Thus, we can form the quotient ring $S / Ker(\phi)$, which is naturally isomorphic to $Im(\phi)$, viz. the image of ϕ (which is a subring of T).

The effective generation of self-referential structural metaphoras via logical conjugation in the context of some algebraic genus, implicated by the action of regular equivalence relations within this genus, provides a powerful methodological device for the resolution of a wide range of problems. Moreover, a self-referential structural metaphora may be combined with another type of metaphora, for instance a genus to species metaphora.

As an example, we may consider the case of the genus of (finite) groups. We have already seen previously that a regular equivalence relation on a group corresponds to a normal subgroup of this group, interpreted as an internal homeotic perspective. More precisely, this homeotic perspective constitutes the regular equivalence kernel of the homomorphism from the group to its corresponding quotient group. A natural problem arising in this context refers to the possibility of decomposition of a group into a finite series of non-further decomposable groups (simple groups) using the method of division with respect to internal homeotic perspectives, namely, with respect to normal subgroups. This is the problem of solvability of a group-theoretic structure, which has been first posed in the context of Galois theory. If solvability is attainable, then the initial group can be thought of as being decomposed into a finite series of irreducible group layers (factor groups) adjoined to each other in a proper way.

This problem can be successfully tackled by means of the conjugation strategy, if we combine the previously explained self-referential structural metaphora with a genus-to-species metaphora between the genus of multiplicative groups and the species of the integers. In the context of the latter metaphora, if a group corresponds to an integer, then a normal subgroup corresponds to a divisor of this integer and the associated quotient group corresponds to the quotient of the integer by the divisor. Furthermore, a non-further decomposable group (simple group) corresponds to a prime integer number, and finally, the notion of decomposition of a group into a finite series of simple groups using the method of division with respect to normal subgroups corresponds to the Euclidean algorithm for divisibility of the integers.

1.11 LOGICAL CONJUGATION VIA A *GNOMON*: *HOMEOTIC* CRITERION OF IDENTITY

It is instructive to emphasize that the appropriate operational implementation of all different manifestations of the logical conjugation strategy rests only on two prerequisites:

First, the ability to induce a meaningful stratification into different hypostatic domains or levels which can be connected by means of encoding and decoding bridges. In the general case, we may think of these levels as structural ones. The stratification may even involve substructures of an initially given structure, delineated according to a specific characteristic and adjoined to the initial structure, as separate levels. The latter is particularly suited to the resolution of self-referential problems through a cyclical conjugation process by means of the reciprocal and reflexive techniques of descending and ascending.

Second, the ability to establish a congruence, or a homeotic relation among the stratified levels. It is precisely the ingenuity of a homeotic criterion that provides the seed for the successful implementation of the logical conjugation strategy. Put differently, an effective analogia or metaphora subsumed by logical conjugation requires an appropriate criterion of homeotics among stratified levels in order to operate. We point out that the notion of metaphora literally means transference or transportation. Thus, logical conjugation can be conceived as a logical transportation process involving at least two separate levels according to a specific homeotic relation among these levels. We also note that metaphora may refer to transportation of information or structure or matter or energy or whatever else this notion can refer to, whereas the logical conjugation strategy via which it takes place is indifferent to its particular qualifications. This provides the sought for universality in the application of logical conjugation to different fields.

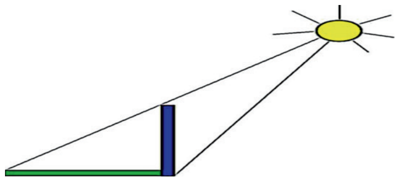
From the above, we deduce that what is crucial for the logical conjugation method is the establishment of some appropriate homeotic criterion operating among the stratified hypostatic levels. Then, based on this homeotic criterion it becomes more tractable to devise appropriate encoding and decoding bridges reciprocally connecting all different levels and effectuating a metaphora. It is interesting to note that from the present viewpoint the notion of homeotics bears a logical function although it is usually introduced and implemented by topological means. At least, it is important to stress that a homeotic criterion is independent of local metrical spatiotemporal notions of distance. For this reason, it can operate non-locally or among different scales. The ubiquity of a homeotic criterion is that it establishes some particular measure of invariance among the stratified levels. This measure can be expressed as an arithmetic invariant, like a ratio or a fraction, or even in structural terms like a group or groupoid. The essential thing is that inter-level connectivity and congruence obtained by metaphora, requires a homeotic criterion in order to be expressed via the logical conjugation strategy, just as the homeotic criterion rests conversely on such inter-level connectivity.

In standard mathematical terminology, what we call a homeotic criterion appears in a variety of different formulations, which are unified conceptually from our perspective. This unification is facilitated by means of logical conjugation and its net effect, which is metaphora according to some qualification, and ultimately serves as an effective means of coping with complexity and self-reference. For instance, a homeotic criterion may be expressed in the simplest possible manifestation as a relation of homeothesis or proportionality of integer magnitudes as in the original Thalesian conception. It may also be expressed as a relation of similarity between two square matrices, where the homeotic criterion is the representation of the same linear transformation with respect to two different bases of a vector space. In this case, the logical conjugation strategy resolves the problem of diagonalization via the method of eigenvalues.

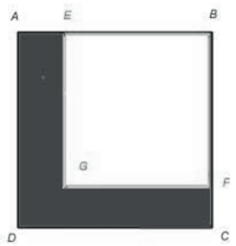
In the field of differential topology and differential geometry a homeotic criterion is provided by the notion of a *local homeomorphism* or *local diffeomorphism* correspondingly. We may note parenthetically that from the perspective of logical conjugation the notions of topological or differential manifolds defined by descending to simpler spaces like the Euclidean ones and then ascending back via the method of gluing from the local to the global level, are solely needed for the formulation of the metaphora process of differentiation, called *covariant parallel transport*, and giving rise to the invariants of curvature. Finally, a homeotic criterion may be literally expressed in standard algebraic topological terms, namely by means of homology and cohomology theory. In broad terms, *homology theory* establishes invariant measures of topological similarity in terms of a series of groups stratified into different scales or dimensions. The topological similarity is defined by means of classifying chains of connectivity into two classes, called *cycles* and *boundaries* correspondingly. More precisely, two cycles are homologically equivalent if they differ by a boundary. The dual theory, called *cohomology theory*, is based correspondingly on the notion of cochains of connectivity, which are classified respectively into *cocycles* and *coboundaries*. In this case, two cocycles are cohomologically equivalent if they differ by a coboundary. For example, in the case of *de Rham* cohomology theory, the cocycles are represented as *closed differential forms* and the coboundaries as *exact differential forms*.

A natural question arising in this context is the following: Notwithstanding the technicalities involved, for example in the setting of homology and cohomology theories of various forms, is there a guiding concept that lends itself to a proper and efficient depiction of a homeotic criterion? In other words, what is the common thread between the homothesis equivalence relation and the more sophisticated algebraic-topological homology equivalence relation which renders them both amenable to the logical conjugation strategy?

We argue that the common conceptual thread for establishing a proper homeotic criterion is provided by the use of a gnomon. The intuitive idea of a gnomon also makes more easily conceptualized the quite abstract notion of a homeotic algebraic kernel. The best definition of the notion of a gnomon has been given by the great mathematician *Heron* of Alexandria in the following terms: A gnomon is that entity which, if it is adjoined to some originally given entity, results in a new augmented entity becoming homeotic, or partially congruent, or even similar, to the original one. In order to understand the depth of this simple-seeming definition of a gnomon it is necessary to start from its initial conception in the context of the Thalesian theory of homeothesis. In this context, the gnomon is, literally speaking, the part of the sundial that casts the shadow.

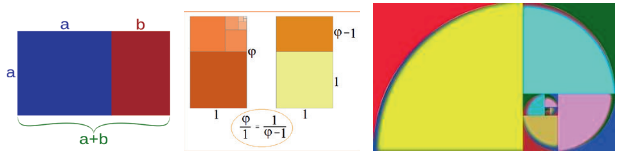


We can easily see that it is exactly the adjunction of the gnomon to the pyramid, which induces a homeothetic equivalence relation between the level of objects and the level of their shadows with reference to their magnitudes at the same time of the day, and consequently makes logical conjugation operative for the determination of the non-directly accessible magnitude of the height of the pyramid in terms of proportion. In its simplest possible form the general process of adjoining a gnomon in order to obtain a relation of homeothesis may be visualized as follows:



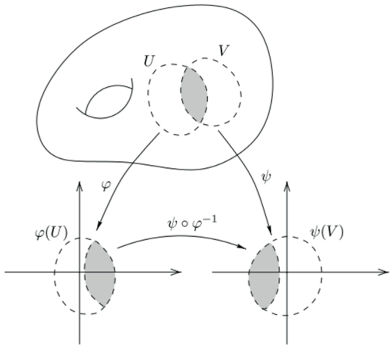
Formally, the relation of homeothesis is an equivalence relation, and thus induces a partition spectrum consisting of equivalence classes standing for the blocks or cells of this partition. The quotient structure obtained by factoring out this equivalence relation incorporates a new homeotic criterion of identity in comparison to the initial one, which is precisely characterized in terms of the chosen gnomon of homeothesis. In other words, the notion of logical identity is relativized with respect to the gnomon, such that the unit element of the quotient structure expresses equivalence modulo the gnomon.

In the case of homeothesis or proportionality of magnitudes, the metaphorical aspect of logical conjugation may be easily visualized in terms of a recursive or periodic application of a gnomon. This leads naturally to the dynamical notions of gnomonic growth or unfolding and reciprocally gnomonic subdivision or folding by means of logical conjugation. A particular well known example is provided by the function of the golden mean gnomon, depicted graphically as follows:

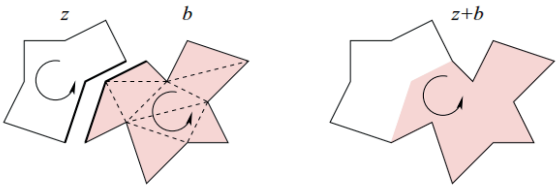


The conclusion obtained from the analysis of the notion of a homeothetic gnomon can be extrapolated to more complex situations, where a more general homeotic criterion is required for the effective application of logical conjugation. The abstraction consists in thinking of a gnomon as a means to indicate, or discern, or distinguish, or to set a boundary. The function of a gnomon is again to induce a certain type of modularity incorporating a homeotic criterion of identity.

For instance, in the case of a manifold, the gnomon is a local Euclidean space and the homeotic criterion is subsumed by the notion of a local homeomorphism. The modularity type is expressed by the gluing conditions of local Euclidean patches adjoined homeotically to a globally intractable space endowing it with the structure of a manifold. The logical conjugation strategy is used as a means to resolve a difficult problem for manifolds in terms of simpler problems, which can be solved at the level of local Euclidean patches and their amalgamations. Equivalently put, this logical method conjugates a complex problem at the manifold level to a simpler problem at the local Euclidean level where it can be directly resolved. The efficiency of logical conjugation rests on the fact that we are able to descend and ascend between these levels due to the homeotic criterion enforced by the associated gnomon.



Finally, it is worth explaining the notion of gnomon employed in standard homology theory, as it is conceptualized in algebraic topology. In this case, the role of a gnomon is played by the notion of a boundary. We recall that chains of connectivity in homology theory are classified in terms of cycles and boundaries. Intuitively, a boundary at some dimension is a bounding chain of a higher dimensional topological form, whereas a cycle stands for a non-bounding chain. Visually, non-bounding chains may be thought of in terms of holes or punctures or higher dimensional cavities, whereas boundaries may be thought of in terms of filled, and thus bounding chains. The basic idea of a boundary as a gnomon, establishing the homeotic criterion of homology, such that logical conjugation can operate, is that adjoining a boundary to a cycle gives a topologically similar or homologous cycle. Thus, two cycles differing by a boundary belong to the same homology equivalence class as depicted visually below.



In this sense, homology equivalence classes, which are actually abelian groups due to the algebraic operations involved in composing chains and orienting boundaries, enfold the invariant information of holes and cavities of topological forms. We emphasize again that these group invariants are obtained solely by the logical conjugation strategy on the basis of the homological criterion of identity set up by the notion of a gnomonic boundary.

TRANSCENDENTAL CIRCULATION: EXPONENTIATION AND LOGARITHMIZATION

2

2.1 EXTENSION BY INVERSION: FORGETTING AND REMEMBERING

One basic characteristic of the process of extension of the algebraic structure by means of metaphora through the scaffolding of sets is that the conceptually inverse bridges employed to achieve conjugation bear some particular meaning that is worth focussing on. More concretely, the encoding bridge is actually a *forgetful* bridge (*"lethe"*) in relation to to the initial algebraic structure, whereas the decoding bridge is one of *recollection* (*"anamnesis"*) that re-establishes the algebraic structure at the initial level. The reason is that the appropriate homeotic criterion of identity is established in terms of an equivalence relation at the level of set elements, following which, the algebraic structure is enforced in a suitable way. The other basic characteristic is that the extended algebraic structure, does not discard, but rather incorporates the initial one. This means that the restriction of the structure extended from our initial one serves to ensure agreement with it. This is clear already from the algebraic treatment of the Thalesian problem that required the extension of the integers into the rationals in order to solve the pertinent proportionality equation between magnitudes and their shadows. In the extended algebraic structure of the rationals the integers are qualified as a particular type of rationals, such that the restriction of the rationals to the integers is feasible.

From an algebraic viewpoint the metaphora realizing the extension of the integers to the rationals addresses the issue that division is not possible within the domain of the integers. It becomes possible only via the extension of the integers to the rationals, since the latter assume a group structure with respect to the operation of multiplication, which is absent in the case of the integers. In this sense, this extension in entailed by the necessity of determining the algebraic domain, where the inverse operation to multiplication becomes a total operation and can be performed without obstruction. The perspective of inverting an algebraic operation, which anyway proves indispensable in solving algebraic equations, is very fruitful and elucidating in understanding the emergence of arithmetics and algebraic structures. From this perspective, a structural algebraic metaphora provides the means to evade the obstacle of inverting an operation in an initially specified algebraic domain.

2.2 TWOFOLD INVERSION OF POWERS: ROOTS AND LOGARITHMS

A very important case presents itself in the consideration of the notion of taking powers. It is well known that the notion of power is defined by recursion on the operation of multiplication. The complexity in the notion of a power is that it involves two numerical entities assuming different operational roles. More concretely, we have the base of the power and the power itself, such that the operation of raising the base to a power is not a commutative operation, i.e. the result is not invariant where the roles of bases and powers are exchanged. In this sense, the non-commutativity appearing for the first time algebraically in the procedure of raising a base to a power, requires the consideration of two distinct inverses; one referring to the base, and the other referring to the power. If we call this non-commutative operation with respect to the base and the power the operation of *exponentiation*, then its inversion is twofold: Inverting with respect to the base is the procedure called *root extraction*, whereas inverting with respect to the power is the procedure called *logarithmization*.

Therefore, two distinct types of conjugation are needed in order to invert exponentiation. The first, referring to the powers with respect to a base obliges us to extend the field structure of the rationals to the field structure of the reals. As a result, logarithmization becomes a total operation in the domain of real numbers. The second, referring to the roots, necessitates the extension of the field structure of the rationals to the field structure of the complex numbers, if we include the roots of negative numbers. Both of these inversions are unified in the field domain of the complex numbers under the notion of the complex logarithm. It is important to highlight that both of these inversions are not purely algebraic, but entrain topological arguments for the effectuation of the respective metaphoras. The first requires an argument of continuity, whereas the second requires additionally a topological argument of evasion of the obstacle of multiple connectivity, to which we will come back later.

At this stage, it is worth considering first, real logarithmization in functional and algebraic terms. If we consider that b is any positive base different from the unit 1, then the exponentiation equation $b^y = x$, where $x > 0$, is solved in terms of y by logarithmization, i.e. $y = \log_b x$. Equivalently, the power y is expressed as the real logarithm of x in the base or root b. It is clearly not allowed to take the real logarithm of zero or a negative number. If we think of b^y as a function of y, then this function is a continuous (and differentiable) function of the variable y, whose inverse is the continuous (and differentiable) real logarithm function $x \mapsto \log_b x = y$. Note the intervention of this topological qualification required for the performance of the required inversion that requires the explicit consideration of the irrationals besides the rationals, in other words, the meaningful inclusion of limit processes, in order to achieve the extension to the domain of the reals. The real logarithm function is characterized as the unique monotonically increasing function from the positive reals to the reals, such that:

$$\log_b b = 1$$

$$\log_b(y_1 \cdot y_2) = \log_b y_1 + \log_b y_2$$

Note that the real logarithm function converts multiplication of positive reals to addition of reals and it is order preserving.

A natural question emerging in the functional context regarding the real logarithm function is how to express the procedure of raising to a power independently of the base employed. For this purpose, we define the exponential function $\exp : x \mapsto \exp(x)$, from the reals to the positive reals, i.e. the value is never zero and never negative, characterized by the property that

$$\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2)$$

meaning that it converts addition of reals to multiplication of positive reals. Then, the problem of raising to any power a with respect to a base $\circlearrowright$, where $\circlearrowright$ is thought of as a variable, is resolved by regarding the exponential and logarithm functions as inverse bridges between the group theoretic domains of the positive reals with respect to multiplication and the reals with respect to addition. More concretely, these inverse bridges, are inverse homomorphisms between these two groups, and thus, constitute the group of the reals under addition isomorphic to the group of the positive reals under multiplication. In other words, the real exponential function and the real logarithm function are not only inverse functions, but more important, they are inverse group homomorphisms.

In this fashion, we may define the real logarithm function $\log : \mathbb{R}^+ \to \mathbb{R}$ as the group homomorphism from the multiplicative group $(\mathbb{R}^+, \cdot)$ to the additive group $(\mathbb{R}, +)$ since $\log(y_1 \cdot y_2) = \log y_1 + \log y_2$ is satisfied for any positive reals y_1 and y_2. Inversely, the real exponential function $\exp : \mathbb{R} \to \mathbb{R}^+$ is a group homomorphism from the additive group $(\mathbb{R}, +)$ to the multiplicative group $(\mathbb{R}^+, \cdot)$ satisfying $\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2)$.

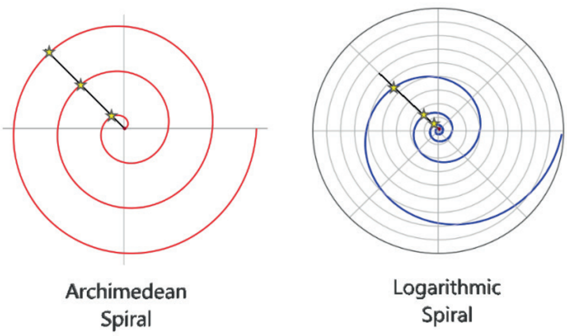
As such these two group homomorphisms are inverse to each other; they establish an isomorphism between these two different group structures. The most important consequence of this isomorphism is that the additive group structure of all real numbers, i.e. of the values of the logarithm function under addition, is indistinguishable from the multiplicative group structure of the positive reals, i.e. of the values of the exponential function under multiplication. Consequently, the difficult operation of raising to a power can be conjugated to the easy operation of multiplication by metaphora from the additive group of the reals to the multiplicative group of the positive reals, where $\exp$ and $\log$ play the role of the inverse bridges. Symbolically, we have:

$$\circlearrowright^{a} = \exp[a]\log(\circlearrowright) = \exp[a]\exp^{-1}(\circlearrowright)$$

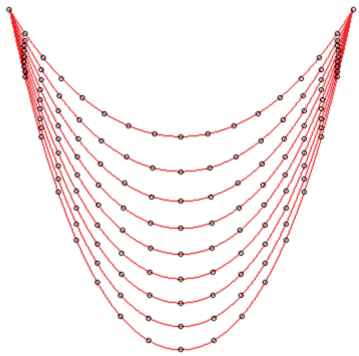
Conversely, the capacity of the above metaphora to solve the problem of raising to a power by conjugating it to multiplication is equivalent to the group isomorphism induced by the inverse bridges identified with the real exponential and the real logarithm function.

2.3 TRANSCENDENTAL *GNOMONS*: BRIDGING THE HARMONIC WITH THE GEOMETRIC

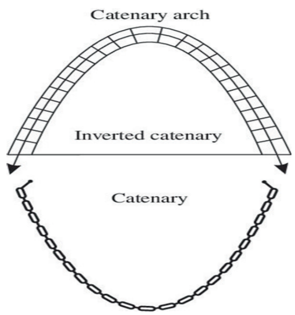
Both the exponential bridge and its inverse logarithmic bridge are characterized by self-similarity. Thus, they can be conceived in gnomonic terms. More concretely, since both of them are transcendental functions they act as inverse bridges between the transcendental or harmonic domain and the geometric domain, which is to say that the exponential is a bridge from the geometric to the harmonic, and inversely, the logarithm is a bridge from the harmonic to the geometric domain. If we consider the well-known example of the logarithmic spiral, it clearly provides an example of gnomonic growth, which is encountered in the natural world, for instance in the case of the Nautilus shell. The logarithmic or equiangular spiral differs from the *Archimedean* spiral in the sense that the distances between successive windings are not constant, but they increase in geometric progression.



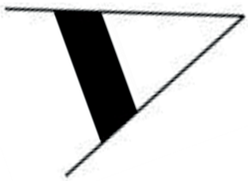
Coming to the exponential bridge, if we consider the arithmetic mean of the exponentials $\exp(x)$ and $exp(-x)$, i.e. $\frac{\exp(x)+\exp(-x)}{2}$, then we obtain a well-known curve, called the *catenary curve*, which can be also thought of in gnomonic terms. The origin of this curve is physical, and more precisely, it is the solution to the least action problem referring to a chain in a gravitational field. Put simply, the catenary curve composed by the arithmetic mean of two exponential bridges according to the above, is the natural shape of a hanging chain under the pull of gravity.

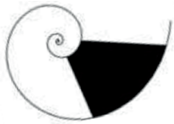


The inverted shape of the catenary is the well-known *catenary arch* in architectonics with myriad of applications. The catenary arch by its specification through the real exponential bridges stands by itself without any support, defying in a sense the pull of gravity as the inverse of the shape assumed by a hanging chain.



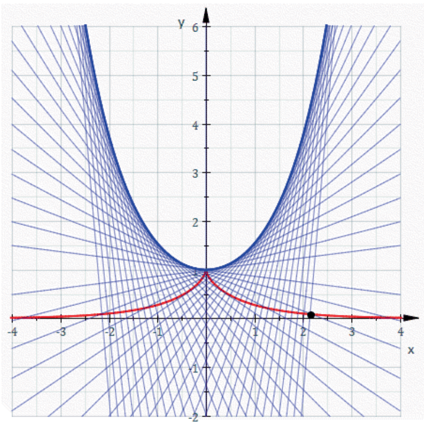
The most interesting aspect of these transcendental gnomons, which is absent from the initial rational conception of the geometric ones, is the appearance of curvature. Moreover, the pattern of gnomonic growth is not a linear trapezium as in the former case, but an angular trapezoidal sector, depicted for comparison below.



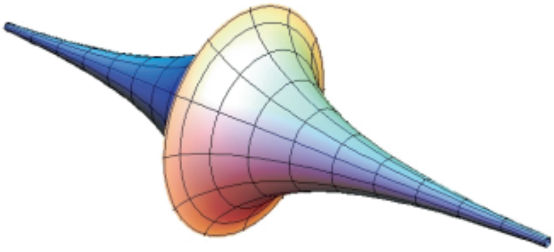


2.4 INTRINSIC GEOMETRIC CURVATURE

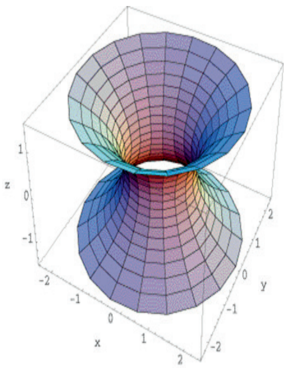
The blueprint of the different types of local curvature in two dimensions is already evident by considering the catenary curve. The geometric way of detecting the local curvature involves the consideration of the tangent and the normal at a point. The normal may be thought of as the radius of a circle at the specified point, whereas the tangent is the orthogonal to the normal, identified with the tangent of the circle at this point. We now imagine another curve that bears the inverse specification of tangents and normals, whereby the former tangents are the normals of the new curve and the former normals are the tangents of the new curve. Then, we obtain a geometric inversion with respect to the local curvature referring to these two curves. If we apply this to the case of the catenary, then we obtain another curve called the *tractrix* as depicted below:

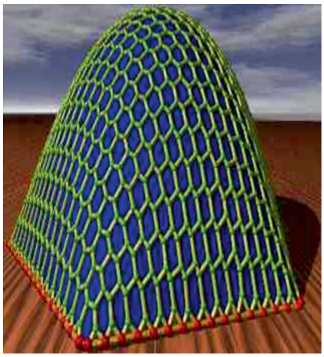


The surface of revolution emerging by rotating the tractrix about its asymptote is a *pseudosphere*, which is a surface with constant negative intrinsic curvature, characterized as a hyperbolic surface. The analogia with the sphere comes from the fact that a sphere has constant positive curvature $1/R^2$, where R is the radius of the sphere, whereas the pseudosphere has constant negative curvature $-1/R^2$. They can be treated on an equal footing by considering the radius of the pseudosphere as an imaginary radius, i.e. iR, such that its curvature becomes the negative magnitude $-1/R^2$. This is a non-trivial step that requires an imaginary metaphora between the harmonic and the geometric domain culminating in the role of the imaginary unit, a subject to which we will come later.

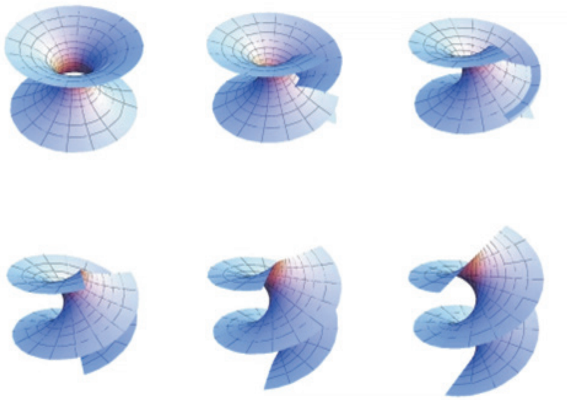


The revolution of the catenary around an axis can be performed in two ways, both concavely and convexly. The surface of revolution obtained in the first case is a *catenoid*, while in the second it is a *catenary dome*. The catenoid is a minimal surface; it occupies the least area when bounded from above and below, e.g. by two circular rings. Because of this fact, it has mean curvature zero everywhere. As such it should be thought of as the curved abstraction of the plane, which is also a minimal surface considered as a surface of revolution. The catenary dome should be thought of as the optimal correction to the shape of an ideally symmetric spherical dome when subject to acceleration due to gravity.





Topologically the catenoid is non-simply connected due to the hole it bears in the middle; if we make a cut, then it can be deformed periodically to a simply-connected helicoid, which is also a minimal surface, although not a surface of revolution. In particular, it occupies the least area when bounded sideways by two helices. In this way, the catenoid becomes locally isometric to the helicoid. A two-dimensional entity could not locally distinguish the catenoid from the helicoid. The fact that this locally isometric deformation exists is a strong motive to explore the implications of the exponential and logarithmic bridges when extended to the imaginary and complex number domains. The crucial observation is that after half a period a mirror image of the same helicoidal surface arises, which may be qualified topologically as a twisting. For example, we may think of a belt as a toy model whose two sides are coloured differently. The closed belt is an approximation to the region around the equator of the catenoid. If we open the belt and move the left end up and the right end down we have an approximate model of a helicoid. On the other side, if we move the left end down and the right end up we obtain the mirror or twisted image of the former helicoid.



Note that the rotation axis of both the helicoid and its mirror is orthogonal to the equator of the catenoid, since there is a $\pi/2$ rotation counterclockwise or clockwise in relation to the equator. This is a strong indication about the role of the imaginary unit from a transcendental viewpoint.

The extension of the real exponential function to the imaginary domain takes place via the complex exponential function $\exp:\mathbb{R}\to S^1$, where S^1 denotes the unit circle, whose elements are described by Euler's formula as $e^{i\theta}=\cos\theta+i\sin\theta$. Note the appearance of the imaginary unit, which is interpreted as a rotation by $\pi/2$ making the imaginary axis orthogonal to the real axis in the domain of the complex numbers. Given together with the imaginary unit, there is always its mirror image, described as its complex conjugate. Since the unit circle is coordinatized by means of the imaginary unit, we think of this circle as an imaginary ring. Its emergence will be elaborated as we go on. At this stage, it is useful not to adopt the more conventional *geometric*, but rather to favour the *harmonic* interpretation. Simply expressed, the imaginary ring is actually a harmonic ring, i.e. it descends not from the domain of geometry but from the transcendental domain of harmonics. Notwithstanding this fact, the image of the ring in the geometric domain of forms may be visualised as a circle, more precisely, as a circular shadow of a harmonic entity. The latter is expressed transcendentally through the complex exponential function as its imaginary power.

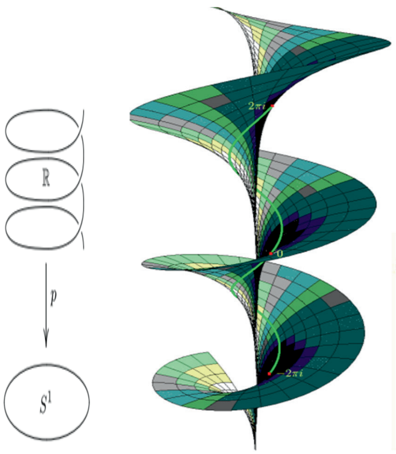
For the consistency of this metaphorical interpretation it is necessary to explicate the qualification of a harmonic entity as well as its expression as an imaginary power. The intuition comes from the dual consideration of the helicoid along with its mirror image as constituting a harmonic entity. Firstly, the helicoid unfolds continuously by parallel translation of its tangent planes, and after half a period of rotation a mirror image of the same helicoidal surface arises. We may think of the helicoid together with its mirror image as helical waves propagating in opposite directions such that the mirror image is the reflection of the first. This is possible if these helical waves are bounded from above and below for temporal length of one period so as to give rise to a helical standing wave. Here, this condition is equivalent to the requirement that within this bounded interval the helical wave is in unison with its reflection, its mirror image. Being in unison means that they are consonant in the fundamental harmonic frequency corresponding to the frequency ratio $1:1$, which in turn, would correspond to an angular temporal interval of one whole period 2π.

2.6 HARMONIC SERIES OF A TEMPORAL HELICAL CHORD AND FREQUENCY SPECTRUM

It is instructive to highlight the difference between a vibrating straight chord whose length is spatial, and a vibrating helical chord whose length is temporal as in the preceding. The unison ratio in the former case corresponds to a zero length spatial interval, whereas it corresponds to a 2π temporal interval in the latter case. Notwithstanding this fact, we are able to establish the whole *harmonic series* in the helicoidal case, such that there is an inverse relationship between frequency and temporal extent or duration. The visual imaginary ring in this context is the unit circle descending from the harmonic domain of relations between a variably bounded helicoid and its mirror image into the visible geometric domain as its observable shadow. It serves to spatialize temporal extents by means of the imaginary unit and its conjugate to allow for twofold directionality. The spatialization records temporal extents in a twofold imaginary axis, qualified by both a positive and a negative direction as usual, as simultaneously extended imaginary spatial lengths at the present of the emergent shadow. Equivalently, these spatialized extents can be viewed as angular sectors of the imaginary circle via the complex exponential function. In this manner, being in unison in the harmonic context of a helical standing wave has a shadow in the visible geometric domain quantified by the imaginary spatial length $2\pi i$, which is identical to the period of the complex exponential function. Alternatively, through the complex exponential function, being in unison corresponds to the whole 2π angular sector of the circumference of the imaginary ring.

It is worth pondering on some specific characteristics of the harmonic domain that make it different from the visible geometric one. If we think ontologically in terms of substances, then in the harmonic domain the twisted or mirror image, or simply the reflection is of the same substance as the original, since it can interact and interfere with it to produce a standing helical wave bounded from above and below. The latter is not traveling in space at all. In contrast, it resolves time in terms of the harmonics series and the concomitant harmonic ratios of frequencies, i.e. by means of *consonances* and *dissonances*. As such a standing helical wave in the context of its resonating environment is not an ontological entity in physical space, although it has a shadow quantified through the imaginary ring. Its most crucial aspect is that it resolves time periodically in terms of the harmonics, in such a way that time and frequency are reciprocally correlated. Thus, in the same way that time is spatialized via the imaginary axis, to give an imaginary length, frequency is spatialized orthogonally to the former as speed or momentum. What really matters is the orthogonal placement of frequency and spatialized temporal duration due to the intervention of the imaginary axis. As such, the opposite convention of indexing frequencies as imaginary quantities and spatialized temporal intervals as real is also valid and acceptable. Keeping in mind the above correspondences it becomes evident that the exponential function is qualified as eigenfunction of the differentiation operator, as well as a kernel of an integral transform, for example, the *Fourier* transformation between functions of these reciprocal variables. Note that an arbitrary angle in its expression as a power in the complex exponential is a product of the reciprocal variables.

Keeping the former convention, we identify the stairs of any bounded portion of the helical wave, unfolding orthogonally to the imaginary ring that constitutes its shadow, or present epiphaneia determined by the bounds of the resonator, with the harmonic series, being able to induce any harmonic ratio. In this manner, the harmonics are qualified as *powers* for the actualization of consonances and dissonances. The negative harmonics, setting up the whole frequency spectrum, correspond to the harmonic series of the reflection. Therefore, the whole frequency spectrum of a bounded helical standing wave does not bear any *ontological* role, but its role may be thought of as *teleological*. More precisely, the whole harmonic series depicted by the helical stairs is the *entelecheia* of the standing wave that accompanies the transcendental domain of time, as expressed in terms of the complex exponential function.



Considered structurally, which means here group-theoretically, the complex exponential is a group homomorphism from the additive group $(\mathbb{R},+)$ to the multiplicative group $(S^1,\cdot)$ satisfying $\exp(i(\theta_1+\theta_2))=\exp(i\theta_1)\cdot\exp(i\theta_2)$. The homeotic criterion of identity is encapsulated in the kernel of this group homomorphism, which is $2\pi\mathbb{Z}$. Note that the homeotic criterion of identity is established in terms of the angular temporal interval of one whole period 2π times the harmonic series, which belong to the group of the integers. In this sense a single moment of time, identified with the present of the imaginary ring shadow, making it a unity, is resolvable homeotically by the whole spectrum of harmonics, so that consonances can occupy this moment.

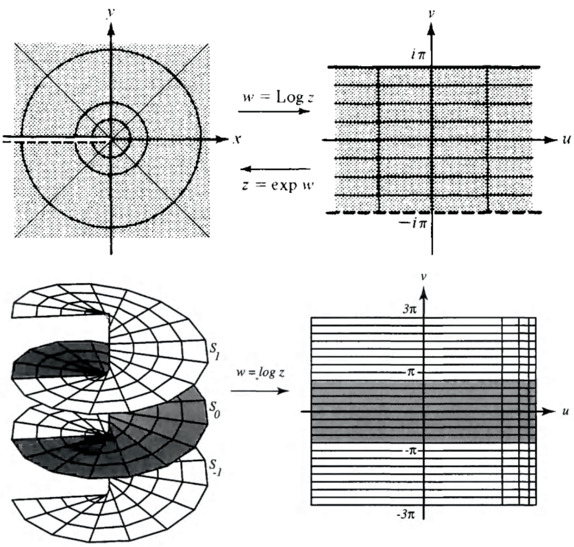
The existence of this homeotic kernel $2\pi\mathbb{Z}$ of the complex exponential group homomorphism $e^i:(\mathbb{R},+)\to(S^1,\cdot)$ has a price. The price is that the complex exponential is not invertible globally, but only locally. This broaches the significance of the domain of sheaves into which we will encounter later on. At present, the fact that it is not possible to have a well-defined global notion of a complex logarithm as the inversion of the complex exponential entails the novel phenomenon of branching. In other words, the projection from the helix to the circle, although it bears well defined local sections inverting it locally, does not possess a global inverse. We may assert that branching is the geometric way to engage with the issue of homeotic consonance in the harmonic or transcendental domain. Topologically, the latter gives rise to what is called multi-connectivity. Branching is the geometric way to evade multi-connectivity by a process of cutting, bounding, and unfolding, until everything becomes simply connected.

Considering the complex logarithm, we realize that an inverse homomorphism from the multiplicative group $(S^1,\cdot)$ to the additive group $(\mathbb{R},+)$ can be defined only locally, i.e. by restricting the values of the angle within a period, i.e. from $-\pi$ to $+\pi$, $-\pi<\theta\leq\pi$, or from 0 to 2π, $0\leq\theta<2\pi$, which depicts a branch by cutting. The meaning of the branch is that the complex logarithm is single-valued within this branch. The whole issue arises from the multi-valuedness of the angle, because the complex exponential has the same value for angle θ, and $\theta+2k\pi$, where k is an integer. This is precisely what is encapsulated in the homeotic kernel of the complex exponential group homomorphism that is understood, as established above, by the nature of the helicoid. Epigrammatically, we may say that if the harmonic domain is associated with multiplexing and knotting, the geometric domain is associated with branching and weaving. Topologically, the main theme is connectivity, and the metaphora pertains to the unfolding of harmonic multi-connectivity into geometric simple-connectivity.

2.8 CANONICS: *METAPHORA* FROM HARMONICS TO GEOMETRY VIA TOPOLOGY

Since the harmonic and the geometric domain incorporate different principles of organization, we may consider the transcendental exponential and logarithmic functions not only from a gnomonic standpoint, but more accurately, from the perspective of canonics. The notion of a canon incorporates the requirements for an analogia or metaphora between two structurally, or organizationally different, domains that can communicate to each other covariantly by conjugation, which means by the enforcement of appropriate encoding and decoding bridges. The notion of canonics emanates from Pythagoras' vibrating monochord that embodied the idea of descending from the harmonic to the geometric domain and ascending back. Conceptually it refers to the transfiguration of an acoustic chord to an optical fiber involving the instantiation of a scale that is able to transform acoustic frequency ratios to visual length intervals. The conception of relative frequencies as powers that can be perceived by the ear implies that the bridge from the harmonic to the geometric is of a logarithmic nature. In other words, the logarithm function transfigures frequency ratios to length intervals, since it converts division to substraction. Inversely, the transfiguration from the geometric to the harmonic domain is of an exponential nature. This fact has been implicated in the impossibility of setting up a rational scale of musical intervals. In turn, this bears the consequence that the discovery of the irrationals does not come from the geometric domain, most typically presented via the Pythagorean theorem, which targets the incommensurability of the diagonal with the sides of an orthogonal triangle, but derives rather from harmonics. The notion of the equally-tempered, based on the equipartition of musical intervals, ending up on the chromatic geometric scale, is a geometric solution to evade this problem at the price of sacrificing the pure harmonics.

The conceptualization of the imaginary ring set out here, allows the exemplification of canonics from the viewpoint of complex geometric function theory, especially as pertaining to the complex exponential and complex logarithm functions. More precisely, the complex exponential is an encoding bridge from the geometric to the harmonic domain, whereas the complex logarithm is a decoding bridge from the harmonic to the geometric domain, which is actually inverse to the former only locally, giving rise to the phenomenon of branching. The complex logarithm bridge of this metaphora may be thought of as the means of unfolding harmonic multi-connectivity into geometric simple-connectivity. From a structural algebraic viewpoint, the complex exponential defined in terms of a group homomorphism from the additive group $(\mathbb{R},+)$ to the multiplicative group $(S^1,\cdot)$ is extended now to a group homomorphism from the additive group of complex numbers $(\mathbb{C},+)$ to the multiplicative group of non-zero complex numbers $(\tilde{\mathbb{C}},\cdot)$. Considering the complex logarithm, an inverse homomorphism from the multiplicative group $(\tilde{\mathbb{C}},\cdot)$ to the additive group $(\mathbb{C},+)$ can be defined only locally, which obliges us to restrict the values of the angle within a period. The period in question is either, from $-\pi$ to $+\pi$, $-\pi<\theta\leq\pi$; or from 0 to 2π, $0\leq\theta<2\pi$, which depicts a branch where the complex logarithm is continuous and single-valued.

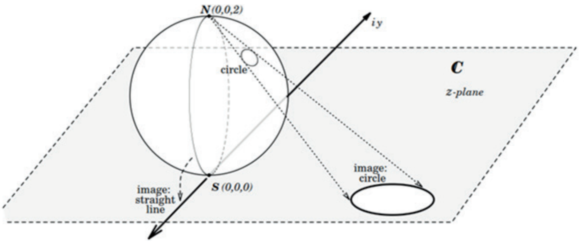


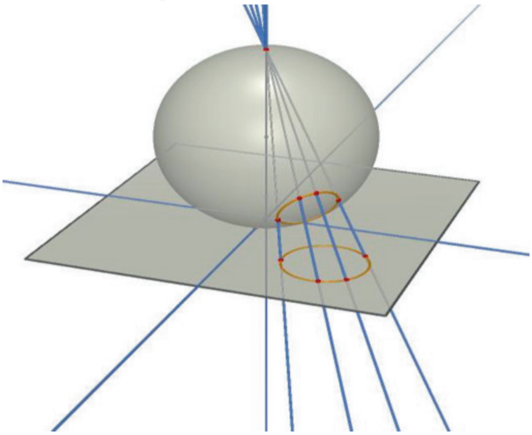
The harmonic multiple-connectivity is encapsulated precisely in the homeotic kernel $2\pi\mathbb{Z}$ of the complex exponential group homomorphism that is intrinsic to the nature of the helicoid. Consequently, the homeotic criterion of identity is expressed in terms of the angular temporal interval of one whole period 2π times the harmonic series, being identified structurally with the multiplicative group of the integers. We stress that the integers in this setting are manifested as powers. Topologically these powers are utilized for counting the number of windings. Thus, topologically, the above homeotic criterion of identity amounts to a homological criterion of identity. More concretely, the multiplicative group of the integers plays the role of the first homology group of the topological circle, e.g. the first topological structural invariant of multiple-connectivity expressed by means of a commutative group. Not only this, but additionally, in the case of the topological circle, the first homology group is isomorphic with the first *homotopy* group, or *fundamental* group, since both are identified with the multiplicative commutative group $\mathbb{Z}$. In this manner, the homeotic criterion of identity, serves not only as a homological criterion of identity, but also as a homotopic criterion of identity in relation to the topological circle. Of course, the imaginary ring endows the topological unit circle with the complex structure by which the complex exponential and logarithm functions are defined.

2.9 SPHERICS: THE PLANISPHERE PROJECTION

Given the identification of the topological unit circle endowed with the complex structure with the imaginary ring, culminating in the polar grid scaffolding of the complex plane, via the Euler representation, we realize that this grid actually descends from the sphere endowed with the complex structure, called currently, the *Riemann sphere*.

This is elucidating, since the original framework of Pythagorean harmonics and canonics was called *spherics*. A simple geometric perspective offering a glimpse to spherics consistent with our interpretation of metaphora is incorporated in the *stereographic projection* of a sphere onto a plane. This projection is described first in Ptolemy's *Planisphaerium*, called originally, the *planisphere projection*. This projection is defined on the whole surface of the sphere with the exception of a single point (usually taken as the North pole) that is identified with the locus of projection, or the point through which light rays enter the sphere, propagate through it, until eventually they emerge out of the sphere by crossing it at a point, which is mapped one-to-one on a point on the plane. The stereographic projection is bijective and smooth with the exception of the single point of projection. It preserves neither distances nor areas. Its major characteristic though is that it preserves oriented angles between any two paths on the sphere, hence it is not only isogonal but also *conformal*. It is precisely this characteristic emerging out of the metaphora from the sphere to the plane through the stereographic projection that provides the crucial insight on what geometrically qualifies the complex structure emanating from the imaginary ring, as we are going to clarify below. At this point, if we think of this projection as the encoding bridge from the sphere to the plane, it is just as important to consider the decoding bridge from the plane to the sphere.





The main observation is that the further out on the plane a point is, the closer its inverse image point on the sphere is to the North pole. But no point on the plane has as its inverse image the North pole itself. Rather, as a sequence of points move out towards infinity on the plane, their inverse images tend towards the North pole on the sphere. Therefore in the setting of this metaphora, the notion of infinity on the plane entrains the North pole of the sphere as its inverse image, so that there is a continuous one-to-one correspondence between the plane together with infinity and the sphere. In this sense, the sphere is homeomorphic topologically with the compactification of the plane emerging by the addition of a virtual point, called the ideal point at infinity.

2.10 STEREOGRAPHY: THE CONFORMAL QUALITY OF COMPLEX STRUCTURE

Next, the basic concept that we intend to pursue is that the conformal quality of the stereographic projection is equivalent to endowing both the plane and the sphere with a complex structure. For this purpose, it is worth pondering in more detail on the conformal character of this projection. The crucial aspect is that the projection preserves on the plane the angles at which paths on the sphere cross each other, and more precisely the angles at a crossing point between the tangent vectors of these paths at the crossing point. On the other side, the stereographic projection does not preserve area, i.e. the area of a region on the sphere is not generally equal to the area of its stereographic projection onto the plane. A precise geometric way to understand this phenomenon can be expressed by means of the notion of intrinsic Gaussian curvature.

More concretely, since the sphere and the plane have different intrinsic curvatures, there cannot exist a projection from the sphere to the plane that preserves both oriented angles and areas, since in that case, the curvature would be preserved. Therefore, a projection from the sphere to the plane can be either conformal or area-preserving, but not both simultaneously. According to the preceding the stereographic projection is only conformal. In consequence, circles on the sphere that do not pass through the North pole, i.e. the locus of projection, are projected to circles on the plane, whereas circles on the sphere that *do* pass through the North pole are projected to straight lines on the plane. Equivalently, these lines may be thought of as circles through the virtual point at infinity, or as circles of infinite radius. Inversely, all lines on the plane being transformed to circles on the sphere by the inverse of the stereographic projection meet at the North pole. In particular, parallel lines, which do not intersect on the plane, are transformed to circles, tangent at the North pole, whereas intersecting lines are transformed to circles intersecting transversally at two points on the sphere, one of which is the North pole. The *loxodromes* on the sphere, by which we mean the paths of constant compass bearing on the sphere, or the paths having invariant angle with the corresponding parallels of latitude and meridians of longitude, project onto paths intersecting radial lines on the plane in an equiangular way, i.e. they project onto logarithmic spirals on the polar grid.

The naturality of the complex differentiable structure on the sphere emerges as follows through the stereographic projection: We notice that although the stereographic projection from a single point on the sphere to the plane fails to map this single point of projection from the sphere to the plane, nevertheless if we consider two simultaneous projections from different points of the sphere to the plane, e.g. the first one from the North pole and the second from the South pole of the sphere, then the entire sphere can be mapped conformally on two copies of the plane. The first one may be thought of as tangent to the South pole, while the second as tangent to the North pole of the sphere. Clearly this double stereographic projection contains redundant information about the sphere.

The idea is that each copy of the plane is a local patch of the sphere, which actually covers the whole sphere with the exception of the projection point, such that each patch is identified with the inverse image of the corresponding projection. In this manner, it is evident that two distinct but overlapping patches afford a complete covering of the sphere in their descriptive terms. The implicated metaphora of structure from the plane to the sphere through these two locally covering patches is that they exchange information compatibly about the sphere, or else they are compatible on their overlaps. Each copy of the plane is endowed with a complex structure induced by the imaginary ring in two distinct ways.

For simplicity, we may identify both copies of the plane with the equatorial plane of the sphere and induce the complex structure in the first case by the complex parametrization $\alpha = W + iV$, while the second is given by the complex parametrization $\beta = W - iV$. Then, a transition map from one patch to the other, i.e. from the α-parameterized copy of the complex plane to the β-parameterized copy of the complex plane, both identified with the equatorial plane of the sphere, asserts how these two copies are glued together by restriction to their overlapping regions. The gluing takes place by the identification of each non-zero complex number α of the first copy with the non-zero complex number $\frac{1}{\beta}$ of the second copy, and conversely. In this sense, what plays the role of origin in the α-parameterized copy, assumes the role of infinity in the second β-parameterized copy, equally and conversely. The important thing is that the transition maps so-defined from one complex patch to the other are holomorphic maps. This means that a transition map as specified above is complex differentiable on its domain of definition.

Thus, we obtain a holomorphic atlas on the sphere endowing it with a complex differentiable structure, i.e. the sphere becomes a one-dimensional complex manifold, called the Riemann sphere.

It remains to show the equivalence of the *complex* structure on the sphere with the *conformal* characterization of the sphere emanating from the stereographic projection. For this purpose, let us denote the complex structure by J, such that analogously with the action of the imaginary unit on the complex plane, it rotates vectors in each tangent plane at a point of the sphere by $\pi/2$. Then, if we consider the stereographic projection from the North pole of the sphere, denoted by P, the conformal characterization in complex differentiable terms with respect to a tangent vector X at a point of the sphere, amounts to the prescription that $dP(JX) = idP(X)$, i.e. dP is a complex linear map.

Equivalently, this informs us that the conformal projection map is a non-degenerate *holomorphic* map; a complex differentiable map such that the differential never becomes zero. Henceforth, the complex structure J on the sphere is equivalent to the conformal characterization of the sphere by the stereographic projection, in the precise sense that the differential map dP is complex linear. This implies a further equivalence: The operation of first rotating a tangent vector at a point of the sphere by J followed by the operation of pushing forward this vector from the sphere to the plane by dP is indistinguishable from the operation of first pushing forward the tangent vector from the sphere to the plane by dP followed by the operation of rotating the latter by the imaginary unit i. In other words, these two distinct operations commute, such that on the complex plane of projection $dP(JX) = idP(X)$, called the *Cauchy-Riemann* equation.

Thus, the complex structure on the sphere making it a complex manifold is equivalent to the conformal characterization of the sphere via the stereographic projection, which in turn, is algebraically equivalent to the commutativity of the operations of imaginary rotation and pushing forward by the differential of the projection with respect to the plane of this projection. Note that if P is the stereographic projection from the North pole of the sphere, then dP is the complex linear map from the tangent plane at a point of the sphere to the plane of projection to be thought of as pushing forward tangent vectors from the former to the latter.

It is important now to stress that the metaphora induced by the stereographic projection of the sphere should not be considered as independent from the original framework of harmonics and canonics that made this same metaphora necessary in the first place. This means that the actual depth of the stereographic projection, together with the threefold correspondence established previously, can be properly appreciated from a temporal standpoint, rather than a spatial one. The objective is to think of the sphere in temporal or chronological terms so as to achieve an insight on the temporal status of the projection plane. What really matters is the fact that the projection plane is capable of representing the sphere completely by inverse stereographic projection, if and only if we employ two distinct complex-parameterized copies of it, standing for projection planes with respect to the North and South pole of the sphere correspondingly, and so long as they are amalgamated compatibly -continuously and holomorphically- together, on their overlapping regions. With these conditions, the projection plane, identified with the equatorial plane of the sphere, binds together antipodally the northern with the southern hemisphere, such that the imaginary ring in the patch corresponding to the projection from the North pole is glued together antipodally with the imaginary ring in the patch corresponding to the projection from the South pole, and both are identified with the equator of the sphere.

Thus, in view of the doubly articulated stereographic projection, the projection disk bounded by the equator of the sphere with the complex structure, i.e. in its function as an imaginary ring, is conceptualized temporally as an epiphaneia of the present, which binds together what is included inside the ring with what is outside the ring by means of inversion with respect to this ring. This is geometrically termed *circle inversion*, meaning geometric inversion with respect to the equator in our context, which is accomplished by means of complex inversion followed by reflection on the W-axis, where the latter is simply complex conjugation. Taking into account that complex inversion is actually physically implemented by a π-rotation about the W-axis, we conclude that what appears on the epiphaneia of the present is a binding of a point inside the projection disk with a point outside it obtained from the first via a π-rotation about the W-axis followed by reflection in the W-axis.

There are now two interrelated issues that we have to elucidate in order to make viable the sought after temporal interpretation. The first issue targets the nature of the above binding; what is the physical process corresponding to the shadow of this binding that is accomplished on the epiphaneia of the projection disk bounded by the imaginary ring of the equator of the sphere? The second issue targets the nature of what we call the present in the context of the temporal interpretation of the stereographic projection. It turns out that these two issues are not independent of each other. We just have to focus carefully on the domain of harmonics in order to make sense of the implicated geometry on the epiphaneia of the disk.

What is actually involved is the helicoid together with its mirror image as constituting a harmonic entity. We recall that the helicoid unfolds continuously by parallel translation, orthogonally and away from the epiphaneia, and after half a period of rotation a mirror image of the same helicoidal surface arises. In this way, we think of the helicoid together with its mirror image as helical waves propagating in opposite directions such that the mirror image is the reflection of the first. This is possible if these helical waves are bounded from above and below for a temporal length of one period so as to give rise to a helical standing wave. Equivalently, within this bounded interval the helical wave is in unison with its reflection, its mirror image. Being in unison means that they are consonant in the fundamental harmonic frequency corresponding to the frequency ratio $1:1$, pertaining to an angular temporal interval of one whole period 2π. Therefore, the pertinent kind of binding appearing on the epiphaneia of the disk expresses the harmonic resonances of the helical standing wave in question, whence the bounding of this standing wave is provided by the sphere in terms of the temporal length of its periods of rotation.

The above leads to a novel meaning in relation to the notion of the present. A single moment of time is resolvable homeotically by the whole harmonic series capable of being instantiated within the posed bounds at this moment. Moreover, the harmonics persist diachronically, that is for each conceivable present, being able to induce any harmonic ratio in that present. In this way the harmonics are qualified as powers for the actualization of consonances in each present. Consequently, the diachronic, harmonically persistent present manifesting on the epiphaneia, constituted by the invariants of the helical standing wave homeorhesis, that is the points of homeostasis depicted by the harmonics, is identified with the bounding imaginary ring, which in turn is the equator of the sphere endowed with the complex structure by virtue of the conformality of the stereographic metaphora. Hence, in the context of the present, the harmonics appear through the complex roots of unity of the imaginary ring equator.

2.12 *ARCHIMEDEAN* SPIRAL: *METAPHORA* FROM THE CIRCULAR TO THE LINEAR

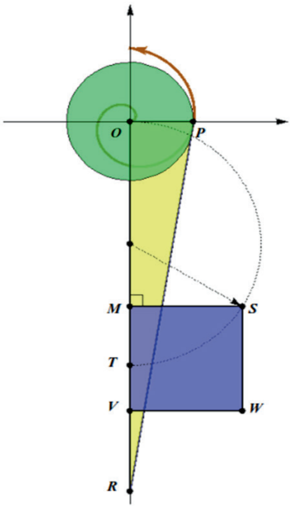
There are two fundamental questions that have to be addressed in the preceding framework. The first concerns the clarification of the precise manner that the sphere bounds the helical standing wave. This is fundamental because it elucidates the temporal interpretation of the sphere in terms of its periods of rotation. The second concerns the problematic of how the present can be thought of as the imaginary axis of the complex plane in its rectangular manifestation in relation to a possible conformal projection of the sphere derived from the stereographic projection that manifests in the polar grid of the projection plane.

Both of these questions can be dealt with in a satisfactory manner if we pay attention to the significance of the major problem of ancient Greek mathematical enquiry, i.e. the problem of squaring the circle. We view this problem as a problem of natural communication, and for this reason the method proposed by *Archimedes* to address it bears great significance. We call this method Archimedes' metaphora because the Archimedean method does not supply a constructible solution to this problem by straightedge and compass. Instead, Archimedes having realized that there does not exist a constructible solution, he invents a metaphora from the circular to the linear domain. It is this metaphora that deserves a proper emphasis and appreciation.

The problem of squaring the circle, refers to the instantiation of a square that has the same area as that of the disk bounded by a circle. In the first stage, Archimedes considers an isomorphic problem. Namely, the problem of geometrically unfolding the perimeter of a circle to a linear length. This more fundamental problem conceptually can be cast isomorphic to the original as follows: If the geometric unfolding of the perimeter of a circle to a linear length is possible, then the area of a circle can be made equivalent to the area of an orthogonal triangle whose sides are given by the radius of the circle and the perimeter of the circle.

The main problem arises from the irrationality of π, which is actually a transcendental number. For every conceivable circle of some radius, π is an invariant characterizing the perimeter through the radius. The incommensurability of the circular domain with the linear domain is precisely captured by the irrationality of π. In the "Measurement of the Circle" Archimedes devised an ingenious approximation to the perimeter of the circle involving the method of exhaustion by means of inscribed and superscribed polygons. This is in effect to march toward the perimeter both from inside and outside using polygonal approximations involving up to 96 sides. In relation to the pertinent problem of squaring the circle, Archimedes devised the means of metaphora from the circular domain to the linear domain in terms of the Archimedean *spiral*. In other words, the Archimedean spiral is introduced relationally with respect to these incommensurable domain; in our terms as a bridge of metaphora from the circular to the linear and inversely.

The spiral is conceived in physical terms by Archimedes. He considers a point particle, located initially at the centre of the circle, which starts to move uniformly from the centre to the periphery of the circle along the radius. Simultaneously, Archimedes considers that the radius rotates uniformly counterclockwise around the centre of the circle. Thus, the particle movers according to the composition of these two uniform motions, the first linear, and the second circular. The composition of these two uniform motions is a non-uniform motion, which describes the trajectory of the considered particle. It is this trajectory that bears the geometric form of the Archimedean spiral. The spiral is devised as a means of metaphora from the circular to the linear domain, more precisely, as a means of geometric linear unfolding of the perimeter of the circle into a measurable linear length. This is accomplished by realizing the tangent to the spiral after one turn, i.e. at the point where it intersects the circle after one turn. Archimedes showed that the tangent line to the spiral at this point crosses the vertical axis at a point whose distance from the origin is exactly $2\pi r$, where r is the radius of the circle. As a result, the tangent to the spiral at the point of its intersection with the circle corresponding to a 2π rotation, accomplishes the required unfolding of the perimeter into a linear length, which is provided by the distance of the point of intersection of this tangent with the vertical axis from the origin. The important thing to notice is that the recording of this linear length corresponds to the time needed by the particle to complete one turn of its spiral trajectory; to all intent the perimeter of the circle is unfolded as a temporal length. This form of temporal unfolding is periodic and can be analogously recorded for all higher turns of the spiral. Note also that the radius of the spiral at each point of the trajectory of the particle is determined by the angle with respect to the horizontal axis.



According to the above, the perimeter of the circle is unfolded into the linear length $2\pi r$, which is recorded at the vertical axis as the vertical side of an orthogonal triangle whose horizontal side is the radius r of the circle. Then, the area of this triangle is half of the area of the parallelogram having the same sides, which is clearly $2\pi r^2$. Thus, the area of the circle is the same as the area of the above triangle, i.e. πr^2.

2.13 HARMONIC RESOLUTION OF TIME: IMAGINARY IMPRINT OF THE *POLYSTROPHIC* SPIRAL

It is elucidating to attempt an interpretation of Archimedes's method in the terms of the imaginary ring. For simplicity, we may consider the circle as the unit circle in the complex plane. Then, the two-dimensional Archimedean spiral unfolds the unit circle into the imaginary axis by the above procedure. The question is how we qualify the imaginary axis in this setting. There are two stages to address this issue. The first refers to the conception of the imaginary axis as a spatialized temporal dimension through the intervention of the imaginary unit. Accordingly, the spatialized temporal length corresponding to the time needed for the completion of one turn of the spiral becomes imaginary, and thus, negative if squared. The second stage takes into account the periodicity that is implicit in the successive turns of the spiral. More precisely, Archimedes' method allows the unfolding of the perimeter of the circle multiple times recorded by the turns of the spiral, which means the Archimedean spiral is *polystrophic* and not only *monostrophic*. This fact forces the conception of time in this setting as a helix in three dimensions unfolding orthogonally to the complex pane, and which is projected epimorphically on the imaginary ring, i.e. on the unit circle on the complex plane, endowed with the polar grid via Euler's coordinatization. This is nothing else than the exponential group homomorphism $\exp : \mathbb{R} \to S^1$, whose kernel is $2\pi\mathbb{Z}$. Thus, topologically the winding number counts the integer number of turns around the origin, which is excluded from the complex plane.

If we consider Archimedes' spiral as the means of metaphora from the circular to the linear domain, according to the preceding, the counterclockwise oriented spiral is the encoding bridge, whereas the inversely oriented clockwise spiral is the decoding bridge. Thus, we may invoke harmonic considerations in our setting, as pertaining to the helical conception of time, by means of a helix and its mirror image, if bounded appropriately. The leading idea again is that a single moment of time, identified with the present of the imaginary ring shadow, can be resolved homeotically by the whole spectrum of harmonics, giving a precise meaning to the polystrophic quality of the Archimedean spiral in the two-dimensional projection.

The effect of this manoeuvre through the domain of harmonics is that we can now qualify the integers as a quantum spectrum of frequencies on the imaginary axis of the complex plane via branches of the complex logarithm. We emphasize that this is viable due to the fact that both the complex exponential and its local inversion in terms of a branch of the complex logarithm are conformal, meaning they preserve oriented angles. In light of this fact, the homeotic criterion of identity can be imprinted on the imaginary axis in terms of the angular temporal interval of one whole period 2π times the harmonic series. Consequent on the periodic resolution of time in terms of the harmonics, time and frequency become reciprocally correlated, and represented orthogonally to each other. Time in the form of the *helix* in three dimensions unfolds orthogonally to the complex plane. The helix considered together with its mirror image give rise to a helical standing wave bounded by temporal intervals of integer periods. The latter projects down to the complex plane with the origin removed on an annular strip of the polar grid. Applying a corresponding branch of the complex logarithm transforms this strip conformally to a rectangular region on the complex plane. The imaginary axis is marked in this way by the harmonic frequencies corresponding to the integer number of cycles per unit of time, where the latter is taken to correspond to the temporal length of one whole period 2π.

Henceforth, it is instructive to note that the bounding in the complex analytic setting takes place via logarithmic branch cutting, which stems from the fact that no global complex logarithm function exists by which to invert the complex exponential function. Notwithstanding this fact the restriction to branches preserves the conformal character mapping annular strips to rectangular regions on the complex plane and inversely. As a side remark, we come to an understanding of the finite topological coverings of the circle by itself corresponding to all different integer powers as emanating from the universal unfolding of the circle by the helix, considered together with the above described process of bounding that reveals the harmonics. Therefore, the topological winding number physically descends from the harmonics, which are qualified as powers for the actualization of consonances, i.e. harmonic ratios. Finally, it is the action of the logarithm through its single-valued branches that transforms these ratios into spatialized spectral intervals measured on the imaginary axis, as outlined above.

2.14 HARMONICS OF THE *ARCHIMEDEAN* SCREW AND THE EQUIAREAL PROJECTION

An adequate mechanical model that captures all of the above aspects is the Archimedean screw in three-dimensional space. If we consider a finite portion of the screw, then the projection of the screw onto the plane thought of as perpendicular to the central axis of the screw, depicts an annular strip of the polar grid on the complex plane with the origin removed. Again, if we apply the complex logarithm this strip is transformed conformally to a rectangular region on the complex plane. We may now imagine the bounding of such a finite portion of the screw by an open cylinder.

This is particularly elucidating in relation to Archimedes's method of determination of the surface of a sphere on the basis of his method of unfolding the perimeter of the circle into a linear length. We recall that the latter leads to the conclusion that the area of a circle is equal to the area of an orthogonal triangle whose sides are equal to the radius of the circle and the perimeter of the circle respectively. Since the area of the sphere is $4\pi r^2$, this area is the same as the area of four circles of the same radius r, or equivalently, the same as the area of four orthogonal triangles fitting compatibly together, whose big side is $2\pi r$ and small side is r, where r is identified with the radius of the sphere.

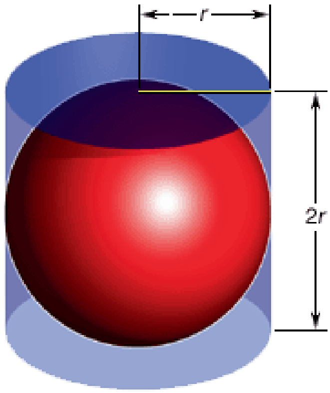
Necessarily these circles should be considered as great circles passing through the North and South pole of the sphere, so as their radius is the same as the radius of the sphere. The unfolding of any such circle into a linear length equals the equatorial length of the sphere, or the length of the perimeter of the equator, given by $2\pi r$. Thus, all four orthogonal triangles should have a big side equal to the equatorial length of the sphere $2\pi r$ and small side equal to the radius of the sphere.

It follows directly that these four triangles fit together in a plane region divided in two halves by the horizontal equatorial line of length $2\pi r$, such that the small side of each triangle r equals the vertical side of a half of this plane region. Conclusively, this plane region has a horizontal side equal to the equatorial length of the sphere $2\pi r$ and a vertical side equal to $2r$. Each horizontally conceived half divided by the equatorial line has sides $2\pi r$ and r respectively. In each half there fit two orthogonal triangles sharing the same diagonal of sides $2\pi r$ and r respectively. Thus, the area of the sphere equals the area of these four orthogonal triangles, each one of which equals πr^2.

In order to obtain a proper insight on the above Archimedean method of determining the area of a sphere it is significant to realize what it implies. Precisely, it implies that there exists an *equiareal* *projection* of a sphere onto a cylinder, which we call the Archimedean projection. Note that this projection of the sphere is not conformal, as is the stereographic projection, but it does preserve areas. It is realized as a horizontal radial projection emerging by placing the sphere within an open cylinder touching it along the equator. In this sense, the Archimedean projection should be considered as complementary to the stereographic projection of the sphere onto a plane.

Topologically, we may easily see that if we cut the sphere along a meridian passing through both the North and South pole of the sphere, we unwrap the sphere onto an open cylinder of height $2r$. The area of this cylinder equals the area of a plane region on which it can roll for the temporal length corresponding to one rotation, $2\pi r$, identified with the equatorial length of the sphere.

Hence, the sphere excluding its North and South pole can be projected in an area-preserving manner onto a open cylinder, and vice versa. Rolling the cylinder as above, we obtain an equiareal projection of the sphere on a planar region whose horizontal side is the equatorial length of the sphere $2\pi r$ and whose vertical side is $2r$, together with a rectangular and straight weaving grid of meridians and parallels.



The inverse equiareal projection from the open cylinder to the sphere is very important in relation to the first of the previously posed questions concerning the clarification of the precise manner that the sphere bounds a helical standing wave. It was conceivable simply as a finite portion of an Archimedean screw, characterized by its harmonics. As we mentioned before, this is fundamental because it elucidates the temporal interpretation of the sphere in terms of its periods of rotation, recorded eventually in terms of spectral lengths on the imaginary axis of the complex plane due to the Archimedean linear unfolding of the equatorial circle of the sphere. Since the latter bounds the epiphaneia of the present by the conformal stereographic projection of the sphere onto the plane, the inverse equiareal projection reveals the resolution of the present into the pertinent harmonics according to the bounding role of the sphere. For these reasons, all bearing on the posited temporal interpretation, the conformal stereographic projection of the sphere onto the plane cannot be considered independently from the equiareal Archimedean projection of the sphere onto the cylinder unrolling onto another plane.

2.15 TEMPORAL DIASTASIS: SYNTHESIS OF CONFORMAL WITH EQUIAREAL *METAPHORA*

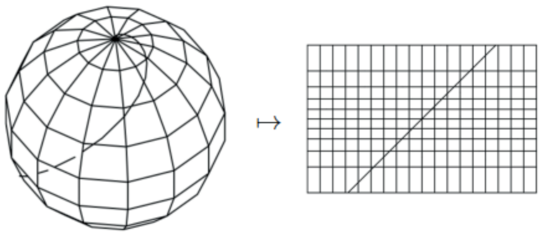
We argued that the present manifesting on the epiphaneia is expressed by the equator of the sphere, endowed with the complex structure by virtue of the conformality of the stereographic metaphora, and thus identified with the bounding imaginary ring of the horizontal disk of this projection. If we think in terms of Leon Battista Alberti's veil metaphora of *Renaissance perspectivism*, then the veil is the epiphaneia of the metaphora, whereas the eye corresponds to the projection point, i.e. either the North pole or the South pole in the case of the stereographic projection. If we identify the veil with the equatorial disk of the sphere, the interesting thing is that the sphere, which plays the role of the scene in this metaphora, lies both in front and behind the veil.

We recall that the pertinent kind of binding appearing on the epiphaneia of the disk descends from the harmonic domain; it expresses the harmonic resonances of a helical standing wave, whence the bounding of this standing wave is provided by the sphere in terms of the temporal length of its periods of rotation. It is precisely this fact that is encoded in the Archimedean equiareal projection of the sphere onto the cylinder, according to the preceding.

We emphasize that the consequence of this synthesis combining the stereographic with the Archimedean metaphora from the sphere to the plane and inversely, implies that the present is resolvable homeotically by the whole harmonic series capable of being instantiated within the pertinent temporal spherical bounds. Additionally, since the harmonics persist diachronically, that is, for each conceivable present, they are qualified as powers for harmonic resonance in each present.

Hence, in the context of the present, the harmonics appear through the complex roots of unity on the imaginary ring equator. The diachronic persistence of the harmonics, elevating them to invariants from a homology-theoretic topological viewpoint, has the effect that they appear as points of *stasis* geometrically. In this sense, the temporal helical axis of unfolding perpendicular to the epiphaneia bears the meaning of a temporal *diastasis*. As we have shown, any appropriate finite bounded portion of this diastasis can be transfigured conformally, by means of a branch of the complex logarithm on the imaginary axis of the complex plane equipped with the rectangular grid, as a spatialized spectral interval. This settles completely the second question concerning the problematic of how the present can be thought of in terms of the imaginary axis of the complex plane in its rectangular manifestation.

A significant observation regarding the complementary roles of the stereographic and Archimedean projections of the sphere is that the achieved synthesis actually pertains to any conformal and equiareal projection of the sphere. We should nevertheless recognize that a projection of the sphere on the plane that is both conformal and equiareal is not possible due to the *curvature* of the sphere. Reciprocally, the geometric characteristic of the curvature can be explicated from the synthesis of these two complementary types of metaphora, which are both indispensable in order to derive the notion of the present introduced here, e.g. from the temporal interpretation of these projections. For instance, we may consider another conformal projection of the sphere on the plane different from the stereographic one, but bearing the same oriented angle-preserving character, called the *Mercator projection*.



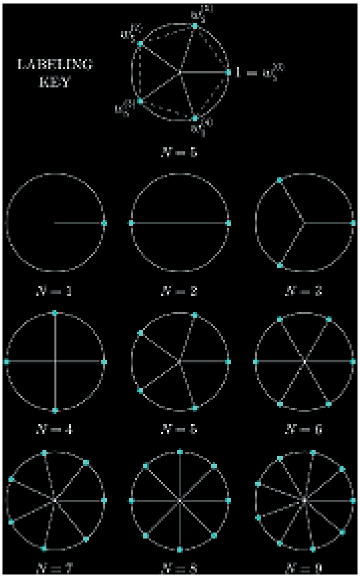
In this projection parallels of latitude on the sphere correspond to horizontal lines and meridians of longitude to vertical lines on a rectangular grid. The importance of this projection is that loxodromes on the sphere correspond to straight lines, making this projection very useful in navigation. Note that in the context of the stereographic projection loxodromes appear as logarithmic spirals whose center is either the North or the South pole. Loxodromes are not defined at the poles of the sphere, but they spiral from one pole to the other. They may be thought of as winding around each pole an infinite number of times as they approach it, but the distance they cover is finite. The Mercator projection is a conformal projection that maps the unit sphere within rectangular strips of width 2π excluding the North and South pole of the sphere, such that loxodromes appear as straight lines.

Using both the Mercator and stereographic projections we can easily realize the conformal character of the complex exponential and the complex logarithm. For this purpose, first we consider the inverse Mercator projection from the plane to the sphere, and then the stereographic projection from the sphere to the plane. Both of them are conformal, thus their composition is conformal as well. This composition can be expressed in terms of the complex exponential, which maps a rectangular strip of width 2π to an annulus on the complex plane with the origin removed, since the pole is excluded from the Mercator projection.

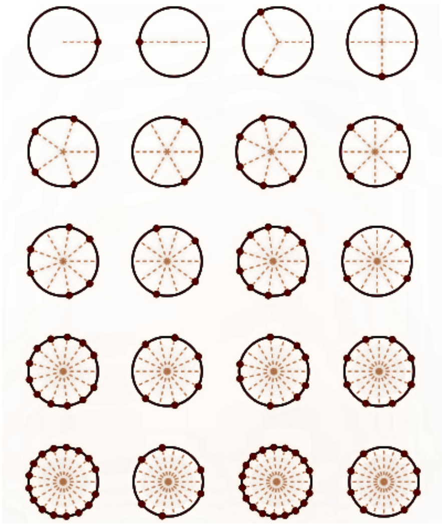
This argument shows that the complex exponential is a conformal map through the considered conformal projections of the sphere. The same argument can be used in reverse to show that the complex logarithm, through its principal branch for instance, is also conformal from the plane of the stereographic projection to the plane of the Mercator projection. With this approach, our basic conclusions pertaining to the synthesis of any conformal and equiareal projection of the sphere, thought of as complementary types of metaphora from the sphere to the plane, are confirmed for the proposed temporal interpretation of these projections.

2.16 CYCLOTOMY: COMPLEX ROOTS OF UNITY ON THE IMAGINARY RING

We recall that since the harmonics persist diachronically they bear the status of powers for harmonic resonance in each present. Therefore, in the context of the present, the harmonics appear through the complex roots of unity on the imaginary ring equator. There are always n different complex n-th roots of unity, that is, complex numbers whose n-th power is equal to unity, equally spaced around the perimeter of the unit circle in the complex plane. Since, they are equally spaced they constitute a well-tempered scale on the epiphaneia of the present. Roots of unity are manifested geometrically as the vertices of a regular polygon that connects them together.



Of particular importance are the primitive roots of unity. More precisely, on the unit circle with n equally spaced rays, there is now a mark on ray k, denoting a primitive root of unity, if and only if k and n are relatively prime, having no common divisors other than 1.



An equally-tempered scale marking the unit circle leads to *cyclotomy* and is manifested geometrically on the epiphaneia in terms of regular polygons inscribed in the unit circle. We consider now the temporal helical axis of unfolding perpendicular to the epiphaneia bearing the meaning of a temporal diastasis, as above. It is important to examine the means of subdivision of this diastasis.

For this purpose, it suffices to adopt a topological standpoint and consider the integer winding numbers of this helical diastasis evenly covering the circle on the epiphaneia. We recall that finite bounded portions of this diastasis qualify these windings in terms of harmonics, corresponding to finite covering spaces of the circle, and expressed as powers in the complex analytic setting. First, we point out that the square power in relation to the unit circle corresponds to doubling the angle, and so on for all higher integer powers. Let us consider the finite double covering of the circle by the circle. This corresponds spectrally to doubling the frequency, and thus by inversion dividing the unit circle into half. Similarly, if we consider the finite triple covering, it corresponds to tripling the frequency, thereby dividing the circle into three parts.

We treat all higher integer powers analogously, and by inversion, that is in terms of the roots of unity, we are able to subdivide the circle. The cyclotomy corresponds spectrally to the generation of regular polygons. The deeper the resolution of the cyclotomy is, the higher the number of vertices appearing equi-distantly on the unit circle, giving an ever-higher number of sides of the inscribed regular polygon. In this manner, the harmonic subdivision of the unfolding helical diastasis, is manifested geometrically as regular polygons inscribed in the circle. The further this subdivision is pursued by ascending to higher harmonics, resolving the circle in a more refined way, the higher the number of polygonal sides inside the circle.

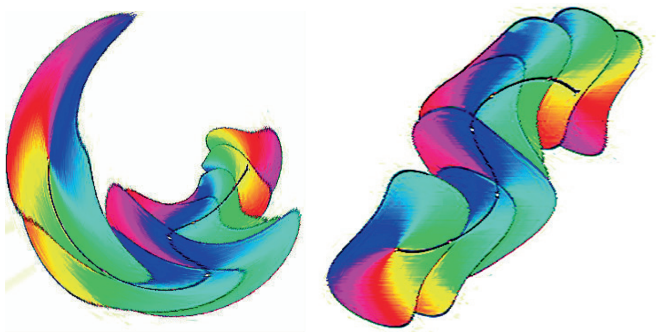
2.17 SHEAVES: *METAPHORA* FROM DESIGN TO ARCHITECTURE AND COLUMN CANONICS

The previous analysis provides an ideal starting point in order to think of the relation between design and architecture. The basic idea is that design descends from the domain of harmonics, whereas architecture descends from the domain of geometry. In this light, they can be characterized as reciprocally related to each other. Equivalently, there exists a metaphora from design to architecture, and vice versa, which should be though of as a metaphora from the domain of harmonics to the domain of geometry, as mediated through the topological level. From this perspective, the notion of a purely geometric design as well as its antipode, i.e. the notion of a purely harmonic architecture appear as degenerate conceptions that ignore the metaphora from the one to the other. Our distinction will be elaborated, in particular, by considering the fundamental notion of a *column* in terms of the proposed metaphora.

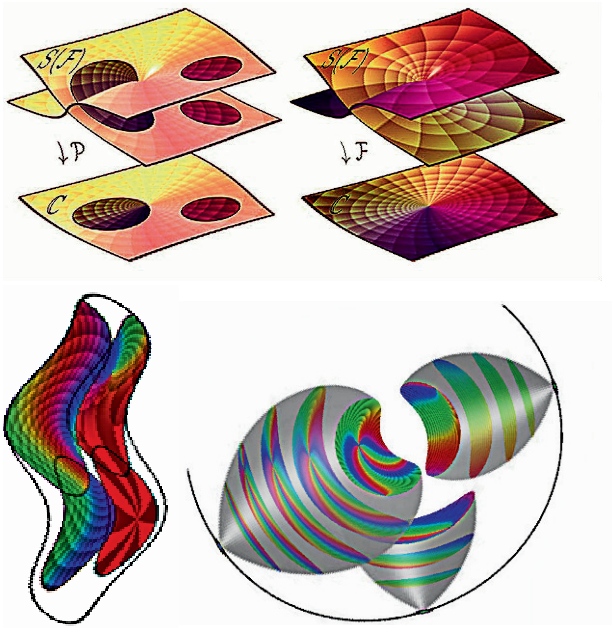
For this purpose, it is preferable to start working at the topological domain, considered as the mediating level between harmonics and geometry. The basic topological distinction is the part-whole, or more concretely, the local-global distinction. In the case of design, the global is envisioned in its totality, and the local parts are organized in a way to fulfill the global. The basic constraint of design is tantamount to the restriction of the local parts by the global. Thus, the local parts are passive, and the global is active restricting the local parts appropriately. Reciprocally, in architecture the local parts are selected in the form of atomic elements, which have to be jointly organized together geometrically so as to open up a new space.

In this sense, the basic constraint of architecture is how a local part can be extended by joining it together with another compatible local part, and so on toward the global. In consequence, in the latter case, the local parts are active, and the global is passive. Of course, in order that the opening up of a new space becomes possible, the local parts cannot be assembled randomly together. Rather their assemblage in space should conform to some pre-conceived global vision topologically, which allows the opening up of such a new space. In short, what is required is a metaphora from the domain of design to the domain of architecture.

If we characterize topologically the architectural assemblage that achieves the opening up of a new space as a sheaf, then the natural communication scheme between these domains, requires that at the level of design this assemblage bears the character of a *presheaf*. The difference between these notions is that in the case of the presheaf only the *global-to-local* compatibility is required, whereas in the case of the *sheaf*, the reverse *local-to-global* compatibility must also hold. Note that the first does not necessarily imply the second.



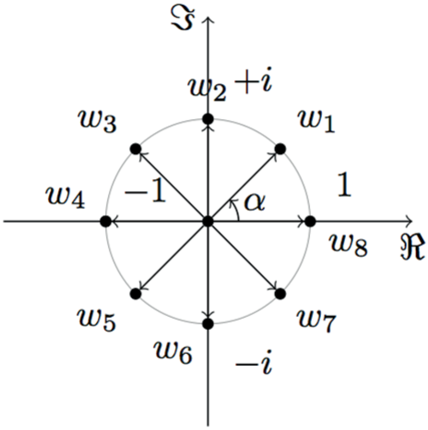
The idea that design, as conceptualized above, descends from harmonics is based on the fact that design is characterized by finality, or better, by *entelecheia*, since it carries its final purpose implicitly within itself according to the Aristotelian conception. Reciprocally, architecture is characterized by elementarity, and *efficient causality* in the organization of the local parts towards the global, subject to the constraints of geometry. Thus, architecture, although operating within the domain of geometry, implicating simple connectivity, is capable of opening up a new space only if the atomic elements are able to achieve a certain consonance with the whole, which in turn, presupposes the envisioning of such a whole: in other words the metaphora from design to architecture.



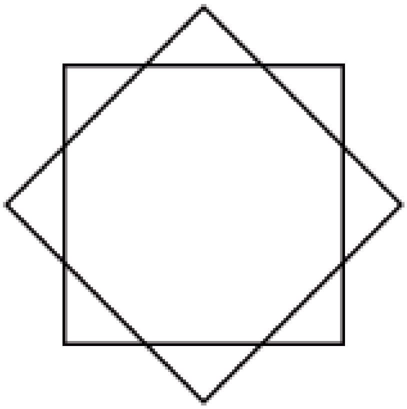
In this sense, the action of design may be though of as the action of a resonator that accomplishes the harmonic resonance of the actively envisioned global with the local through powers or spectral frequencies. Also in this sense, the assemblage of the active atomic elements geometrically should comply with the spectral compatibility of these elements in order that the opening up of a new geometric space in agreement with the harmonically envisioned global becomes feasible.

Note that beyond the topological level, in architecture, the active local parts can be joined together by admissible geometric transformations, in particular by translation and rotation, which in three-dimensional space can combine together in the form of a screw, a simply-connected finite portion of a helicoid. In the case of design, the envisioned global may be thought of harmonically as bearing the characteristic of the imaginary ring. We emphasize that this is not a *visual*, but a *harmonic* characterization, which means that the active global in design is not an already formed geometric entity, since this would contradict the essence of the metaphora. The active global in its harmonic conception should be able to restrict the suitability of the local parts only on the condition of harmonic resonance with them at each present, i.e. diachronically. This condition can be met only if the local parts can be qualified as complex roots of unity, which they synchronize as a polygonal totality with the global via powers up to a certain depth of spectral frequential resolution.

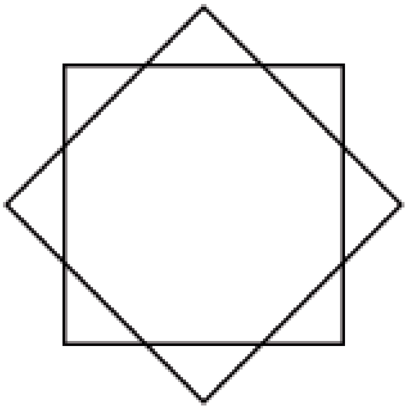
As a particular example of the above metaphora from design to architecture, together with their topological characterization in terms of the notions of a presheaf and a sheaf respectively, we consider the case of a column. We are going to examine the active global in design through the 8-th complex roots of unity, displayed below.

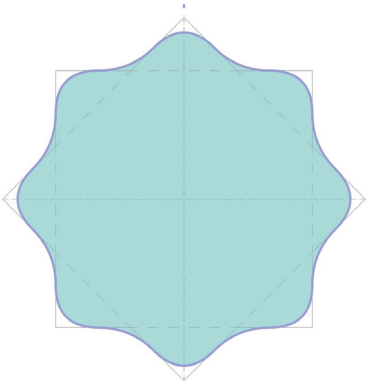


The basic idea is that the active local parts in this architectural implementation of the harmonics instantiated by design synchronize as a polygon of eight sides with the global via corresponding powers up to the depth of this spectral resolution. These active local parts can be amalgamated by the admissible geometric transformations of translation and rotation, which combine to a helicoidal screw in three dimensions. If we assume that these atomic elements admit the geometric form of graphs of primitive roots of unity, expressed as polygons, then in the considered case of the 8-th complex roots of unity, we instantiate a geometric template consisting of the superposition of a square together with another tilted square, together comprising an octagon.

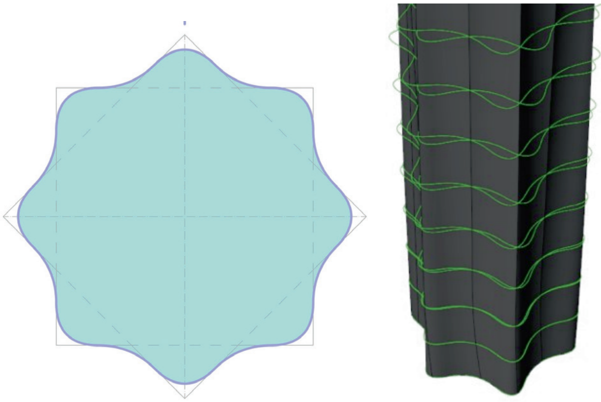


In this manner, the active local parts in architecture give rise to a geometric template. We consider the screw motion of this template along the temporal diastasis orthogonally to the superscribing imaginary ring marked by the 8-th complex roots of unity. Since we view the template as a superposition of two squares, it is instructive to consider the geometric template as an 8-star, whose sharp nodes may be rounded. The idea is that this star constitutes the basis of a column. Note that the 8-star template as a constellation is amenable to a screw motion with respect to two orientations; it can ascend according to the counterclockwise orientation and descend back according to the clockwise orientation, thereby providing the *ichnography* of a helical standing wave comprised of the harmonics under consideration.

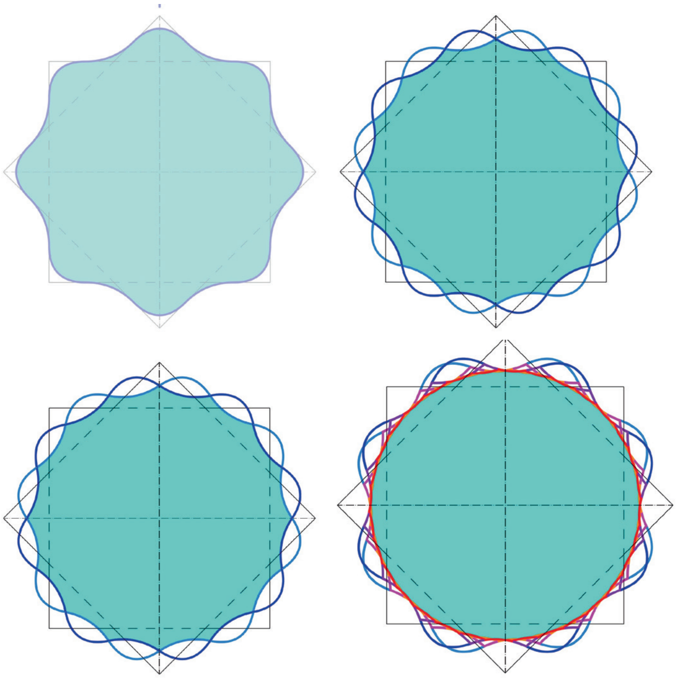




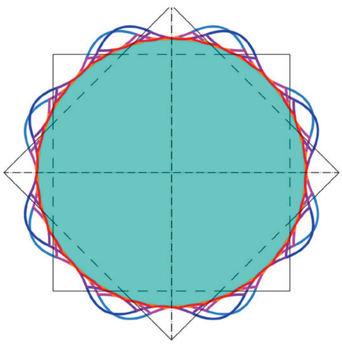
Therefore, the double screw motion of the 8-star template in three dimensions gives rise to a weaving pattern of two oppositely oriented helicoids, which topologically they overlap compatibly on the harmonics. Thus, they comprise a sheaf that admits the precise geometric manifestation of a part of a column.



Note that the above realized part of a column bears 8 striations. Clearly, it emerges from a quite low spectral resolution of the global imaginary ring involving only eight roots. We bear in mind that a pretty faithful approximation, according to the original construction of Archimedes, would involve 96 striations. Thus, either we increase the resolution implementing more and more striations, all at once, following geometrically the same pattern of double screw motion, or we treat the above realized part of a column as a branch of the whole column.



The latter is possible, since the whole column comprises a sheaf, and thus, we may legitimately consider its amalgamation through joining together distinct branches compatibly. We emphasize that joining branches together furnishes a resembling a tree. The basic notion is that when different logarithmic branches, constituted according to the preceding, join together, the number of roots doubles at each joint section. This branching tree-like structure of a column is iterated until the global section approximates a circle, completing in this sense the metaphora from design to architecture.



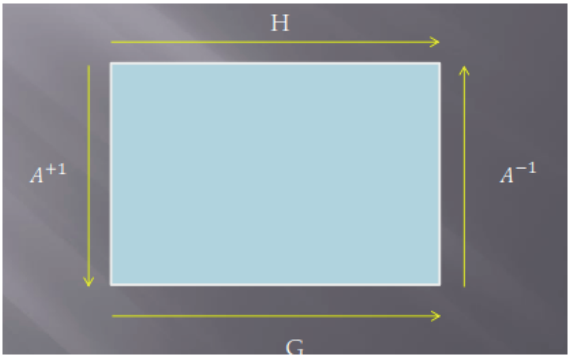
3

TEMPORAL BONDS: TRIPODAL SPECTRAL ARCHITECTONIC SYNTAXIS OF TIME

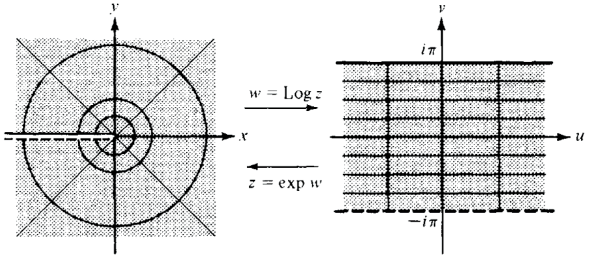
3.1 HARMONICS TO GEOMETRY: ENCODING LOGARITHM-DECODING ALGORITHM

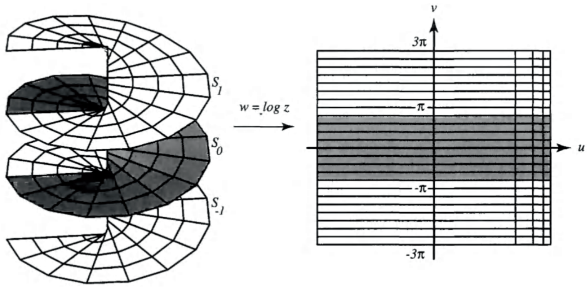
It is elucidating to consider in more detail the temporal aspects of the metaphora from harmonics to geometry along the lines already established. The temporal helical axis of unfolding perpendicular to the epiphaneia bears the meaning of a temporal diastasis. Any appropriate finite bounded portion of this diastasis can be transfigured conformally, via a branch of the complex logarithm on the imaginary axis of the complex plane equipped with the rectangular grid, as a spatialized spectral interval. In the context of each present, the harmonics appear as powers through their corresponding action on the complex roots of unity, in accord with the cyclotomy of the circle, or equivalently, the imaginary ring in the complex analytic setting. The diachronic persistence of the harmonics, elevates them to topological invariants that manifest as points of stasis geometrically. In this sense, a point of stasis bears temporal depth, since harmonic resonance is qualified in terms of such points diachronically. The crucial issue is that a spatialized spectral interval is not merely a geometric interval, but carries topological and harmonic information that manifests in discrete or quantum terms with respect to the temporal unfolding diastasis. More precisely, the implicated geometry is the one arising out of a cohomological spectrum, congredient with the sheaf localization structure. We qualify this geometry simply as spectral geometry to avoid further technicalities at this stage. The objective is to foreground the metaphora from harmonics to geometry in relation to the temporal aspects of this metaphora, not to focus exclusively on the terms of the complex analytic function-theoretic setting of our previous discussion.

The motivation for this reflection follows from the idea that the complex analytic function-theoretic setting is based on and presupposes for its consistency a certain notion of generalized number domain; the domain of complex numbers, which is an algebraically closed field. In a well-defined sense, the metaphora from harmonics to geometry cannot be consistently implemented without reference to the arithmetic cosmos of the complex number field. We recall that we employ two distinct types of metaphora in order to invert exponentiation. The first, referring to the powers with respect to a base necessitates the extension of the field of the rationals to the field of the reals, so that logarithmization becomes a total operation in the domain of real numbers. The second, referring to the roots, calls for the extension of the rationals to the complex numbers, to accommodate the roots of negative numbers. Both of these inversions are unified in the field domain of the complex numbers under the notion of the complex logarithm. But, in turn, the complex logarithm inverts complex exponentiation only locally, namely by considering a branch of this multi-valued function corresponding to the interval between two consecutive harmonics. As a result, the inversion of the logarithmic encoding bridge from harmonics to geometry, i.e. the decoding bridge from geometry back to harmonics, requires the algorithmic instantiation of the harmonics as powers acting on the roots of unity. Therefore, the notion of *arithmos* pertains to the metaphora from harmonics to geometry in terms of the *logarithm* notion in the encoding direction, and in terms of the *algorithm* notion in the decoding direction.



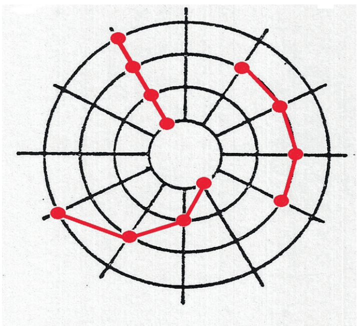
In the above setting, arithmos via its dual connotation as a logarithm and algorithm correspondingly, in relation to the encoding of harmonics to geometry and, inversely to the decoding from geometry back to harmonics, provides the bridges of the metaphora from harmonics to geometry. We may assert that the domain of harmonics directs the global choreography, the domain of arithmetics in its double role, as previously, directs the scenography and the orthography respectively, while the domain of spectral geometry directs the ichnography. The latter interprets a point of stasis as the trace of its temporal depth in each present. Taking into account the concomitant weaving pattern on the geometry depicted by the rectangular grid of the complex plane via the branched action of the logarithm, we realize that it descends from the epiphaneia conceived in its polar Euler representation bounded by the imaginary ring.





3.2 HARMONICS-ARITHMETIC-GEOMETRY: THREEFOLD *METAPHORA* AS A *STATIC TRIPOD*

According to our argument, the spectral geometry on the epiphaneia manifests itself like the spider's web, called *"arachne"* in ancient Greek. It is telling that in nature the spider ascends and expands its web by means of the logarithmic spiral, i.e. following the *geometric progression*, whereas it descends back to the center and stabilizes its web along a radius, i.e. following the *arithmetic progression*.



The ancient Greek naming is very interesting since all three of the involved notions -armonia, arithmos, and arachne- bearing the correspondence with harmonics, arithmetic, and spectral geometry respectively, emanate from the same linguistic root. We argue that this phenomenon is not accidental, but it is intimately correlated with the personification of Time in mythology. This provides a very elucidating conceptual insight on all the preceding. Before going to this matter, it is significant to point out a fundamental symmetry property pertaining to the metaphora from harmonics to geometry through arithmetics. This symmetry property, which is of a topological origin, is that the round trip at the level of harmonics via the level of geometry under the arithmetic bridges can equivalently and isomorphically be conceived as a round trip via the level of arithmetic under geometric bridges.

The latter is feasible if the level of arithmetic is identified simply as a spectrum characterized by the integer frequencies, whereas the encoding and decoding geometric bridges are identified on the epiphaneia in terms of the ascending logarithmic spiral and the descending Archimedean spiral respectively. In the latter setting, the domain of harmonics still directs the global choreography, the domain of the geometry in its double role, as above, directs the scenography and the orthography respectively, and the domain of arithmetic directs the ichnography.

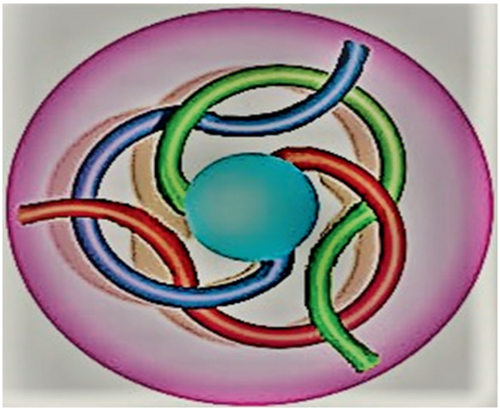
We realize that in the initial setting arithmetic is exemplified through its algebraic or operational aspect, whereas in the latter setting it is exemplified through its spectral discrete aspect. Analogously, in the initial setting geometry is exemplified through its spectral, quantum qualification, in the form of points of stasis, whereas in the latter setting it is exemplified through its kinematic qualification whose forms are spiral progressions. It is not difficult to realize that there exists a further symmetry, which can be considered a metaphora from arithmetic to geometry via encoding and decoding bridges belonging to the domain of harmonics. The encoding bridge is the extraction of the roots of unity, whereas the decoding bridge is the utilization of the harmonics as powers for synchronization of the points of stasis with the unity. We conclude that there exists a threefold canonical metaphoric communication among the domains of harmonics, arithmetic, and geometry, such that any one of them gives rise to a pair of encoding/decoding bridges with respect to a metaphora establishing the canonics between the other two. In this case, we say that these three domains constitute a *static tripod*, in accordance to the ancient Greek term.

3.3 TRIPOD OF TIME: WINDING-MEASURING-BOUNDING

Since a tripod of this nature and function is balanced on the notions of armonia, arithmos, and arachne, according to the above, it is worth revisiting briefly the cosmogonical context of the pre-socratic philosophy, where these threefold canonical communication relations were first conceived and explicated. The notion of the static tripod is of *Delphic* origin, where it stands stably over the *omphalos* symbolizing the primordial obstacle. In this sense, the static tripod is the generator of a threefold stable communication relation embracing the obstacle.

The very first abstract tripod incorporating the characteristics described above, is the tripod [*thauma*(state of wonder)/*ananke* (necessity)/ *aletheia*(truth)]. *Thaumazein* is the act of looking upward to the sky and wondering in the attempt to unveil what is true. The encoding and decoding bridge between these two levels, of thauma and aletheia is enacted by necessity. Note the bidirectionality of necessity in this metaphora, which is not restricted to some type of efficient causality, but it incorporates the entelecheia inversely as well. Moreover, aletheia is not identified with what we call today logical truth. Aletheia refers to unveiling what is diachronically true, where the latter is identified with what should not be forgotten after being unveiled; in other words aletheia is tantamount to *mnemosyne*, the faculty of memory and remembrance, personified by the mother of the nine muses.

Since thaumazein is the act of looking upward and aletheia is meaningful with respect to *lethe* which comes first, we can easily consider them as bidirectional bridges between the two other legs of the tripod, thereby completing the circuit of the threefold metaphoric communication. We may think of the tripod as follows:



Concomitantly, we think of each leg of the above tripod as a *melos*, meaning a breaking in the continuity associated with the *omphalos* in the middle. Each melos in its rendering as a bidirectional bridge with respect to the other two, may be thought of in terms of an umbilical cord emanating from the omphalos, serving as a means of establishing its diachronic identity. A melos is distinguishable as such, i.e.

independently from its participation in the tripod, by bounding and cutting the cord, breaking the continuity in this way, to be re-established by its re-habilitation within the tripod that re-activates its cord. The re-activated cord bears the mark of its breaking as a root of its unity that allows a melos to synchronize with its cord via powers, to be thought of as harmonic frequencies giving rise to a spectrum that translates a cord into a chord.

Each distinguishable melos of the initial tripod gives rise to another tripod embracing the obstacle that its associated discontinuity imposes. The state of wonder gives rise to the tripod [philosophy/ mathematics/ architectonics], which is resolved further to the tripod [*logos*/ *orthos doxa*/ *arche*], to the tripod [*ratio*/ *arithmos*/ *architecture*], and to the tripod [*armonia*/ *arithmos*/ *arachne*], we encountered previously.

The aletheia as mnemosyne gives rise to a triad of tripods capturing the metaphoric communication among the nine muses, the first composed of [epic poetry/ lyric poetry/ hymn], the second of [choral dance/ comedy/ tragedy], and the third of [music/ history/ astronomy]. We come finally to necessity, the melos from which the tripod of time emerges out of its spindle and the thread it spins as an *axis mundi*, i.e. as a cosmic helical axis of unfolding perpendicular to the epiphaneia.

The tripod of time consists of [*Clotho/ Lachesis/ Atropos*], together called the *Moirai*. Clotho is the one who winds the thread, Lachesis is the one who unfolds and measures the thread, and Atropos the one who bounds and cuts the thread. Especial care is needed in comprehending the tripod of time specifically, as a means of threefold metaphoric communication among Clotho, Lachesis, and Atropos, and not as a literal spatial representation of different actions. The key is especially provided by the domain of harmonics, without which a literal geometric representation becomes untenable and distorting.

In particular, the discrete marking or quantization of the thread by Lachesis in terms of the integer periods of winding around by Clotho is not accidental, but it is subordinate to the bounding of the thread by Atropos, so as to reveal the harmonics. In this manner, the winding of the thread as a temporal diastasis of unfolding is complemented by the harmonics of a bounded helical standing wave appearing in the caduceus (*kerykeion*) of Hermes, the father of the Moirai and personified God of communication, which in turn, qualifies the measurement of the thread by Lachesis as a spectral measurement. This brings us back to the tripod [harmonics/ arithmetic/ geometry], which has been already analyzed in detail. It is an amusing realization that the tripod of time harbours such an articulating relevance for these mathematical considerations pertaining to metaphora and natural communication.

3.4 SINGLE TEMPORAL DIASTASIS: CHANGE OF TIME AS CHANGE OF PHASE

It is worth examining the major difference between the notion of a temporal diastasis, i.e. the notion pertaining to a helical temporal axis of unfolding perpendicular to the epiphaneia, and the standard notion of a real temporal dimension totally ordering events. An interesting analogia is emerging from the domain of music pertaining to the difference between *harmony* and *melody*. Whereas melody refers to the sequential horizontal ordering *of* notes, harmony refers to the vertical, consonant or synchronized listening *to* the notes at each *present*.

We recall that the concept of the epiphaneia can either be thought of as a disk in the complex plane equipped with the polar grid and bounded by the unit circle, identified with the imaginary ring, or under a conformal transfiguration via a branch of the complex logarithm, as a rectangular strip on the complex plane, where the imaginary axis bears the information of spatialized spectral intervals. The important thing to keep in mind is that the imaginary ring or the imaginary axis, model the present diachronically. The reason is that a spatialized spectral interval is not merely a geometric interval, but bears topological and harmonic information that manifests in discrete or quantum terms with respect to the temporal unfolding diastasis.

To recapitulate, at each present, the harmonics lead to the cyclotomy of the circle, bearing the status of powers for harmonic resonance in each present. This is why the meaning of the *imaginary axis*, as an axis spatializing time spectrally in each present, is very different from a real dimension ordering events. In the latter case, events appear as geometric points on this axis, which are sequentially totally ordered from the past to the future on a geometric linear continuum. In the former case, the geometric points on the imaginary axis are points of stasis bearing temporal depth, characterized by their power to synchronize or resonate with the present. Notice that from the perspective of the present, and given that the same helical temporal axis applies to each present isomorphically, the difference between past and future is a difference in orientation, which is to say that past and future appear symmetrically with respect to the present differing in orientation.

From a topological viewpoint, we have the phenomenon of multiple-connectivity of the past with the future with respect to the present. In particular, the past can be connected to the future in a multiplicity of possible ways in the present, if we take into account the symmetric appearance of roots of unity in the present differing in orientation, as well as the significance of the primitive roots of unity for this purpose. We argue that the notion of a temporal diastasis presents interesting results even in the case, where we restrict our attention to the spectral interval referring to a single harmonic. Within such an interval change of time may be simply thought of as a continuous change of phase. If we assume that the same interval applies for each present, then we have to distinguish only two cases. The first case applies when the continuous rate of change is the same for each present, whereas the second applies when this continuous rate of change differs from present to present.

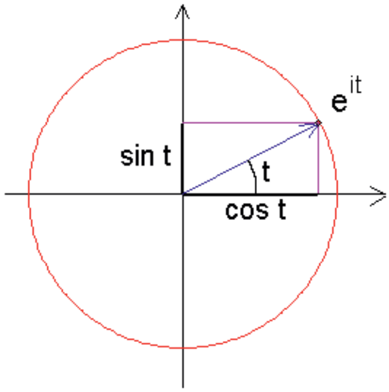
3.5 CONSTANT RATE OF PHASE CHANGE AND CONTRACTION OF LENGTH

First, let us consider a kinematical model of Special Relativity in the complex plane, where the real horizontal axis is a spatial axis, whereas the imaginary axis is a spatialized temporal axis. According to this theory, the maximal speed of electromagnetic signal transmission is defined by the speed of light c, which is constant in all directions. Moreover, the spatiotemporal metric relations are constant at every point-event leading to the group of Lorentz transformations as the kinematical symmetry group of the theory. The metric measuring distances is expressed as $dS^2 = dx^2 - c^2dt^2$, which takes the form $dS^2 = dx^2 + (icdt)^2$ on the complex plane, where the imaginary unit i is used in the conversion of the temporal factor into the spatialized form, which the metric relation refers to.

Note that the temporal metric factor appears in imaginary spatialized form by the adoption of the speed of light c through the intervention of the imaginary unit i. The metric relation $dS^2 = dx^2 + (icdt)^2$ on the complex plane is subordinate to the upper bound in geometric information signaling defined by the speed of light c, and thus, it pertains to phenomena approximating that speed. Furthermore, this type of continuous temporal unfolding at very high speeds takes place at a constant rate in each present given by the speed of light c.

We would like to study the connectivity between the past and the future in the present in this case. The basic idea is not only that the speed of light is constant, but that the chrono-geometric relations induced by this upper bound are constantly the same at each present, since the metric is not variable. This means that the imaginary ring, thought of as the unit circle with the complex structure, by normalizing the speed of light to unity, persists isomorphically at each present regarding both its shape, and the continuous rate of change upon it. It follows that, from the viewpoint of each present the difference between future and past is a difference in orientation only, subsumed by the imaginary unit and its complex conjugate.

Thus, in this case, change of time from the view of the present amounts to continuous change of phase, and this is isomorphically the same in each present for both the past and the future, differing only in orientation with respect to the present. As a consequence, we obtain the Lorentz contraction of lengths in the direction of motion. If we consider motion along the horizontal real spatial direction at a high speed below the speed of light, then spatial extension in the real horizontal linear dimension by 1 unit of length will appear contracted with respect to the present at “0”, i.e. with respect to the imaginary ring centered at “0”, since it amounts to a change of phase on the unit circle equal to the passage of spatialized time, measured by the angle with respect to the real horizontal dimension. Hence, the length contraction (depending on the speed of motion) with respect to the present at “0” is just the projection on the horizontal linear spatial dimension of the corresponding phase change on the unit circle. Interestingly, at the speed of light, the length is contacted to zero with respect to the present at “0”.



3.6 VARIABLE RATE OF PHASE CHANGE AND METRIC ANHOLONOMY

In the case of General Relativity, the spacetime metric becomes variable from point-event to point-event, depending on the distribution of matter in its vicinity. In this way, General Relativity is reducible to Special Relativity only in the infinitesimal vicinity of every point-event. Thus, the metric, and therefore, the chrono-geometric relations are not constant as in Special Relativity, but become variable. In turn, the variability of the metric requires that a standard of comparison is required at each point-event. This gives rise to the infinitesimal process of *parallel transport*, induced by a connection, involving small round trips around each point according to a prescribed rule of parallelism, characterized by the metric-compatibility of the connection. These round trips detect the change of orientation of a transported vector, expressed by means of a relative geometric phase factor, called the *metric anholonomy* of the connection, explicating locally the curvature associated with uneven matter distributions.

In this case, in the infinitesimal vicinity of any point-event the metric assumes the form, $dS^2 = dx^2 + (icdt)^2$, but this form is not retained constantly as we move from point-event to point-event. Thus, the rate of phase change is not constant between the past and the future with respect to the present. Concomitantly, although change of time amounts to continuous change of phase from the view of the present, the rate of phase change is differentiated between the past and the future. In this manner, past and future differ not only in *orientation* with respect to the present, but there is also a relative *geometric phase difference* between them, i.e. their metric anholonomy.

We conclude that the notion of a temporal diastasis presents interesting results even in the case, where we restrict our attention to a single spectral interval, qualified metrically. Within such an interval change of time corresponds to a continuous change of phase, and assuming that the same interval applies for each present, then, either the continuous rate of change is the same for each present, or the continuous rate of change differs from present to present giving rise to a relative geometric phase factor.

Actually, the analogous type of effects considered in the case of Special Relativity and General Relativity, that have been concisely discussed above from our perspective, may be thought of as emerging naturally in seemingly unrelated contexts, just by abstracting the relevant constraints. In the kinematical case of Special Relativity the constraint emanates from the constancy of the speed of light as an upper bound characterizing the propagation of electromagnetic signals. The crucial thing is that this upper bound speed is used as a universal metrical factor for the spatialization of time in the present along the imaginary axis on the complex plane in two dimensions. In this way, another type of constant speed pertaining to an upper bound for a different kind of propagation characterized metrically as above, would also correspond to a change of time as a change of phase, giving rise to an analogous effect of length contraction in the present referring to the direction of propagation.

Let us consider the case of propagation of an army in the battlefield. Initially, the notion of an armored vehicle was simply conceived as a means of protecting the infantry following it. As such the upper bound in the speed of propagation was set by the infantry. The strategic transmutation of the role of an armored vehicle into a unit of armored vehicles moving independently from the infantry amounts to a change in the syntaxis of time in the battlefield. This is because the speed of propagation of the army is altered by the upper bound set by the unit of armored vehicles. Thus, at speeds of propagation near this upper bound, change of time amounts to change of phase in the battlefield, where time in the present is spatialized on the imaginary axis due to this upper bound. Then, assuming that the continuous rate of change remains constant for each present, we obtain the phenomenon of length contraction in the direction of motion of the army in the battlefield from the view of the present of the resting infantry. In a similar fashion, effects of local curvature out of a relative geometric phase factor may appear in the present at the battlefield. This is the case if the continuous rate of change of phase does not remain constant for each present. For example, consider the case of a cavalry unit in comparison to an armored unit. Although change of time amounts to continuous change of phase from the point of view of the present in both cases, the rate of phase change is differentiated between the past and the future, e.g. at the present where a cavalry unit is substituted by an armored unit. In terms of this example, past and future do indeed differ not only in orientation with respect to the present, but there is also a relative geometric phase difference between them, which locally curves the battlefield to the advantage of the armored unit. In this case, we may think of the change of time as a *"massive"* one in analogy to the geometrized gravitational effect.

3.7 PAIR OF LINKED TEMPORAL DIASTASES: DOUBLY-PERIODIC SPECTRAL WEAVING

From a topological viewpoint, the most interesting phenomenon accompanying the notion of a temporal diastasis is the multiple-connectivity of the past with the future with respect to the present. We examined previously only the metrical case involving a single spectral interval, where change of time corresponds to a continuous change of phase in the ring of the present. Additional challenges naturally arise in the case of listening and comprehending a piece of music, or in the case of reading a book and trying to understand its content.

Clearly, the process of acquiring meaning and understanding cannot be reduced to the sequential order of time. More precisely, at each and every present, we rather instantiate temporal bonds between the memorized past and the anticipated future, where the present plays the role of a modulus for this type of bond. Due to topological multiple-connectivity these bonds allow information amalgamations irrespectively of any notion of distance or proximity in the text.

In particular, things in the very far past may form a temporal bond in the present with anticipated things in the very near future. If both the past and the future comply to the same temporal diastasis, in our terms if the ring in the epiphaneia of the present persists, through unfolding on the same helical axis perpendicular to the epiphaneia, then a temporal bond in the present can be formed from the primitive roots of unity on the persisting ring between the past and the future. In other words, there are roots from the past, which are relatively prime with anticipated roots in the future. Both past and future roots are elicited bidirectionally to the same diachronically persisting ring of the present, as seeds bearing corresponding powers, capable of forming a bond in the present. This bond is articulated in the same ring by the demarcation of the primitive roots. As such, change of time amounts to the relative phase difference that is subordinate to new root primitivity. This is how, the present is synchronized with the whole constellation of consonant spectral intervals, whence the primitive roots account for temporal bonds.

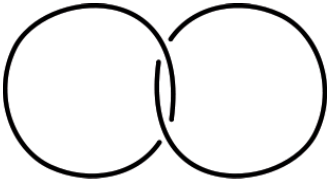
We will now think of the general case, where the same temporal diastasis does not persist diachronically, in the sense that change of time in the present cannot be accounted for in terms of a relative phase difference. This possibility already presents itself where we have two different temporal diastases that act jointly, in the sense of imposing two different periodic rules. We have to be careful to distinguish two possibilities in this setting.

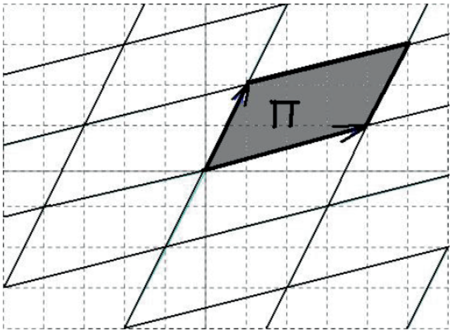
The first refers to the case that these temporal diastases are characterized by different, but directly interlinked rings in the present. The second refers to the case that these temporal diastases are unlinked in any direct way to each other, but at some present they get linked together through a third ring, such that if any of these three rings is eliminated then everything becomes totally unlinked. The latter case will present the major interest for us, since here, thinking from the view of the present, change of time is not merely a change of relative phase, but a change of circle. Additionally, this case depicts the depth of a temporal bond, and a posteriori captures the essence of a tripod based on a diachronically stable natural communication constituted by means of tripartite metaphora.

Before we study in detail the synergetic metaphora that gives rise to a change of circle, it is worth making some remarks concerning the first paradigm, where two directly linked temporal diastases are acting jointly at the same place. Recall that in the case of a single temporal diastasis, meaning a helically unfolding temporal axis, the imaginary ring, or its conformal logarithmic transfiguration to the imaginary axis of the complex plane serves to model the present on the epiphaneia. In those cases where we have another temporal diastasis directly linked with the former, then the only way that we can model it in the context of the two-dimensional epiphaneia is in terms of the orthogonal axis to the former one. These diastases are linearly independent over the real numbers and the whole complex plane tesselated by rectangles, whose orthogonal sides correspond to the periodic rules of the two diastases, i.e. their spectral measures, models the present.

In this manner, $\mathbb{Z}^2$ gives rise to a lattice on the complex plane, and we obtain instead of spectral measures on the imaginary axis, spectral measures on the integer lattice on the complex plane. The functions defined on the complex plane bearing this spectral lattice are doubly periodic, called *elliptic functions*. Analogously to the trigonometric functions culminating to the simply-periodic complex exponential function parameterizing the circle, the doubly-periodic elliptic functions parameterize two directly linked circles. In this fashion, note that complex analytically the unit circle is identified with the imaginary ring in the first case. In the second, due to the direct chain linkage, only one of the rings is thought of as imaginary. This is also clear if we think of the parametrization of the spectral lattice taking place in the complex plane bearing one imaginary and one real dimension. If we denote these two periods by ω_1, and ω_2, then their ratio is imaginary.

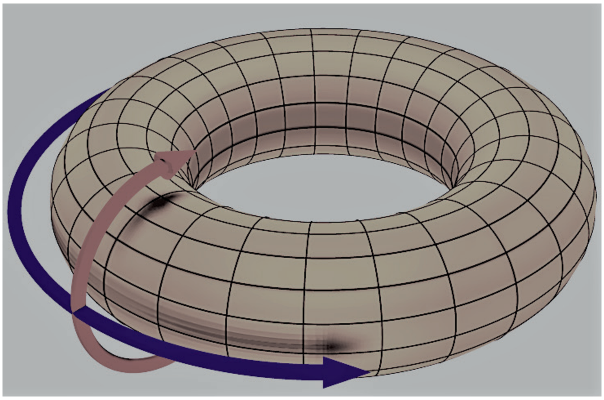
The fundamental parallelogram on the complex plane is the one with vertices 0, ω_1, ω_2, $\omega_1+\omega_2$.





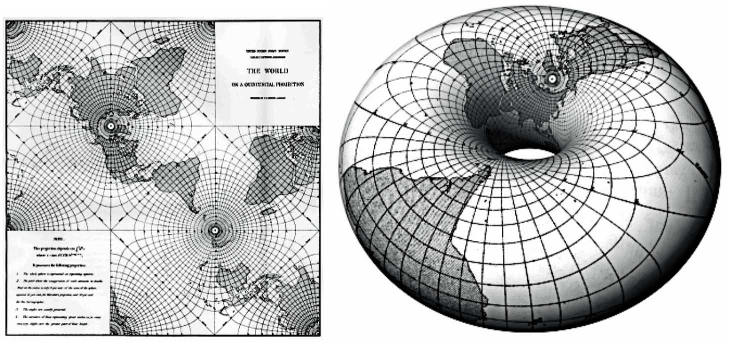
3.8 TORICS: THE QUINCUNCIAL PROJECTION

From a topological perspective, the above case of the two directly linked chain-like temporal diastases, descends from the torus. Topologically, we may think of them as two linked cycles on the torus constituting the homology basis of its first homology group.



Analogously to the case of a single temporal diastasis descending from the sphere, two directly linked temporal diastases descend from the torus. Note that the torus is topologically different from the sphere, since it bears the central hole in the middle, i.e. it has *topological genus* 1 in comparison to the sphere whose genus is 0. So once more, we may think of the topological genus temporally, as emanating from the notion of a temporal diastasis. If we think of this link in terms of the sphere, then we would need a sphere bearing four poles. This is explained complex-analytically by saying that the torus constitutes a *double branched covering* of the sphere.

The above has been conceived by Charles Sanders Peirce as a way of mapping the sphere on the epiphaneia, called the *quincuncial projection*. If we think of the above as the fundamental parallelogram on the complex plane, we see that it bears two poles whose residues cancel each other. Considering the elliptic functions determined by their values on a fundamental parallelogram as above, we realize that they can not be holomorphic, since in that case they would be constant as bounded functions; called for this purpose, *meromorphic* functions instead.



It is interesting to compare the stereographic with the quincuncial projection. The latter enfolds two copies of the sphere bearing four branch points. These can be identified with the points where the displayed axis pierces the torus. The two different projections on the epiphaneia of the complex plane are correlated as follows: Using the inverse quincuncial projection we obtain a conformal mapping of the fundamental region on the torus, which then can be projected to the Riemann sphere by means of the branched double cover of the sphere by the torus. Then, the stereographic projection of the sphere accomplishes the sought after metaphora.

3.9 PAIR OF NON-DIRECTLY LINKED TEMPORAL DIASTASES AND TEMPORAL BONDS

Next we consider the general case of two temporal diastases, which have no direct link to each other, but at some present they get linked together indirectly giving rise to a temporal bond. On the epiphaneia, this type of linking of two rings is possible only through a third ring, such that if any of these three rings is eliminated then all linkage is completely lost. The first important thing to realize is that a temporal bond cannot be constituted within two dimensions; an additional dimension is needed.

Alternatively, we have to consider two unlinked imaginary rings, each of which lies on its own copy of the complex plane. The feasibility of a temporal bond in three imaginary dimensions implies that two imaginary rings can be amalgamated together with respect to a third imaginary ring, otherwise they remain unlinked. Thus, a temporal bond cannot be modelled on the complex analytic plane, requiring rather the four dimensional *quaternionic* analytic setting, where three of the four axes are imaginary. In much the same way that a single imaginary axis is meaningful together with a real axis orthogonal to it, three imaginary axes are meaningful together with a real axis being orthogonal to all three of them. Note that the three imaginary axes are orthogonal to each other, but any one of them emerges from the product of the other two. As such this product is not commutative, showing that the algebraic and analytic modelling of a temporal bond is non-commutative. This means that the articulation of a temporal bond is based on non-reversible temporal actions. It is due to this irreversibility that the bond becomes stable. Since we do not intend to enter the domain of non-commutative quaternionic analysis, we call the site involving three imaginary axes a *crystalline* site.

We are going to restrict ourselves to the simplest topological rendering of a temporal bond not involving any analytic considerations. Keeping the temporal perspective, and in particular, thinking in terms of the present, change of time is not merely a change of relative phase, but a change of circle, where each circle is viewed here as a topological circle. Since the circle at present is assumed to amalgamate together the two other unlinked a priori circles, which entails the linking should take place in modular relation to the circle of the present, it is necessary to examine the conditions that make this feasible. We note that topologically no distinction pertains between the circles with respect to the temporal before and after relation that enters only by identifying a particular circle with the present, according to our previous analysis. In other words, topologically speaking, a temporal bond is threefold symmetric with respect to three circles linked in this way. The distinction comes from identifying one circle as the circle of the present, so that the other two can be temporally amalgamated together in modular relation to the marked one. The topological distinction that matters is whether two circles are directly linked or not, but even when not is the case, if they can be linked through a third one appropriately.

3.10 CHANGE OF TIME AS A SYNERGETIC CHANGE OF CIRCLE IN THE LIVING PRESENT

In the course of this problematics, the change of time as a change of circle from the past to the future in modular relation to the present, should be thought of as a synergetic change of circle belonging to the conceptual domain of *synectics* in the *Aristotelian* meaning of this term. At a first stage, we may conceptualize a synergetic change of circle as a higher connectivity interface, binding cohesively together the past with the future, independently of their metrical linear distance, in modular relation to the present, considered as their unity. At the event level, such a symmetric treatment of the past and the future with respect to the present is not possible. But the connectivity interface we examine currently does not refer to the level of events, where past and future stand in an asymmetrical ordered relation with respect to the present.

This interface capable of giving rise to a temporal bond mediates the metaphora of both the past and the future with respect to the present at this present. In this sense, the metaphora of the past to the future through the present via change of circle by means of a temporal bond should qualify these terms as three different circles capable of being linked together. Since the linking is meaningful in the present where a change of circle takes place, a metaphora of both the past and the future must be in play at this present. Of course, this metaphora is not in the nature of a metaphora of events that would be in any case impossible. The mental strain concerns especially the notion of the future appearing absurd from a standard perspective if not conceived through its metaphorical essence.

More precisely, one of the main functions of the human brain is to act as a metaphora from the future to the present, in the sense of gnomonically anticipating and envisioning the future. This is possible exactly because of the symmetry of a metaphora in terms of encoding and decoding bridges. Thus the vision of the future in relation to a change of circle from the past that takes place in the present is qualified topologically as the metaphora of this envisioned circle in the present. The same actually holds symmetrically for the past through the faculty of memory, which functions topologically as the metaphora of the circle of the past in the present. Consequently, memory and vision are symmetrically articulated with respect to the circle of the present, from the viewpoint of gnomonics and natural communication, although it is not obvious at all how it is possible to form a bond in the present that will allow the concomitant change of circle from the past to the future.

Since we have established, that this bond pertains to a higher connectivity interface in the present it cannot be expressed at the level of events. In other words, from the naive view of a simply-connected linearly ordered one-dimensional continuum of events a temporal bond amounts to a wormhole that can connect very distant events non-locally, something impossible given only the conceptual apparatus of, say, a *Markov chain* connecting these events. Essentially, such wormholes qualify as metrically non-local bridges of connectivity that can instantaneously bind the very far past with the very near future. Although this is paradoxical if considered at the ordered event level, the human brain utilizing mental metaphora performs it unproblematically.

The issue is that the metaphora takes place not at the level of events, but at the level of topological *germs*, or simply elicited *seeds*, of events that bear a *meteoric* nature. For this reason, the present does not play the role of a pathetic point on a line ordered by means of succession, but its role becomes energetic in the sense of binding the remembered past with the gnomonically or canonically envisioned future that precisely characterizes its invisible depth. The claim is that this notion of the present is a living one, whereas the former is a dead one. It is the same with the articulation of some given history at any present as a living or as a dead one.

In other words, the qualification of the living present is metaphorically equivalent to its ability to give rise to temporal bonds. It is clear from this discussion that this does not apply to any past and to any envisioned future in the present. Put differently, there are certain conditions that allow a change of circle in the present from a circle of the past to a circle of an anticipated future. From the standpoint that harmonics takes prior place to topological considerations, these circles may be though of as unlinked chords bearing their harmonics as elicited seeds in the present, and the issue is how they can be tuned together in the present so as to give rise to a temporal bond through which change of circle takes place.

The notion of a temporal bond is qualified as follows: First, it should not be conditioned by relations of metrical proximity of the elicited seeds from the past and the future in the present, where a change of circle can take place; Second, a temporal bond between elicited seeds from the past and the future, always bears a modular relation with the present acting as their unity. Equivalently, a temporal bond should not be thought of as a direct pair-wise gluing, but as a modular gluing, that takes place in relation to the present, and together with the present; Third, a temporal bond induces a synergetic circle change, if and only if the pertinent elicited seeds from the past and the future, in their capacity for amalgamation together in modular relation to the present, are both relatively prime with respect to the present. Equivalently, they should be neither analyzable nor localizable to any other common factors with respect to the present; Fourth, a temporal bond as a synthetic unity that takes place in the present, characterized symmetrically and bidirectionally by relative primeness, or primitivity, with respect to the present, specifies the syntaxis of a compounded temporal unfolding encapsulated by synergetic circle change in the present; Fifth, a temporal bond should be a Tripodal link of least action, since it corresponds to an inseparable tripartite correlation, which cannot be analyzed to any pairwise correlations.

It is crucial to make some further remarks in relation to the property of primitivity, or relative primeness, required bidirectionally from the elicited seeds in their capacity to enter into a modular gluing relation with the present, characteristic of a temporal bond. This notion is analogous to the one in integer modular algebra conceived by Gauss, where the absolute notion of an integer prime number becomes relativized with respect to a modulus. The significance of this generalization, in the case of integer modular systems, is that any integer can assume the role of a prime, but only in relation to another integer acting as a modulus. The idea here is that the quality of relative primeness in relation to the present is crucial for the realization of a temporal bond. In more intuitive terms, the assertion is that, relatively to the present, a seed from the past becomes spectrally spontaneously recognizable, and thus, capable of being elicited in the present, without any possible factorization through anything else. The same, symmetrically, holds for an envisioned seed in the anticipated future in its capacity to enter into a temporal bond with a seed from the past in the present, which acts as the modulus of unity for this Tripodal link.

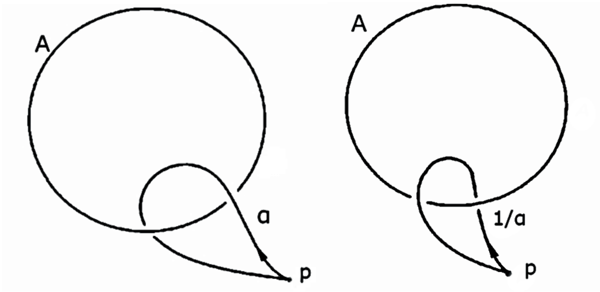
It is instructive to recall that the realization of such a temporal connectivity interface becomes effective only on the condition of topological non-degeneracy of the temporal unfolding between the past and the future in the present, acting as their modulus of unity.

Equivalently, the different winding stairs of the helices corresponding to the a priori unlinked temporal diastases should be spectrally distinguishable in the present. Topologically, this spectral distinguishability amounts to quantized action. In other words, the winding stairs indexed by the group of the integers, count quanta of action. Again, spectral distinguishability should be always relativized with respect to the pertinent present, where a temporal bond is instantiated as a least action solution to the indirect linking of the past with the future.

The most important consequence of relative primeness conceived this way, is that the pertinent seeds of the past and the future entering into a temporal bond, become, both symmetrically, relationally inverse with respect to the present, and relationally conjugate with respect to each other, in the present. In turn, the present is qualified through an Archimedean fulcrum relative to these seeds or, more precisely, relative to their respective a priori unlinked topological circles. Thus, seeds from the past and the future become eliciting seeds in their power to enter into a temporal bond in the present, if and only if they can be leveraged metaphorically to the present, relationally to each other, with respect to the fulcrum of the present.

3.12 RESONANCE OF TEMPORAL CHORDS: THE TRIPODAL LINK OF A BOND

If we consider a seed either in the past or in the future, it becomes spectrally spontaneously recognizable from the fulcrum in the present, i.e. not factorizable through any other simpler common factor, by means of a loop, and more precisely, a simple tame closed curve, which is based at the fulcrum. This means that it starts and ends at the fulcrum, and passes through the topological circle corresponding to its temporal diastasis. Since we refer to a seed, it is better to consider the whole equivalence class of such loops that can be continuously transfigured to each other. It is enough to recognize a single representative of this class by means of a based loop at the fulcrum. If we denote the relevant topological circle by A, then a based loop at the fulcrum passing through A, may admit two distinct orientations: If the loop passes through A with direction away from the fulcrum $(+)$, it is denoted by α, whereas if it passes with direction toward the fulcrum $(-)$, it is denoted by α^{-1}. Therefore, oriented loops based at the fulcrum of the present and crossing unlinked topological circles encapsulate the reflexive principle of seed recognition from the past and the future, depending on their orientation.



In other words, reflexive recognition of a seed in the past or in the future, in its power to enter into a temporal bond in the present, is not enough for the establishment of the bond. What is required additionally is the metaphoric leveraging of these seeds to the fulcrum of the present by utilizing the property of relative primeness, so that they become elicited seeds in the present capable of modular amalgamation. This condition leads to the notion of a temporal chord in the present, through which the expression of a temporal bond becomes explicit.

Consider a recognized seed from the past, identified either with the based oriented loop at the fulcrum, $\alpha^{+1} := \alpha$, or with α^{-1}, by means of crossing the topological circle A, depending on the orientation. Analogously, consider a recognized seed in the future, identified either with the based oriented loop at the fulcrum, β^{-1}, or with β, by means of crossing the topological circle B, depending again on the orientation. For instance, if α and β^{-1} are recognized, they both become eligible to be elicited seeds in the present by metaphoric leveraging with respect to the fulcrum. For this purpose, they should be relationally conjugate to each other by the requirement of relative primeness with respect to the present. This means that β and β^{-1} should play the role of bidirectional bridges for the leveraging of α, and also symmetrically that α and α^{-1} should play the role of bidirectional bridges for the leveraging of β^{-1}. It is elicited seeds that give rise to temporal chords in the present. Therefore, a temporal chord in the present is enunciated by interpolating a recognized seed from the past, for example, α, between the bridges β and β^{-1}, i.e. by metaphora through β and β^{-1}, which leverages the seed α with respect to the fulcrum, e.g. $\beta\alpha\beta^{-1}$.

The significance of temporal chords in the present is that they can resonate together harmonically in the present. More precisely, a temporal chord emanating from a recognized seed in the past can be fused together with a temporal chord from a recognized seed in the future by harmonic resonance in the present. The latter is instantiated by means of new topological circle in the present capable of amalgamating together the circles A and B in a non-pairwise fashion.

For this purpose, we consider a recognized seed from the past, identified with the based oriented loop at the fulcrum, α, by means of crossing the topological circle A in the prescribed orientation, and symmetrically, a recognized seed from the future, i.e. an envisioned seed in the present, identified with the based oriented loop at the fulcrum, β, by means of crossing the topological circle B also in its respective prescribed orientation. Note that A and B are a priori unlinked corresponding to two different temporal diastases. These oriented fulcrum-based loops α and β can be composed, either in the order $\alpha\beta$, or in the order $\beta\alpha$, and these compositions are non-commutative. In this manner, composed actions of seed recognition from unlinked temporal diastases are order irreversible with respect to the fulcrum.

Let's consider the composition in the order $\alpha\beta$. The basic idea is to extend this composition in consecutive stages and express it in terms of temporal chords with respect to the fulcrum. If we adjoin by composition α^{-1} to $\alpha\beta$, we obtain the temporal chord $\alpha\beta\alpha^{-1}$, which amounts to the metaphora of β with respect to the fulcrum, utilizing the bridges α and α^{-1} for this recognized seed in the future. Next, we adjoin β^{-1} to the temporal chord $\alpha\beta\alpha^{-1}$, to obtain $\alpha\beta\alpha^{-1}\beta^{-1}$, which can be read either as the composition of the temporal chord $\alpha\beta\alpha^{-1}$ with β^{-1} or as the composition of α with the temporal chord $\beta\alpha^{-1}\beta^{-1}$, due to the associativity property of non-commutative composition. In the same way, by continuing the process of adjoining as above, the objective is to generate a cycle based at the fulcrum. A cycle is generated when the process of adjoining ends with the composition $\alpha\beta$ that has been utilized at the initial stage. A cycle based at the fulcrum is generated by the resonance of a temporal chord from a recognized seed in the past with a temporal chord from a recognized seed in the future, as follows:

$$\alpha\beta \to \alpha\beta\alpha^{-1} \to \alpha\beta\alpha^{-1}\beta^{-1} \to \alpha\beta\alpha^{-1}\beta^{-1}\alpha \to \alpha\beta\alpha^{-1}\beta^{-1}\alpha\beta.$$

In the above process, the cycle generated is given by

$$C=\alpha\beta\alpha^{-1}\beta^{-1},$$

since starting from the ordered non-commutative composition $\alpha\beta$ we ended at:

$$(\alpha\beta\alpha^{-1}\beta^{-1})(\alpha\beta)=C(\alpha\beta).$$

The cycle

$$C=(\alpha\beta\alpha^{-1}\beta^{-1}):=[\alpha,\beta],$$

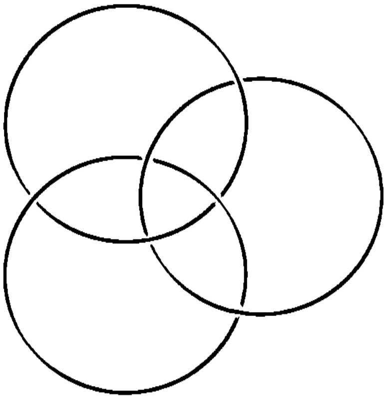
i.e. the commutator of α and β, is generated by the resonance of the temporal chord $\alpha\beta\alpha^{-1}$ with the temporal chord $\beta\alpha^{-1}\beta^{-1}$. Therefore, the new topological circle C leads to the indirect gluing of the topological circles A and B in modular relation to the present, identified thereby, as the circle of the present. It is only through the Tripodal link instantiated in the present by C that a synergetic change of cycle from A to B takes place.

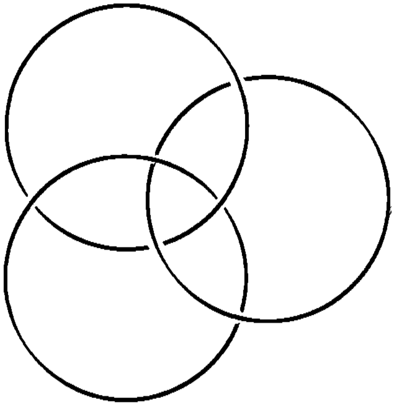
The topological circle $C=(\alpha\beta\alpha^{-1}\beta^{-1}):=[\alpha,\beta]$, involves four crossings of the circles A and B, more concretely, two of the circle A and two of the circle B, with opposite orientations and in an alternating order. Notice that the structure of C does not depend on what we consider as an initial composition, like $(\alpha\beta)$ in the case presented. If we consider any other initial composition from all possible ones, we will again arrive at a cycle of the same structure, i.e. to a resonance of a "temporal chord" from the "past" with a "temporal chord" in the "future" with respect to the fulcrum. In other terms, the compositional structure of the topological circle C, amalgamating A and B in modular relation with respect to the present, is the invariant of resonance between a temporal chord from the past with a temporal chord from the future, that qualifies change of time as a synergetic change of cycle from A to B in the present, and giving rise to a temporal bond.

The crucial observation is that a temporal bond induces a particular type of topological linking of the cycles A, B and C, which we call a *Tripodal topological link*. Equivalently, a synergetic change of cycle from A to B in the present is tantamount to a Tripodal link of the cycles A, B and C, qualified by the property that if any one of the cycles is removed from this link the remaining two come completely apart.

Topologically, a Tripodal link may be simply thought of as an interlocking family of three loops, such that if any one of them is cut, then the remaining two become completely unlinked. Each loop is considered a tame closed curve. The property of tameness means that the closed curves considered can be deformed continuously and without self-intersections into polygonal curves, which are those formed by a finite collection of straight-line segments.

Moreover, a loop, as a topological object, discloses the following properties: First, a loop is not separated into two pieces by cutting it at a point, which is rather achieved by cutting it at two points; Second, a loop is an intrinsically one-dimensional object though of as a figure in three dimensions; Third, a loop is bounded, i.e. it is contained in some sphere of sufficiently large radius. Moreover, a loop is called *knotted* if it cannot be continuously deformed into a topological circle in three dimensions without self-intersection. Therefore, each one of the three interlocking loops of the Tripodal link should be considered as an *unknotted* tame closed curve. We refer to them simply as loops keeping in mind that each one of them is unknotted. To sum up, in terms of loops, a Tripodal link is depicted as the configuration displayed on the left below, which is to be contrasted with a different type of configuration consisting of three interconnected loops displayed on the right.

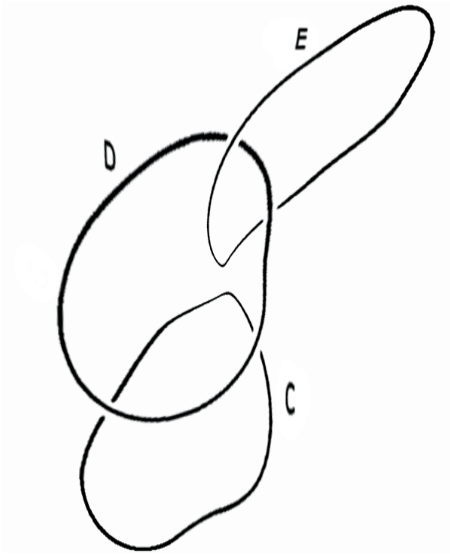




The Tripodal link configuration of loops on the left is such that if any of the loops is cut at a point and removed, then the remaining two loops become completely unlinked. In contrast, the configuration on the right is such that each loop actually links each of the other two.

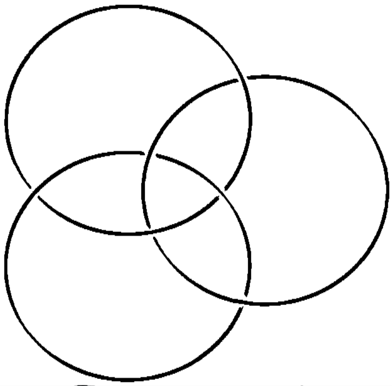
The topological notion of a link pertains to the connectivity among a collection of loops. In general, an N-link is a collection of N loops in three dimensions, where N is a natural number. Regarding the connectivity of a collection of N loops, the crucial property is that of splittability of the corresponding N-link. We say that an N-link is *splittable* if it can be deformed continuously in three dimensions, such that part of the link lies within B and the rest of the link lies within C, where B, C denote mutually exclusive solid spheres (balls) in three dimensions.

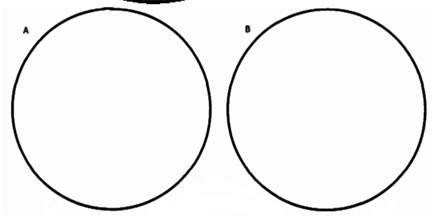
Intuitively, the property of splittability of an N-link means that the link can come at least partly apart without cutting. Complete splittability means that the link can come completely apart without cutting. On the other side, non-splittability means that not even one of the involved loops, or any pair of them, or any combination of them, can be separated from the rest without cutting. As an illustration, we consider the following 3-link:



The above 3-link consists of three loops, denoted by C, D and E. Clearly, this is a splittable 3-link, which is not completely splittable. As can easily be seen in the above figure, the loops D and E cannot be split apart without cutting. Notwithstanding this fact, it is a splittable 3-link because the loop C can be separated from the rest without cutting. Thus, the above 3-link can come at least partly apart, and therefore is splittable.

The property of splittability of a topological link, is adequate to completely characterize the Tripodal link. First, the Tripodal link is a 3-link, since it is consists of 3 loops. Second, the connectivity of this 3-link in terms of the splittability property, implies that the Tripodal link is a non-splittable 3-link, such that every 2-sublink of this 3-link is completely splittable. More precisely, it is a non-splittable 3-link because not even one of the three loops, or any pair of them, can be separated from the rest without cutting. A 2-sublink is simply any sub-collection of two loops obtained by removing the loop that does not belong to this sub-collection. Since, the Tripodal link is characterized by the property that if we erase any one of the three indirectly interlocking loops, then the remaining two loops become unlinked, it follows that every 2-sublink of the initial 3-link is completely splittable, according to the figure below:

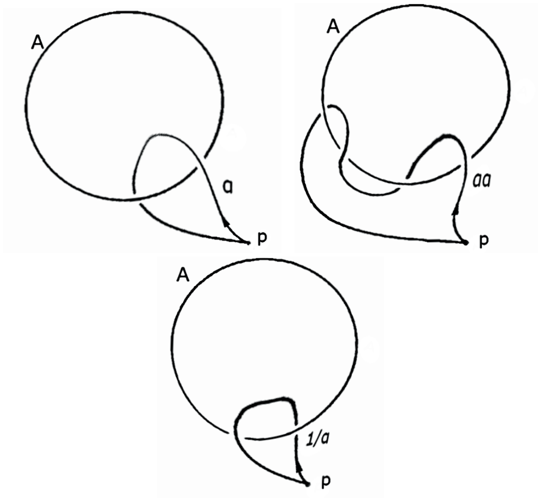




3.14 THE FREE NON-ABELIAN GROUP STRUCTURE OF ORIENTED BASED LOOPS

Our objective is to discover an appropriate algebraic structure capable of encoding the connectivity type of the Tripodal link, that is to say we seek a metaphora from the domain of topology to the domain of algebra, which will enable us to disclose the essence of a Tripodal link.

First, we consider an unknotted tame closed curve in three dimensions. Since any such curve can be continuously deformed to a topological circle it is enough to consider such a circle in three dimensions, denoted by A. Second, we think of a based oriented loop which may pass through this circle a finite number of times, each one with a prescribed orientation. A based loop means simply that it starts and ends at a fixed point p. The orientation of the loop is defined as follows: If it passes through the circle one time with direction away from p it is denoted by α^{+1}, whereas if it passes one time with direction towards p it is denoted by α^{-1}. Thus, in the algebraic symbols of the generic type "χ" we encode: First, the passage or not of a based loop through a circle A, which qualifies or not the naming of the loop by the corresponding symbol α. Second, the number of times that this loop passes through the circle A, which is encoded as a power of the symbol α. Third, the orientation of the loop with respect to p, which is encoded by a "$+$" sign if a passage through the circle takes place away from p and by a "$-$" sign if a passage takes place towards p.

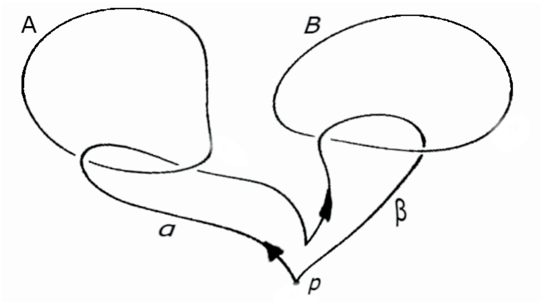


The first figure from the left depicts a loop based at p, beginning at p, then passing through the circle A once directed away from p, then curving around the circle A, and finally returning to p. According to the above, this loop in relation to the circle A should be denoted by α^{+1}, which we write simply as α. Note that any other loop with the same behavior can be continuously deformed to the loop α. Thus, the algebraic symbol "α" actually denotes a partition block, i.e. the equivalence class $[\alpha]$ of all loops of the kind α, passing through the circle A once with the prescribed orientation. Any loop in the block $[\alpha]$ can be continuously deformed to an equivalent one in the same class. Taking into account this remark, we still keep using the symbol α as above, where α is thought of as a representative of the equivalence class $[\alpha]$.

In the middle figure, we have a loop based at p, such that; it starts at p, then passes through the circle A twice directed away from p, then it curves around the circle A, and finally returns to p. This loop, in relation to the circle A, should be denoted by $\alpha^{+2} = \alpha \circ \alpha$, which we write simply as α^2.

In the last figure from the left, we find a third loop based at p, such that; it starts at p, then curves around the circle A, then passes through the circle A once directed towards p, and finally it returns to p. This loop, in relation to the circle A, should be denoted by $\alpha^{-1} = 1/\alpha$.

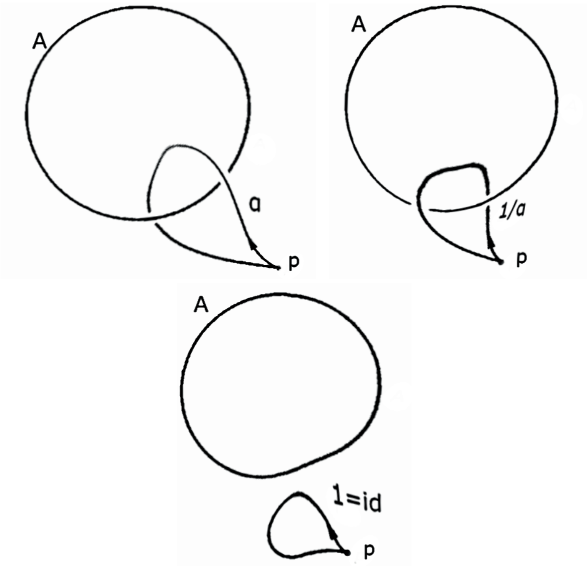
Next, we need to consider the composition of based oriented loops of the generic type "χ" in relation to circles of the generic type "X". The composition of two loops is viable if both of the loops are based on the same point p. Then, the composed based oriented loop should also be a loop of the same generic type in relation to the two circles of the composing ones.



In more detail, let us consider two based oriented loops, which are both based at the same point p. Taking into account the orientations, we denote the first loop by α (in relation to the circle A) and the second loop by β (in relation to the circle B). Then, we can define their composition denoted by $\alpha \circ \beta$ respecting the order of tracing the loops, viz. we first trace α, and then we trace β. Thus, the rule of composition produces a based oriented loop $\alpha \circ \beta$ in relation to the circles A and B, which is interpreted as follows: It starts at p, then passes through the circle A once directed away from p, then it passes through the circle B once directed away from p, and finally returns to p. We note that it is allowed topologically to remove the end of α and the beginning of β from the base point p, and then join them together at a nearby point. We interpret the composition rule $\alpha \circ \beta$ as the multiplicative product of the oriented loops α and β based at the same point p, which we denote simply as $\alpha\beta$. This establishes the closure of the elements of the generic type "χ" under multiplication as above.

We note also that the above multiplicative product is not commutative, i.e. $\alpha\beta \neq \beta\alpha$. This is due to the fact that the rule of composition of based oriented loops at a point is order dependent, such that $\alpha \circ \beta \neq \beta \circ \alpha$. The underlying topological reason is that the based oriented loop $\alpha \circ \beta$ cannot be continuously deformed to the based oriented loop $\beta \circ \alpha$, meaning that the order-dependence of the composition rule makes the corresponding multiplicative product non-commutative. Notwithstanding this fact, multiplication is an associative operation, i.e. $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$, so we skip the parentheses in multiple compositions of based oriented loops.

Next we look for the existence of a neutral element, and inverses with respect to this product operation. Clearly, for each based oriented loop α, there exists the inverse loop α^{-1}, such that both compositions $\alpha \circ \alpha^{-1}$ and $\alpha^{-1} \circ \alpha$ give as their multiplication product the loop based at the same point that does not pass through any circle at all. Thus, we call the latter loop the neutral element, or equivalently the multiplicative identity 1, such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$. We verify immediately that $1\alpha = \alpha 1 = \alpha$, where the equality sign is interpreted as an equivalence of based oriented loops under continuous deformation, according to the preceding.



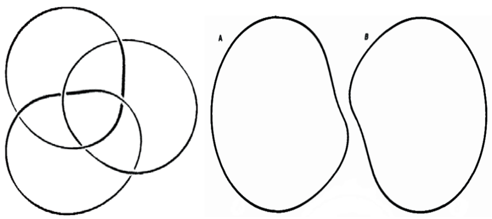
We conclude that the set of symbols of the generic type "χ" representing based oriented loops in relation to topological circles X, endowed with the non-commutative multiplicative product expressing the ordered composition of loops based at the same point, bears the algebraic structure of a non-commutative group, denoted by Θ. Since this group is generated by two non-commuting elements, and there are no further relations imposed on its algebraic structure, Θ is identified with the non-abelian group in two generators.

3.15 FROM TOPOLOGY TO ALGEBRA: ENCODING-DECODING THE TRIPODAL LINK

Using the multiplication operation we may form any permissible string of symbols in the group Θ, which can be shortened into an irreducible form by using only the standard group-theoretic relations $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, $1\alpha = \alpha 1 = \alpha$, $\alpha\alpha = \alpha^2$ and so on. Two arbitrary strings of symbols, i.e. words in the group Θ, are equal if they can be brought into the same irreducible form in Θ, meaning that the corresponding product loops, are equivalent under continuous deformation.

The property of irreducibility of a string of symbols in the group Θ, which amounts to the irreducibility of a product loop in Θ, is the leading idea for the algebraic encoding of the Tripodal link in terms of the group structure of Θ. Note that any multiplicative concatenation of symbols in the group Θ, when translated in product loop terms is always thought of in relation to corresponding circles, forming the collection of all circles that a product loop is associated with.

We proceed by investigating what kind of topological information the property of irreducibility of a string of symbols in the group Θ encodes in algebraic terms. We will show that algebraic irreducibility encodes the topological property of non-splittability of a link. Bearing in mind that a link corresponds generally to a collection of loops. The topological connectivity of a link is expressed by the property of splittability. In particular, the Tripodal link is a non-splittable 3-link, such that every 2-sublink formed by erasing one of the three loops of this 3-link is completely splittable.



The idea is to encode the Tripodal link group-theoretically in terms of an appropriate product loop in the group Θ, which is associated with two circles A and B. Note that erasing this hypothetical product loop would leave the two circles unlinked, since that removal results in a completely splittable 2-sublink. In algebraic terms, this situation depicted by the above figure on the right is described by the neutral element, i.e. the identity 1 of the group Θ. Hence, complete splittability of this 2-sublink is encoded by the identity 1 of Θ. For reasons of symmetry, the same behavior appears if we erase any of the circles A or B, since the neutral element of Θ is unique. Nevertheless, in order to prove it algebraically we need the explicit formula describing the product loop in the terms of elements of Θ.

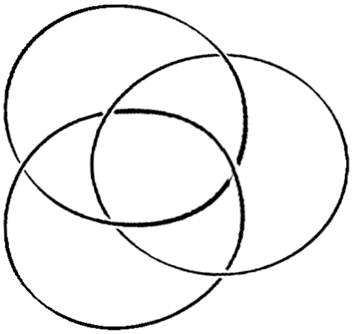
At the next step, since the product loop should be expressed in relation to the circles A and B, it is necessary to involve at least a string of symbols consisting of α, β and their group inverses α^{-1}, β^{-1} in some specific order, which does not involve any consecutive appearance of $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$, since all of the latter are reduced to 1. The reason for the appearance of both α, β, and their group inverses α^{-1}, β^{-1}, lies in our expectation that erasing any of the circles A or B would collapse the product loop to the neutral 1. This is the desired case referring to the Tripodal link because every 2-sublink is completely splittable. If the circle A is erased, for example, then in the sought after product loop formula both instances of α and α^{-1} should be deleted, since both α and α^{-1} have a meaning with respect to A. The same holds symmetrically for β and β^{-1} in relation to the circle B. Finally, since the fact that every 2-sublink of the Tripodal link is completely splittable is encoded algebraically by reducibility to the neutral element of Θ, the requirement is that the non-splittability of the total 3-link should be encoded by the irreducibility of the product loop formula.

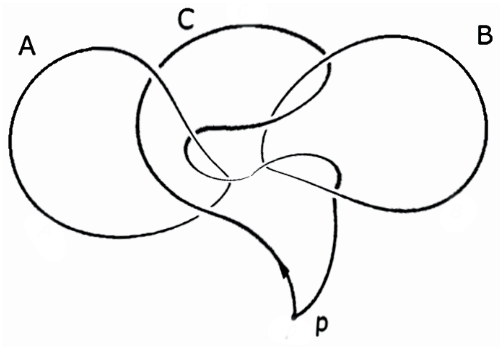
We conclude that only one combination of symbols exists that fulfills our requirements, namely:

$$\gamma = \alpha\beta^{-1}\alpha^{-1}\beta.$$

Thus, the irreducible formula $\alpha\beta^{-1}\alpha^{-1}\beta$ represents the loop γ as a product loop composed by the ordered composition of the four based oriented loops $\alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$. We call the product loop γ the *"Tripodal loop"* and the formula or multiplicative string $\alpha\beta^{-1}\alpha^{-1}\beta$ the

"Tripodal loop formula". The algebraic irreducibility of $\alpha\beta^{-1}\alpha^{-1}\beta$ in the group Θ encodes the non-splittability of the Tripodal link. Deletion of both α and α^{-1} (corresponding to removal of the circle A) reduces the formula to the identity 1 (and the same happens symmetrically for both β and β^{-1} in relation to the circle B). Thus, every 2-sublink of the Tripodal 3-link is completely splittable.





In the above figure, we imagine that we continuously pull apart the two upper topological circles of the Tripodal link displayed on the left. Then, we obtain the configuration on the right, which is interpreted in group-theoretic terms as a product loop, that is, the irreducible "Tripodal loop" associated with these two circles. Hence, we have a geometric representation of the "Tripodal loop formula". The algebraic irreducibility of this formula $\alpha\beta^{-1}\alpha^{-1}\beta$ in the group Θ encodes the non-splittability of the 3-link in the Tripodal topological configuration. If we cut the "Tripodal loop", or remove any of the circles A or B, we obtain a completely splittable 2-sublink. The "Tripodal loop formula" reads as follows: First, it passes away from p through A (represented by α); Second it passes towards p through B (represented by β^{-1}); Third it passes again towards p through A (represented by α^{-1}); Fourth, it passes away from p through B (represented by β).

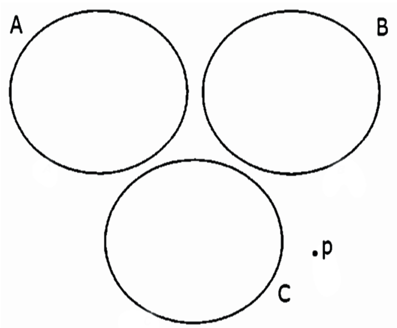
Thus, the topological information of the Tripodal link has been completely encoded in terms of the algebraic structure of the non-commutative multiplicative group Θ. In this way, we have obtained a bi-directional bridge between the topological connectivity model of the "Tripodal rings" expressed in terms of links and the algebraic algorithmic information model expressed in terms of the structure of the group Θ. This is of fundamental significance because it allows the translation of a hard topological problem into algebraic terms, which becomes the encoding of the problem in group-theoretic terms, where it can be solved quite easily, and then inversely, the decoding of this solution into topological terms, which provides the solution of the topological problem posed initially. An illustration of this powerful method, which generalizes the case of the "Tripodal link" to higher non-splittable links whose all sublinks are completely splittable, in analogy to the Tripodal case, will be presented as we progress.

3.16 HIGHER TEMPORAL BONDS FROM THE TRIPODAL LINK

It is instructive to clarify that the algebraic structure of the group Θ is not only restricted to the typical Tripodal configuration, explained in the previous section, but it can encode the topological information of higher links since we are free to construct product loops composed of any number of factors according to the composition rule we have defined. This presents the challenge of using the group Θ in order to solve the harder topological problem of identifying a non-splittable 4-link, all of whose 3-sublinks are completely splittable. Clearly, this problem constitutes the immediate higher generalization of the Tripodal link, which involves a non-splittable 3-link for which all 2-sublinks are completely splittable. The main interest in such a generalization lies in the intuition that the Tripodal link acts as a kind of a building block for the substantiation of higher order links of this type.

The method we will follow in order to attack this topological problem is the use of the bi-directional bridge between topology and algebra we have established in this context. Namely, we will translate the problem in terms of the algebraic structure of the group Θ, we will try to solve it in group-theoretic terms, and then decode the solution back into topological terms. Intuitively, the notion of a link involves the gluing conditions among its constituents. It is precisely these gluing conditions that are expressed algebraically in terms of the group Θ, as the fundamental case of the Tripodal link has revealed by means of the "Tripodal loop formula" $\gamma = \alpha\beta^{-1}\alpha^{-1}\beta$ in relation to the circles A and B.

The starting point is the analogous one to the standard Tripodal link case. Namely, since all 3-sublinks of the sought after non-splittable 4-link are completely splittable we will consider three circles A, B, C and look for a product loop composed of the products of α, β, γ and their group inverses α^{-1}, β^{-1}, γ^{-1}, in some specific order, which does not involve any consecutive appearance of $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$, $\gamma\gamma^{-1}$, $\gamma^{-1}\gamma$, because all of them are reduced to the identity 1. The crucial point again is that the product loop formula should reduce to 1 in the group Θ in case of removal of any of the circles A, B, or C, which is encoded algebraically by the deletion of all instances of both α, α^{-1}, or β, β^{-1}, or γ, γ^{-1}, which follows whenever A, or B or C respectively are erased. This is again the algebraic encoding of the fact that every 3-sublink of the total non-splittable 4-link should be completely splittable. Clearly, the non-splittability of the 4-link is again encoded by means of irreducibility of the product formula describing this 4-link.



Algebraically, this problem can be solved quite easily. The most elegant solution, which also trivializes the algebraic encoding of even higher links of this type, is to use the Tripodal link, viz. the algebraic "Tripodal loop formula" $\alpha\beta^{-1}\alpha^{-1}\beta$ in the group Θ as a building block and iterate it self-referentially. We will explain how this works for the case at issue. First, by inspecting the "Tripodal loop formula" $\alpha\beta^{-1}\alpha^{-1}\beta$ we realize that it can be written as the commutator in the group Θ, that is defined as follows

$$[\alpha,\beta^{-1}]=\alpha\beta^{-1}\alpha^{-1}\beta.$$

This means that the commutator $[\alpha,\beta^{-1}]$ of the elements α and β^{-1} in the group Θ producing the "Tripodal loop formula" encodes algebraically both the gluing condition of the non-splittable 3-link as well as of the completely splittability of all 2-sublinks, according to the preceding analysis. We may also re-define the element β^{-1} as b, viz. $\beta^{-1}:=b$ in the group Θ in order to obtain the commutator:

$$[\alpha,b]=\alpha b\alpha^{-1}b^{-1}$$

in the group Θ equivalently. Thus, the idea of using the Tripodal link as a building block for analogous links of a higher type means employing the group commutator iteratively as an encoding device for these higher links of the same type. Therefore, in the case of a total non-splittable 4-link all 3-sublinks of which are completely splittable that involves the gluing of the three circles A, B and C of the above figure by a "higher Tripodal loop" we proceed as follows:

First, we glue the circles A and B by the standard "Tripodal loop" and then we glue this product analogically with C. Algebraically speaking, the first step is simply the commutator $\xi=[\alpha,b]=\alpha b\alpha^{-1}b^{-1}$. The first iteration of this procedure, which involves the gluing of the product ξ with γ (in relation to the circle C), reads simply as the commutator of ξ with γ. We conclude that a "higher Tripodal loop" that solves the problem is given in the structural terms of the group Θ simply as follows:

$$\delta=[\xi,\gamma]=[[\alpha,b],\gamma].$$

If we expand this formula, by using the definition of the group commutator as well as the group theoretic relation

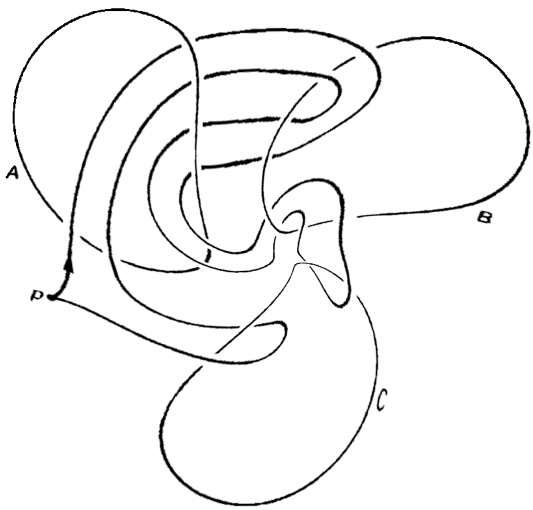
$$(\chi\psi)^{-1}=\psi^{-1}\chi^{-1},$$

where χ, ψ may stand for arbitrary strings of elements of the group Θ, we obtain the following unfolded expression for the "higher Tripodal loop formula":

$$\delta=[\xi,\gamma]=[[\alpha,b],\gamma]=\{\alpha b\alpha^{-1}b^{-1}\}\gamma\{b\alpha b^{-1}\alpha^{-1}\}\gamma^{-1}.$$

From the above expanded “higher Tripodal loop formula” it also becomes clear how the Tripodal link becomes a building block via terms of the form $\lambda\mu\lambda^{-1}\mu^{-1} = [\lambda, \mu]$ for expressing higher order links of the Tripodal type. We can also see that deletion of all incidences of any of the symbols (which involves the simultaneous deletion of the inverse symbol as well, as we have seen) reduces the formula to the identity 1 in the group Θ.

As a final step, we decode the obtained algebraic solution back into topological terms by using the inverse bridge, and the obtained topological solution of the problem of finding a non-splittable 4-link whose all 3-sublinks are completely splittable by means of “Tripodal building blocks” is illustrated as follows:



We conclude with the observation that although the topological solution of the problem is quite hard to obtain in a straightforward manner, as evidenced by the above figure, the same problem can be solved quite easily by using the algebraic structure of the group Θ, and in particular, the notion of the group commutator and its iterations. It is a remarkable fact that the Tripodal link is encoded in terms of the commutator of Θ. In this way, the Tripodal link can be efficiently used as a building block for the encoding of higher-order links of the type described above, by iterating the formation of commutators for product loops.

3.17 DEPTH OF HIGHER TEMPORAL BONDS: NESTED STACKING OF TRIPODAL LINKS

In the previous Section we proposed the idea of using the Tripodal link as a building block for analogous links of a higher type by making higher order iterations of the group Θ commutator. We have explained already how this method works in the case of a total non-splittable 4-link all 3-sublinks of which are completely splittable. The crucial insight is that the group commutator acts as an encoding device for these higher links of the same type in two ways: First, the commutator provides the gluing scheme of link-formation by means of "Tripodal loops". Second, due to fact that deletion of all incidences of any of the involved symbols reduces the commutator to the identity 1 in the group Θ, the commutator also encodes the information of complete splittability of any remaining sublink after removing any of the constituents of the total non-splittable link.

In order to proceed more efficiently, we need to systematize our terminology as follows: The notion of the commutator of the simple oriented based loops a, b, that is $[a,b]$, is used as synonymous to the algebraic "Tripodal loop formula" in the group Θ and it is decoded in topological terms as the concept of a Tripodal link, equivalently identified as a Borromean link. We denote the latter by $\Sigma(3,2)$ meaning that it is a total non-splittable 3-link all 2-sublinks of which are completely splittable. In this way, the symbol $\Sigma(4,3)$ denotes a total non-splittable 4-link all 3-sublinks of which are completely splittable.

By induction, the symbol $\Sigma(N,N-1)$, where $N \geq 3$, denotes a total non-splittable N-link all $(N-1)$-sublinks of which are completely splittable. We have shown that a $\Sigma(4,3)$ link can be constructed in terms of the "Tripodal link building block" simply by one iteration of the commutator formation. This means that starting with three symbols a, b, c, we first glue a with b together by means of the commutator $[a,b]$, and then we glue their glued product $[a,b]$ with c to obtain the stacked commutator $[[a,b],c]$. This final glued product gives the required fourth symbol in the group Θ_2, which decodes topologically as a $\Sigma(4,3)$ link. In an analogous manner, by iterating the commutator formation twice, starting with four symbols a, b, c, d, we obtain a $\Sigma(5,4)$ link. The same process can be clearly repeated inductively, so that we finally can construct any $\Sigma(N,N-1)$ link by means of Tripodal building blocks, or more precisely, "Tripodal connectivity units", where $N \geq 3$. We may summarize this process in the following table:

Borromean Link	Σ(3,2)	[a, b]	Gluing of a with b	3-link
1st Iteration	Σ(4,3)	[[a, b], c]	Gluing of [a, b] with c	4-link
2nd Iteration	Σ(5,4)	[[[a, b], c], d]	Gluing of [[a, b], c] with d	5-link
By Induction N≥3	Σ(N,N-1)	Repeat process with (N-1)-symbols.	Gluing via commutator stacking	N-link

We note that the process of iterating the commutator formation in the group Θ, so as to obtain any link of the form $\Sigma(N, N-1)$, can be realized as an algorithmic procedure of commutator stacking in consecutive nested levels. Semantically, this procedure may be thought of as an operation of self-referential unfolding. The reason is that if we start iterating the commutator formation from level-0 (Tripodal link $\Sigma(3,2)$) which involves simple loops, then already at level-1 (link $\Sigma(4,3)$), the symbol $[a,b]$ in the composite stacked commutator $[[a,b],c]$ plays a dual role: First, it is the symbol of a loop, namely the product "Tripodal loop" of a and b, and second, it is the symbol of a gluing operator acting on a and b. Thus, the unfolding from level-0 to level-1 takes place self-referentially by identifying a loop as an argument of the stacked commutator at level-1 with the result of a gluing operator at the previous level-0. Clearly, the same phenomenon repeats at all higher levels.

It is instructive to explain in more detail the algebraic operation of commutator stacking. Recall that a commutator of two symbols a and b produces a new symbol $[a,b]$ in the group Θ, where $[a,b]$ denotes the gluing of a and b together to produce a new symbol, such that the triad of symbols a, b and $[a,b]$ constitute a Tripodal link of the type $\Sigma(3,2)$. Thus, a $\Sigma(3,2)$ link involves a commutator in 2 symbols standing for the gluing operator of these two symbols according to the Tripodal constraint. Similarly, a $\Sigma(4,3)$ link involves a stacked commutator in 3 symbols. The commutator is stacked because first we have to glue a with b, and then we have to glue their product $[a,b]$ with c in order to produce a new symbol $[[a,b],c]$, such that the tetrad of symbols a, b, c and $[[a,b],c]$ constitute a $\Sigma(4,3)$ link.

We stress again that deletion of any of the symbols involved in the stacked commutator collapses it to the unity of the group Θ, meaning that erasing any one of them causes the rest to come apart. Thus, by induction a $\Sigma(N, N-1)$ link involves a stacked commutator in $(N-1)$ symbols, where $N \geq 3$. For convenience, we call it a stacked commutator of order $(N-1)$. Note that the order of the stacked commutator in any link of the form $\Sigma(N,N-1)$ coincides with the number of symbols that separate if we remove any symbol from the total non-splittable N-link. For example, a $\Sigma(7,6)$ link is expressed via a stacked commutator of order 6, meaning that it should be a commutator in 6 symbols of the form $[[[[[a,b],c],d],e],f]$. For reasons of simplicity, we define a stacked commutator of order $(N-1)$ as a "Tripodal stack" of order $(N-1)$.

3.18 TEMPORAL MULTIPLICATION: CHAINS OF TRIPODAL LINKS

First, we introduce another definition to our series for terminological convenience. This refers to the characterization of a link of the general form $\Sigma(N,K)$. A link of the form $\Sigma(N,K)$ is defined as a link of N loops in 3-d space, such that each K-sublink is completely splittable, but each $(K+1)$-sublink, $(K+2)$-sublink, $\ldots$, $(N-1)$-sublink up to the N-link itself, is non-splittable. For example, a $\Sigma(7,3)$ link is a link of 7 loops, such that each 3-sublink is completely splittable, but each 4-sublink, 5-sublink, 6-sublink and the 7-link itself, is non-splittable. The natural question emerging in this context is if it is possible to express a general link $\Sigma(N,K)$ in terms of "Tripodal building blocks", or equivalently "Tripodal functional units" encoded algebraically by the gluing operator of symbols, that is, by the commutator in the group Θ. We already know the answer in case that $K=(N-1)$. Namely, we have shown that the algebraic operation of commutator stacking of order $(N-1)$ is enough to express any $\Sigma(N,N-1)$ link. In other words, an arbitrary $\Sigma(N,N-1)$ link is simply a "Tripodal stack" of order $(N-1)$. So we need to consider what happens in the general case, where $K \neq (N-1)$.

We will show in the sequel that there exists another natural operation on "Tripodal building blocks", which is described by taking an appropriate product of commutators in the group Θ. Intuitively speaking, this natural operation should express a procedure of Tripodal extension in length, or simply the formation of a "Tripodal chain" of some appropriate length. In order to motivate the notion of a "Tripodal chain" it is necessary to start with the simplest example of this type, namely the $\Sigma(4,2)$ link. This is a link of 4 loops, such that each 2-sublink is completely splittable, but each 3-sublink and the 4-link itself, is non-splittable. From this definition, we immediately deduce that if we remove any loop from a $\Sigma(4,2)$ link we obtain a 3-sublink which is non-splittable. Moreover, since each 2-sublink is completely splittable, we deduce that if we remove any loop from a $\Sigma(4,2)$ link we actually obtain a $\Sigma(3,2)$ link, viz. a Tripodal link. Furthermore, if we remove any two loops from a $\Sigma(4,2)$ link the remaining two fall completely apart because again each 2-sublink of a $\Sigma(4,2)$ link is completely splittable. Therefore, by encoding this information in the group Θ, we attack the problem as follows: Consider three symbols a, b, and c. We seek a formula expressing the fourth symbol, such that deletion of all incidences of any of the symbols a or b or c causes the formula to reduce to the "Tripodal loop formula" (that is the commutator of the remaining two symbols), whereas deletion of all incidences of any two of the three symbols, viz. (a,b), or (a,c), or (b,c) causes the formula to reduce to the unity 1.

It is instructive to emphasize that the algebraic encoding of the problem referring to a $\Sigma(4,2)$ link paves the way to its solution. The problem is whether it is possible to express a $\Sigma(4,2)$ link in terms of "Tripodal building blocks", that is, in terms of suitable operations on commutators in the group Θ. By the defining properties of a $\Sigma(4,2)$ link, if a formula in three symbols a, b, c actually existed fulfilling the two requirements laid out in the previous paragraph, and also expressed exclusively in terms of commutators built from these three symbols, then it would be true that the $\Sigma(4,2)$ link can be constructed in terms of "Tripodal building blocks". Now, considering the symbols a, b, and c, we may construct the "Tripodal stack" of order 3, viz. the stacked commutator formula $[[a,b],c]$. Clearly, although this expresses a $\Sigma(4,3)$ link as we have seen in the previous Section, it is not an appropriate formula to express a $\Sigma(4,2)$ link because deletion of any of the three symbols causes the formula to reduce to 1. What we need is another operation, which hopefully can involve only commutators and have the desired properties. A simple observation is that given three symbols a, b, and c, we may construct out of them three distinct commutators, namely $[a,b]$, $[a,c]$ and $[b,c]$. Since each of these commutators gives a new symbol in the group Θ, we may take their product which is also a new symbol in the group Θ.

Notice that each of the commutators $[a,b]$, $[a,c]$, $[b,c]$, gives separately a Tripodal link. Thus, their product $[a,b][a,c][b,c]$ is actually a composition of three separate "Tripodal links" in the group Θ:

$$\rho = [a,b]\circ[a,c]\circ[b,c],$$

which gives rise to a "Tripodal chain" of length 3. The formation of this "Tripodal chain" ρ provides the sought after operation on "Tripodal building blocks" to express a $\Sigma(4,2)$ link, and therefore solve the posed problem. We can immediately see this as follows: First, we notice that deletion of any one of the symbols a, b, c, in the "Tripodal chain" ρ of length 3, $[a,b]\circ[a,c]\circ[b,c]$, reduces this chain to a Tripodal link. For instance, if we delete the symbol a, what remains is the Tripodal link $[b,c]$, and analogously for the other two cases. Second, we notice that deletion of any two of the symbols a, b, c, reduces this chain to unity. Hence, we conclude that the "Tripodal chain" of length 3, defined by the product of commutators $[a,b][a,c][b,c]$, provides the formula for the fourth symbol ρ in the group Θ, such that the defining properties of a $\Sigma(4,2)$ link are satisfied, and moreover, this link is expressed in terms of "Tripodal building blocks". An interesting observation that we will put to use as we progress is that the length of the "Tripodal chain" solving the problem is given by the number of combinations of 2 symbols out of 3, where a combination is simply the formation of the commutator of 2 symbols in this case.

3.19 PRIMAL ROLE OF THE TRIPODAL LINK IN ALL HIGHER TEMPORAL BONDS

Regarding the possibility of expressing arbitrary links in 3-d space of the general form $\Sigma(N,K)$ in terms of "Tripodal building blocks", or equivalently "Tripodal connectivity units" we have proved up to present the following: First, the algebraic operation of commutator stacking of order $(N-1)$ is enough to express any $\Sigma(N,N-1)$ link. In other words, an arbitrary $\Sigma(N,N-1)$ link is simply a "Tripodal stack" of order $(N-1)$. For instance, a $\Sigma(4,3)$ link is simply a "Tripodal stack" of order 3. Second, we have shown that the expression of a $\Sigma(4,2)$ link requires the consideration of another operation on "Tripodal building blocks", which is interpreted as the operation of extension of length 3, called the formation of a "Tripodal chain" of length 3. Based on these findings, the next question posing itself naturally in this context is if these two operations on "Tripodal building blocks", namely the formation of "Tripodal stacks" of some suitable order and the formation of "Tripodal chains" of some suitable length are adequate in order to express any arbitrary link in 3-d space of the general form $\Sigma(N,K)$.

This would be certainly of significance in our understanding of the whole universe of links, because it would prove that any $\Sigma(N,K)$ link can be constructed by means of "Tripodal connectivity units" via the combinatorial formation of "Tripodal stacks" and "Tripodal chains". Moreover, due to the algebraic modelling scheme instantiated structurally by the non-commutative group Θ, the process of analysis and synthesis of arbitrary links in terms of prime elements, which is to say in terms of "Tripodal connectivity units" would be implementable algorithmically, and thus at hand as a valuable tool for making evaluations and predictions.

Before we consider the general case of a $\Sigma(N,K)$ link, it will embellish our intuition to examine the case of a $\Sigma(5,3)$ link. The reason is that a $\Sigma(5,3)$ link has enough complexity so as to pave the way for the treatment of the general case of a $\Sigma(N,K)$ link. From the definition of a $\Sigma(5,3)$ link, the crucial observation is that if we remove any of the loops, what remains is a $\Sigma(4,3)$ link, which we already know is expressed by means of a "Tripodal stack" of order 3, by the stacked commutator formula $[[a,b],c]$ in 3 symbols. Thus, in order to express the formula of a $\Sigma(5,3)$ link, if we consider 4 symbols a, b, c, d, we require a formula such that deletion of any of them causes it to reduce to one of a $\Sigma(4,3)$ link, to a "Tripodal stack" of order 3.

The important concept solving this problem is based on the observation that we can form "Tripodal chains" of arbitrary length using "Tripodal stacks". In the particular case of a $\Sigma(5,3)$ link considered, since we require that deletion of any of the four involved symbols a, b, c, d, reduces the formula to a "Tripodal stack" of order 3, we just need to form a "Tripodal chain" of "Tripodal stacks" of order 3, where the length of the chain should be 4. This is explained easily by the fact that the length of the "Tripodal chain" is given by the number of combinations of 3 symbols (which is the number of symbols involved in a "Tripodal stack" of order 3) out of 4 symbols a, b, c, d. We immediately conclude that the sought after formula expressing a $\Sigma(5,3)$ link is given by the "Tripodal chain" of length 4, composed by "Tripodal stacks" of order 3, and described explicitly by the following formula:

$$\chi = [[a,b],c] \circ [[a,b],d] \circ [[a,c],d] \circ [[b,c],d].$$

In more detail, we see that the above formula is given by the composition of 4 "Tripodal stacks" of order 3 (since they involve 3 symbols each), and thus produces a "Tripodal chain" of length 4, such that deletion of any of the four involved symbols a, b, c, d, reduces this chain to a "Tripodal stack" of order 3 as required. Thus, we have completely resolved the problem of a $\Sigma(5,3)$ link in terms of prime "Tripodal connectivity units".

Now, having understood in detail the case of a $\Sigma(5,3)$ link, we are ready to state the following theorem:

An arbitrarily complex link of the general form $\Sigma(N,K)$, where $1 \leq K \leq N$, can be enunciated solely in terms of Tripodal links, by means of combining nested stacking and multiplicative chaining of Tripodal links of appropriate depth order and length respectively.

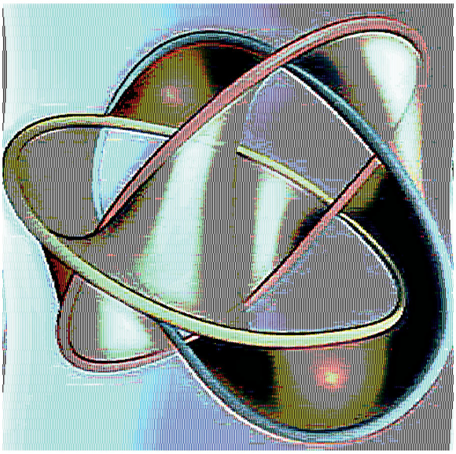
We consider an arbitrarily complex link of the general form $\Sigma(N,K)$, where $1 \leq K \leq N$, and prove that it can be constructed solely in terms of "Tripodal building blocks" within the group Θ. For any K, we already know that the link $\Sigma(K+1,K)$ is expressed by means of a "Tripodal stack" of order K. Next, we consider $(K+1)$ symbols in Θ, and we wish to construct a $\Sigma(K+2,K)$ link. The crucial observation is that if we remove any topological circle from a $\Sigma(K+2,K)$ link, what remains is a $\Sigma(K+1,K)$ link. Thus, we treat this case in complete analogy to the case of a $\Sigma(5,3)$ link, discussed previously. More precisely, we form a "Tripodal chain" out of "Tripodal stacks" of order K, where the length of this chain is given by the number of combinations of K symbols out of $(K+1)$ symbols. The formula expressing this "Tripodal chain" provides the sought after $(K+2)$ symbol. Now, we consider $(K+2)$ symbols, and we wish to construct a $\Sigma(K+3,K)$ link. We just have to form a "Tripodal chain" out of "Tripodal stacks" of order K, where the length of this chain is given by the number of combinations of K symbols out of $(K+2)$ symbols. The formula expressing this new "Tripodal chain" provides the sought after $(K+3)$ symbol in Θ. We continue the same process of formation of new "Tripodal chains" of appropriate combinatorial length composed by "Tripodal stacks" of order K, stage by stage, until we reach N. This completes the proof of the theorem that an arbitrarily complex link of the general form $\Sigma(N,K)$ can be constructed solely in terms of "Tripodal building blocks", or equivalently, "Tripodal connectivity units".

We may consider as an application of this theorem the case of a $\Sigma(7,4)$ link. The link $\Sigma(5,4)$ is expressed by means of a "Tripodal stack" of order 4. Next, we consider 5 symbols, and we wish to construct a $\Sigma(6,4)$ link. Let us call these symbols a, b, c, d, e. Next, we form a "Tripodal chain" of "Tripodal stacks" of order 4, where the length of this chain is given by the number of combinations of 4 symbols out of 5 symbols, which is 5. Let us denote by f the new symbol provided by this "Tripodal chain" of length 5. Thus, we have constructed a $\Sigma(6,4)$ link. Now, we consider these six symbols a, b, c, d, e, f, and we wish to construct a $\Sigma(7,4)$ link. We just have to form a "Tripodal chain” of “Tripodal stacks” of order 4, where the length of this chain is given by the number of combinations of 4 symbols out of 6 symbols, which is 15. The product formula expressing this new “Tripodal chain” of length 15, provides the sought after 7th symbol. Therefore, we have constructed a $\Sigma(7,4)$ link by means of prime “Tripodal connectivity units” using only the combinatorial formation of “Tripodal stacks” and “Tripodal chains”.

3.20 IMAGINARY SURFACE OF TEMPORAL COHESION AND ENTANGLEMENT

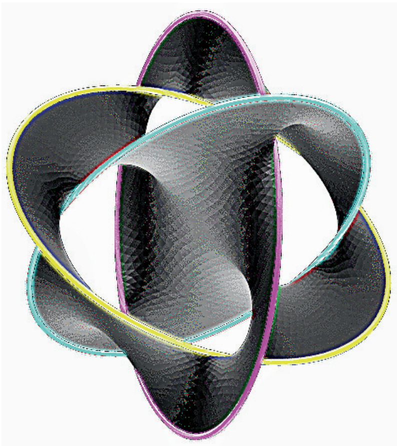
The modeling of a prime temporal bond as a topological “Tripodal link” enables us to comprehend the process of “synergetic cycle change”, effected by the “modular gluing” of a seed from the “past” with an anticipated seed in the “future” with respect to the “present”, upon establishment of this temporal bond. This threefold metaphora provides the necessary topological means to elucidate how a holographic boundary of temporal cohesion can be adjoined to 3-d spatial space in the present that this bond is realized. The adjunction of this synectic boundary of temporal cohesion in 3-d space permit the holographic connectivity and entanglement of a seed from the “past” with an envisioned seed in the “future” topologically, independently of their proximal distance. This is achieved by demarcating an imaginary oriented compact and connected surface of temporal cohesion in the present.

We consider the compact, connected and oriented surface with boundary to constitute the “Tripodal link”. This surface is visualized as follows:



Thus, the imaginary surface of temporal cohesion generated by a prime temporal bond is equivalent to a torus bearing three punctures (corresponding to the *aphaeresis* of three disks). The significance of this imaginary surface of temporal cohesion instantiated by a temporal bond is that it gives rise to a global curvature topological effect characterized as a minimal surface. In other words, it is the least-action connectivity solution, and thus, the most economical solution to the modular amalgamation instantiated by a prime temporal bond.

A simple method to visualize this surface spatially is to consider the "minimal surface" formed by a soap film, when three wire rings linked together as the "Tripodal link" are immersed into a solution of soapy water and then taken out. This surface is a "least-action" solution to the shape that a soap film acquires in this case, since it minimizes the area. Interestingly enough, every point in this "surface of cohesion" is locally similar to a saddle, i.e. its local curved geometry is of the hyperbolic type, whereas its global topology is of the toroidal type.



4 UNDECIDABILITY-UNCERTAINTY: INDIRECT LOGICAL CYCLES AND COMPUTATION

4.1 INDIRECT LOGICAL *METAPHORA*: GÖDEL'S FIRST INCOMPLETENESS THEOREM

The conceptual essence of *Kurt Gödel's* first incompleteness theorem may be summarized in the assertion that if a formal system containing arithmetic, meaning any arithmetic structure endowed with the operations of addition and multiplication, is consistent, then it contains undecidable propositions, namely statements whose truth or falsity cannot be expressed within the language of this formal system. According to Gödel, the reason for the existence of undecidable propositions in a formal system containing arithmetic is that a complete epistemological description of a language A cannot be given in the same language A, because the concept of *"truth"* of sentences of A cannot be defined within A. Thus, the "truth" of the propositions of a language cannot be expressed in the same language, while *provability*, which is an arithmetic relation can. In a nutshell, true $\neq$ provable.

Gödel's proof of the first incompleteness theorem is based on the explicit construction of an arithmetical formula that asserts its own non-provability, and thus, it is undecidable within the language of its formal system. From our viewpoint, the particular interest in Gödel's proof stems from three factors:

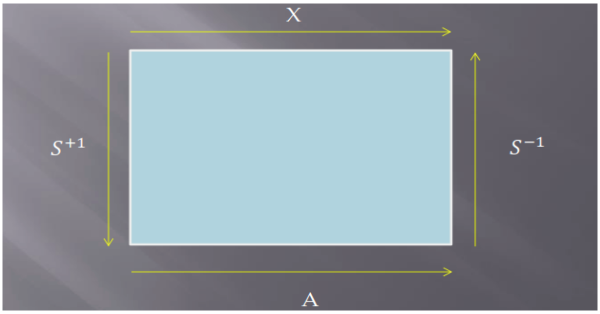
1. According to a remark of Gödel himself, there exists an analogia between his undecidable proposition within a formal system containing arithmetic and classical semantic paradoxes, like the Liar paradox. The analogia is based on the existence of direct *self-referentiality*;
2. The correct solution of these semantic paradoxes derives from the method of proof that Gödel devised in order to evade the logical obstacle of direct strong self-referentiality within the language of his formal system. Concisely put, this method of proof involved an argument requiring a stratification into two *hypostatic* levels, one of which is called the mathematical level and the other the *metamathematical* level. In other words, the circumvention of self-referentiality required a metaphora into another level of hypostasis, such that the direct obstruction is avoided by dint of ascending to another level and then descending back. In this way, *indirect* self-reference, leads to a well-defined legitimate statement and not to a paradox.
3. The process of ascending from the mathematical to the metamathematical level and then descending back, or the other way round symmetrically, effecting indirect self-reference, and thus eventually, producing a legitimate statement asserting its own unprovability, required a metaphora, i.e. the instantiation of

encoding/decoding bridges for translating between these two levels. This is precisely the role of "Gödel's numbering" or "Gödel's ordering" idea.

From our viewpoint, "Gödel's numbering" is actually *"Gödel's gnomon"*, utilized as a means to indicate or label propositions at both the mathematical and the metamathematical level, in such a way that a certain type of *homeotic equivalence* can be established between these two levels. In other words, the role of "Gödel's gnomon" is to logically conjugate the intractable problem of direct, strong self-referentiality at one level of hypostasis by a definite tractable process at the other level of hypostasis, where the latter is qualified in terms of Cantor's diagonalization method, as we shall show below.

Henceforth, the key to understanding Gödel's argument from our view, consists in delineating the stratification of the argument into levels and identifying the "gnomon" which induces an appropriate "homeotic criterion" permitting the "metaphora by logical conjugation", or else, descending and ascending between these levels. Gödel's argument requires a stratification into two levels: the mathematical level involves general propositions about numbers and the metamathematical level involves general propositions *about* general propositions about numbers.

Gödel's argument refers to a true proposition at the metamathematical level, whose truth is established by "logical conjugation" through the mathematical level. It is clear that this argument involves an indirect self-reference, which is legitimate since it is arrived at by descent to and re-ascent from the mathematical level, as we have stressed previously. Gödel's gnomon is a gnomon of numbering or ordering and it is utilized to establish encoding and decoding reciprocal translation bridges between these two levels. In terms of Gödel's theorem, the possibility of establishing a true proposition at the metamathematical level, is proved by descending to the mathematical level, such that a particular argument can be formulated by means of an infinite closure operator, qualified in terms of *"Georg Cantor's diagonalization"*, which is then transferred back to the metamathematical level by means of ascending the inverse bridge to prove the theorem.



Since the alphabet of arithmetic is countable, it is possible to instantiate a fixed schema of numbering or ordering, which assigns a unique positive integer to every legitimate arithmetic formula. The same schema can be extended to order finite strings of arithmetical formulas. Of course, many such appropriate schemas of ordering or numbering exist, but the essential idea is that by fixing any one of them the function of ordering or numbering can be carried out. For example, we may fix the ordering gnomon provided by the natural numbers' sequence, such that every arithmetic formula and every finite string of arithmetic formulas is assigned a unique number in this sequence, called its Gödel number. It follows directly that in the way described the ordering structure of the natural numbers may be adjoined to the structure of an arithmetic. In particular, the proof of an arithmetic formula K constitutes a finite string ending with K itself, and thus proofs are naturally assigned Gödel numbers in the ordering.

Gödel starts his argument by considering the proposition $p(x, y)$ at the metamathematical level stating the following:

"$p(x, y)$: x is the Gödel number of an arithmetic formula whose proof has Gödel number y".

Then, still at the metamathematical level, he considers the associated proposition,

$$\forall y \neg p(x, y),$$

which reads as follows:

"$\forall y \neg p(x, y)$: No number y is the Gödel number of a proof of the arithmetic formula whose Gödel number is x".

The last proposition simply means that the x-th formula in our ordering schema is not provable.

The crucial idea in the last proposition boils down to the fact that the variable x is a free variable. Then, the natural question to ask is the following: Is the proposition

$$\forall y \neg p(x, y)$$

at the metamathematical level Gödel-numberable itself? Equivalently stated, does Gödel's gnomon apply to this proposition? This is the crux of the matter because, as we already know, a gnomon is effective if it induces a "homeotic criterion" to the structure it is adjoined to, that permits the descent and ascent between the metamathematical and the mathematical level.

Clearly, such a criterion is feasible in the present case, only if Gödel's gnomon actually assigns a unique number to the proposition $\forall y \neg p(x, y)$, where x is a free variable.

We realize from this reasoning why the major part of Gödel's paper is devoted to showing that the aforementioned proposition is indeed Gödel-numberable. Let us denote the Gödel number of the metamathematical level proposition

$$\forall y \neg p(x, y), \text{ where } x \text{ is free,}$$

by the number ξ at the mathematical level. The criterion can now be implemented using Gödel's gnomon by applying Cantor's diagonalization process at the mathematical level in order to achieve closure. This simply amounts to substituting the free variable x in the proposition $\forall y \neg p(x, y)$ by the definite number ξ to obtain now at the mathematical level the proposition

$$\forall y \neg p(\xi, y),$$

which means that the concrete ξ-th formula in our ordering schema is not provable.

The role of Gödel's gnomon is enunciated as follows: If we apply this gnomon the metamathematical level proposition

$$\forall y \neg p(x, y), \text{ where } x \text{ is free,}$$

is precisely mirrored at the number ξ at the mathematical level. This means that the above metamathematical level proposition is symmetrical, and thus equivalent, to a certain arithmetic formula at the mathematical level whose sequential number is ξ modulo the gnomon employed.

Henceforth, the metamathematical level proposition $\forall y \neg p(x,y)$, where x is free, is symmetrical modulo the gnomon, and thus homeotically identical, with the ξ-th arithmetic formula in the ordering induced by the gnomon at the mathematical level.

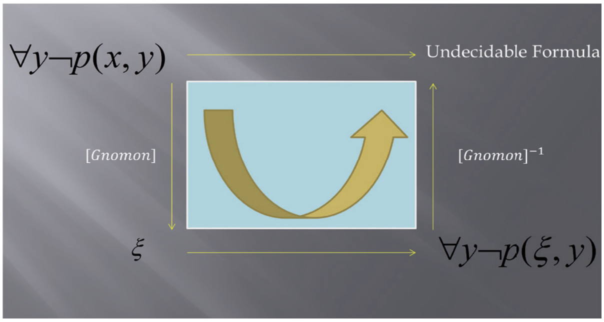
It is important to notice that the process of Cantorian diagonalization at the mathematical level involves a reflexive action, since we feed this fixed ordering number ξ as an argument in the place of the free variable x of $\forall y \neg p(x,y)$.

In this manner, we obtain a legitimate proposition at the mathematical level

$$\forall y \neg p(\xi, y),$$

which states that the concrete ξ-th formula in our ordering schema is not provable, since no number y is the Gödel number of a proof of the arithmetic formula whose Gödel number is ξ.

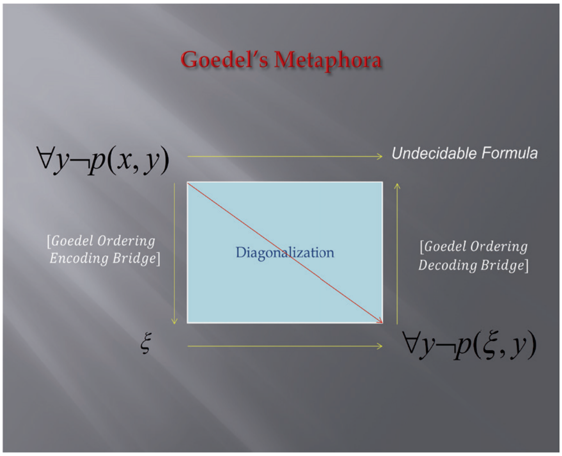
Finally, using the homeotic criterion established by Gödel's gnomon in reverse, we ascend back to the metamathematical level, where we finally obtain a proposition that ascertains its own unprovability. It is precisely this proposition that expresses Gödel's incompleteness theorem itself, since this proposition is undecidable given the consistency of our arithmetic.



Gödel's gnomon and the previously described metaphora by means of logical conjugation between the metamathematical and mathematical levels is operative with respect to the whole structure of an arithmetical formal system, that is with respect to both the additive and multiplicative structure of an arithmetic system. In case that only the additive structure is considered, Gödel's gnomon does *not* induce a homeotic criterion between the metamathematical and mathematical levels, and it can be shown that the incompleteness theorem is not valid.

Epigrammatically, Gödel's gnomon effects an indirect self-reference at the metamathematical level by means of the descent to and re-ascent from the mathematical level. The indirect self-reference is conducted through the mathematical level by utilizing the infinite closure operation of Cantor's diagonalization process. In other words, employing Gödel's gnomon renders indirect self-reference feasible by conjugating the initial intractable problem to an infinite closure operation, and thus circumventing it appropriately.

The far-reaching consequence of the above is that Turing's argument, according to which, the halting problem inherent to a universal Turing machine is undecidable, should be viewed as the computational variant of Gödel's first incompleteness theorem. The reason is that Turing's argument can be also considered as a logical conjugation argument of the same form, meaning that indirect self-reference at the level of a universal Turing machine is feasible by conjugating it to the infinite closure operation of Cantor's diagonalization. "Turing's gnomon" is similarly a gnomon of ordering or numbering programs by means of the natural numbers' sequence.



4.2 INDIRECT FORCING IN LOGIC: GENERIC FILTERS AND CONTEXTUALIZED TRUTH

Since the formulation of Gödel's first incompleteness theorem, it is well known that for any comprehensible list of set theoretic axioms, there will be statements neither provable nor unprovable from those axioms. What is really interesting, however, is how many of the most natural questions about sets are not decidable by the standard axioms of Zermelo-Fraenkel-Choice (ZFC) set theory, and how many ways of deciding these questions there are available.

In this state of problematics, *Paul Cohen* managed to establish the independence of the Axiom of Choice (AC) from Zermelo-Fraenkel (ZF) and the independence of the *Continuum Hypothesis* (CH) from ZFC. This became possible by developing a novel technique, called forcing, that constitutes a far-reaching generalization of the logical notion of implication, which he used for extending a standard model of set theory. Cohen's extension method constitutes another significant case of the paradigmatic schema of analysis we developed in the previous Section in relation to Gödel's first incompleteness theorem. This schema involves a process of indirect self-reference through extension to a new logical level of hypostasis by a metaphora providing bidirectional bridges for translating between the initial standard model of set theory and some novel model of set theory internally distinguishable from the former one.

Epigramatically, Cohen discovered the precise means of operationally extending a standard model of set theory to some other admissible one bearing specific properties without altering the ordinals. The central technical innovation was based on the notion of a *forcing condition*, through which satisfaction for the extension could be approached in the ground standard model.

In general terms the method of forcing consists in the instantiation of a novel model of set theory from a standard model by the adjunction of certain sets with particular properties via "forcing conditions" encoding information about those sets. Intuitively, some conditions are stronger than others, and this serves as a criterion for partially ordering them within the ground standard model. In this state of affairs, proving general results about how the elements of a partially ordered set P force certain conditions to hold allows one to prove statements referring to the novel constructed models, without looking closely at the forcing conditions themselves anew in each particular case. It is this generality that lends the method its efficiency and universality.

Let us start with a ground model V of set theory (countable and transitive) and consider a specific partially ordered set (poset) P in V. The set P is to be interpreted as a partial order of forcing conditions, ordered by means of their strength. In this way, a nonempty poset $(P,<)$ in induces a notion of forcing, whose elements will play the role of "forcing conditions". In general, if p, and $p<q$, we say that p is stronger than q. That is, p represents a stronger condition than q. It is customary for the poset to have a largest element, denoted 1, such that all elements of P are stronger than 1. The elements p and q

are compatible if there exists an element r in $(P,<)$ stronger than both p and q, otherwise they are incompatible.

It is important to note that the poset and its ordering must be elements of V. However, the idea is to utilize the poset $(P,<)$ as a partial order of forcing conditions, so as to construct and eventually adjoin to V certain sets that are not already in V. In particular, by adjoining a "generic set" G to the ground model V, a new model $V[G]$ can be consistently formed by extension.

A dense subset of P is defined as a set D such that for all $p \in P$, there is $q \in D$ stronger than p. A filter on a poset P is a nonempty subset F of P such that: (a) if $p \in F$ and $p \leq q$, then $q \in F$ and (b) if p, $q \in F$ there is $r \in F$ extending both p and q. If this filter intersects every dense set D in the ground model V, then it is called a "generic filter".

Considering that the subset $G \subseteq P$ is a *"generic filter"*, meaning that G contains members from every *dense* subset of P in V, one proceeds to build the forcing extension of the ground standard model V by G, denoted by $V[G]$, such that $V \subseteq V[G]$ by requiring closure under all elementary set-theoretic operations.

Consequently, the forcing extension has adjoined the *"ideal"* object G to V, in much the same way that one might build an algebraic extension of a ring by means of an ideal. In particular, every object in $V[G]$ has a name in V and is constructed algebraically from its name and the generic filter G. Remarkably, the forcing extension $V[G]$ is always a model of ZFC. Nevertheless, the crucial point is that it can exhibit different set-theoretic properties in a way that can be precisely controlled by the choice of the poset of forcing conditions P.

More precisely, one may proceed by defining the forcing relation on a proposition ϕ, denoted by p ë ϕ, which holds whenever every generic filter G containing the forcing condition p implies $V[G]$ ë ϕ. For this purpose, it is important to consider the complete Boolean algebra B, which functions as the completion of the selected poset of forcing conditions P. We say that a condition $p \in P$ forces ϕ if and only if p is less or equal to the "Boolean value" of ϕ in B.

It is worth recalling that in a standard model of set theory, all propositions are evaluated as "true" or "false", meaning that they are strictly evaluated with respect to the two-valued Boolean algebra $\{0,1\}$. In comparison, referring to a non-standard model, propositions can take values on any element of the Boolean algebra under consideration. The concrete meaning of this logical maneuver will be examined subsequently. Presently, if we take for granted the notion referring to the *"Boolean value"* of ϕ in B, the notion of forcing implicates that if the proposition ϕ is being forced by p, then it is going to be also forced by any condition that is stronger than p.

The fundamentally significant elements of the method of forcing, pioneered by Cohen, are the following:

1 The forcing extension of the ground standard model V by G, namely $V[G]$, satisfies the axioms of ZFC set theory.
2 Every proposition ϕ that holds in $V[G]$, is forced by some condition p in P.
3 The forcing relation p ë ϕ is definable in the ground model for fixed ϕ.

Of course, the crux of the matter, is the delineation of an appropriate generic filter that actually accomplishes the required extension. In the case of a countable and transitive ground model V, for any chosen *partial order* P of elements from V, to be interpreted as forcing conditions, there exist only countably many dense subsets of P, which may be enumerated externally as D_0, D_1, D_2, and so on. Then, we may pick in order any condition $p_0 \in D_0$, then $p_1 \in D_1$ below p_0, and so on. With this procedure, and using the method of diagonalization, we can construct a descending sequence $p_0 \geq p_1 \geq p_2 \geq \ldots$, such that $p_n \in D_n$. Then, the filter G generated by this sequence is generic, and therefore, suitable for the construction of the forcing extension $V[G]$.

The restriction that V stands for a countable and transitive model ground model of set theory can be effectively lifted. In this case, it is not possible to delineate a generic filter as described above, and consequently, the method of "Boolean values" seems to provide the most general approach, to the semantics to which we now turn our attention.

Cohen's main innovation lies in the distillation of the notion of forcing by means of a chosen partial order in the ground model, and the positing of a generic filter in this partial order containing elements not already grasped in the ground model. This innovation made it possible to secure suitable properties of a novel set, which emerged through the extension of a standard model by a generic filter, without having distinguished all of the members *ab initio*. According to this conception, the generic set G will not be determined completely, but in spite of this, properties of G will be completely determined on the basis of very incomplete information about G. This is phenomenally contradictory, because how could one decide whether a statement about G is true, before we have determined G itself? The seeming contradiction stems from the standard conception of truth in terms of evaluations of a priori distinguishable elements with respect to the two-valued Boolean algebra $\{0,1\}$. The method of forcing requires a re-conceptualization of the notion of truth in a novel way. Actually this novelty re-enforces the ancient conception of truth as "aletheia". In this manner, truth emerges temporally, in the precise sense of being unveiled, through a process of percolation through the generic filter.

The main idea is that in the ground model V, the devised set of forcing conditions P, where each condition is the carrier of partial information toward an eventual generic filter G, is precisely ordered according to the potential amount of information. In the case of a countable ground model V, a complete sequence of stronger and stronger conditions p_0, p_1, p_2, $\ldots$ is applied, so that every proposition, or its negation, is forced by some member of this sequence. Therefore, it is owing to this sequence that a generic object G is eventually manifested bearing the appropriate properties.

In the general case, a new meaning is elucidated in relation to the notion of truth, which may be described as follows: Working inside the ground model V, we consider the set of elementary conditions P which forces a given set either to lie in the generic set G, or not in G. Because of the fact that forcing is defined in the ground model V, we can examine all the possibilities of assigning sets of elementary conditions P, which force the members of A to lie in an arbitrary B. This set of elementary conditions is the "truth value" of the statement. As we mentioned above, the notion of truth in this context bears the meaning of "aletheia", in the sense that the "truth value" is identified with the set of conditions that unveil the statement. Conclusively, in a standard model of set theory a subset of X is determined by a two-valued function on $\{0,1\}$ applied to the members of X. In a non-standard model, a subset is determined by a function taking its values in the subset of the elementary conditions. Since these values are all in the ground model V, quantification is possible over all possible truth values.

The precise connection with the idea of evaluating propositions in the complete Boolean algebra B obtained by the completion of the poset of forcing conditions P emerges in the following manner: The subsets of the set of elementary conditions P, which determine the truth or falsity of each statement, are thought of as elements of a

Boolean algebra. Then, a Boolean-valued model of set theory arises, in the sense that the truth values are identified with the elements of the concomitant Boolean algebra, which is different from the bivalent $\{0,1\}$. The notion of forcing is expressed by saying that the set P forces a statement S, if no extension of P forces the negation of S".

In the above setting, the essential idea is that the conceptual maneuver of evaluating statements by means of the Boolean algebraic completion B of the chosen set of forcing conditions P in the ground model V, provides the means to embrace the obstacle; namely, that the elements of the generic set G are not specified *ab initio*. This gives rise to a non-standard Boolean-valued model of set theory, denoted by V^B. In other words, a Boolean-valued model, besides the *crystallized* elements, contains elements that are "partially or locally distinguishable", where the extent of their distinguishability in the *percolation* process of *unveiling* is provided by the "truth value" they are assigned in B, thought of as a domain of truth values, and not in the bivalent domain $\{0,1\}$. This logical maneuver in the specification of the Boolean-valued set V^B allows us to think of it as a set of fluid potential members in the process of eventual crystallization.

In this state of affairs the role of the generic set G is precisely to determine which elements of V^B will eventually be crystallized giving rise to a novel standard model of set theory satisfying the axioms of ZFC. Henceforth, the generic set G plays the role of Cohen's gnomon for extending a standard ground model of set theory to a novel standard model obeying ZFC, which is nonetheless internally distinguishable from the former one. This is accomplished by means of indirect self-reference through the level of Boolean-valued models of set theory as follows:

1 The encoding bridge is from the level of standard ZFC models of set theory to the level of non-standard Boolean-valued models obtained by completion of any chosen partial order of forcing conditions in the ground standard model we are starting with;

2 At the level of a Boolean-valued model V^B, an appropriate equivalence relation is formulated by utilizing Cohen's gnomon, i.e. a generic set G in B. This establishes a homeotic criterion of symmetry for the members of the Boolean-valued set V^B in the sense of a common logical measure provided by G giving rise to a *partition spectrum*. In particular, the generic set G is qualified as a *generic ultrafilter* in B. The ultrafilter characterization means that:

a $1 \in G$,

b 0 not $\in G$,

c if $\kappa, \lambda \in G$, then $\kappa \wedge \lambda \in G$,

d if $\kappa \in G$, and $\kappa \leq \lambda$, then $\lambda \in G$,

e For all $\xi \in B$, either $\xi \in G$ or $\neg\xi \in G$.

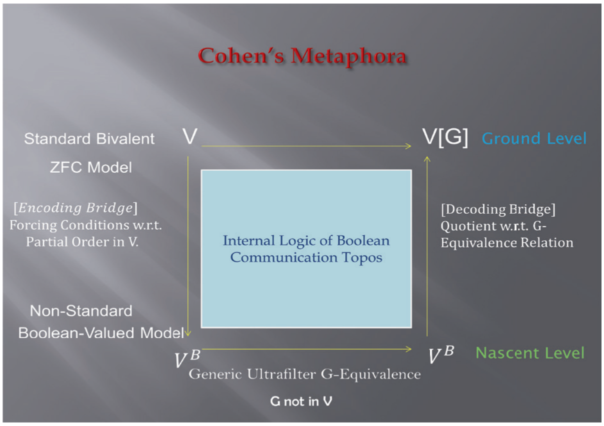
Then, the common logical measure implied by a generic ultrafilter G, i.e. the symmetry, or homeotic criterion, for elements of the Boolean-valued model V^B with respect to the gnomon G, is formulated in terms of the following equivalence relation:

$$\text{For all } x, y \in V^B,$$
$$x \sim_G y \text{ iff } \|x = y\|^B = \xi \in G,$$

where, $\|x = y\|^B$ denotes the *"Boolean value"* characterizing the extent to which $x = y$ for all $x, y \in V^B$.

Since, the generic set G (not required to be in V) is an ultrafilter in B, the equivalence classes of elements of V^B, i.e. the blocks of the corresponding partition spectrum, with respect to the above symmetry criterion provided simply by means of membership in G or not, that is, in a bivalent manner, are the candidates for forming the elements of the new standard model of set theory, constituted or crystallized by the forcing extension of the ground standard model V by G, i.e. $V[G]$.

3 The decoding bridge is from the level of Boolean-valued models of set-theory back to the level of standard ZFC models of set theory. The standard ZFC model obtained by imposition of the above equivalence relation on elements of V^B with respect to Cohen's gnomon G is the quotient set of equivalence classes V^B / G, which is bivalent. The ingenuity of Cohen's proof rests on the requirement of generic status for his gnomon, in the sense that the quotient set V^B / G is actually a standard ZFC model, if the utilized ultrafilter G is a generic one. Actually, if G is a generic ultrafilter then $V^B / G := V[G]$ is the smallest standard ZFC model containing both V and G.



Finally, it is noteworthy that Cohen's method of forcing has been adapted in category-theoretic language. The relevant context is the topos of *sheaves* over a partially ordered set bearing the semantics of Cohen's forcing conditions. In this manner, it turns out that a non-standard Boolean-valued model of set theory can be equivalently represented as a sheaf over the considered "complete Boolean algebra" completion. Consequently, the internal logic of this topos of sheaves can be adequately depicted by means of the above diagram of metaphora between the indicated levels, from the viewpoint of natural communication.

4.3 ALGORITHMIC COMPLEXITY AND THE HALTING PROBABILITY

In the context of *algorithmic* or program-size *complexity* theory, *Gregory Chaitin* came to focus on the implications of Gödel's first incompleteness theorem, which finally lead to a refinement of the former in a computational context. The algorithmic complexity of a string is essentially defined by the length of the shortest program that generates this string and then halts. In this way, a finite string is characterized as random if its complexity is equal approximately to its length. There are strings with arbitrarily large algorithmic complexity and the problem of program-size complexity is undecidable. In this context, Chaitin's incompleteness theorem states that given a consistent arithmetic, there exists a number C depending upon that arithmetic, such that any proposition of the form "the program-size complexity of the string s is greater than C" is not provable. Thus, since there are true instances of such propositions, it follows that there are propositions of the above form which remain undecidable within the context of the given arithmetic.

Chaitin's argument constitutes a refinement of Gödel's first incompleteness theorem because it involves a metaphora extension in depth. First, Chaitin's gnomon is based on counting the number of bits in a program, whence the homeotic criterion is applied for *self-delimiting* programs, defined as strings having the property that one can tell where they end. Second, Chaitin's program-size counting gnomon is modified probabilistically, by a deeper stage logical conjugation at the measure-theoretic level involving the probability $P(x)$ that a program will give a number x at the higher level, while preserving the same homeotic criterion as applied to self-delimiting programs.

This is called the *algorithmic probability* of x, and a summation of probabilities over all possible outputs x yields the halting probability $\Omega = \sum_x P(x)$, where Ω is interpreted as a random infinite sequence of bits. In particular, the halting probability Ω is a random real number. The most intuitive conception of randomness is tied to the notion of absence of predictability. In other words, if one knows the first n-bits of a random sequence it is not possible to predict the next $n+1$-bit. Here, the central objects of our attention are elements of the continuum $\{0,1\}^N := 2^N$.

Elements of 2^N may be viewed either as infinite sequences of bits (infinite strings) or as sets of natural numbers, which can be identified with their characteristic functions. We denote the set of finite binary strings as $2^{[N]}$. The set $2^{[N]}$ can be canonically identified with N, so that subsets of N may be thought of as sets of strings. We also denote the length of a finite string σ by $|\sigma|$. Using finite binary strings, we may define a topology on 2^N as follows: First, we define the extension of a finite string σ by the *clopen* set $E(\sigma) = \{x \in 2^N : \sigma = [x]_{|\sigma|}\}$, where $[x]$ denotes the operation of restriction. Second, we consider clopen sets of the form $E(\sigma)$, where σ is a finite binary string, as the base of a topology on 2^N, where each $E(\sigma)$ is a basic clopen set, to be thought of as an interval in the continuum. In particular, we may identify 2^N with the interval of real numbers $[0,1]$ by associating each real number with its usual binary representation. If we regard μ as the *Lebesgue measure* on $[0,1]$, then we have that $\mu(E(\sigma)) = 2^{-|\sigma|}$. Now, we expect that non-random sequences form a set of measure zero. Intuitively, using the above defined topology, we require that the extensions of longer and longer initial segments σ of a string $x \in 2^N$ become arbitrarily small. In this manner, *random* sequences are defined from a complementary viewpoint measure-theoretically on the basis of the fact that non-random sequences should form sets of measure zero.

Next, if we recall the intuitive conception of randomness as related with the absence of predictability, we may require that there is no algorithm α which can ever compute, and thus uniformly measure, $[x]_{|\sigma|}$ from any sorter string. Here an algorithm is considered as a function $\alpha : 2^{[N]} \to \{0,1\}$.

The above idea constitutes, in effect, a complexity measure based on program-size. The notion of program-size complexity introduced by Chaitin to this effect, regards σ as a self-delimiting program, i.e. as a program delimited by an end-marker. Clearly, no extension $\acute{\sigma}$ of a self-delimiting program can be a self-delimiting program, since the end-marker will not be in the right place. If $\psi : 2^{[N]} \to 2^{[N]}$ is a partial recursive function with prefix-free domain, which means computable by a self-delimiting reference *universal Alan Turing machine*, the Chaitin complexity of σ, or the algorithmic information content of σ is defined by $I(\sigma) = min\{|\tau| : \psi(\tau) = \sigma\}$. This is the length of the shortest program τ of the self-delimiting universal Turing machine that outputs σ. Then, we define an infinite sequence $x \in 2^N$ to be random if all its extensions have high Chaitin complexity, capturing in this way the above intuitive conception of randomness.

More precisely, an infinite sequence x is random if and only if there exists a constant k, such that $(\forall n)[I([x]_n) \geq (n-k)]$. The infinite sequences that satisfy this condition form a set of measure one, and thus random sequences form a set of measure one. This result is in good compatibility with the measure theoretic characterization of non-random sequences as sets of measure zero derived in the previous paragraph. In this sense, the characterization of random sequences according to Chaitin or program-size complexity is in agreement with the measure-theoretic characterization completing the logical conjugation.

Chaitin's incompleteness theorem constitutes not only a refinement of Gödel's first incompleteness theorem due to the deeper stage logical conjugation at the measure-theoretic level, or equivalently via the program-size complexity level, but it also contains the seeds of two powerful generalizations: The first comes from an even deeper level conjugation via the level of generic sets and Cohen's forcing conditions, based on an effective analogical relation between the notions of *random* sets and *generic* sets. The second comes from an interpretation of the constant involved in the definition of random sequences in terms of an uncertainty relation between two logically conjugate domains in the spirit of *Werner Heisenberg's uncertainty principle* in quantum mechanics.

4.4 BOOLEAN-VALUED SETS: FROM RANDOMNESS TO GENERICITY

Regarding the first of the previously stated issues, the deeper stage metaphora takes place through the hypostatic level of Boolean-valued sets. The concomitant logical conjugation utilizes the effective analogical relation between random sets and generic sets. Both of them can be formulated as Boolean-valued models of set theory, or equivalently as variable sets, called sheaves, over a Boolean algebra. In the first case, the Boolean algebra is identified with the Borel algebra of clopen sets, which comprehends both closed and open sets, defined above, modulo the sets of measure zero (non-random sequences). In the second case, it is identified with the Boolean algebra of Cohen forcing conditions. In this manner, the proposed deeper level logical conjugation is based on the consideration of random sequences as Cohen forcing conditions with respect to a Boolean measure algebra, in the context of a Boolean-valued model of set theory containing a consistent arithmetic. Intuitively stated, the sets in this Boolean-valued model, or equivalently the sheaves over the Boolean measure algebra, are to be thought of as sets, whose elements are not evaluated to the two-valued Boolean algebra 2, but are evaluated on the clopen sets of the Boolean measure algebra.

The basic idea can be put as follows: Let us think that we start from a standard model of set theory, which we agree to call constant sets. The elements of constant sets are characterized by valuations in the two-valued Boolean algebra 2. Then we adjoin a multiplicative encoding bridge from the level of constant sets to the level of variable sets, which in this case are the sets varying over a Boolean algebra. From *Marshall Harvey Stone's representation theorem* for Boolean algebras, the *spectral* representation of a Boolean algebra is a totally *disconnected* and *compact* Hausdorff space, called the Stone space. Then, we are able to think of the pertinent variation in terms of measurable functions over this space.

If we arrest the variation at a point of this space, i.e. at a principal ultrafilter of the associated Boolean algebra, then we force a homeotic criterion of identity, by the stipulation that two functions are equivalent if their measurable values agree at this point. Thus, after having identified the partition spectrum, representing the equivalence classes induced by this criterion, we can ascend back to the level of constant sets. In other words, the quotient set obtained is a standard set at the initial level. If we arrest the variation at an ideal point of the Stone space instead, i.e. at a non-principal ultrafilter of the Boolean algebra, then a new possibility arises. More concretely, if we apply the same homeotic criterion for ideal points, we obtain a new quotient set at the level of constant sets, which is an extension of the constant set we started with, called a *Boolean ultrapower* of this set. The Boolean ultrapower is a new constant set, which is internally indistinguishable from the initial set we started with.

In this light, Cohen's forcing employing the gnomon of generic filters is a refinement of the method of evaluation at ideal points aiming toward the construction of new constant sets internally distinguishable from the set we started with. Instead of ideal points, one considers a partially ordered set P of forcing conditions. Arresting the variation with respect to these forcing conditions, one obtains a generic distinguishable extension of the initial set at the level of constant sets, such that a proposition is true in the generic extension if and only if it is forced by some generic forcing condition in P. Note that the generic set of forcing conditions is not contained in the initial constant set, and thus Cohen's forcing requires logical conjugation through the deeper level of variable sets. Moreover, Cohen's method of forcing via some generic set is equivalent to forcing with respect to a Boolean algebra, which in the present case is identified with a Boolean measure algebra. This is why, the notion of random sets involved in applying Chaitin's gnomon may be interpreted by logical conjugation via the notion of generic sets underpinning Cohen's gnomon.

4.5 QUANTUM UNCERTAINTY AND COMPLEMENTARITY

In quantum mechanics, Heisenberg's uncertainty relation involves a limit or bound, which is defined in terms of *Max Planck's constant*, pertaining to the simultaneous determination of two *conjugate observables*, for example, the position and momentum of a quantum system.

We note that observables in quantum mechanics are defined as self-adjoint operators, bearing thus a spectral resolution in terms of *projection operators*. In this way, each observable is associated with a complete Boolean algebra of projection operators obtained by its spectral decomposition. If two observables commute, then they can be resolved by means of a common Boolean algebra of projectors. In other words, a *commutative* algebra of observables is logically characterized by means of the Boolean algebra of projectors (idempotent elements of the commutative algebra), which simultaneously resolve all the observables belonging in this algebra. The non-commutativity of observables like the position and the momentum of a quantum system, quantified by means of Heisenberg's uncertainty principle, signifies the fact that there does *not* exist a universal Boolean algebras of projectors resolving all the observables in quantum mechanics.

Thus, the internal logic of a quantum system is not a Boolean logic of projection operators, but a globally non-Boolean *amalgam* of local Boolean patches, where each patch covers the extent of a maximal commutative algebra of simultaneously measurable observables. Non-commutative observables like position and momentum belong to two different Boolean patches, which cannot be amalgamated together simultaneously under a bigger Boolean patch. Notwithstanding this fact, a position observable can be transformed to a momentum observable by means of a unitary transformation and conversely; these are the well-known *Joseph Fourier's transform* and its inverse. Hence, the position and momentum Boolean patches constitute two conjugate logical domains, which *cannot* be subsumed under a universal Boolean domain, and thus are *complementary* in the standard terminology.

These conjugate Boolean domains correspond to conjugate Boolean projection-valued measure algebras. Note that each Boolean algebra of projectors gives rise, using Cohen's gnomon in this context, meaning logical conjugation through the level of variable sets as above, to a generic set of forcing conditions. Then, a proposition is true in the generic extension, obtained as we have seen, if and only if it is forced by some generic forcing condition. This is suited to understanding the *measurement process* of an observable in quantum mechanics, where a proposition refers to the result of a measurement on this observable and the generic forcing condition corresponds to the projection operator of a measurement device which clicks upon registration of this result.

The difference in comparison to the previous case, appearing for the first time in quantum mechanics, is that distinct local generic sets of forcing conditions corresponding to conjugate observables exist, which cannot be subsumed under a universal global generic set. Hence, in a well-defined sense, which can be made precise using the theory of sheaves, the logical treatment of quantum mechanics requires a *localization* of Cohen's gnomon of forcing, with respect to local Boolean domains, thereby giving rise to generalized *local models* of set theory called *topoi*. In turn, this logical localization with respect to conjugate Boolean valued sets gives rise to the phenomena of *contextuality* in quantum theory. We interpret Heisenberg's uncertainty principle as setting the bound (in terms of Planck's constant) of the simultaneous determination of two conjugate observables with respect to the same Boolean domain of measurement. This is expressed in terms of the standard deviations in the expectation values of conjugate observables in the form $\delta x \cdot \delta p \geq \hbar / 2$, where $\hbar := h / 2\pi$ in the case of position and momentum observables. Each of these observables is considered as a Boolean homomorphism from the *Emile Borel measure algebra* of the real line (where the results of measurements are recorded) to the corresponding Boolean patch containing the respective projections in the spectral resolution of these observables.

4.6 UNCERTAINTY IN A SELF-DELIMITING UNIVERSAL TURING MACHINE

Let us now examine if Chaitin's gnomon can be presented in a form giving rise to an uncertainty relation between two conjugate Boolean domains. The first Boolean domain we consider is the domain of random real numbers in the continuum $[0,1]$. Bearing in mind that we identify 2^N with the interval of real numbers $[0,1]$ by associating each real number with its binary representation. Moreover, if we regard μ as the Lebesgue measure on $[0,1]$, we have that $\mu(E(\gamma)) = 2^{-|\gamma|}$, where γ is a finite binary string, to be thought of as a program of a self-delimiting universal Turing machine ψ. For an output χ of this machine, we have immediately that the probability of χ is given by:

$$P(\chi) := \mu(\chi) = \sum_{\gamma:\psi(\gamma)=\chi} 2^{-|\gamma|}$$

Chaitin's $\Omega = \sum_{\chi} P(\chi)$ is a random infinite sequence of bits, and thus a *random real* in $[0,1]$ of Lebesgue measure one. It is interpreted as the *halting probability* of ψ, defined as the probability that ψ halts when its binary input is chosen randomly bit by bit, such as by flipping a coin. In practice, we may only compute finitely many digits of Ω.

The second Boolean domain we consider is the domain of program-size complexity. If $\psi : 2^{[N]} \to 2^{[N]}$ is a partial recursive function with prefix-free domain, that is, computable by a self-delimiting universal Turing machine, the Chaitin or program-size complexity of χ, or even the algorithmic *information content* of χ is defined by:

$$I(\chi) = min\{|\gamma| : \psi(\gamma) = \chi\}$$

The complexity measure $I(\chi)$ is the length of the shortest program γ of the self-delimiting universal Turing machine that outputs χ. Moreover, an infinite sequence x is random if and only if there exists a constant k, such that:

$$(\forall n)[I([x]_n) \geq (n-k)]$$

The infinite random sequences that satisfy this condition form a set of measure one, and thus for Chaitin's Ω we obtain:

$$(\forall n)[I([\Omega]_n) \geq (n-k)]$$

The above inequality is interpreted clearly as an uncertainty relation pertaining to the conjugate Boolean domains of random real numbers in $[0,1]$ and program-size complexity length measures. Since it is an *uncertainty* relation between two *conjugate* Boolean domains, these domains cannot be embedded in a universal Boolean domain simultaneously subsuming both of them. Thus, the constant k is interpreted as setting the *bound* of the simultaneous determination of two conjugate observables, viz. the random real Ω in $[0,1]$ and the program-size complexity length measure I.

4.7 LOGICAL CONJUGATION CYCLES

We have shown previously that both Heisenberg's and Chaitin's logical conjugation methods give rise to uncertainty relations between two conjugate or complementary Boolean domains which cannot be subsumed under a common universal Boolean domain simultaneously with absolute precision. Moreover, if we consider each Boolean domain separately we may interpret it as a Boolean algebra of generic forcing conditions, descend to the level of Boolean valued sets, then apply a Cohen-type criterion of homeotic identity with respect to these forcing conditions, and finally ascend back to the initial level of constant sets, obtaining in this manner a generalized model internally distinguishable from the one we started with. The latter reflects the intervention of a suitable measurement procedure for obtaining information with respect to an observable logically classified by this Boolean domain. The logical classification takes place via the procedure of spectral resolution in terms of a Boolean algebra of projectors in the context of operator functional analysis, or more generally according to, the procedure of measurability in terms of a Borel measure algebra, which can even be projection-valued. The important point to be emphasized is that the Cohen-type strategy of logical conjugation cannot be implemented simultaneously with respect to two complementary Boolean domains.

A natural question arising in this context is if it possible to implement the strategy of logical conjugation in such a way that circumvents the above obstacle. We may think of each logical Boolean domain as giving rise to a separate gnomon of conjugation. If we consider two complementary Boolean domains, we cannot apply the method of logical conjugation with respect to both of them simultaneously, but the possibility remains of *composing* these two gnomons in an appropriate way. Since we consider these two gnomons as complementary in a precise sense, justified by the existence of an uncertainty relation as above, then the most economical hypothesis is to assume that each gnomon may conjugate the complementary one. In other words, the hypostatic levels between which each gnomon operates should function as the encoding/decoding bridges of the complementary gnomon.

We may explicate this idea in more detail as follows: We recall that the method of logical conjugation expressing a metaphora requires, first, a certain stratification into different levels and, second, the delineation of encoding/decoding bridges between these levels in order to be able to descend and re-ascent. Each Boolean domain of discourse provides a natural stratification as well as a natural descending/ascending bridging between the strata, which can be conceptualized in accordance with Cohen's gnomon. But, what if there is *no* intrinsic way of *distinguishing* between *strata* and *bridges*? Reciprocally put, the distinction between strata and bridges is meaningful only under the specification of a Boolean domain. If two complementary gnomons pertaining to two complementary Boolean domains are utilized simultaneously the only way that logical conjugation can function is by reversing the role of strata and bridges with respect to these two gnomons, such that a closure is achieved. Algebraically, the only way that these two complementary gnomons may be amalgamated together simultaneously is by temporarily *suspending* the rigid distinctions between strata and bridges, and just *iterating* the process of logical conjugation with respect to the composition of these two gnomons until we reach a *closure*. The closure corresponds to a non-trivial *logical cycle* of compositions. It turns out that the formation of this cycle is equivalent to composite logical conjugation where the levels of one gnomon correspond to the bridges of its complementary gnomon. We present this simple algebraic argument as follows:

A logical conjugation is generally expressed in the symbolic form

$$X = S \circ A \circ S^{-1}$$

which defines X to be conjugate to A under S, where S^{-1} is considered to be the conceptual inverse of S. Now we consider the first two symbols of the conjugation $S \circ A \circ S^{-1}$, that is, $S \circ A$, as a string, and extend this string by adding new symbols at the end, such that every three consecutive symbols pertain to a logical conjugation, or equivalently, establish a metaphora. We iterate this operational procedure until we generate a cycle, which means until the last two symbols are $S \circ A$ again that we started with. In more detail we obtain successively:

$$S \circ A \dashrightarrow S \circ A \circ S^{-1} \dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \dashrightarrow$$
$$\dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \circ S \dashrightarrow S \circ A \circ S^{-1} \circ A^{-1} \circ S \circ A$$

Since the iteration has produced the string $S \circ A \circ S^{-1} \circ A^{-1} \circ S \circ A$, where the last two symbols are $S \circ A$ again, our initial terms, we have generated a closure, viz. a non trivial *conjugation cycle* that in linear sequential unfolding reads as follows:

$$S \circ A \circ S^{-1} \circ A^{-1} := [S, A] := \circlearrowleft (S, A) := S \circlearrowleft A$$

By a slight abuse of notation we may identify the complementary gnomons by the symbols S, A correspondingly, whence their composition or gluing is denoted by the conjugation cycle $S \circlearrowleft A$. Note that the order of composition cannot be reverted, viz. $S \circlearrowleft A \neq A \circlearrowleft S$, hence the operation of composition of complementary gnomons is non-commutative. Thus, it is significant to impose an orientation on the conjugation cycle, which reflects the specified cyclic order of composition.

In the case that S, A are elements of a non-commutative group, the composition $[S, A]$ is referred as the commutator of S, A. In this case the symbols S^{-1} and A^{-1} stand for the group-theoretic inverses of S, A respectively. This observation leads to the conjecture that the complementarity of conjugate Boolean domains pertains to their Boole group theoretic structures, or else it is of a group-theoretic origin. A Boole group is a group-structure on the topological spectrum of a Boolean algebraic domain. Thinking of two complementary Boolean group domains as local patches of a non-abelian global structure the notion of a conjugation cycle provides a natural method of logically gluing them together simultaneously.

4.8 SOLVABILITY VIA NILPOTENCY: CIRCUMVENTING NON-COMMUTATIVITY

Before we examine the aspects of this logical gluing by conjugation cycles of complementary gnomons it is instructive to start from a reciprocal viewpoint and leverage the existing knowledge about the structure of groups. This will provide the method to locate the existence of complementary gnomons from a group-theoretic perspective. The central notion of significance for our problem has to do with the *Galoisian* notion of *solvability* of a group. In particular, the understanding of *Évariste Galois'* theory of groups by the strategy of logical conjugation uses the gnomon of solvability. This will be discussed in more detail as we go on, but for the time being it is enough to convey the basic idea.

The triumph of Galois theory is based on the theorem that a polynomial equation is solvable by radicals if and only if the corresponding Galois group of the equation is solvable. Now a general group is solvable if it can be derived by the method of group *extensions* of *Abelian* (commutative) groups. Reciprocally, a solvable group is a group whose *derived series* terminates in the trivial subgroup. Intuitively, the derived series is a stratification into group levels together with a descending staircase among these strata formed by identifying each subgroup in the descending series with the *commutator* subgroup of the previous one. In turn, the commutator subgroup of a group is the group generated by all the commutators of this group. The importance of the commutator subgroup of a group rests on the fact that it provides the most economical way, i.e. it is the smallest normal subgroup, such that the quotient of the initial group by the commutator subgroup is an abelian group. Thus, a group is solvable if by descending into lower and lower subgroup strata by division with the commutator subgroup we end up with the trivial subgroup that completely annihilates the complexity of the group we started with.

It is well-known that all Abelian groups are solvable, as well as that all *nilpotent* groups are solvable. The first is trivial, but the second is very important, for example, in quantum mechanics. It is worth explaining the latter in more detail. A nilpotent group is one that may be thought of as an almost-Abelian group, in the sense that the commutator subgroup is almost trivial. For instance, we know that in quantum mechanics we have complementary Boolean algebraic domains, like those pertaining to position and momentum. The bounded form of these conjugate observables, called the *Hermann Weyl form*, are constrained to obey the canonical commutation relations expressed by means of the infinitesimal Planck's constant, and hence almost commute. These give rise to a nilpotent group, called the *Heisenberg group*. The Heisenberg group is of fundamental importance in quantum mechanics and essentially constitutes the solvability of the theory in group-theoretic terms. In other words, the non-commutativity induced by any two conjugate or complementary Boolean domains in quantum mechanics is *circumvented* in an almost-commutative manner by the nilpotency of the Heisenberg group, and its attendant solvability. This circumvention is technically possible in all cases where we have at our disposal the structure of a *vector space* equipped with a *symplectic form*. In other words, the structure of a nilpotent group, induced symplectically, transforms the intrinsic insolvability of two conjugate domains into a solvable case. From the perspective of logical conjugation this amounts to considering conjugation cycles as *infinitesimally* small, and thus, behaving like covariant *differential* operators in a precise differential geometric sense.

The above analysis requires the investigation of the source of intrinsic insolvability in groups. It is enough to consider the case of finitely generated linear groups, i.e. matrix groups, which are used as a concrete representation of abstract groups. In this case, according to a well-known theorem of *Jacques Tits*, called the Tits alternative, a finitely generated linear group is either virtually solvable, meaning that it contains a solvable subgroup involving a finite descending staircase, or it contains a non-Abelian (non-commutative) *free* subgroup in *two* generators. Thus, we are able to locate the free group in two non-commuting generators, denoted by Θ_2, as the actual *source* of intrinsic insolvability. From the view of logical conjugation, Θ_2 should be associated with non-trivial and non-reducible logical conjugation cycles between two complementary Boolean domains. The only way that non-solvability can be leveraged or circumvented is through nilpotency, as in the case of the Heisenberg group in quantum mechanics. We bear in mind that uncertainty relations will always pertain between the observables of two complementary Boolean domains. If the associated constant of interrelation can be made either infinitesimally small or reciprocally very big, then the resultant logical conjugation cycles vanish in higher order iterations and the complexity is reducible. It is not an accident that both of our fundamental physical theories, to wit the theory of relativity and quantum mechanics involve this type of constants between conjugate Boolean domains. Thus, from the perspective of logical conjugation, the free group in two generators is the *source* of logical conjugation cycles and the group-theoretic property of nilpotency is the "golden mean" between non-commutativity and commutativity.

4.9 CANONICS FROM THE LOGICAL TO THE TEMPORAL DOMAIN

Therefore, it is of high priority to focus our attention on the fundamental significance of the non-Abelian free group Θ_2. The surprising and counterintuitive result is that the non-Abelian free group in two generators contains *copies* of all other non-Abelian free groups in any finite number of generators as *finite* index subgroups! Thus, the complexity of non-reducible logical conjugation cycles and their iterations generated by two complementary (in some appropriate sense) gnomons subsumes the *whole* complexity we may get from *any* number of obstacles! A way to qualify this proposition, whose reference is algebraic and derives from logical considerations, is to consider the representation of Θ_2 in three dimensions.

From this representation, we obtain a valuable and novel connection between logic and time, thereby providing the canonics from the logical domain to the temporal domain. More specifically, the notion of a logical conjugation cycle is mirrored on the notion of a tripodal link that expresses the quality of a prime temporal bond between two temporal diastases. In the same way that a logical conjugation cycle amalgamates two complementary Boolean domains simultaneously, a temporal bond amalgamates two unlinked temporal diastases in the present. Moreover, and most importantly, the nilpotency condition of solvability transferred to the temporal domain through bonds, provides the origin of geometric differential calculus under the algebraic commutation rule of two infinitesimal flows at a *fulcrum* point, being *bounded* at this point. We will examine this path later on in detail, together with its ramifications, starting from *Hermann Grassmann's* theory of extension and its relation with *Gottfried Wilhelm Leibnitz's* infinitesimal analysis, and culminating with *sheaf cohomology*.

The canonics from the logical to the temporal spectral domain, through the group Θ_2, is particularly important in relation to algorithmic information theory, and more generally, the concept of programs and computability. There are two reasons on which we base our claim. The first is the fact that elements of Θ_2 can be assigned complexity lengths. Since every element of Θ_2 can be uniquely expressed as a freely reduced word in the generators and their inverses, we may simply define the length of an element as the number of terms in this freely reduced expression. This notion of length has the property that the length of an element equals that of its inverse element in this group. The second is related to the fact that, although the group Θ_2 has *exponential* growth rate, a deep theorem of *Mikhail Gromov* shows that a nilpotency circumvention, in agreement with the preceding, reduces the growth rate to a *polynomial* one, and thus proves economical for computational purposes.

4.10 QUBIT COMPUTABILITY: SELF-DELIMITING PROGRAMS AS CYCLES ON THE SPHERE

The initial motivation of this investigation is based on the profound idea conceived by Chaitin, according to which, a key technical point that must be stipulated in order for Ω to make sense is that an input program must be self-delimiting. Its total length in bits must be given *within* the program itself. Chaitin points out essentially that this seemingly minor point, which paralyzed progress in the field for nearly a decade, is what entailed the redefinition of algorithmic randomness. Real programming languages are self-delimiting, because they provide constructs for beginning and ending a program. Such constructs allow a program to contain well-defined subprograms, which may also have other subprograms nested in them. Because a self-delimiting program is built up by concatenating and nesting self-delimiting subprograms, a program is syntactically complete only when the last open subprogram is closed. In essence the beginning and ending constructs for programs and subprograms function respectively like left and right *parentheses* in mathematical expressions.

If programs were not self-delimiting, they could not be constructed from subprograms, and summing the halting probabilities for all programs would yield an infinite number. If one considers only self-delimiting programs, not only is Ω limited to the range between 0 to 1 but also it can be explicitly calculated in the limit from below.

Our main interest in this section focusses on the metaphora considering the beginning and ending constructs of self-delimiting programs and subprograms in analogy to the left and right parentheses in mathematical expressions. It is true that our linear representation of strings or words implicates the self-delimiting property by means of left and right parentheses. A natural generalization would be to *complete* each such pair of parentheses in the 1-dim line to a *circle* in the 2-dim plane, or equivalently the 1-dim *complex line*. This extremely simple generalization generates two conjugate domains immediately, where each one of them corresponds to the choice of orientation on the circle. If we do not impose any orientation on a circle, it is as though we are working in the *modular* arithmetic $\mathbb{Z}_2$, that is, we recover the bit representation of linear strings. Even better, we may complete each pair of parentheses in the 1-dim line to a circle in the one-point *compactification* of the 1-dim complex line, i.e. on the 1-dim *complex projective* space, or equivalently the *Bernhard Riemann sphere* S^2. Can we imagine representing self-delimiting programs by means of *circular strings* on the sphere S^2?

The choice of the sphere S^2 is not accidental. Without loss of generality we may consider the unit sphere S^2, that is imply the normalization according to which all points lying on the sphere are of unit distance from the origin. The unit 2-sphere S^2 constitutes the space of pure states, or equivalently *rays*, of a 2-level quantum mechanical system, called currently a *qubit*. The unit 2-sphere may be thought of as embedded in the usual 3-dimensional space $\mathbb{R}^3$. The *David Hilbert space* of normalized unit state vectors of a qubit is the 3-sphere S^3, and thus the unit 2-sphere is considered as the base space of the topological *Heinz Hopf fibration*:

$$S^1 \rightarrowtail S^3 \twoheadrightarrow S^2$$

We note that each pair of *antipodal* points of S^2 corresponds to mutually orthogonal state vectors. The north and south poles are chosen to correspond to the standard orthonormal basis vectors $|0\rangle$ and $|1\rangle$ correspondingly. In the case of a *spin*-$\frac{1}{2}$ system, these simply correspond to the spin-up and spin-down states of this system.

We consider the unit sphere S^2 as the set of points of 3-dimensional space $\mathbb{R}^3$ that lie at distance 1 from the origin. Then, the non-commutative group $SO(3)$ denotes the group of *rotation operators* on $\mathbb{R}^3$ with center at the origin, viz. linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ represented as 3×3 matrices with determinant one. These are called *orthogonal* matrices, characterized by the fact that their columns form an orthonormal basis of $\mathbb{R}^3$. Rotations around an axis going through the origin are the isometries of 3-dimensional Euclidean space $\mathbb{R}^3$ leaving the origin fixed. Note that a 3×3 orthogonal transformation preserves the inner product for any pair of vectors in $\mathbb{R}^3$, and moreover it is an *isometry* of $\mathbb{R}^3$ that takes the unit sphere S^2 to itself.

In this context, we ask the following question: Does there exist a *representation* of the non-Abelian free group in two generators Θ_2 on the unit sphere S^2, which lifts to a unitary representation on S^3? We recall that the existence of such a representation would imply the *action* of non-trivial logical conjugation cycles on S^2 and S^3 respectively. Moreover, these logical conjugation cycles would be representable by means of the Tripodal link topology. Such a representation definitely exists if we are able to locate a subgroup of the non-commutative group $SO(3)$, which is *isomorphic* to Θ_2.

We will show further on that this is indeed the case. The proof is based on the observation that there actually exist rotation operators A and B about two *independent* axes through the origin in $\mathbb{R}^3$ generating a non-commutative subgroup of $SO(3)$, which is isomorphic to the free group Θ_2. In other words, there exists an isomorphic copy of Θ_2 in $SO(3)$ generated by two independent rotations A and B. The term independent refers to the requirement that all rotations performed by sequences of A and B and their inverses are distinct strings in Θ_2.

Actually, we realize that *most* pairs of rotations in $SO(3)$ are independent in the above sense, so that even picking A and B randomly would do. For instance, one could consider two counterclockwise rotations A and B about the z-axis and the x-axis respectively of the same angle $arccos(3/5)$. The proof is based on showing that no reduced string in the symbols A and B and their inverses collapses to the identity transformation (3×3 identity matrix). Intuitively, if we choose two counterclockwise rotations A and B about the z-axis and the x-axis of the same angle, then this specific angle needs to be an irrational number of degrees. More precisely, given an initial orientation, if the specified angle *is* an irrational number of degrees, then none of the distinct strings of rotations in Θ_2 performed by sequences of A and B and their inverses can give back the initial orientation. Thus, no reduced word in Θ_2 collapses to the identity transformation.

The existence of an isomorphic copy of Θ_2 in $SO(3)$ has the following consequence: Each rotation belonging to the non-commutative free subgroup Θ_2 of $SO(3)$ fixes two points in the unit sphere S^2, namely the intersection of S^2 with the axis of rotation passing through the origin. If we take the union of all these points, they form a countable set of points. This reveals not only that an action of Θ_2 on the unit sphere S^2 (as a subgroup of $SO(3)$ generated by A and B) must exist, but that this action is actually free on S^2 *modulo* the countable set of fixed points K.

Thus, we can partition $S^2 \setminus K$ into a disjoint union of orbits for the action of Θ_2. If we choose a base point for an orbit we may identify this orbit with Θ_2 due to the freeness of the action. Moreover, if a countable collection K of points as above is removed from S^2 they can be restored by rotations around an axis through the origin which has zero overlap with K. In this way, the action of the group Θ_2 via strings of rotation operators allows us to *resolve* the whole unit sphere S^2. The crucial point again is that the algebraic irreducibility of the *commutator* $[A,B]$ of the rotations A and B generating an isomorphic copy of Θ_2 in $SO(3)$ expresses a non-trivial logical conjugation cycle. In turn, such a logical conjugation cycles express the fundamental property of topological Tripodal non-splittability, or non-separability, of these three rotations belonging to the subgroup of $SO(3)$ that is isomorphic with Θ_2.

Most important, this interpretation provides a topological justification of the fact that one cannot specify a finitely additive rotation-invariant probability measure on all subsets of the unit sphere S^2 simultaneously. In the same vein of ideas, if we consider S^2 embedded in 3-dim space $\mathbb{R}^3$, we deduce that it is not possible to specify a finitely additive measure on $\mathbb{R}^3$ that is both translation and rotation invariant, which can measure every subset of $\mathbb{R}^3$, and which gives the unit ball a non-zero measure. This explains why the Lebesgue measure, which is countably additive and both translation and rotation invariant, and gives the unit ball a non-zero measure, cannot measure every subset of $\mathbb{R}^3$. Thus, it has to be carefully restricted to only measuring subsets that can be Lebesgue measurable.

According to the preceding analysis, since the group of rotation operators $SO(3)$ contains an isomorphic copy of the free non-commutative group Θ_2 is unsolvable.

An immediate consequence of the above is that the group of 2×2 *complex unitary* matrices with unit determinant $SU(2)$ is also unsolvable, that is it also contains an isomorphic copy of Θ_2. The reason is that topologically, the simply-connected special unitary group $SU(2)$ is a *covering* space of the non-simply connected group of rotations $SO(3)$, and in particular it is a *double cover*. More concretely, there exists a two-to-one surjective homomorphism of groups:

$$\Delta : SU(2) \twoheadrightarrow SO(3)$$

whose kernel is given by $Ker\Delta = \mathbb{Z}_2 = \{+1,-1\}$.

Hence, it follows that there must be an isomorphic copy of Θ_2 in $SU(2)$. More precisely, if A and B are rotations generating an isomorphic copy of Θ_2 in $SO(3)$, and $\Delta : SU(2) \twoheadrightarrow SO(3)$ is the covering projection, then $\overline{A}$ and $\overline{B}$ generate a free subgroup of the form Θ_2 in $SU(2)$, for any $\overline{A}$ and $\overline{B}$ with $\Delta\overline{A} = A$ and $\Delta\overline{B} = A$. Since $SU(2)$ is a double cover of $SO(3)$ there can only be exactly two elements of the form $\overline{A}$, namely U and $-U$ such that $\Delta U = \Delta(-U) = A$ (the same holds for $\overline{B}$ respectively).

We conclude that there exists a representation of the group Θ_2 on the unit sphere S^2, which *lifts* to a unitary representation on S^3.

The representation of the group Θ_2 on the unit sphere S^2 is given by the free subgroup of rotations of $SO(3)$ generated by A and B according to the above. Concomitantly, this representation lifts to a *unitary* representation on S^3 by the free subgroup of unitary operators of $SU(2)$ generated by $\overline{A}$ and $\overline{B}$.

Thus, the Hilbert space of normalized unit state vectors of a qubit or of a spin-$\frac{1}{2}$ system carries a unitary representation of the group Θ_2. This means that the algebraic irreducibility of the commutator $[\overline{A},\overline{B}]$ of the unitary operators $\overline{A}$ and $\overline{B}$ generating an isomorphic copy of Θ_2 in $SU(2)$ expresses non-trivial conjugation cycles. Moreover, since the action of the group Θ_2 by strings of rotations in two generators allows S^2 to resolve, such that the same lifted action resolves S^3 as well, by strings of corresponding unitary operators, we can make a conclusion. It is the Tripodal link topological connectivity - by means of conjugation cycles – that is transferred through these actions to the space of rays S^2 and the space of unit state vectors S^3 of a qubit. This is the crux of the *non-classical* behavior of a qubit and the problem arising here is whether the existence of non-trivial conjugation cycles can be turned to a *novel* computational possibility.

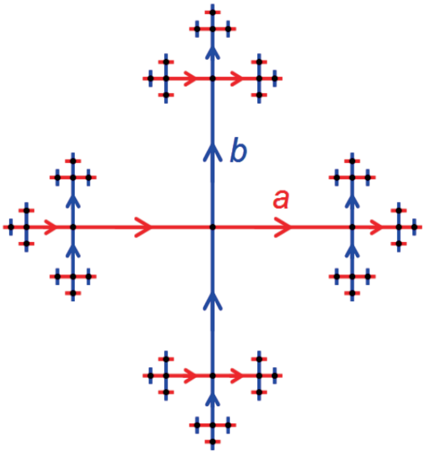
We will outline the first steps towards implementing such a computational paradigm. For this purpose, our guiding principle will be the implementation of Chaitin's uncertainty relation. We recall that the form of Chaitin's uncertainty relation reads:

$$(\forall n)[I([\Omega]_n) \geq (n-k)]$$

where the constant k is interpreted as setting the bound of the simultaneous determination of two conjugate observables, viz. the random real Ω and the program-size complexity length measure I.

We have shown that isomorphic copies of the group Θ_2 exist within the group of rotation operators $SO(3)$ and the group of unitary operators $SU(2)$ leading to the realization of logical conjugation cycles, or equivalently Tripodal loops, on S^2 and S^3 respectively. Since the group $SU(2)$ is a subgroup of the group $GL(2,\mathbb{C})$, the matrix group of 2×2 matrices with complex coefficients, and of the group $SL(2,\mathbb{C})$, viz. the group of 2×2 matrices with complex coefficients and unit determinant, they also contain a copy of the group Θ_2. So we are going to identify two complex matrices acting as the generators of this copy of Θ_2 using Chaitin's uncertainty relation in the present setting. For this purpose, we assume the existence of a positive integer N playing the role of string length measure, such that for all $m \geq N$, the powers G^m and H^m, where G, H are complex matrices, generate a copy of Θ_2. This is possible using the method of *dominant eigenvalues* and *dominant eigenvectors* of matrices.

We observe that for this purpose we have to *diagonalize* these matrices, a technique which is also based on logical conjugation. In particular, we look for two matrices G and H, such that: G has the dominant eigenvalue μ corresponding to a dominant eigenvector u. This means that the eigenspace of $G-\mu I$ is 1-dimensional and all other eigenvalues of G have modulus less that $|\mu|$. Similarly, let H take the dominant eigenvalue ν corresponding to a dominant eigenvector v. Finally, we denote the dominant eigenvalues and corresponding dominant eigenvectors of G^{-1} and H^{-1} by ρ, w, and υ, z, respectively. Next, we consider the dominant eigenvectors as points on the 1-dim complex projective space, viz. equivalently on S^2. Then, the dominant eigenvalues/eigenvectors implement the requirement that there exist disjoint open sets containing the points u, v, w, z, denoted by U, V, W, Z, respectively, such that: There is some $m \geq N$ with the property that, G^m sends each of these open sets to U, and correspondingly for the others, viz. H^m to V, G^{-m} to W and H^{-m} to Z. Now, we think of a *finite state computer*, with four states labelled by U, V, W, Z and an *alphabet* $G^m := a$, $H^m := b$, $G^{-m} := a^{-1}$, $H^{-m} := b^{-1}$ and transitions rules as described above. It is clear that the matrices a and b now *generate* a copy of the free group Θ_2, and thus we obtain logical conjugation cycles for the formation of strings using our alphabet with the prescribed transition rules.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CIRCUMVENTING COMPLEXITY: MOTIF OF INFORMATION AS ANADYOMENE

5

If the history of humankind is viewed from the standpoint of the "obstacle-oriented approach" to complexity, pertaining to localizing and embracing obstructions of any particular type, a naturally arising problematics complicates the border-lines of the most persistent and covariant forms of this stance. The character of temporal persistence and covariance is meant to serve as a criterion for the *Platonic "ousia ontos ousa"* of history, or "being that truly is". Surprisingly enough, this criterion filters out only two kinds of entities, namely myths and theorems. At a first encounter with this provocative claim, there seems to be a contradiction, but this is only apparent, stemming mainly from a certain type of pre-occupation with "target-oriented" methods instead of "obstacle-oriented ones". Indeed, both *myths* and *theorems* become noematic if they are obstacle-embracing, they are just different types of metaphora in relation to communication. Myths and narratives constitute an *embodied* symbolic process of metaphora around a localized obstacle. Mathematical theorems constitute an *abstract* symbolic process of metaphora around a localized obstacle. Although "myth" and "math" deceptively seem to occupy opposite sides of the linear spectrum forced by "analytic reason", they are, in fact, *antipodally* inter-related and topologically may be glued together in the *projective* geometric rooting of this linear spectrum, *centered* around the notion of an obstacle.

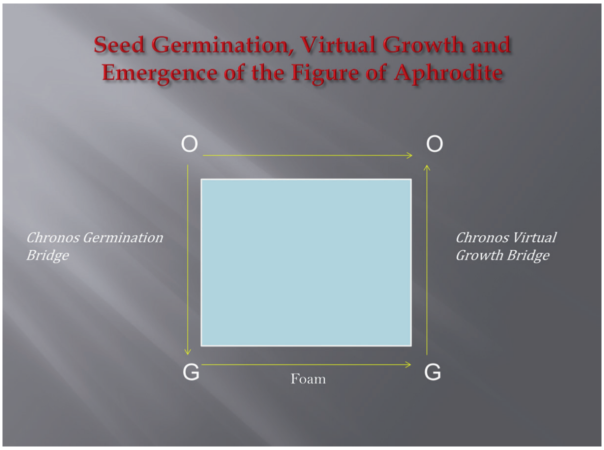
In order to articulate this claim, we will consider the dual pair of concepts consisting of ambiguity and information in relation to communication. On the side of the myth, we will scrutinize the narrative referring to the genesis of Goddess *Aphrodite* according to *Hesiod's Theogony*. On the side of the mathematical logos, we will scrutinize the theorem referring to the genesis of the roots of a polynomial equation by radicals according to *Galois' Theory* and its far-reaching articulations. The remarkable conclusion is that in both cases information emerges as *"anadyomene"* from another level of hypostasis, being in communication with the level where the initial obstruction is localized by means of bidirectional bridges.

Following the narrative of Hesiod's Theogony, the genesis of Goddess Aphrodite took place in an extraordinary manner. She emerged in the form of a fully grown female figure rising out of the sea foam (*aphros*). Her name Aphrodite, or *Aphrogeneia*, according to Hesiod, originates from the fact that she grew and formed amid the sea foam. The epithet *anadyomene* uncovers the metaphora of Aphrodite's genesis. This is a process of circulation between the Heavenly level (*Uranus*), and the Earthly level (*Gaia*), which is meant to embrace the obstacle of heterogeneous, chimerical and intrinsically incongruent constituency of these two levels. The initiator of this obstacle-embracing communicative process is the Titan *Chronos*, who stands for the personification of *Time* as a unitary circular flow reciprocally bridging these two levels into a *germinal syzygy*.

According to the myth, Chronos forced the transposition of the sperm of Uranus into the sea water, and out of this syzygy a white foam spread around the locus of germination in the sea from the immortal flesh. In this foam there grew the maiden Aphrodite virtualized, to emerge as a fully grown in-formed spectral figure by inverse transposition to the level of Gods, thereby completing the temporally circulating metaphorical process of anadyomene initiated by Chronos. It is important to highlight that the two levels of the myth are *not* hierarchically ordered, meaning that there is no relation of *hypotaxis* or subordination of one level to the other.

On the contrary, the two levels exist autonomously in *parataxis* and only the unitary circular flow of Chronos binds them through reciprocal bridges. The bridge from the Heavenly to the Earthly level is the germination bridge, whereas the reciprocal or conceptually inverse one is the virtual growth bridge. The latter is characterized as virtual because the growth of the figure to emerge as Aphrodite is always surrounded by the sea foam, and thus, the growing figure is indistinguishable and inseparable from the foam.

This is crucial for conceiving, both the symmetry implicated by the circular unitary flow-action of Chronos, and the emergence of Aphrodite as a fully grown in-formed figure only after breaking the symmetry of this inter-level communicative flow. Thus she becomes separated from the sea foam and distinguished or discerned spectrally from it at the level of Gods.



5.2 AMBIGUITY: FROM GALOIS GROUPS TO RESOLVING SPLITTING FIELDS

Galois' monumental work focus on the resolution of the problem of algebraic solvability of *polynomial* equations, i.e. equations of the form $p(x)=0$. Of most interest is Galois' paper entitled "Memoir On the Conditions for the Solvability of Equations by Radicals". In this algebraic treatise, he proves that *no general formula* for the *roots* of a fifth, or any higher degree polynomial equation can possibly be found if we employ the usual algebraic operations of addition, subtraction, multiplication, division, and applying radicals, i.e. taking roots of the $n-th$ degree.

The first novelty of Galois' method consists in the introduction of the *group* concept, which he devised as a *structure* G that encodes the *permutations* of the roots of a polynomial $p(x) \in \mathbb{F}[x]$, which map each root onto a conjugate of it, where $\mathbb{F}$ is a field, called the *ground* field of coefficients.

The second novelty of Galois' method consists in employing the structure of the above group G together and in relation to the structure of its invariant subgroups under conjugation, in other words, its self-conjugate or *normal* subgroups. The underlying reason is that the notion of structural divisibility of a Galois group becomes possible only with respect to its normal subgroups.

The third novelty of Galois' method consists in the notion of *group solvability*. The triumph of Galois theory is based on the theorem that a polynomial equation is solvable by radicals if and only if the corresponding Galois group of the equation is solvable. A solvable group is a group whose derived series terminates in the trivial subgroup. Intuitively, the derived series is a stratification into group levels together with a descending staircase among these strata formed by identifying each subgroup in the descending series with a normal subgroup of the previous one. Thus, a group is solvable if by descending into lower and lower subgroup strata by division with normal subgroups we end up with the trivial subgroup.

In this state of affairs, the Galois group G corresponding to a polynomial $p(x) \in \mathbb{F}[x]$, where $\mathbb{F}$ is the ground field, can be though of as a non-numerical, but *structural measure* of complexity of this polynomial. In turn, the repeated process of division involved in setting up the derived series of G, by means of normal subgroups, can be thought of as a process of complexity *reduction* until the trivial subgroup of G is eventually reached. It is important in this frame of thinking to clarify what we mean by a structural measure of complexity as this is encoded in the Galois group of a polynomial $p(x) \in \mathbb{F}[x]$. The general notion of a group is associated with a criterion of *symmetry* under its action. What is the implicated notion of symmetry in this case, and how can it be associated with a measure of complexity?

The answer lies in the observation that the Galois group G, being the group of permutations of the roots of a polynomial $p(x) \in \mathbb{F}[x]$, which maps each root onto a conjugate of it, where $\mathbb{F}$ is the ground field, actually encodes these $\mathbb{F}$-transformations of the roots of $p(x)$ that *cannot* be distinguished from the *resolution capacity* afforded by the ground field $\mathbb{F}$. In other words, the Galois group G of $p(x)$ is an *ambiguity group* with respect to $\mathbb{F}$, and this ambiguity structure amounts to a measure of complexity of $p(x)$ in relation to the resolving means afforded by $\mathbb{F}$. Henceforth, the symmetry encapsulated by the Galois group G of $p(x)$ in relation to $\mathbb{F}$ can be reduced only by possible successive symmetry reductions, by the process of divisibility with normal subgroups in a descending fashion, until the total symmetry constituting the structural measure of $\mathbb{F}$-complexity is completely reduced.

Reciprocally, the same effect can be achieved by successive extensions of the ground field $\mathbb{F}$ to larger fields that increase the resolution capacity afforded by the ground field itself, until eventually all roots can become distinguishable. More precisely, given a polynomial $p(x) \in \mathbb{F}[x]$ of degree n, it can be shown that a minimal extension of $\mathbb{F}$, exists, called the *splitting field* $\mathbb{F}^p$ of $p(x)$, in which all n roots of $p(x)$ can be distinguished (counted with multiplicity). The splitting field $\mathbb{F}^p$ of $p(x)$ constitutes in this way the smallest field extension of $\mathbb{F}$ over which the polynomial $p(x)$ splits or decomposes into linear factors.

For instance, let us consider the classical case of the polynomial $p(x)=x^2+1$, i.e. $p(x)\in\mathbb{R}[x]$, where $\mathbb{R}$ is the ground field of coefficients of $p(x)$. We know that this polynomial does not have any roots in the field of real numbers $\mathbb{R}$. In particular, the two roots of the polynomial equation $p(x)=0$, namely i and $-i$, do not belong into the ground field of coefficients of $p(x)$, but can be located in the minimal field extension of $\mathbb{R}$ into $\mathbb{R}\bigoplus i\mathbb{R}\cong\mathbb{C}$. In any case, we also know that the field $\mathbb{C}$ is algebraically closed. In other words, if $\mathbb{D}$ is an algebraic extension of $\mathbb{C}$, then $\mathbb{D}=\mathbb{C}$. Conceptually, the roots i and $-i$ are not distinguishable from the resolution capacity afforded by the ground field $\mathbb{R}$, so that the minimal field extension from the ground field $\mathbb{R}$ to the *algebraically closed* field $\mathbb{C}$, i.e. $\mathbb{R}\to\mathbb{C}$, is required to make these two roots distinguishable.

Let us explore now, how the Galois group of the minimal field extension $\mathbb{R}\to\mathbb{C}$ emerges. There are two equivalent ways to describe it. The first one is the original method devised by Galois, to describe the Galois group of the field extension $\mathbb{R}\to\mathbb{C}$ as a group of permutations of the roots of the polynomial equation $p(x)=x^2+1=0$, which map each root onto a conjugate of it. The second one is the method devised by *Richard Dedekind*, i.e. to consider the group of *automorphisms* of the splitting field $\mathbb{C}$, obtained by the minimal field extension $\mathbb{R}\to\mathbb{C}$, which leaves the ground field $\mathbb{R}$ fixed. We note that an automorphism of this form is a one-to-one and onto homomorphism from $\mathbb{C}$ to itself that leaves $\mathbb{R}$ fixed, or equivalently, invariant. The connection between these two equivalent viewpoints, comes into force if we interpret the Galois group of permutations of roots as a subgroup of a *symmetric group*. In general, thinking about the Galois group of a polynomial with degree n as a subgroup of the symmetric group S_n captures the original viewpoint of Galois, and provides the connection with Dedekind's reformulation of Galois theory.

The symmetric group on n letters is realized as a group of permutations of these n letters, so may identify a symmetric group element σ with the corresponding permutation. In general, associating to each element of the Galois group its permutation on the roots of the polynomial, viewed as a permutation of the subscripts of the roots of the polynomial when we list them in a particular order as α_1, α_2, $\ldots$ α_n is an injective homomorphism from the Galois group to the symmetric group S_n. Two different choices for indexing or ordering the roots of the polynomial can lead to different subgroups of the symmetric group S_n, but they will always be conjugate subgroups.

In general, let α_1, α_2, ... α_n be the n distinct roots of $p(x)=0$ in the minimal field extension of $\mathbb{F}$, i.e. in the splitting field $\mathbb{F}^p$ of $p(x)$, in which all distinct n roots of $p(x)=0$ can be distinguished. Then the Galois group G permutes these roots. Now, any polynomial expression in these roots, which is left invariant under any permutation of the roots, and thus is a symmetric polynomial expression thereof, is located in the ground field $\mathbb{F}$. The basic examples of such expressions are the elementary symmetric functions in the roots. It is precisely in this sense that the n roots of $p(x)=0$ are *indistinguishable* from the resolution capacity afforded by the ground field of coefficients $\mathbb{F}$, that is, in terms of all polynomial expressions symmetric with respect to $\mathbb{F}$ in the roots of $p(x)=0$. In other words, and considering the simple case of two roots, the existence of at least one non-symmetric polynomial expression with respect to $\mathbb{F}$, meaning a polynomial expression not preserved by the permutation of these roots, would provide the means to distinguish between them.

We are ready now to come back to our initial problem; how to describe the emergence of the Galois group of the minimal field extension $\mathbb{R} \to \mathbb{C}$. From Dedekind's perspective, only two automorphisms of the splitting field $\mathbb{C}$ exist, obtained by the minimal field extension $\mathbb{R} \to \mathbb{C}$, which leaves the ground field $\mathbb{R}$ invariant, namely the *identity* automorphism $z \mapsto z$, and the *complex conjugation* automorphism $z \mapsto \bar{z}$. In particular, if σ is such an automorphism of $\mathbb{C}$, then it is completely determined by the action of σ on i, $\sigma(i)$; i.e. if $\sigma(i)=i$, then $\sigma(z)=z$ for all $z \in \mathbb{C}$, whereas if $\sigma(i)=-i$, then $\sigma(z)=\bar{z}$ for all $z \in \mathbb{C}$. Thus, the Galois group of $\mathbb{C}$ over $\mathbb{R}$ is a group consisting of two elements, namely complex conjugation and the identity map. From Galois' perspective, the Galois group of the field extension $\mathbb{R} \to \mathbb{C}$ is the group of permutations of the roots of the polynomial equation $p(x)=x^2+1=0$, which map each root onto a conjugate of it. There are only two permutations, namely the identity permutation and the complex conjugation permutation, and clearly any polynomial expression in the roots i and $-i$, which is left invariant under any permutation of these two roots, and thus provides a symmetric polynomial expression in these roots, is located in the ground field $\mathbb{R}$. Thus, the ground field $\mathbb{R}$ remains *invariant* under the action of the Galois group G consisting of the identity and the complex conjugation permutation, and therefore, the two roots i and $-i$ are *indistinguishable* from the resolution capacity afforded by $\mathbb{R}$.

We conclude that the Galois group $G \cong Z_2$ of the polynomial equation $p(x) = x^2 + 1 = 0$ is an ambiguity group with respect to $\mathbb{R}$, and this ambiguity structure, induced by the two possible permutations of the roots in the present case, defines a measure of complexity of $p(x)$ in relation to the resolving means afforded by $\mathbb{R}$. This group-theoretical measure of complexity expresses that the two roots i and $-i$ are indistinguishable from the resolution capacity afforded by $\mathbb{R}$. They eventually become distinguishable only by a process of *field extension*, which terminates in the minimal field extension $\mathbb{R} \to \mathbb{C}$, where $\mathbb{C}$ is identified as the splitting field of $p(x)$. In turn, the ambiguity group Z_2 is identified as the group of automorphisms of the splitting field $\mathbb{C}$, which leaves the ground field of coefficients $\mathbb{R}$ invariant, and thus fixes it.

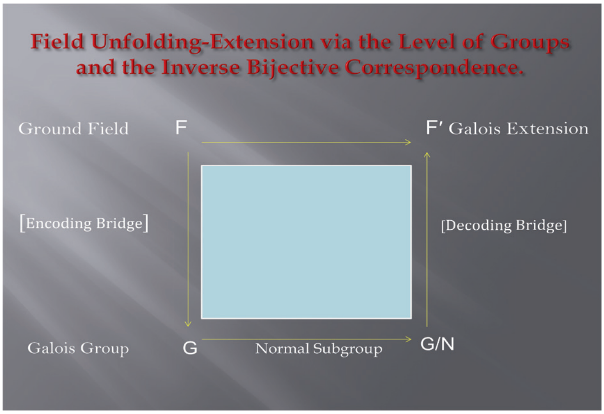
Let us now consider a general polynomial equation of the form $p(x) = 0$. The proper starting-level to settle the question referring to the idea of algebraic solvability of $p(x) = 0$ by radicals is the theory of algebraic fields. More precisely, if we start with the ground field of coefficients $\mathbb{F}_0 := \mathbb{F}$, we may produce an *unfolding sequence* of successive field extensions $\mathbb{F}_0$, $\mathbb{F}_1$ ... $\mathbb{F}_N$ by the *adjunction* of surds, so that eventually $\mathbb{F}_N$ emerges as the splitting field for $p(x)$. The theory of algebraic groups, on the other side, due to their multiplication and division properties, set the proper level to examine the structure of such extensions by surds, as follows: When the field $\mathbb{F}_{i+1}$ arises from $\mathbb{F}_i$ by the adjunction of the surd $\alpha^{1/\wp}$, then the corresponding group G_{i+1} is a normal subgroup of the group G_i, with an Abelian factor group. Eventually, one obtains the principal theorem of Galois theory, formulated as follows:

The polynomial equation $p(x) = 0$ is solvable by radicals if and only if its Galois group G is solvable, that is if a descending series of groups exists, such that G_{i+1} is a normal subgroup of the group G_i with an Abelian factor group, and G_0 is G, while G_N consists of the identity alone.

We stress that the Galois group $G_0 := G$ is identified with the group of those automorphisms of the splitting field $\mathbb{F}_N$ for $p(x)$ that preserve the ground field of coefficients $\mathbb{F}_0 := \mathbb{F}$, and thus they fix it. According to our interpretative scheme, the descending series of groups starting from the Galois group $G_0 := G$ until the trivial group G_N is eventually reached, represents all the successive stages of symmetry reduction. That is to say all the successive stages of complexity or ambiguity reduction with respect to the ground field $\mathbb{F}_0 := \mathbb{F}$ until all the roots of $p(x) = 0$ become distinguishable in the inverse ascending series of field extensions from the ground field $\mathbb{F}_0 := \mathbb{F}$ to the splitting field $\mathbb{F}_N$.

We conclude that an inverse correspondence obtains between the descending series of groups starting from the Galois group $G_0 := G$ and terminating at the trivial group G_N with the ascending series of fields starting from the ground field $\mathbb{F}_0 := \mathbb{F}$ and terminating at the splitting field $\mathbb{F}_N$ with respect to any polynomial equation of the form $p(x) = 0$ solvable by radicals. The crucial points to notice are the following:

- i As the Galois group is the group of all those automorphisms of the splitting field for $p(x)$ that preserve the ground field of coefficients, in the same way a subgroup of the Galois group is a group of automorphisms of the splitting field for $p(x)$ that preserve an intermediate extension of the ground field of coefficients;
- ii The internal symmetry or structural complexity reduction of the Galois group takes place by division with a normal subgroup. Intermediate extensions of the ground field corresponding to a normal subgroup of the Galois group are called Galois field extensions;
- iii There is a one-to-one correspondence between Galois field extensions and normal subgroups of the Galois group;
- iv The Galois group -with normal subgroups structure- corresponds bijectively in an inverse manner to the ground field -with Galois extensions- structure.



5.3 EQUIVALENCE: FROM TRANSFORMATION GROUPS TO GEOMETRIC KINDS

The central conceptual aspect of the so called *Erlangen program*, conceived by *Felix Klein*, is expressed by the thesis that the objective content of a geometric theory is captured by the *group* of *transformations* of a space. The crucial insight of Klein's program is that transformation groups constitute an algebraic encoding of a criterion of *equivalence* for geometric objects. Moreover, a transformation group determines the notion of what it is to be a meaningful property of a concrete geometric figure. Therefore, from the Erlangen perspective, a geometric figure may be conceived from an abstract algebraic viewpoint as a *manifold* acted upon *transitively* by a group of transformations. The decisive aspect of the criterion of equivalence that a transformation group furnishes is its use in characterizing *kinds* or types of geometric figures and not particular instances of these figures.

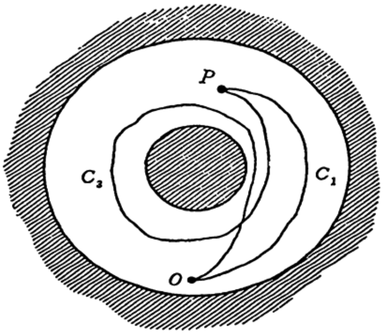
The above leads to the idea that *geometry*, in an abstract sense, refers to kinds of figures, which are specified by the transformation group of the space. Each kind can have infinitely many instantiations, thus the same *geometric form* may be manifested in many different ways, or else assume multiple concrete realizations. This reveals an important ontological dimension of Klein's program, since a transformation group of a space provides an efficient criterion to abstract a geometric kind from particular geometric instantiations, whereas the specific details of these instantiations, irrespective of their features as instances of a geometric kind, is irrelevant. In the light of this, a geometry is specified by a group and its transitive action on a space, which remarkably can be presented in a purely algebraic way as a group *homomorphism* from the transformation group to the group of automorphisms of the underlying space. Conceptually speaking, the form of a geometric theory is encoded in the transitive action of a respective transformation group. Different particular geometric configurations are the same *in form* if and only if they share the same transformation group. In other words, the transitive group action provides a precise characterization regarding matters of geometric equivalence.

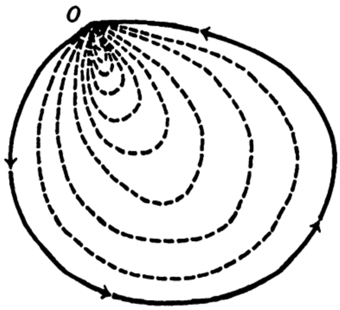
Mathematically, the above thesis is expressed as the principle of *transference,* or principle of *isomorphism*, induced by a transitive *group action* on a space. A transfer of structure is taking place by means of an isomorphism providing different equivalent models of the same geometric theory. Philosophically, an *Aristotelian* conception of space underpins this thought, according to which space is conceived as being matter without form. The form is brought about by the action of a concrete transformation group. Still, more important, the space itself may be considered as the *quotient* of the transformation group over a closed subgroup of the former. A change in algebraic form, or else, a change of transformation group signifies a change in geometry, in the sense that the equivalence criterion encoded in the group action is altered.

Thus, moving from a group to a larger one amounts to a change in the resolution unit of figures, expressed as a relaxation of the geometric equivalence criterion involved in the procedure. In effect, the criterion of equivalence serves as a powerful classification principle for geometries in relation to group hierarchies. A crucial aspect of the Erlangen program is that it does not specify which underlying manifolds exist as spectra of corresponding observable algebras, but deals with the possible existence of geometric structures on these manifolds in relation to the action of form-inducing transformation groups upon them. This leads naturally to a *bidirectional relation* of dependent-variation between transformation groups and geometric structures on manifolds. This bidirectional relation conveys the information that two spaces cannot have different transformation groups without differing as geometric structures, whereas the converse is clearly false.

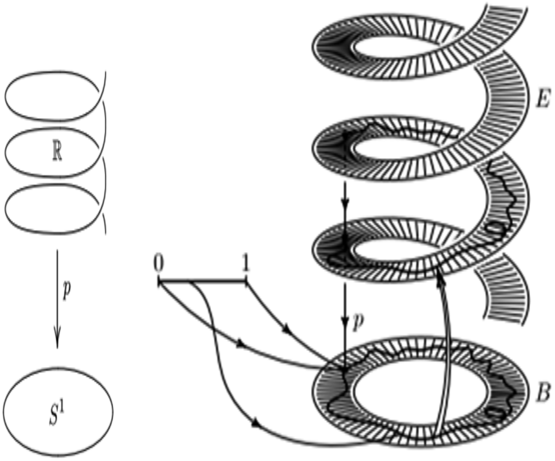
5.4 MULTI-CONNECTIVITY: FROM OBSTACLES TO THE FUNDAMENTAL GROUP OF LOOPS

In the early 19th century, *Jules Henri Poincaré* attempted to probe the *connectivity* problem of a topological space by using paths, and in particular, *loops* based at a point of this space. This approach gave rise to *homotopy* theory, and in particular, led to the notion of the *fundamental group* of a topological space. The fundamental group at a point of a space is defined in terms of the set of *based loops* at this point modulo homotopies. The notion of homotopy is based on a homeotic criterion of identity of based loops, which is expressed in terms of invariance under *continuous* distortion and shrinking.





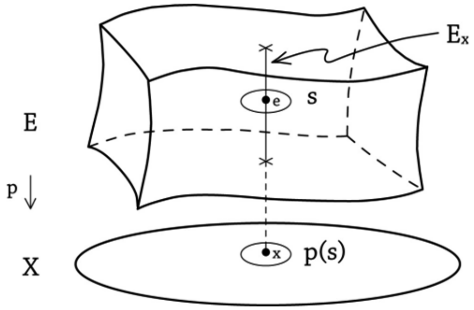
The most basic example is the example of the *unit circle* S^1. If we consider *homotopy classes* of based loops *winding* around the circle, then they are classified by the number of times winding around the circle. Thus, the fundamental group of the circle is the additive group of the *integers* $\mathbb{Z}$. If we think of the real line $\mathbb{R}$ as a *helix* in 3-d space covering the unit circle S^1 depicted in 2-d space below, then the continuous surjective group homomorphism $p:\mathbb{R} \twoheadrightarrow S^1$ is the *covering projection*, which is given by the *exponential map*. In other words, the exponential map wraps the real line anticlockwise around the unit circle. We notice that the real line $\mathbb{R}$ is *simply* connected, thus it is a *universal* covering space of S^1. Moreover, the symmetries of $\mathbb{R}$ leaving the circle S^1 invariant is given by $t \mapsto t+k$, where k in $\mathbb{Z}$ is the winding number, obtaining in this manner the group isomorphism $S^1 \cong \dfrac{\mathbb{R}}{\mathbb{Z}}$.

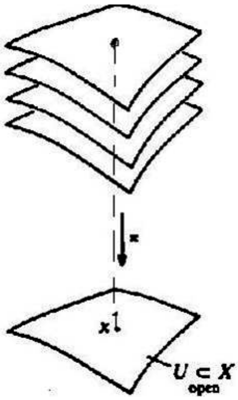


In general, the set of equivalence classes of based loops with respect to continuous distortion and shrinking can be always endowed with a multiplicative group structure under the operation of composition of paths. If the topological space is *path-connected*, meaning that any two points may be joined by a path, then the isomorphic class of the fundamental group does not depend on the selection of the base point, since the respective fundamental groups at two different base points can be made isomorphic. A path-connected space is always connected. The crucial thing is that it is *simply-connected*, and thus a *geometric space*, if it has a *trivial* fundamental group.

5.5 DISCRETE FIBRATION: FROM COVERINGS TO THE MONODROMY ACTION

A *covering space* of a base topological space is a *local homeomorphism*, such that for each point on the base space, the inverse image of an open set containing this point is a *disjoint union* of open sets in the covering space lying over the base, each of which is mapped homeomorphically on this open set, as it is displayed schematically below. In particular, if the base space X is connected, then the *fibers* of the covering space projection $p: Y \twoheadrightarrow X$ are all homeomorphic to the same *discrete* space I, such that locally, Y is isomorphic with $X \times I$.





The most important examples of covering spaces arise from group actions on topological spaces. Let G be a group acting continuously from the left on a topological space Y. The action of G is *even* if each point $y \in Y$ has some open neighborhood U such that the open sets gU are pairwise disjoint for all $g \in G$.

We recall that if a group G acts from the left on a topological space Y, one may form the quotient space Y/G whose underlying set is the set of orbits under the action of the group G and the topology is the finest one that makes the projection $Y \twoheadrightarrow Y/G$ continuous. If Y is connected, then the qualification of the action of G on Y as even, according to the above, makes the surjective projection $Y \twoheadrightarrow Y/G$ a covering one, or equivalently, Y becomes a covering space of Y/G. For example, let the group $\mathbb{Z}$ act on the space of the reals $\mathbb{R}$ by translations $x \mapsto x+n$. In this case, we obtain $\mathbb{R}$ as a covering space of the space of orbits $\mathbb{R}/\mathbb{Z}$, where the latter is homeomorphic to the circle S^1.

Given a covering projection $p: Y \twoheadrightarrow X$, where the base X is locally connected, we may consider the group of automorphisms of the covering space Y compatible with the projection p, denoted as $Aut(Y \mid X)$. We note that for each $x \in X$, the fiber $p^{-1}(x)$ is mapped onto itself under the (left) action of the group $Aut(Y \mid X)$. The important thing is that if the covering space Y is connected, then the action of the group $Aut(Y \mid X)$ on Y is an even action. Not only this, but inversely, if G is a group whose action is even on a connected space Y, the group of automorphisms of the covering space Y compatible with the projection $Y \twoheadrightarrow Y / G$, i.e. $Aut(Y \mid X)$ is identical with G.

The objective posed by *Alexander Grothendieck* was to explore how Galois's theory can admit a natural instantiation in this context. The initial simple observation is that in the Galois theory of field extensions if we think of the base field as a base point, then a finite separable extension of this field should be thought of as a discrete set of points mapping to this base point. Galois theory then furnishes this discrete set of points with a continuous action of the Galois group which leaves the base point fixed. In this state of affairs, it is a naturally emerging idea to consider as a base not just a point but a more general topological space. The role of field extensions would then be played by certain continuous surjections of the type discussed above, called *covering space projections*, whose fibers are discrete spaces.

The main thing to highlight in Grothendieck's perspective on Galois theory is that he follows the original Galois conception of the Galois group of a field extension as a group of permutations of the roots of a corresponding polynomial equation, which map each root onto a conjugate of it. Thus, there exists an action of the Galois group of the considered field extension via permutation of the roots of the corresponding polynomial equation. Grothendieck observed that this action is characterized by two important properties. First, it is a transitive action, and second, it is a continuous action if the Galois group is viewed as a topological group. Based on these two properties of the action of the Galois group on the finite set of roots of the polynomial equation, he proved the following correspondence, setting up the ground for generalizing the initial context of application of Galois theory:

There is a one-to-one correspondence between isomorphism classes of finite (separable) extensions of a base field and finite sets equipped with a continuous and transitive action of the Galois group by permutations.

The generalization amounts to introducing the notion of a Galois covering space $p : Y \twoheadrightarrow X$ as analogous to the notion of a Galois extension of a base field, which is enforced by considering the even action of the group of topological automorphisms $Aut(Y \mid X)$ as analogous to the action of the corresponding Galois group in the field case. In this manner, a connected covering space $p: Y \twoheadrightarrow X$ is qualified as a Galois covering space if and only if the group $Aut(Y \mid X)$ acts transitively and continuously on each fiber of $p: Y \twoheadrightarrow X$ playing the role of a topological Galois group in this setting. Equivalently, we have a Galois covering space if and only if each orbit of the quotient $Y / Aut(Y \mid X)$ is identical to a whole fiber of $p: Y \twoheadrightarrow X$.

Then, the fundamental theorem of Galois theory in the case of (finite) field extensions, according to which, the Galois group-with normal subgroups structure- corresponds bijectively and inversely to the ground field-with Galois extensions structure, can be transcribed in the topological setting as follows:

The topological Galois group -with normal subgroups structure- corresponds bijectively in an inverse manner to the base topological space -with Galois covering spaces structure.

The next step is to examine how the covering principle, fits and gets naturally unified with the Galois metaphora in the context of the topological generalization of Galois theory by Grothendieck, according to the above. Ultimately, the covering principle targets the eventuation of the universal covering space, characterized uniquely by the property of simple-connectivity. This is the key notion in the present case if it can be qualified appropriately through some group theoretic structure that can assume the role of a topological Galois group. Recalling from Poincaré's metaphora that a path-connected topological space is simply-connected if it has a trivial fundamental group, the eventuation of the universal covering space can be transcribed group-theoretically thanks to the notion of the fundamental group of the base space, as targeting the annihilation of the fundamental group of the base.

The annihilation of the fundamental group of the base space can be interpreted as a process of complexity contraction, or symmetry reduction, in the Galois theoretic sense, if and only if there exists an even, transitive and continuous action of the fundamental group on each fiber of the universal covering space $p: Z \twoheadrightarrow X$. If this is the case then the fundamental group of the base space actually plays the role of a topological Galois group, whose symmetry is reduced in successive stages by division with normal subgroups corresponding to Galois covering spaces; Ultimately the whole symmetry is completely reduced by unfolding to the universal covering space characterized by simple-connectivity, which thus bears a trivial fundamental group.

In other words, the maximal ambiguity engulfed in the fundamental group of the base space at a marked point, interpreted as a structural measure of multiple-connectivity characterizing the homotopic complexity of the base at this point due to its prospective qualification as a topological Galois group, can be entirely eliminated by unfolding to the simply-connected universal covering space of the base topological space.

It remains to examine if there actually exists an even, transitive and continuous action of the fundamental group on each fiber of the universal covering space $p : Z \twoheadrightarrow X$, qualifying it as a topological Galois group. The fact that such an action actually exists is due to the two most important properties of a covering space, namely the *path-lifting* property and the *homotopy-lifting* property from the base to the fibers of a covering space.

The main idea is that both paths and loops (belonging to a homotopy class of the fundamental group) on the base space can be *lifted uniquely* from the *base* to the *fibers* of a covering space. In particular, if we consider a based loop at a marked point of the base space, then its unique lift on a covering space is not necessarily a loop, but it is always going to be a path whose starting and ending point belong to the same fiber of the covering space that projects to the marked point of the base, where the loop is based. Since such a based loop is an element of a homotopy class in the fundamental group at the marked point, the transition from the starting to the ending point of the fiber over the marked point induced by the lift of this based loop, amounts to an even, transitive, and continuous action of the fundamental group at the marked point on the fiber of the covering space over this point. This action is called the *monodromy action* of the fundamental group (at the marked point) on the fiber of the covering space over this point.

Grothendieck proved that the monodromy action actually gives rise to a *functor* from the category of covering spaces over the base space X, where a marked point has been depicted, towards the category of sets equipped with a left action of the fundamental group at this point. This takes place by sending a covering space $p : Y \twoheadrightarrow X$ to the fiber $p^{-1}(x)$ over this marked point, called the *fiber functor* at the marked point. Essentially the fiber over a marked point encapsulates the global connectivity depth of this point in the presence of topological obstacles.

There is a subtle indirect self-referential metaphora regarding the notion of a *point* in a topological space that is worth explicating. More precisely, the notion of a point in a topological space is only *implicitly* assumed, and thus it needs to be articulated through the basic topological characteristic, which is *connectivity* in the presence of obstacles. The metaphora from the topological domain of connectivity to the algebraic domain of the fundamental group, culminating in the notion of a covering space, pertains to the level of points by means of the monodromy action that eventually indirectly articulates a point through its fiber.

The fiber functor at a marked point of the base space, induces the following categorical equivalence:

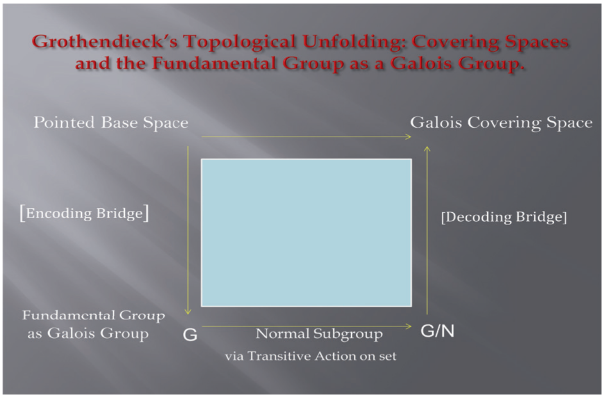
There exists an equivalence between the category of covering spaces over the base space X and the category of sets equipped with a left action of the fundamental group at the marked point, such that connected covering spaces correspond to sets with a transitive left action of the fundamental group at the marked point, and Galois covering spaces correspond to coset spaces of normal subgroups of the fundamental group at the marked point.

In this way, we obtain the following unification of the covering principle with Galois theory in the current topological setting:

The fundamental group of a base topological space (where a marked point has been depicted) -with normal subgroups structure-corresponds bijectively in an inverse manner to the base topological space -with Galois connected covering spaces structure-.

The only condition for the validity of Grothendieck's theorem, establishing the role of the fundamental group as a Galois group in the setting of connected covering spaces, is that the base space is *connected* and each point of this space has a *basis* of simply connected open neighbourhoods.

Now, if the Galois covering space is a universal simply-connected covering space of the base space X where a marked point x has been depicted, denoted by means of the covering projection $p : \widehat{X_x} \twoheadrightarrow X$, then it corresponds to a set equipped with a *left transitive action* of the whole fundamental group at this point $\pi_1(X,x)$, or else a $\pi_1(X,x)$-set. This action is of the *Galois-type* since it induces permutations of all the elements of the universal covering space fiber projecting to the marked point x in the base. Thus, conceptually, it is naturally isomorphic with the group of *automorphisms* of the *universal* covering space $\widehat{X_x}$ compatible with the projection p, denoted as $Aut(\widehat{X_x} \mid X)$. In this way, the Galois action of $Aut(\widehat{X_x} \mid X)$ of the universal covering space fiber projecting to the marked point x in the base is identified with the monodromy action of the fundamental group (at the marked point) on the fiber of the universal covering space over this point.



5.6 LOCALIZATION: SHEAVES AND THE CIRCULATION FROM THE LOCAL TO THE GLOBAL

The concept of a *sheaf* is based on two fundamental pillars. The first refers to the notion of a locally defined element of an algebraic structure, whence the second refers to the gluing or pasting conditions of these locally defined elements together. The concept of locality is introduced by means of an appropriate topology, or more generally, in terms of an appropriate covering system. The sheaf is understood as ubiquitious to address the precise manner in which locally defined elements, organized in terms of the structure of groups, or modules, or algebras, or even sets, can be topologically extended from the *local* to the *global*. They are thus collated compatibly into global elements over a partially ordered covering system, like the one defined by the open covers of a topological space.

In this sense, a sheaf may be thought of as a continuously variable algebraic structure, whose continuous variation is expressed in terms of its sections over the local covers of a global topological space. The existence of the latter is only implicitly assumed, since the actual objective of a sheaf is the topological articulation of the *points* of this space through the compatible amalgamation and extendibility of the sections from the local to the global. Thus, essentially the notion of a sheaf targets the nature of points in a topological space in the presence of obstacles to connectivity.

The amalgamation, this process of gluing, is conducted by the formation of appropriate equivalence classes, called *germs of sections* of a sheaf, which are defined over a partially ordered family of covers of the implicitly assumed global topological space. A *section* of a sheaf, which is defined over a local cover may be thought of as a partially defined, continuously variable functional relation, whose degree of specificity depends on the spectral resolution afforded by the corresponding cover.

One of the most interesting aspects of the sheaf concept in comparison to the former set-theoretically articulated notion of a topological space is that a sheaf actually captures the model of a topological space in its *spectral* becoming. Given that a topological space, for instance a manifold, is physically utilized to represent the event or state structure of a physical theory corresponding to the evaluation of observables, its associated sheaf model pertains to its actual taking place, i.e. to its actual constitution by amalgamating compatibly local observables into global ones. This is because the underlying notion of a global topological space, referring to its structure of points, both standard and singular, is initially only implicitly assumed, and then, step-by-step induced indirectly by completion via the germs of the sheaf. This culminates in extracting the invariant information pertaining to these germs after their integration.

In this sense, we should emphasize that a sheaf model of a space does not have a local structure defined by its points, which are assumed to be absolute and pre-existing according to former set-theoretic lines of reasoning. In contrast, the local structure is considered as being intrinsically continuous with respect to the partially ordered covers of this space. A closed partially ordered family of intersecting local covers, capable of gradually unraveling the invariant information that germs bear at a point, acquires the meaning of a temporal order. The algebraic structure of germs at a point subsumes in this way all the contextual information in relation to this point.

The sheaf completion property at a point is tantamount to the integration of these germs, though of as expressing differentially the local or infinitesimal variations around this point. Eventually, this is precisely the spectral resolution process that characterizes not only the underlying point itself sheaf-theoretically, but also its *genetic* constitution and variation in relation to the considered *temporal order*. This is because the global characterization of a point, for example of a physical state or an event, is the final result, the *ichnography*, or technically the *trace* of a continuous unfolding process of genetic constitution in terms of granular elements. These granules are precisely the germs of sections around that point, each one of which bears the resolution capacity of their cover-horizon. Therefore, germs incorporate all their compatible subcovers within the pertinent temporal order.

Since *Jean Leray's* initial conception of the notion of a sheaf, and *Grothendieck's* articulation of the same concept, the various examples and applications of sheaves have come to play a major role in such diverse fields of mathematics as several complex variables, algebraic geometry, algebraic topology, and differential geometry. This is especially true for Grothendieck's contribution, which generalized the notion of sheaves beyond the realm of topological spaces, making it applicable over general category-theoretic sites. In sum, categories equipped with covering systems, called Grothendieck topologies, has created a wealth of new models enriched with the power of the methods of homological algebra.

The algebraic topological ubiquity of the sheaf concept is based on the realization that, since it is possible to localize standard algebraic notions, such as homomorphism, kernel, image, subobject and quotient object sheaf-theoretically, in such a way that these concepts have essentially the same meaning as in abstract algebra, one can interpret them from a categorical standpoint, and infiltrate all the constructions of homological algebra through sheaf theory. The resulting category of sheaves has the same classical properties as the category of Abelian groups, or the category of modules. More precisely, one can define for sheaves direct sums, direct products, tensor products, inductive limits, and all other related concepts.

For this reason, the apparatus of sheaf theory is able to penetrate into various fields of mathematics providing an effective algebraic tool especially in those areas which ask for global solutions to problems whose hypotheses are local. This is due to the fact that there is a natural definition of the *cohomology* of a site, that is, in the simplest case of a topological space with coefficients in a sheaf. In particular, Grothendieck's insights and formulations led to the crystallization of the idea that the natural argument of a cohomology theory is a *pair* consisting of a global *topological space*, or more generally a *category-theoretic site*, together with a *sheaf of coefficients* defined over it, rather than just the space itself. In point of fact, and since the topological space is only implicitly assumed, the major role is played by the means of measuring and spectrally resolving this space over its local covers, i.e. by the pertinent sheaf of coefficients. This realization has been transferred to the field of complex analysis, and more recently to the field of differential geometry by the development of the geometric theory of vector sheaves, which are equipped with a connection, according to the framework of *Anastasios Mallios*, called *Abstract Differential Geometry* (ADG). Since then, it is a common *topos* that the sheaf gauge of algebraic-topological localization and extendibility sought-after is always provided by sheaf cohomology.

Cohomology has been invented as an efficient algebraic technique of assigning global *invariants* to a topological space, or more generally, to a categorical site, for the purpose of capturing group-theoretically its most important shape-related characteristics in a homotopy-invariant way. In particular, the cohomology groups encode the global obstructions for extending sheaf sections from the local to the global level, for example, extending local solutions of a differential equation to a global solution.

For instance, *Georges de Rham* cohomology theory measures the extent that *closed differential forms* fail to be *exact*, and thus, it qualifies the obstruction to integrability group-theoretically, since according to the *lemma of Poincaré*, every closed differential form is *locally* exact. The de Rham theorem asserts that the homomorphism from the de Rham *cohomology ring* to the differentiable *singular cohomology ring*, which is given by the *integration* of closed forms over differentiable singular cycles, is a ring isomorphism. The sheaf-theoretic understanding of this deep result came after the realization that both the de Rham cohomology and the differentiable singular cohomology are actually *special* isomorphic cases of sheaf cohomology with values in the constant sheaf of the reals. In particular, it has been also clarified that the de Rham cohomology of a differential manifold depends only on the property of *paracompactness* of the underlying topological space.

Aside from the cohomological machinery associated with the algebraic theory of sheaves, the general process of transition from the constant to the variable takes place by substituting global rigid set-theoretic algebraic structures with localized continuous sheaf-theoretic algebraic structures. It is in this precise sense that a sheaf may be thought of as a continuously variable set, whose continuous variation is enacted over the employed local covers of a topological site, such that local sections bear the capacity to interlock together in their extendibility from the local to the global. These local covers are required to obey certain topological closure conditions, which in fact, generalize the definition of a topology formulated in terms of open sets covering a classical topological space.

From a physical standpoint, we are forced to consider the conditions of localization of physical observability, to elevate observables from the *topos* of *sets* to an appropriate *topos* of *sheaves* with respect to a covering system of a global site. The idea is that the global is not directly accessible, and thus, information can only be qualified and quantified sheaf theoretically, i.e. in terms of pasting conditions, from local measurements taking place over the covers of this site. Intuitively speaking, this amounts to a new type of relativization pertaining to the local behavior of observables as opposed to their classical point behavior due to the obstacle erected by objective indistinguishability, or intrinsic randomness, or non-subjective uncertainty.

It is useful here to recall that in the guiding case of a topological space, the notion of a topology provides the means to talk about what a *continuous* function is between topological spaces. A function is said to be continuous if and only if the inverse image of every open cover of the range is an open cover of the domain topological space. This formulation is an attempt to capture the intuition that there are no breaks or separations in a continuous function. This being so, it is instructive to highlight the following:

i The definition of a topology on a space is solely used for the formalization of what a continuous function is on that space,
ii the continuity of a function is a property which is determined locally, that is only by reference to the open covers of a space.

This means that due to the variability of open covers in the topology the property of continuity of a function should respect the inverse algebraic operations of restriction and unique extension with respect to the open covers of a covering system. Thus, a continuous function can be restricted consistently to open subcovers of any open cover in the topology and inversely extended by gluing uniquely together all its local restrictions. This is the crucial conceptual insight in relation to the notion of continuity that is incorporated in the technical definition of a sheaf.

The above insight referring to the precise formulation of the property of continuity may be generalized in two directions: Firstly, instead of open covers of a topological space we may consider generalized covers under the constraint that they collectively obey topological closure conditions analogous to the ones used for open covers. Secondly, instead of functions varying continuously over local covers, we may consider generalized functional relations, which are precisely the sections of a sheaf. We note that local sections of a sheaf depict functional relations relatively to a local cover. From this viewpoint, a sheaf is essentially the totality of its sections, comprehending both the local and the global in case that the latter actually exist.

Thus, in terms of sections, what actually matters is their consistent interrelation as well as their respective pairwise interlocking properties with respect to the local-global distinctions subsumed by the underlying covers. In particular, the operation of restriction of sections is meaningful with respect to *nesting* local covers under intersection, whereas the operation of extension or gluing of sections is meaningful with respect to compatible *pairwise intersections* of sections over their respective local covers, followed by *extendibility* of a section by another section over the union of their covers. We note that if only the compatibility property of sections under restriction for nested local covers is satisfied, then we obtain a weaker structure called a *presheaf*. In cases where the compatibility property of sections under extension is also satisfied, then we obtain a *separated presheaf*. Only in case that the operation of extension results in a *unique* gluing of sections, that is local sections can be uniquely glued together, a separated presheaf becomes a sheaf.

As a result, in general, there will be more locally defined or partial sections than globally defined ones, since not all partial sections need be extendible to global ones. Nevertheless, a compatible family of partial sections uniquely extends to a global one, or in other words, any presheaf uniquely defines a sheaf. More precisely, for the leading example of *localization* with respect to the open covers of a topological space we have the following basic notions:

A presheaf $\mathbb{F}$ of sets on a topological space X, is constituted as an *information structure* in relation to this space, as follows:

i For every open set U of X, there is defined a set of elements denoted by $\mathbb{F}(U)$; and

ii For every inclusion $V \infty U$ of open sets of X, there is defined a restriction morphism of sets in the opposite direction:

$$r(U \mid V) : \mathbb{F}(U) \to \mathbb{F}(V) \quad (1)$$

such that:

a $r(U \mid U)$ = identity at $\mathbb{F}(U)$ for all open sets U of X; and

b $r(V \mid W) \circ r(U \mid V) = r(U \mid W)$ for all open sets $W \infty V \infty U$.

Usually, the following simplifying notation is used: $r(U \mid V)(s) := s|_V$.

A presheaf $\mathbb{F}$ of sets on a topological space X, is defined to be a sheaf if it satisfies the following two conditions, for every family V_a, $a \in I$, of local open covers of V, where V open set in X, such that $V = \cup_a V_a$:

i Local identity axiom of a sheaf: Given $s, t \in \mathbb{F}(V)$ with $s|_{V_a} = t|_{V_a}$ for all $a \in I$, then $s = t$; and

ii Gluing axiom of a sheaf: Given $s_a \in \mathbb{F}(V_a)$, $s_b \in \mathbb{F}(V_b)$, $a, b \in I$, such that:

$$s_a|_{V_a \cap V_b} = s_b|_{V_a \cap V_b}, \quad (2)$$

for all $a, b \in I$, then there exists a unique $s \in \mathbb{F}(V)$, such that: $s|_{V_a} = s_a \in F(V_a)$ and $s|_{V_b} = s_b \in F(V_b)$.

If we consider the partial order of open covers of a topological space X as a category denoted by $\mathcal{O}(X)$, where all arrows are inclusions, then $\mathbb{F}$ denotes the contravariant presheaf/sheaf functor that assigns to each open set $U \subset X$ a set in the category **Sets**. We note that the above definitions hold if instead of presheaves/sheaves of sets we consider presheaves/sheaves of algebraic structures, for example groups, algebras over a field, vector spaces over a field or modules over an algebra.

As the most basic example, if $\mathbb{F}$ denotes the contravariant presheaf functor that assigns to each open set $U \subset X$, the commutative and unital algebra of all real-valued continuous functions on U, then $\mathbb{F}$ is actually a sheaf. This is clear since the specification of a topology on X, and hence, of a topological localization system on $\mathcal{O}(X)$), is solely used for the definition of the continuous functions on X, to be thought of physically as observables on X.

Thus, the continuity of each function can be determined locally. This means that continuity respects the operation of restriction to open sets, and moreover that continuous functions can be collated in a unique manner, as is required for the satisfaction of the sheaf condition. More precisely, the sheaf condition in this case means that the following sequence of commutative algebras is *left exact*;

$$0 \to \mathbb{F}(U) \to \prod_a \mathbb{F}(U_a) \to \prod_{a,b} \mathbb{F}(U_a \bigcap U_b). \quad (3)$$

Let us further assume implicitly that x is a point of a topological space X. Moreover, let T be a set consisting of open covers of X, containing x, such that the following condition holds: For any two open covers U, V, containing x, there exists an open cover $W \in T$, contained in the intersection $U \bigcap V$. We may say that T constitutes a basis for the system of open covers around x. We form the disjoint union of all $\mathbb{F}(U)$, denoted by;

$$\mathbb{D}(x) := \bigcup_{U \in T} \mathbf{F}(U). \quad (4)$$

Then we can define an equivalence relation in $\mathbb{D}(x)$, by requiring that $f \sim g$ for $f \in \mathbb{F}(U)$, $g \in \mathbb{F}(V)$, provided that they have the same restriction to a smaller open cover contained in T. Then we define;

$$colim_T[\mathbb{F}(U)] := \mathbb{D}(x)/\sim_T. \quad (5)$$

It is clear that the above definition is independent of the chosen basis of open covers T, and thus corresponds to an *inductive limit* construction. The inductive limit obtained is denoted by $\mathbb{F}_x$, and referred to as the *stalk* of $\mathbb{F}$ at the point $x \in X$. We identify an element of $\mathbb{F}$ of sort U with a local section of $\mathbb{F}$ over the open cover U.

Then the equivalence relation, used in the definition of the stalk $\mathbb{F}_x$ at the point $x \in X$ is interpreted as follows: Two local sections $f \in \mathbb{F}(U)$, $g \in \mathbb{F}(V)$, induce the same *contextual information* at x in X, provided that they have the same *restriction* to a smaller open cover contained in the basis T. Then, the stalk $\mathbb{F}_x$ is the structure containing all contextual information at x, that is the structure of all equivalence classes.

Moreover, the image of a local section $f \in \mathbf{F}(U)$ at the stalk $\mathbb{F}_x$, denoted by f_x, that is the equivalence class of this local section f, is precisely the *germ* of f at the point x. We deduce that the fibration corresponding to a sheaf of sets $\mathbb{F}$ is a topological bundle, called an *étale bundle*, defined by the continuous mapping $\varphi : F \to X$, where;

$$F = \bigcup_{x \in X} \mathbb{F}_x \quad (6)$$

$$\varphi^{-1}(x) = \mathbb{F}_x. \quad (7)$$

The mapping φ is locally a homeomorphism of topological spaces. The topology in F is defined as follows: for each local section $f \in \mathbb{F}(U)$, the cover $\{f_x, x \in U\}$ is open, and moreover, an arbitrary open cover is a union of covers of this form. Obviously, the same arguments hold in the case of a sheaf of sets $\mathbb{F}$ endowed with some algebraic structure, for example $\mathcal{F}$-algebras (where $\mathcal{F}$ is a field). Finally, the sheaf $\mathbb{F}$ can be canonically identified as the sheaf of *cross-sections* of the corresponding étale bundle F.

We stress that the notion of a sheaf depends on the fact that we require the gluing condition with respect to *all covers* of any local cover. In principle, one could select some covers of a local cover and require the gluing condition only with respect to the selected covers. In this way, the notion of sheaf would be meant with respect to the selected family of covers. On the other hand, there is no restriction in considering only *hereditary* or *genetically descending* covers, that is covers containing *all* their subcovers. More precisely, any cover can be made hereditary (by adding to each cover all its considered subcovers), and compatible (uniquely glued) families of sections on the original cover are in *bijective* correspondence with compatible families of sections on the new one.

This fact provides an important insight on the nature of the topological localization process of observables implicated by the sheaf concept. First, let us think of a local cover as a *partial information* carrier. Then, the idea of considering hereditary covers can be implemented according to a procedure of *refining* the *resolution grain* of information via localization through *sieving*. More concretely, a sieve on a local cover can be thought of as consisting of spectral horizons distributed across different nested layers, such that, every partial information which is compatible with respect to one of these horizons passes through it but not otherwise.

The conception of covering sieves as hereditary covers used for localizing, and thus sharpening, information with respect to intersecting local covers at deeper ordered depths, gives rise to localization systems of global observable algebras which induce a semantic transition of events from the set-theoretic to the sheaf-theoretic level. In this manner, events are just the traces of the process of sharpening observable resolution by the way of sheaf-theoretic localization. The operation which assigns to each local cover a collection of covering sieves satisfying the appropriate transitivity and stability properties under localization into deeper ordered depths, defines a topology, which is technically called a *Grothendieck topology*.

The notion of a Grothendieck topology formulated in terms of covering sieves is significant for the following reasons: First, it elucidates precisely the topological concept of locality in *relational order-theoretic* terms, such that this concept becomes distinguished from its usual spacetime connotation. Second, it permits the *amalgamation* of local information into global by utilization of the notion of sheaf with respect to the defined topology.

In more detail, the extension of observable information from the local to the global takes place through a compatibly glued family of sections over a covering sieve constituted of local covers of sharper and sharper resolution, giving rise to an inductive localization system upon its closure that ia tantamount to the *completion* of the localization process. A sheaf assigns a set of sections to each local cover of a localization system. A selection of sections from these sets, one for each local cover, forms a compatibly glued family with respect to a localization system, if the selection respects the operation of restriction, and additionally, if the sections selected agree whenever two local cover of the localization system overlap. If such a locally compatible selection of sections extends *uniquely* to a global one, then the sheaf conditions are satisfied. We note that in general, there will be more local sections than global ones (if they exist), since not all local information need be extendible to global ones, but a compatible family of local information uniquely extends to a global one with respect to a localization system. It is crucial that sheaf-theoretic localization takes place in terms of continuous granular elements, i.e. the sections of a sheaf, which gradually, and in terms of the temporal order they are associated with, expressed by a *closed sieve* of covers, genetically unfold the becoming of point-events.

It is essential to clarify that the sheaf model of a space takes place in three steps, after the specification of a covering sieve: The first step organizes compatibly the local cover-infiltrated information of sections, such that the latter can be compatibly restricted from the global to the local. This process produces a structure that is only a presheaf. Note that conceptually the *implicitly* assumed global, from the presheaf-theoretic viewpoint, refers to the global in its *potentiality* to *germinate* in the context of some temporal order, and eventually produce facts. The second step involves the process of *functional* localization of the sections of the presheaf. This is necessary because the sections of the presheaf are not *a priori* functional elements. It is accomplished by means of the topological realization of the presheaf of sections as a *display space* constituted by disjoint stalks of germs over each point of the underlying base space. The third step involves the *completion* of the presheaf, or equivalently the completion of its associated display space, such that the germs belonging to each stalk can be evaluated and produce a fact, temporalizing in this way the pertinent point of the stalk, by disclosing *genetically* the temporal order they descent from. In this way, it is realized a sheaf, incorporating all the necessary and sufficient conditions for the *bidirectional* compatibility of information under restriction or reduction from the global to the local, and inversely under unique extension or induction from the local to the global. The latter sheaf-theoretic conceptualization of the global refers now to this term in its *actual propensity* to explicate points globally in terms of events following the germination of sections with respect to the temporal order they participate and descend from.

Clearly, both of these connotations depend on the role of the covers of a covering sieve, since it is true both, that the production of facts is pre-conditioned by their existence, and their interconnection is possible only via the sheaf-theoretic gluing procedure conducted over these covers. Hence, the local covers can be thought of as both, localities of contextualized potentiality (under restriction from the global to the local), and as localities of relativized facticity (under extension from the local to the global), with respect to which point-events are localized, actualized, and eventually, interconnected consistently.

Due to the *dual* articulating role of the term global, in relation to both potentiality and actuality, pertaining to the conception of the sheaf model of a topological space as a *complete* presheaf, according to Leray, a logical scheme of indirect self-referentiality is induced, whence locally all the information is *sieved through* an appropriate family of *local covers*. The uniquely defined sheaf-theoretic extension from the local to the global takes place thanks to the formation of germs, i.e. equivalence classes of locally or partially defined section compatible by our criteria. Due to the process of extensive connection and the formation of *continuous histories* in terms of germs in purely topological terms (without assuming any pre-existing metrical linear sequence of an external time line), the notion of a sheaf resolves the self-referentiality induced by the dual role of the topological global in a *spiral-like* form. The two possible ways of orientation on this spiral-like form represent the inverse processes of ascending (extending from the local to the global) and descending (restricting from the global to the local). Ascent corresponds to indirect self-referential constitution of the global in terms of events, from the local, via the extensive connection of their corresponding germs. This means that the global (in actuality) can be genetically and connectively, that is, *historically*, accessed only through the germs of observables over the corresponding point-events. Descent corresponds to indirect self-referential resolution of the global (in potential) with respect to a multiplicity of local covers, such that the inversely constituted global (in actuality) achieved by evaluating germs, is compatible with the former. The basic idea pertaining to the above may be simply put as follows: Since the evaluation of a section over a cover gives rise to an observed event, this event is simultaneously implicitly correlated with a germ, i.e. with the whole family of sections, which are compatible with the considered one under restriction, or more generally, under pulling back. Thus, we can access the historicity of an event from a topological perspective through the connective extension of a continuous germ, which indirectly correlates this event to all antecedent events.

5.7 ANALYTIC CONTINUATION: RAMIFICATION OF MULTI-VALUED FUNCTIONS

Geometric function theory on a *Riemann surface*, culminating in the thriving field of complex analysis, traces its germ of conception and initial development back to *Bernhard Riemann's* principles for dealing with the notion of a *multiply-extended* or *multi-fold magnitude*. This notion, called simply today a multi-valued function, takes hold because the *analytic continuation* of a given holomorphic function element along different paths on the complex plane, due to the presence of an obstacle, leads to different *branches* of that function.

The basic idea of Riemann, giving rise to the covering principle, consists in the replacement of the initial domain of definition of the function with a *multiple-sheeted covering* of the complex plane, or more generally, of the Riemann sphere, so that this function becomes eventually *single-valued* on the *universal* covering space of the initial domain. In this manner, a multiply-extended variable magnitude on the complex plane is unfolded into a simply or uniformly-extended variable magnitude on the universal covering space. Thus, the covering principle is essentially based on the idea of metaphora from an obstacle-laden domain to the universal covering of this domain. The latter is qualified as the maximal, simply-connected and obstacle-free unfolding space, where uniform extendibility of an equivalence class, a locally-defined germ of a holomorphic function element, becomes feasible.

Uniform extendibility in the universal covering or unfolding space is thought of in terms of the process of analytic continuation of a multi-valued function along *loops* surrounding obstacles. Riemann conceived of an infinitely thin sheet propagating along a loop of this kind such that, when one returns to the starting point, one arrives on a different *sheet* whenever the value of the function obtained by analytic continuation is different from its initial value.

In this way, by performing the analytic continuation along all the possible loops, one may associate a many sheeted smooth compact surface which covers the Riemann sphere with the given multi-valued function, where the former functions become uniform and single-valued. If the covering surface is constructed so that it has as many points lying over any given point in the complex plane (or the Riemann sphere) as there are function elements at that point, then on the universal covering Riemann surface, the analytic function unfolds completely and becomes single-valued. This process of unfolding a multi-valued function by means of *spreading out* into a multiplicity of branches constituting the universal covering is called the *ramification* of a multi-valued function.

Consequently, according to Riemann's covering principle, when a multi-valued function unfolds, the covering surface will also unfold with it. In a region where two or more unfoldings of the function occur, the covering surface will be double or multiple. It will consist of two or several sheets, each one of them corresponding to a *branch* of this function. Around a ramification point of the function, a sheet of the covering surface will unfold to another sheet, and in such a way that, in the neighborhood of this point, the surface may be thought of as a *helicoid* whose axis is perpendicular to the complex plane at that point. But when, after several *windings* around the ramification value, the function takes back its initial value, one must assume that the superior sheet of the surface *connects* to the inferior one by *traversing* the rest of the sheets. At each point of a surface which represents the way it ramifies, the multi-valued function admits a *single* determined value, and may therefore be looked upon as a perfectly determined *function* of the place (of a point) on this surface. It is important to highlight that the different covering branches can be joined together *only* at points, not along lines.

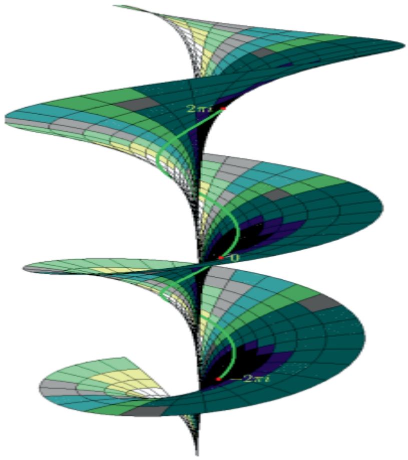
In modern terminology, a covering Riemann surface gives rise to a *holomorphic* map with codomain the complex plane, or the Riemann sphere, which is characterized topologically as a *ramified covering* *projection* or, equivalently, as a *branched covering space* of the latter. In other words, it is a local homeomorphism, such that for each point on the plane or the sphere, the inverse image of an open set containing this point is a disjoint union of open sets in the covering Riemann surface, each of which is mapped homeomorphically on this open set. This holds universally with the *exception* of the ramification points, which are inverse images of branch points on the plane or the sphere. At these points, the various branches of the covering Riemann surface, thought of as a helicoid winding around the various corresponding branch points, are alternated or interchanged in winding cycles.

We note that if we forget the fact that these concrete surfaces are spread out over the complex plane (or the Riemann sphere) as branched covering spaces referring to the process of unfolding of a many-valued function, we obtain the notion of an *abstract* Riemann surface, defined as a two-dimensional real *analytic manifold* equipped with a holomorphic structural *atlas*, which is considered as the natural domain of definition of analytic functions in one complex variable. In this manner, the notion of a concrete covering Riemann surface is used only with reference to a many-valued function. Recalling *Weyl's* suggestive formulation, a concrete Riemann surface is not merely a visual representation of a many-valued function. Quite the opposite, it must throughout be thought of as the *prius*, as the mother earth in which functions, like *plants*, can first of all grow and flourish.

If we recall the complex exponential covering projection from $\mathbb{C}$ to $\mathbb{C}^* = \mathbb{C} - \{0\}$:

$$exp(z) : \mathbb{C} \to \mathbb{C}^*$$

then, a determination of its inverse many-valued function $\log(z)$ is only *locally* feasible, it can take place only in terms of local sections of the exponential covering $exp(z)$. More precisely, each section defined on $\mathbb{C} - \{ray\}$ constitutes an inversion of $exp(z)$ only locally, and thus contributes, to a *local determination* of the logarithm. Each section bears the form $\log(z) + 2\kappa i\pi$, where κ is an integer. The compatible gluing of all these local determinations over their non-trivial overlaps gives rise to the Riemann surface of the many-valued function $\log(z)$, as follows:



The multiply-extended aspect of a many-valued function of one complex variable lead naturally to the first topological considerations regarding the *connectivity* of surfaces. In particular, a piece of a surface is connected when any two points in it can be *joined* by a path. Riemann's method of probing the connectivity of surfaces is based on the notion of *boundary cuts*, which are simple paths joining two points on the boundary. A surface is simply connected if any boundary cut dissects it into *two* simply connected pieces rendering it disconnected.

In general, the type of connectivity of a surface involves the counting of the number of boundary cuts and the number of simply connected pieces obtained after performing these cuts. The basic idea is that if a surface is cut into m simply connected pieces by n boundary cuts, then the number $(m-n)$, for m and n variable, remains constant. According to Riemann, this constant captures the *order* of connectivity of the surface.

With this stipulation, a connected surface is judged to be n*-fold* connected if a system of $(n-1)$ boundary cuts are required to make it simply connected. Of course, these considerations pertain to a surface *with* boundary. In order that they become applicable to a closed surface it is necessary the *metamorphosis* of this surface to another one with a boundary. In particular, if we make a *puncture* at any point of a closed surface, then the analysis takes place by considering a cross section starting from this point and returning to it, giving rise to a *loop*, or closed curve.

Thus, whenever it is possible to draw n closed curves α_1, α_2, $\ldots$, α_n on a surface T, which, either taken separately, or taken together, do not form the complete boundary of part of the surface, but which, joined to any other closed curve, *do* form the boundary of part of the surface, then the surface is. $(n+1)$-ply connected.

The next fundamental notion refers to the topological *genus* of a closed surface, especially concerning its relation with the connectivity order of a closed surface. If $2p$ is the maximum number of closed curves (being allowed to intersect) which may be drawn on a closed surface T without rendering it disconnected, then p is the genus of this surface.

The genus is characterized as a complete homeomorphism invariant, meaning that two (compact and orientable) surfaces without boundary are homeomorphic, if and only if they have the same genus. Moreover, $2p$ is identified as the rank of the *first homology group* of T, i.e. its *first Betti number*. Finally, it is straightforward to obtain the relation of the connectivity order c of a closed surface T with the genus p of this surface, namely $c=2p+1$.

According to *William Kingdon Clifford*, a closed curve on a surface should be thought of as a *circuit*. If it is possible to move a circuit continuously on the surface until it shrinks up into a point, the circuit is called reducible; otherwise it is irreducible. In general, there is a *finite* number of irreducible circuits on a closed surface which are independent. An independent circuit is characterized as one that by continuous motion cannot be made to coincide with a curve made out of the others.

In particular, for a surface bearing p holes there are $2p$ independent irreducible circuits; one around each hole, and one through each hole.

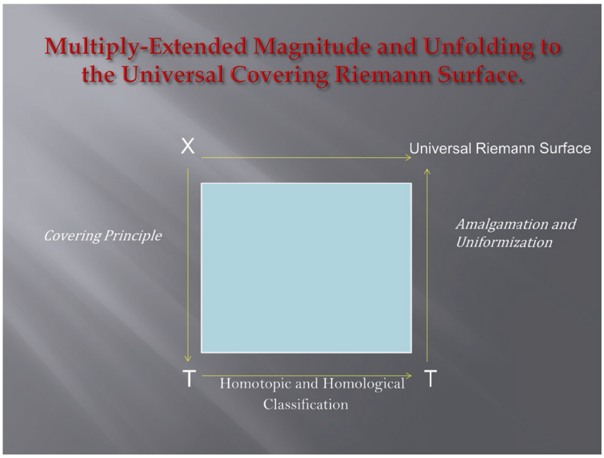
Conclusively, Riemann's metaphora addressing the ramification of a many-valued function, may be summarized as follows:

a eventuation of obstacles until each closed curve that embraces them plays the role of a boundary for a portion of the universal covering surface;
b application of boundary cuts;
c spreading out into all possible distinct covering branches; and,
d amalgamation of all the covering branches to obtain the simply-connected universal covering surface.

In the context of this metaphora, the relation between two different simply connected regions is expressed through Riemann's mapping theorem. More specifically, any two simply connected regions can be mapped *conformally*, that is, in an oriented angle-preserving way, onto one another. In particular, any such simply connected plane region can be mapped conformally onto the unit disc $\mathbb{D}$.

The generalization of the Riemann mapping theorem to what is called the *uniformization* theorem for Riemann surfaces, proved by *Henri Poincaré* and *Paul Koebe*, states that any simply connected Riemann surface is isomorphic, meaning *conformally equivalent* in this context, to the Riemann sphere, to the complex plane, or to the unit disk. The term uniformization accords to the process of unfolding a multiply-extended variable magnitude to the universal covering, simply-connected, Riemann surface of this function, where it becomes uniformly valued. The type of uniformization applicable to a surface with holes depends on the *genus* of this surface. More precisely, those of genus 0 are their own universal covering Riemann surfaces, isomorphic to the Riemann sphere; those of genus 1 are universally covered by the complex plane; and all those of genus at least 2 are universally covered by the unit disk.

The basic idea here is the following: If we think of a hole on a surface as an obstacle, then the emergent simply-connected geometric unfolding of the great majority of these surfaces, involving all cases where we have two or more obstacles, conforms to the norms of neither the spherical, nor the flat geometry, but to those of the hyperbolic geometry.



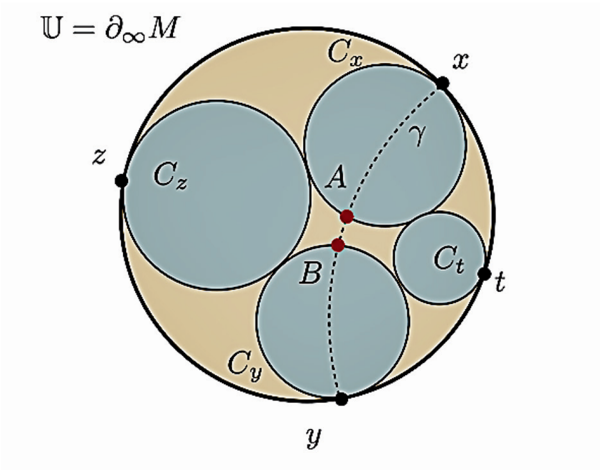
5.8 HYPERBOLICITY: LOGARITHMIC CROSS RATIO ON THE CONFORMAL DISK

The hyperbolic plane bears four different models by means of which it can be explicated. The first is the *upper half* complex plane model, the second is the *Beltrami*-Klein projective disk model, the third is the *Lorentz* hyperboloid model, and the fourth is the *Poincaré conformal* unit disk model in the complex plane. We are going to focus on the conformal disk, and the upper half plane model, in what follows.

The points of the Poincaré disk model of the hyperbolic plane are the points which are *interior* to the *unit disk* in the complex plane, and the lines are the diameters and the arcs of circles which are *orthogonal* to the boundary of the disk. The definition of the distance between two points z_1 and z_2 in the interior of the disk is secured by constructing the *hyperbolic line* joining them. We also take into account that ζ_1 and ζ_2 are the endpoints of the diameter or arc of circle determining the hyperbolic line. Then the distance between the points z_1 and z_2 is:

$$d(z_1, z_2) = |\, log(z_1, z_2, \zeta_1, \zeta_2)\,|$$

where $(z_1, z_2, \zeta_1, \zeta_2)$ denotes the *cross-ratio* of these four points, which is real and positive, so that the logarithm is definable. The *geodesics* in the unit disk are the circles and the lines in the plane that are orthogonal to the unit circle at two points.



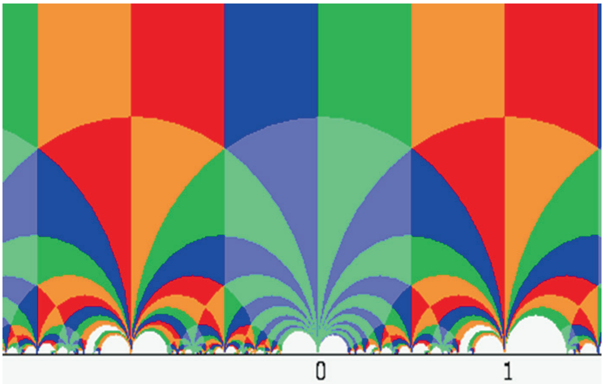
Poincaré assumes the existence of a world *enclosed* in a large *boundary circle* and subject to the following *law*: The temperature in this world is not uniform; it is largest at the center, and it diminishes as one moves away from the center, so that it reduces to absolute zero when one reaches the boundary circle where this world is enclosed. He considers the non-uniform variation of the temperature as follows: Let R be the radius of the limit boundary circle, and let r be the distance from the point under consideration to the centre of this circle. The absolute temperature will be proportional to $R^2 - r^2$. He additionally assumes that, in this world, all bodies have the same coefficient of dilatation, in such a way that the length of any ruler shall be proportional to its absolute temperature. Finally, he assumes that an object transported from one point to another, whose temperature is different, shall immediately reach thermal equilibrium with its new location. A moving object will then become smaller and smaller as it approaches the boundary circle. If this world is finite from the point of view of our customary geometry, it will appear infinite to its inhabitants. In fact, when they intend to approach the boundary circle, they will get colder and become smaller and smaller. The steps they take are therefore also smaller and smaller, so that they can never actually reach this boundary.

In the same geometric context, *Harold Scott Coxeter* remarks that the conformal disk model is an *inversive* model of the infinite hyperbolic plane geometrically, using a circular "nutshell" figure to stand for the *Absolute*. When *Hamlet* exclaims (in Act II, Scene II) "I could be bounded in a nutshell and count myself a king of infinite space" he is providing a poetic anticipation of Poincaré's disk model.

Note that angles in the unit disk model of the hyperbolic plane are measured in the Euclidean way, so that the measure of the angle between two hyperbolic lines is the Euclidean measure of the angle between their tangents. Moreover, it is not adequate to consider only the hyperbolic plane disregarding its boundary. In the disk model, the boundary is realized by the unit circle.

Let us consider now the *upper half plane* model of the hyperbolic space. Here the points of the geometry are the points in the upper half plane, and the lines are either vertical rays from points on the real axis or semicircles with diameter on the real axis. Given two points z_1 and z_2 in the upper half plane, the Euclidean perpendicular bisector of the Euclidean segment joining them meets the real axis at the center of a semicircle through the two points.

If we consider that ζ_1 and ζ_2 are the endpoints of the semicircle, we can define the notion of distance in this model as in the Poincaré disk model, i.e. $d(z_1,z_2) = |log(z_1,z_2,\zeta_1,\zeta_2)|$. Since an *August Möbius* transformation exists, which maps the unit disk model to the upper half plane model of the hyperbolic plane, there is a one-to-one mapping of one model to the other, which preserves all the hyperbolic distances and constitutes these two models of the hyperbolic plane as *isometric images* of one to the other.



5.9 GENUS AND CURVATURE: THE INTRINSIC GEOMETRY OF RIEMANN SURFACES

The Uniformization theorem encapsulates, via the notion of conformal equivalence, the *geometry* of the universal covering space, where the the complete unfolding of a multi-valued function takes place. In particular, since the universal covering is a simply-connected geometric space, the function becomes uniformly-valued on this space. From the algebraic-topological view of the first homotopy or, equivalently, fundamental group, Riemann's metaphora to the universal covering space of a base topological surface bearing holes targets the *annihilation* of the fundamental group of the base by continuous unfolding until trivialization.

Concomitantly, from an information-theoretic perspective, the same metahora may be thought of as a *temporal* process of complexity contraction, or symmetry reduction, in the *Galois* theoretic sense. More specifically, the monodromy action of the fundamental group on the fiber of the universal covering space over a marked point amounts to the Galois group-theoretic *encoding* of Riemann's covering principle utilizing Poincaré's first homotopy group as a Galois group.

According to the Uniformization theorem, the type of uniformization applicable to a topological surface bearing holes depends on the *genus* of this surface. Precisely, those of genus 0 constitute their own universal covering Riemann surfaces, which due to simple-connectivity, are all isomorphic to the Riemann sphere; those of genus 1 are universally covered by the complex plane; and those of genus at least 2 are universally covered by the unit disk.

Poincaré's conception of uniformization is enacted in terms of a metaphora from the *topological* and *complex analytic* domain to the *geometric* domain. In the latter domain only three types of geometric spaces exist distinguishable by their curvature; the three classical geometries of constant curvature. Namely, the Riemann sphere bearing *positive* curvature, the Euclidean (or complex) plane bearing *zero* curvature, and the hyperbolic plane bearing *negative* curvature. Analogously to the case of elliptic functions, i.e. doubly-periodic functions, becoming uniformly valued on the complex plane, another type of functions, becoming uniformly valued on the hyperbolic plane, exist. These functions are called *Fuchsian functions*, since they have been first introduced by *Lazarus Immanuel Fuchs* in relation to the solution of *differential* equations with singular points. In this manner, the Fuchsian functions are to the geometry of the hyperbolic plane what the doubly-periodic functions are to the geometry of the Euclidean plane.

The metaphora from the geometric to the complex-analytic and topological domain emerges through the *action* of a certain group of symmetries or, *automorphisms*, on one of these three classical geometries of *constant curvature*. Geometrically, the group of symmetries is an *isometry* group of transformations that induces a tessellation on the simply-connected universal covering space, identified with one of the three types. In particular, tessellation implies the existence of a fundamental *polygon* that periodically tiles the universal covering space. In the case of doubly-periodic functions, the fundamental polygon is a parallelogram that tiles the complex plane, thus analogously, in the case of Fuchsian functions the fundamental polygon always possesses *more* than *four* sides tiling the hyperbolic plane.

The metaphora from geometry to analysis emerges by the consideration of this group as a group of *conformal* symmetries. In the case of the hyperbolic plane, this group is called a *Fuchsian group*. Finally, the metaphora from geometry to topology emerges by the consideration of this group in terms of the *fundamental group* of the underlying topological surface bearing holes, and which is universally covered by the corresponding tessellated geometric space. Concomitantly, a metaphora also emerges from geometry to algebra by identifying the same group of symmetries as the *Galois group* of covering automorphisms of the universal covering space. In this way, every Riemann surface can be realized as the topological *quotient* with respect to the action of a certain group of conformal symmetries on the three types of geometric spaces.

Concerning all of these three types, the conformal symmetries are Möbius transformations, i.e. *fractional linear* transformations, or else, *homographies*. In more detail, the only group acting on the sphere is the trivial group, so the only Riemann surface that we obtain is the sphere itself with the conformal structure. For the complex plane, the group of conformal symmetries is either cyclic, or a group generated by two independent translations on the complex plane, which captures the elliptic functions. Therefore, the only surfaces that can emerge by taking the quotient are the complex plane itself, considered as equivalent to the sphere bearing one puncture, the sphere bearing two punctures, and the torus. Every other Riemann surface is actually the quotient of the hyperbolic plane by a group of conformal symmetries of the hyperbolic plane. Such a group, a Fuchsian group, must be discrete and should act *properly discontinuously*.

The important idea is that, since a Fuchsian group is *also* a group of isometries of the hyperbolic geometric plane, the geometric structure *descends* under the quotient morphism to the Riemann surface. Due to this property, every Riemann surface comes equipped with a *natural* intrinsic geometry, completing the metaphora from geometry to complex analysis. Note that in the *generic* case, this geometry is *non-Euclidean*, namely it is of the hyperbolic type. Moreover, since the action of a Fuchsian group, as a discrete group of conformal isometries, *does not* have any fixed points, the Fuchsian group is isomorphic with the fundamental group of the Riemann surface, so completing the metaphora from geometry to topology as well.

We conclude that, essentially every Riemann surface, with the exception of the sphere, the plane, the punctured plane, and the torus, arise as *quotients* of the *hyperbolic* plane by a discrete and properly discontinuous group of conformal isometries, inheriting its *own* characteristic hyperbolic geometry by *projection* in this way. Conversely, the hyperbolic plane, instantiates the *universal* covering space of all those Riemann surfaces arising in the above manner, whose *Galois group* of covering automorphisms is identical with the fundamental group of the Riemann surface it covers universally, being in turn, identical with the pertinent *Fuchsian group* of conformal isometries.

Let us consider the subgroup of Möbius transformations, defined on the Riemann sphere, that map the unit disk into itself. The composition of any Möbius transformation in this subgroup with complex conjugation also maps the unit disk onto itself. Thus, we may consider the *subgroup* I of the group of *extended* Möbius transformations (including *antihomographies*) mapping the unit disk onto itself. Since automorphisms in the unit disk belonging to this subgroup map lines and circles to lines and circles and are also conformal, the hyperbolic line between any two points in the interior of the unit disk will be mapped to a hyperbolic line between their respective images. Moreover, since transformations in I *preserve* the cross-ratio of four points on the same line or on the same circle, they preserve the hyperbolic distance between two points in the unit disk. Therefore, the conformal automorphisms of the unit disk are isometries of the Poincaré disk model of the hyperbolic plane, and thus, I is the *isometry group* of the Poincaré disk model of the hyperbolic plane. Notice that the action of the isometry group I on the open unit disk is a transitive group action.

Restricting our focus only on Möbius transformations, that is, on homographies, these symmetries of the extended real line preserving the upper half plane are characterized as linear fractional transformations with real parameters a, b, c, and d, where $ab-cd>0$. This group is isomorphic to the *projective group* $PSL(2,\mathbb{R})$, the group of 2×2 matrices with real coefficients, whose determinant is equal to 1, and each matrix M is considered as equivalent to $-M$. Additionally, it follows that, each of these Möbius transformations is an *isometry* of the upper half plane with its hyperbolic metric that preserves the *orientation* as well. In this model of the hyperbolic plane, the geodesics are the *semicircles* orthogonal to the real line including the vertical semilines.

Coming back to the Poincaré disk model of the hyperbolic plane, a group of conformal isometries I of the disk is a Fuchsian group if it is equipped with the *discrete* topology. Let us consider the action of a Fuchsian group on the hyperbolic disk. We say that a subset of the disk is a *fundamental set* for the Fuchsian group, if this subset contains exactly *one* point from every *orbit* or *fiber* of the Fuchsian group. The idea is to obtain a *fundamental domain*, an open subset of the disk contained in the *closure* of a fundamental set, which under the action of the Fuchsian group is able to tessellate the hyperbolic plane. It is easy to see that a *connected convex hyperbolic polygon* plays the role of a fundamental domain for the action of a Fuchsian group on the disk. Hence, for any given Fuchsian group there is a corresponding convex hyperbolic polygon that tessellates the disk. The *Poincaré polygon* theorem tackles the inverse issue, i.e. given a convex hyperbolic polygon, what are the conditions ensuring the existence of a Fuchsian group for which this polygon functions as a fundamental domain of its action?

For a hyperbolic polygon P a *side pairing* of P is an injective morphism from the set of sides of P to the group of all isometries of the hyperbolic disk, such that for sides s, s', g_s is an isometry with $g_s(s)=s'$, $g_{s'}=g_s^{-1}$, and $g_s(P)$ has an empty intersection with P for all sides s. Then the Poincaré polygon theorem states that considering P a compact and connected hyperbolic polygon of the disk with a side pairing that generates a Fuchsian group of isometries, if every angle of P is equal to $2\pi/n$, for some $n \in \mathbb{N}$, then P is a fundamental polygon for this Fuchsian group.

The process of relating the symmetries of tessellations of the hyperbolic disk to the annihilation of the fundamental group of the underlying Riemann surface is based on the conception of the hyperbolic disk that is being tessellated as the simply connected universal covering surface of this Riemann surface. The latter emerges by taking a quotient topology of the hyperbolic disk based on a specific fundamental domain hyperbolic polygon together with the procedure of side pairing. Thereby, the tessellated hyperbolic disk is the universal covering space of that particular quotient surface and the set of all covering transformations becomes isomorphic to the Fuchsian group that induces this tessellation. It is thus, also isomorphic to the fundamental group of the quotient space, the underlying Riemann surface.

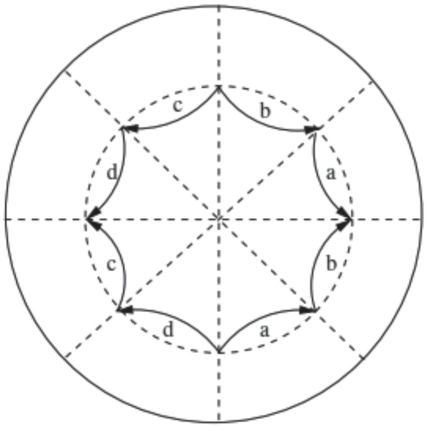
In sum, a Fuchsian group represents a single Riemann surface, and inversely, every Riemann surface (modulo the exception of the cases mentioned in the beginning) is uniformizable by a concrete Fuchsian group acting on the hyperbolic disk up to conformal equivalence. Henceforth, all Riemann surfaces of topological genus greater than or equal to two can be identified with a quotient of the hyperbolic disk by a discrete group of conformal isometries. The precise connection between the conformal and the hyperbolic metric structure is expressed by the *Schwarz-Pick theorem*, asserting that every conformal automorphism of the hyperbolic disk is *contracting* for the hyperbolic metric.

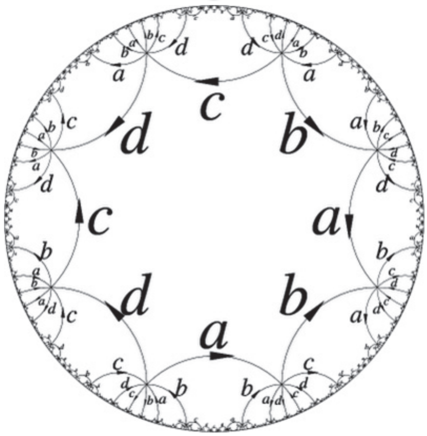
The important notion characterizing Poincaré's metaphora in this setting is that a Fuchsian group used to set up a certain tessellation of the hyperbolic disk by hyperbolic polygons is isomorphic to the fundamental group of the quotient Riemann surface made with that same polygon by isometric side pairings.

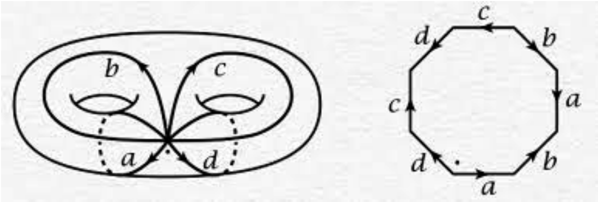
This culminates in the *threefold isomorphic* manifestation of the *same* uniformizing group, comprehending simultaneously, the isometry group inducing a tessellation on the hyperbolic disk, the group of covering transformations of the universal covering space identified with the tessellated hyperbolic disk, and the fundamental group of the underlying quotient Riemann surface. The unifying power of this threefold isomorphic manifestation of the group concept in relation to unveiling the nature of a Riemann surface is illuminated by the realization that the first of these manifestations is of geometric type (in particular, it pertains to the tessellation of the hyperbolic disk by isometries), the second is of the Galois type in a topological setting, and the third is of a homotopic type.

For example, we consider a *regular hyperbolic octagon* O, which is centered in the hyperbolic disk $\mathbb{H}$. We notice that a regular octagon in the Euclidean space has each angle equal to $6\pi/8$, thus its hyperbolic analogue must be less than that. In particular, since O has eight sides, the Poincaré polygon theorem dictates that considering O to be a compact and connected hyperbolic polygon of the disk with a side pairing that generates a Fuchsian group of isometries, every angle of O must be equal to $2\pi/8$, where O is a fundamental polygon for this Fuchsian group, denoted by F. Since the action of F must be free and properly discontinuous, after fixing one vertex of O there can be exactly 8 distinct elements of F that map each vertex of O to the fixed vertex.

Equivalently, this means that a Fuchsian group F that has O as its fundamental domain polygon, can be identified as a group of covering automorphisms of the universal covering space identified with the tessellated hyperbolic disk by hyperbolic octagons. Such a Fuchsian group F is realized by isometric side pairings of O, and is cast isomorphic to the fundamental group of the underlying Riemann surface, obtained by the quotient $\mathbb{H} \twoheadrightarrow \mathbb{H}/F$. The latter is being covered universally by the octagonally tessellated hyperbolic disk, and is identified as a genus two surface Σ_2, isomorphic with the *double torus*.







Since F is realized by isometric side pairings of O, we have a *generator* for each pair of sides, i.e. we have 4 generators, denoted by a, b, c, d. Accordingly, the underlying Riemann surface Σ_2 is manifested in terms of the hyperbolic octagon O with sides identified in pairs according to the *boundary relation* $[a,b]\cdot[c,d]=1$ under the action of F. Thus, in turn, the Fuchsian group F that has O as its fundamental domain, is cast isomorphic with the fundamental group of Σ_2, i.e. the group generated freely by the 4 generators a, b, c, d subject to the boundary relation $[a,b]\cdot[c,d]=1$.

5.10 CONSTELLATORY RAMIFICATION: THE UNIVERSAL COVERING TREE OF A BOUQUET

The abstract algebraic notion of a group is amenable to an elucidating metaphora that crosses from algebra to geometry by the encoding bridge constituted by the action of a group. In the simplest case of a group action on a set, if S_X denotes the *symmetric group* of a set X, i.e. the group of *permutations* of the set X, then the action of a group G on X is equivalent to a group *homomorphism* from G to S_X, denoted by $G \to S_X$.

From this correlation, each element of G gives rise to a permutation of X by acting upon it, in such a way that the composition and the identities are preserved in both groups. The permutation group S_X should be thought of geometrically as the group of *symmetries* of X, thus a group action of G on X provides the means of *realization* of X in terms of the group of symmetries of X. If *no* fixed point emerges under the action of G on X, we say that this action is *free*. Additionally, if we consider the orbit or fiber of the action of G on X at x, i.e. the set $G \cdot x$, and this set reproduces the whole set X under the action of G, then this action is characterized as *transitive*.

Most important is the fact that a group may act on *itself*, and this can happen in two ways. First, a group may act on itself by *left multiplication*, defined by $g \cdot h = gh$. Second, a group may act on itself by *conjugation*, defined by $g \lozenge h = ghg^{-1}$. We recall that the second action distinguishes the *normal* subgroups of a group as well, as those that remain *invariant* under conjugation. Only the first action is free and transitive, in general. Since, the self-action by left multiplication is realized as a group homomorphism $G \to S_G$, every group is realized isomorphically as a subgroup of some symmetric group.

The above metaphora from algebra to geometry is based on the idea of realization of a group in terms of its action on a set, including the cases of self-action. The issue is whether or not we can identify another geometric realization of a group that is more *intrinsic*, in the sense that it is capable of geometrizing the notion of a group itself. For this purpose, we have to consider a group not in terms of its action on a set, but in terms of its action on a *graph*, called the *Arthur Cayley* graph of this group. This task is enunciated by adopting the *combinatorial* perspective on group theory, which refers to the characterization of a group in terms of *generators* and *relations*, or equivalently, in terms of a *presentation* of the group. In the combinatorial setting, there appears first of all, the fundamental notion of a free group.

A *free group* is characterized by a certain number of generators *without any* additional defining relations beyond the existence of an inverse for each generator, according to the general requirements of the notion of a group. Therefore, every morphism of a set of free generators onto a set of elements of any group, is tantamount to a homomorphism of the free group into this group. Consequently, a presentation of a finitely generated group can be expressed as a *quotient* group of a certain free group of finite rank with respect to the *congruence* relations defined among the generators.

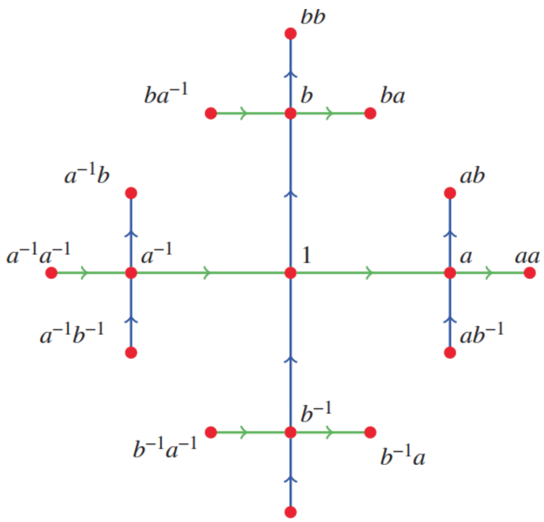
We consider that S is a generating set of a group G. Then the *Cayley graph* of G with respect to the generating set S is a directed, labelled, or even colored, graph $\Gamma(G,S)$ whose set of vertices is the set of elements of G, such that there exists a directed edge from g to gs for every g in G and s in S, labelled by s. Hence, if s_1, s_2, $\ldots$, s_n is an ordered sequence of labels on an edge path in $\Gamma(G,S)$ from the identity 1 to g, then $g = s_1 s_2 \ldots s_n$. Conversely, if $g = s_1 s_2 \ldots s_n$, then an edge path on $\Gamma(G,S)$ from the identity 1 to g exists, whose labels or colors are s_1, s_2, $\ldots$, s_n in this order. Thus, we obtain a correspondence between *edge paths* in the Cayley graph from the identity 1 to g and *words* in the generators representing g in this manner.

Each element of a group G induces a symmetry, i.e. an automorphism of the Cayley graph $\Gamma(G,S)$ of G with respect to the generating set S in the following way: The automorphism Y_g associated with the element g in G is defined on the vertices of the Cayley graph by $Y_g(v)=gv$, identified with the *left action* of G on itself. Since there exists *at most* one directed edge connecting any two vertices, and since no edges connect a vertex to itself, there can be one and only one way to *extend* this action to an automorphism of the entire directed, labelled (colored) graph, i.e. Y_g sends the edge $v \rightarrow vs$ to the edge $gv \rightarrow gvs$. Further, considering that the *word length* of an element g in G with respect to the generating set S is the length of the shortest word in $S\bigcup S^{-1}$ that is equal to g, we derive that the group morphism $G \rightarrow Aut\Gamma(G,S)$ is an isomorphism. In this combinatorial setting, the notion of word length bears a *geometric* signification, in that the word length of g is the *minimum* number of edges in an edge path from the identity 1 to g.

The notion of the free group on *two non-commutative* generators bears a fundamental role. The simplest way to describe a free group is the following: We consider the set of elements $S=\{x_1,x_2,\ldots x_n\}$ in a group G. A word or ordered string $w\in\{S\cup S^{-1}\}^{\pm}$ is said to be freely reduced if it does not contain a substring consisting of an element adjacent to its formal inverse. For instance, the ordered string $w=xyx^{-1}y^{-1}=[x,y]$ is freely reduced, while $z=xy^{-1}yxy$ is not. The group G is a non-Abelian or, equivalently, a non-commutative free group with basis S if S is a set of generators for G and no freely reduced string in the x_i nor their inverses represents the identity of the group. The rank of a free group with basis S is the number of elements of S. We denote a free group of rank 2 by Θ_2. It can be easily shown that all free groups of the same rank are *isomorphic* replicas of each other. So we may identify all of them and talk universally of the non-Abelian free group on two generators Θ_2.

The objective is to characterize the non-Abelian free group on two generators Θ_2 as a group of symmetries, expressed as automorphisms of its Cayley graph with respect to any generating set.

For this, we may consider a and b as the standard non-commuting generators of Θ_2. If we start with the vertex corresponding to the identity 1 in Θ_2, the empty word, then we have to consider *four directed* edges containing 1, namely the edges connecting 1 with the vertices a, b, a^{-1}, and b^{-1}, exhausting in this way all four different elements of Θ_2 with word length 1. For each one of these four acquired vertices, there are going to be *three* new vertices connected to it, or equivalently, three new edges incident to it, according to the following:



Notice that each new vertex is a *reduced word* on the generators a and b, and moreover, since distinct reduced words provide distinct elements of Θ_2, there is *no* vertex that emerges as new more than once in this procedure. Hence, the Cayley graph of the free group Θ_2 with respect to any generating set is actually a *tree*. This is precisely the qualification of a Cayley graph characteristic of *freeness* in two non-commuting generators.

In more detail, the Cayley graph of the free group Θ_2 is a tree whose vertices have *valence* four. From the perspective of this metaphora from algebra to geometry, the free group on two non-commuting generators Θ_2 is characterized as the symmetry group of a 4-valent tree. A 4-valent tree is simply connected topologically, since there are *no* cycles, and can be also endowed with a *metric* making it into a *geometric* space. The considered metric is the path metric, i.e. the metric imposed on the set of its vertices such that the distance between two vertices is the length of the shortest path made through edges connecting these two vertices. Thus, we conclude that a group, together with a generating set, gives rise to a metric space, in such a way that its actions take place through isometries.

The action of the free group Θ_2 on its Cayley graph, on the corresponding 4-valent tree, is an action without any fixed points, meaning that it is a free group action. It turns out that this property characterizes a group *uniquely* as a free group. Equivalently, if a group acts freely on a tree, then this group is a free group. Therefore, the qualification that a group is free amounts to the condition that this group acts freely on a tree, and in sum can completely characterized by its free action as a group on a tree. The consequence of this equivalence is called the *Nielsen-Schreier theorem*, stating that *any* subgroup of a free group is also free.

We recall now that the free group in two non-commuting generators has been utilized for the articulation of a *temporal bond* in terms of a Tripodal link, where the generators of this group are identified with two *temporal actions* whose composition is irreversible, and thus, non-commutative.

We conclude that, if the group Θ_2, i.e. the non-commutative group in two generators is enunciated in terms of its temporal actions, then it is characterized *uniquely* as the symmetry group of a 4-valent tree and conversely. In this sense, the 4-valent tree deciphers the *universal form* of joint unfolding taking place by means of all possible combinatorial compositions of these temporal actions. The 4-valent tree is a simply-connected geometric space, identified in this way, as the universal covering or unfolding space generated by two non-commuting temporal actions. We note that the *growth* of this 4-valent tree is *exponential* and *boundless*. The fundamental role of the free group in two non-commuting generators rests on the fact that a free group in any number of generators bigger than two is actually included as a *subgroup* of Θ_2.

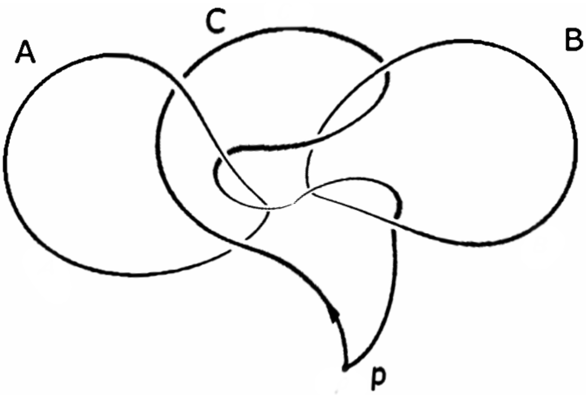
In turn, the 4-valent tree, being simply connected and amenable to the metric structure of its natural path metric, qualifies as a geometric space whose group of symmetries is Θ_2. Taking into account that the action of Θ_2 on this 4-valent tree is free and transitive, it qualifies as a *Galois action*, which in turn, means that Θ_2 is manifested as a group of *covering automorphisms* of this tree in its role as a universal covering space that annihilates the fundamental group of its quotient by this action.

We can elucidate further by considering the free group Θ_2 in its function as the fundamental group of a *bouquet* of two *unlinked* circles (equivalently called a *2-rose*, or a *rose* with two *petals*) whose universal simply-connected geometric covering space is a 4-valent tree.

The fact is that the non-commutative free group in two generators Θ_2, expresses a genuine and non-reducible type of non-commutativity. To grasp this, it is indispensable to stress the behavior of the *group commutator*. The group-theoretic commutator induced by the generators of Θ_2:

$$[\alpha,\beta^{-1}]=\alpha\beta^{-1}\alpha^{-1}\beta$$

produces an irreducible non-commutative and ordered *string* of symbols in Θ_2. This string represents a *based loop* γ as a product loop, which is composed by the ordered composition of the based oriented loops $\alpha\circ\beta^{-1}\circ\alpha^{-1}\circ\beta$. The crucial observation is that deletion of both symbols α and α^{-1}, *reduces* the group commutator to the identity 1. Clearly, the same behavior is encountered symmetrically for both β and β^{-1}.



In particular, the group-theoretic commutator $[\alpha,\beta^{-1}]$ in Θ_2, algebraically encodes the *modular* gluing condition of the based oriented loops α and β^{-1}, which in *three* dimensions corresponds to the

Tripodal link. Because of this, the commutator $[\alpha, \beta^{-1}]$ constitutes the basic element of the fundamental group of a bouquet of two circles, identified with the group Θ_2, from which the non-commutativity of this group is based on. The annihilation of this fundamental group takes place by means of the universal simply connected geometric covering space of the bouquet, identified as a 4-valent tree, whose Galois group of covering automorphisms is precisely Θ_2.

We have deduced that the non-commutative ordered product $\gamma = [\alpha, \beta^{-1}] = \alpha\beta^{-1}\alpha^{-1}\beta$ is not contractible to the identity, due to the homotopic non-deformability of the commutator product loop to a trivial loop. Equivalently, γ belongs to the non-trivial homotopy class of the fundamental group defined on the *complement* of two disjoint, or directly unlinked, topological circles A and B. Conversely, we realize that γ is actually reducible to the identity if α commutes with β^{-1}. Hence, the vanishing of the commutator γ amounts to the *Abelianization* of the fundamental group, which in turn is identified with the Abelian first homology group.

The above provides a crucial insight into the working of *Hurewicz's theorem*, regarding the *interrelation* between the fundamental group and the first homology group in algebraic topology. According to *Witold Hurewicz*, the concept of homotopy is a mathematical formulation of the intuitive idea of a *continuous transition* between two geometrical configurations, whereas the concept of homology gives a mathematical precision to the intuitive idea of a curve *bounding* an *"area"*, or a surface bounding a *"volume"*. The crucial idea is that the basic process of homology theory consisting in *decomposing* a space into smaller pieces with simpler homology structure has *no* counterpart in homotopy theory.

In the preceding setting, the first homology group arises as the Abelian quotient of the fundamental group with respect to its normal subgroup generated by *all* commutators. In this way, reciprocally, non-trivial commutator elements in the fundamental group give a *measure* of deviation from its Abelian shadow manifested by its first homology group. Essentially, the based loops α, β in the fundamental group are viewed as *oriented* and *commuting* 1-cycles representing homology classes, and as such generate the first homology group by means of defining a canonical *free* $\mathbb{Z}$*-module* basis.

Every Riemann surface corresponds to the action of a *unique* discontinuous group of conformal transformations on the Riemann sphere. More precisely, this is a claim regarding the issue of *uniformization* of Riemann surfaces by means of a free, transitive, and discontinuous group action. We have already seen that that a Fuchsian group can be used to set up a certain tessellation of the hyperbolic disk by hyperbolic polygons, where this group is isomorphic with the fundamental group of the quotient Riemann surface emerging from this polygon by isometric side pairings. In this sense, the conformal symmetries effected by a discontinuous group action are actually isometries of the hyperbolic disk.

Of especial interest here is the case of a figure *bounded* by a certain number of *non-intersecting circles* on the Riemann sphere, attention to which was first drawn by *Friedrich Hermann Schottky*, and then by *Christian Felix Klein*, concerning the new light it throws on matters of uniformization.

We recall that homographies or, Möbius transformations, are the conformal symmetries of patterns on the Riemann sphere. More specifically, Möbius transformations are classified into three types as *loxodromic*, *parabolic*, or *elliptic*. Loxodromic transformations have *two* fixed points, one of which may be physically thought of as *attracting* whereas the other one as *repelling*, and are conjugate to scaling by complex numbers except for scaling by unit complex numbers. These ones, whose multiplier is a positive real number, are also called *hyperbolic* transformations. Parabolic transformations have *one* fixed point and are *conjugate* to parallel translations. Elliptic transformations have *two* fixed points and are *conjugate* to rotations.

Since Möbius transformations are the symmetries of patterns on the Riemann sphere, we are interested on the type of pattern obtained, characterized as *simultaneously symmetrical* under the action of *two non-commuting* loxodromic Möbius transformations. The initial set-up consists of a single loxodromic transformation A and a pair of *non-overlapping disks* D and D', selected by the criterion that A maps the *outside* of D to the *inside* of D'.

In this setting, we say that A effects a *pairing* between D and D'. Since, the Möbius transformation A is loxodromic, we consider that the repelling fixed point is inside D, and that the attracting fixed point of the transformation is inside D'. We denote the inverse of A by $A^{-1} := a$. In this manner, we apply the notational convention that the outside of the disk D_A is mapped by a to the inside of the disk

D_a, and the inside of the disk D_A is mapped to the outside of the disk D_a. In the same way, the *bounding circle* C_A of the disk D_A is mapped by a to the bounding circle C_a of the disk D_a; whence the attracting fixed point of a, $F^{+}(a)$, is inside D_a, and the repelling fixed point of a, $F^{-}(a)$, is inside D_A.

We note that successive images of D_A and D_a under respective *iterative* loxodromic actions are *nesting down* toward the attractive and the repelling fixed points of these actions. More concretely, iterative powers of a contract the disk D_a to smaller and smaller disks containing $F^{+}(a)$, whereas iterative powers of $a^{-1} = A$ contract the disk D_A to smaller and smaller disks containing $F^{-}(a)$.

At the next stage, we consider two non-commuting loxodromic transformations a and b which act *jointly* on a constellation of *four* non-overlapping disks, denoted by D_a, D_b, D_A, D_B, where $a^{-1} = A$ and $b^{-1} = B$, according to the generalization of the preceding. Note, that since Möbius transformations always map circles to circles, when any of the four transformations a, b, A, B is applied to any of the four respective disks, all four images are disks themselves. Thus, the transformation a maps the outside of D_A to the inside of D_a, and the inside of D_A absorbs everything except D_a, i.e. $a(D_A) = D_A \cup D_B \cup D_b \cup O$, where O is the region outside the four disks.

Analogous considerations hold for the action of the loxodromic b and its inverse. It is clear that the two non-commuting loxodromic transformations a and b acting jointly on this constellation of four non-overlapping disks, and without imposing any further relations, generate a free non-commutative group, i.e. the group Θ_2, called in this context the *Schottky group* on two generators. Repeated application of all the four transformations, i.e. applying a and b and their inverses to form *words* in the free group Θ_2, leads to an action of this group on the constellation of four non-overlapping disks that is characterized by a *repetitive* pattern at all different *levels* of *magnification*.

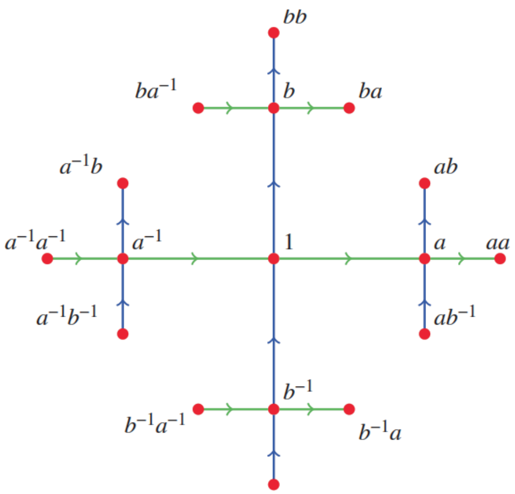
More precisely, each of the disks involved contains three smaller disks, each of which in turn contains three smaller disks, and so on *ad* *infinitum*. As a result, disks within disks come to nest down leaving *invariant* at the *limit* a type of *Cantorian dust*, consisting of those *points* that belong to the disks at every single level of the process. We may think of a *single particle* of dust at the limiting end point of each infinite *chain* of disks nested within each other by the action of Θ_2. In this sense, the *limit-set* of this group action consists of these particles of dust, and hence, this limit-set is the *invariant* pattern under the *action* of Θ_2 in the Schottky setting identified with the *closure* of all attracting and repelling points.

5.12 HYPERBOLIC BOUNDARY: INVARIANT PATTERN OF FREE GROUP'S ACTION

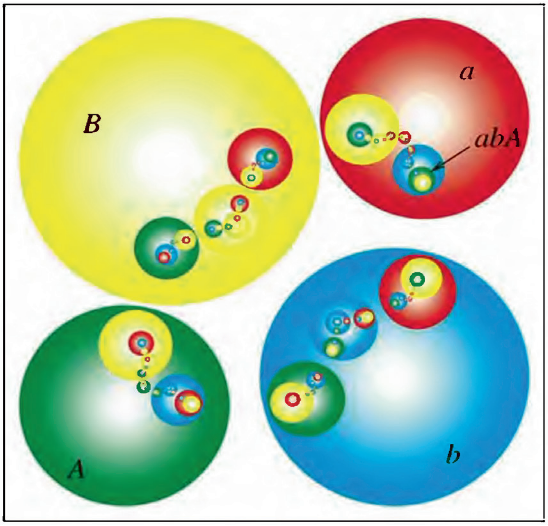
We remind that the non-commutative group Θ_2 is realized by its free action on a 4-valent tree, giving rise to a boundless exponential schema of growth. In turn, this 4-valent tree is simply connected and amenable to the metric structure of its natural path metric. Thus, it qualifies as a geometric space whose group of symmetries is Θ_2. Taking into account that the action of Θ_2 on this 4-valent tree is free and transitive, it qualifies as a Galois action.

Thereby, Θ_2 is realized as a group of covering automorphisms of a 4-valent tree in the role of the latter as a universal covering space annihilating the fundamental group of its quotient by the action of Θ_2. In the present setting, the action of Θ_2 as a free group generated by two non-commuting loxodromic transformations a and b which act jointly on a constellation of four non-overlapping disks on the Riemann sphere may be comprehended by means of the concomitant 4-valent tree. Equivalently, this 4-valent tree is able to record the associated pattern of nesting disks within disks under the action of Θ_2, by explicating graphically the organization pattern of nested disks at different levels of semantic unfolding.

If we consider a and b as the standard non-commuting generators of Θ_2, and we start with the vertex corresponding to the identity 1 in Θ_2 at level 0, i.e. the empty word, then we have to consider four directed edges containing 1, i.e. the edges connecting 1 with the vertices a, b, a^{-1}, and b^{-1}, exhausting in this way all four different elements of Θ_2 with word length 1, at level 1. In the present setting, we may identify these four vertices with the representation of the four disjoint disks D_a, D_b, D_A, D_B respectively. For each one of these four acquired vertices, there are going to be three new vertices connected to it, or equivalently, three new edges incident to it at level 2, according to the following diagram:

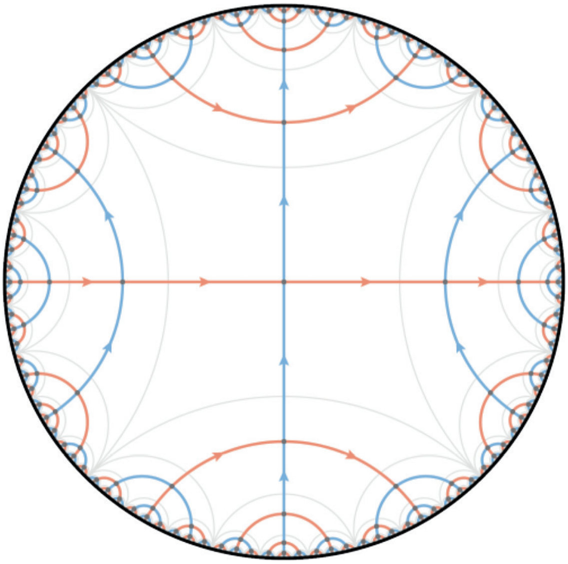


We note that the representation of each new disk is formed by the application of a certain reduced word in the group Θ_2 to one of the initial four disjoint disks D_a, D_b, D_A, D_B. Moreover, the *name* of each new disk at a level *remembers* the original disk at the previous level from which it emamantes, together with the action applied to the latter, so the disk D_{aa} at level 2 is tautologous to $a(D_a)$, and so on. Therefore, the unfolding of the 4-valent tree represents the nesting of disks, for instance the disk $D_{abA} = a(D_{bA}) = ab(D_A)$ at level 3, is inside the disk D_{ab} at level 2, which correspondingly is inside the disk D_a at level 1. In consequence, we obtain a nested chain of disk inclusions $D_{abA} \subset D_{ab} \subset D_a$.



The limit-set of the action of Θ_2 consisting of a type of Cantorian dust, and hence representing, the invariant pattern under the action of Θ_2 identified with the closure of all attracting and repelling fixed points of loxodromic transformations on the four disjoint disks on the Riemann sphere, may be thought of as the *boundary* of the 4-valent representing tree. The region on the Riemann sphere, which is outside the initial four disjoint disks, denoted by O, is the fundamental domain of the group action of Θ_2. The region of the Riemann sphere that is filled up by the copies of the fundamental domain O under the action of Θ_2 is called the *domain of discontinuity* of this group action because all replicas of O remain separated from each other *without* getting stacked.

Each of the loxodromic transformations generators of Θ_2, i.e. a, b, and their inverses a^{-1}, and b^{-1}, *replicate* conformally the fundamental domain O within each of the four involved disjoint disks accordingly. By these means, under the action of Θ_2 the fundamental domain O tessellates the whole Riemann sphere except the limit-set, identified with the boundary of the infinite 4-valent tree of unfolding.



It was Klein's student *Walther von Dyck*, who first realized that Klein's emphasis on Schottky's model, involving a group acting on the Riemann sphere where no reduced nontrivial combination of elements fixes the sphere, corresponds to the action of a free group, providing this group with its name. In this context, what is called *"Klein's criterion"*, furnishes the *conditions* under which a group acting on a set is characterized as a free one. "Klein's criterion" assumes that a and b generate a group acting on a set S. If S has disjoint nonempty subsets S_a and S_b, and $a^n(S_b) \subset S_a$ as well as $b^n(S_a) \subset S_b$, then this group is isomorphic to Θ_2. Klein's criterion has been employed in the context of the group of Möbius transformations, i.e. the conformal automorphisms of the Riemann sphere, characterized as linear fractional transformations with complex coefficients a, b, c, and d, where $ab - cd = 1$. This group is isomorphic to the *projective group* $PSL(2,\mathbb{C})$, the group of 2×2 matrices with complex coefficients, whose determinant is equal to 1, and each matrix M is considered as equivalent to $-M$. Note also that the projective group $PSL(2,\mathbb{C})$ is the group of metrical symmetries, i.e. orientation-preserving isometries of the three-dimensional hyperbolic space. Klein formulated his criterion in order to identify a Schottky group as a free subgroup generated by two loxodromic generators of the projective group $PSL(2,\mathbb{C})$.

A natural question arising in the context of Riemann surfaces is whether it is possible to consider the free non-commutative group on two generators as a *uniformizing group* in terms of its action on the fundamental domain O and the induced tessellation of the Riemann sphere *except* the identified limit-set, i.e. the closure of the attracting and repelling fixed points of the two non-commuting loxodromic generators of this free group, identified as a Schottky group in the setting of our discussion. Notice that the free group on two generators is manifested here as a *discrete subgroup* of $PSL(2,\mathbb{C})$.

This should be compared with the action of a Fuchsian group, i.e. a group of conformal isometries of the extended real line that preserve the upper half plane, identified as a discrete subgroup of $PSL(2,\mathbb{R})$, and used to set up a certain tessellation of the hyperbolic disk by hyperbolic polygons. We emphasize that the same group is isomorphic to the fundamental group of the quotient Riemann surface made with that same polygon by isometric side pairings, culminating on the *threefold isomorphic* manifestation of the *same uniformizing* Fuchsian group: as an isometry group inducing a tessellation on the hyperbolic disk; as the group of covering transformations of the universal covering space identified with the tessellated hyperbolic disk; and as the fundamental group of the underlying quotient Riemann surface.

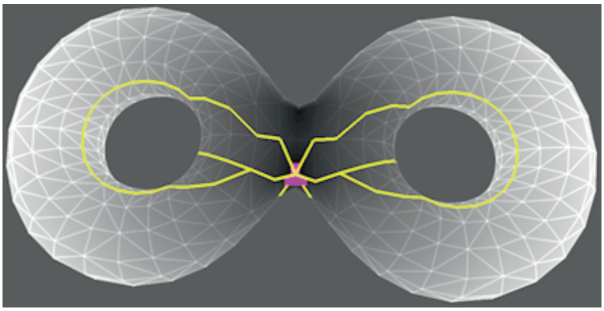
For the purpose of uniformization in the Schottky setting, we consider the region O on the Riemann sphere being outside the four disjoint disks, or equivalently, the fundamental domain of the group action of Θ_2, generated by a, b, where their inverses are $a^{-1}=A$, and $b^{-1}=B$. We may glue each point p on the boundary circle C_a of the missing disk D_a with its symmetrical point $A(p)$ on the boundary circle C_A of the missing disk D_A under the action of the generator a of Θ_2. The same goes for the two other boundary circles C_b and C_B of the missing disks D_b and D_B respectively, under the action of the generator b. The first gluing gives rise to a *handle* on the Riemann sphere, and analogously the second gluing gives rise to *another* handle.

Thus, we obtain a Riemann surface bearing *two* handles, i.e. a Riemann surface of *genus two*, identified with the *double torus*. In this manner, the double torus *admits uniformization* by means of the *action* of the *free* group Θ_2 on the Riemann sphere *without* four disjoint disks. Note that in this case, the action of the free group on two generators Θ_2

is *not* considered on the simply connected universal covering space of the double torus, but on the covering space depicted by the domain of *discontinuity* of this free group action.

More generally, the *retrosection theorem* of Koebe, states that every compact Riemann surface S can be represented as the quotient Ω / Θ_g, where Θ_g is the *free* group on g generators, g is the *genus* of the Riemann surface, and Ω is the domain of *discontinuity* of this free group action. Not only this, but additionally, the image of the $2g$ boundary circles on the Riemann sphere under the quotient morphism may be identified with $g+g$ oriented loops and their inverses in the fundamental group of S.

Note that for a compact Riemann surface of genus g, its fundamental group is generated by $2g$ oriented loops $l_1,\ldots,l_g$ and $\lambda_1,\ldots,\lambda_g$ *modulo* the *commutator* relations $\prod_{i=1}^{g}[l_i,\lambda_i]=1$. Hence, g from the generators of the fundamental group of S, for instance l_1, and l_2, in the genus-two case, may be identified with the *images* of the corresponding *boundary* circles on the Riemann sphere under the quotient morphism $\Omega \twoheadrightarrow \Omega / \Theta_g$. Moreover, the latter is a covering space projection morphism, such that its group of covering automorphisms constitutes the *smallest normal* subgroup of the fundamental group of S containing the generators $l_1,\ldots,l_g$. In this way, the free group on g generators, Θ_g, where g is the genus of the Riemann surface, is identified with the quotient of the fundamental group of S modulo the above-defined normal subgroup.



We conclude that while *Fuchsian uniformization*, utilizes the universal covering space of a Riemann surface with the Fuchsian group corresponding *isomorphically* to the fundamental group of this Riemann surface, *Schottky uniformization*, expressed in terms of the action of the free non-commutative group, as we have seen, corresponds to the *intermediate* covering space obtained by means of the quotient of the Fuchsian group by the normal subgroup generated by *half* of the generators of the fundamental group. Finally, using the fact that Θ_g is a subgroup of Θ_2 for every $g > 2$, we realize the *universal* role of the free non-commutative group in two generators Θ_2 in the uniformization of Riemann surfaces by means of covering spaces.

GEOMETRIC CALCULUS: DIFFERENTIAL FORMS AND INTEGRATION

6

Gottfried Wilhelm Leibniz's notion of infinitesimal analysis, culminating in the development of the theory of differentials, targets the development of a genuine type of *"geometric calculus"* by algebraic means, one based not on artificial choices of coordinates and other subjective conventions, but pertaining to the geometric objects themselves, as an articulation of an *extension* process from the infinitesimal to the global. The notion of a geometric object is conceived in terms of the *spectrum* of a commutative *ring* or, an *algebra* over a field, like the real or complex numbers. The idea of a spectrum, originating from physics, refers to what can be *observed* through this ring by evaluating it into a field of measurement *scales*. In this sense, the ring or, algebra, on which a *differentiation* procedure may be applied, as pertaining to an infinitesimal type of extension, is qualified as an algebraic structure whose elements are observables.

In this line of thought, the *ideal theory* of rings, conceived first by *Richard Dedekind*, provides the necessary additional conceptual and technical metaphorical means for a precise rendering of Leibniz's ideas in modern terms.

A non-empty subset $\mathcal{I}$ of a commutative ring $\mathcal{A}$ is called an *ideal* if the following properties hold:

i $\mathcal{I}$ is an *additive subgroup* of $\mathcal{A}$, i.e. for every x, $y \in \mathcal{I}$, we have $x-y \in \mathcal{I}$;

ii $\mathcal{I}$ is *stable* with respect to *multiplication* with elements of the ring $\mathcal{A}$, viz., for every $x \in \mathcal{I}$ and for every $y \in \mathcal{A}$, the product xy is being *absorbed* in $\mathcal{I}$, viz. $xy \in \mathcal{I}$.

It is clear that for every ring $\mathcal{A}$, the subsets 0 and $\mathcal{A}$ itself are ideals of $\mathcal{A}$, the trivial ideals. Moreover, if $\mathcal{I}$ contains an invertible element of $\mathcal{A}$, then $\mathcal{I} = \mathcal{A}$. Then, the only ideals of a field are 0 and the field itself. Let us now consider a *homomorphism* of rings $u : \mathcal{A} \to \mathcal{B}$, to be physically thought of as a measurement procedure of a ring of observables $\mathcal{A}$, by a ring of measurement scales $\mathcal{B}$. We recall that the *kernel* of u is defined as follows:

$$Ker(u) := \{a \in \mathcal{A} \mid u(a) = 0\} = u^{-1}(0)$$

Then $Ker(u)$ is clearly an ideal of $\mathcal{A}$. The kernel of the homomorphism u depicts the set of observables in $\mathcal{A}$, whose evaluation is the *neutral*

additive element of the rings of scales $\mathcal{B}$. Hence, the ideal $Ker(u)$ of the ring of observables $\mathcal{A}$, can be interpreted geometrically as a *schematization perspective*, with respect to which the measurement procedure $u:\mathcal{A}\to\mathcal{B}$ is effectuated. Obviously, the set of all observables in $\mathcal{A}$ constituting a perspective, identified as the ideal $Ker(u)$ of $\mathcal{A}$, in relation to a measurement procedure $u:\mathcal{A}\to\mathcal{B}$, are being evaluated to the 0 of the ring of scales $\mathcal{B}$.
Epigrammatically, we may say that a geometric schematization perspective related with a measurement procedure of a ring of observables, constitutes the kernel of its evaluation at a ring of scales identified with the *inverse image* of the additive neutral element in $\mathcal{B}$, which is the zero scale.

This basic conceptual point can be further clarified by introducing the notion of a quotient ring. More concretely, if we have at our disposal a ring $\mathcal{A}$ and an arbitrary ideal $\mathcal{I}\subseteq\mathcal{A}$, then, we can define the *quotient ring* $\mathcal{A}/\mathcal{I}$. The essence of this construction amounts to declaring equal to *zero* everything that is in $\mathcal{I}$. We can transform this idea into something precise by remarking that, if every element of $\mathcal{I}$ has to be considered as zero, then two elements x, y of $\mathcal{A}$ whose difference $x-y$ is in the ideal $\mathcal{I}$ have to be considered as *equal*.

This procedure produces a new ring, the pertinent quotient ring as follows: We define the following equivalence relation in $\mathcal{A}$: $x \Diamond y$ if and only if $x-y \in \mathcal{I}$, where $x, y \in \mathcal{A}$. The key feature of the equivalence relation defined above, is that it is compatible with the ring-theoretic operations on $\mathcal{A}$, or else, it is being preserved by addition and multiplication in $\mathcal{A}$, formally, if $x \Diamond x'$ and $y \Diamond y'$, then $x+y \Diamond x'+y'$ and $xy \Diamond x'y'$. Therefore, we can legitimately define two operations on the quotient set $\mathcal{A}/\Diamond := \mathcal{A}/\mathcal{I}$, and hence, $\mathcal{A}/\Diamond$ becomes a ring with respect to these operations, that is, the quotient ring $\mathcal{A}/\mathcal{I}$.

In order to translate the information encoded into the ring $\mathcal{A}$, into the quotient ring $\mathcal{A}/\mathcal{I}$ we employ a morphism of rings $\mathcal{A}\to\mathcal{A}/\mathcal{I}$. In fact, the *canonical* morphism $\iota:\mathcal{A}\to\mathcal{A}/\mathcal{I}$ given by $a\mapsto[a]:=a+\mathcal{I}$ is a *surjective* morphism of rings, the canonical epimorphism. Thus, we finally have:

$$Ker(\iota)=\mathcal{I}$$

The above clarifies the essential idea behind the construction of the quotient ring, of declaring equal to zero everything that is in $\mathcal{I}$. Put simply, when we change the ring from $\mathcal{A}$ to $\mathcal{A}/\mathcal{I}$ by means of the canonical epimorphism ι, then everything in $\mathcal{I}$ goes to *zero*.

Thus, in relation to the intended interpretation, we form the following conclusion: a geometric schematization perspective related with a measurement procedure of a ring of observables $\mathcal{A}$, identified as an ideal $\mathcal{I}$ of $\mathcal{A}$, constitutes the kernel of its evaluation at a ring of scales, identified, in turn, as the quotient ring $\mathcal{A}/\mathcal{I}$.

From then on, based on the information encoded in the quotient ring $\mathcal{A}/\mathcal{I}$ we can easily obtain the following classifications: an ideal $\mathcal{I}$ is *maximal* if and only if $\mathcal{A}/\mathcal{I}$ is a *field*, whereas an ideal $\mathcal{I}$ is *prime* if and only if $\mathcal{A}/\mathcal{I}$ is an *integral domain*, or equivalently, a ring without divisors of zero. Obviously, every maximal ideal is a prime ideal.

Intuitively, a maximal ideal is an ideal which is second only to the entire ring. A prime ideal can be conceived as a *measure* of the *complexity* of a ring. This is based on the observation that if a ring $\mathcal{B}$ is a field, then there is *only one* prime ideal, namely the zero ideal, since it is the only ideal besides $\mathcal{B}$, and moreover, it is prime since $\mathcal{B}$ is an integral domain. Thus, prefixing the ring of measurement scales to a field, as in classical physical theories, where the field is that of the real numbers, we assume a *zero complexity* of scales. A scale of zero complexity is unable to register a stable irreducible process, or a process of indistinguishability between the observed and the means of observation, and hence, it coordinatizes the geometric spectral point-states, corresponding to maximal ideals in the ring of observables, in terms of numbers. Hence, the state-manifolds constructs of classical theories are geometric spaces built on the assumption of zero complexity of measurement scales, and consequently, constitute a kind of an arithmetic continuum, which, incorporates the hypothesis of a *pre-assigned* demarcation boundary between the observed and the observational means.

If we consider instead the ring of the integers $\mathcal{Z}$ as a ring of measurement scales, the prime ideals are the maximal ideals $p\mathcal{Z}$, where p is a prime number, as well as the zero ideal. Then, it is not a coincidence that prime integer measurement scales coordinatize stable and not further reducible spectral perspectives, represented geometrically as *pure* states. More generally, the notion of a prime ideal in an abstract ring of measurement amounts to a stable, and not further reducible spectral perspective, playing exactly the role of a measure of *irreducible* complexity.

In this line of reasoning, any ideal in a ring expresses a stable perspective geometrically, capable of being *decomposed* or *factorized* into perspectives of irreducible complexity enunciated by prime ideals. Equally from this viewpoint, the algebraic theory of factorization of rings into prime or even primary ideals deserves special attention. We point out that the notion of a *primary* ideal generalizes that of a prime ideal, in the sense that it is characterized by a quotient ring that is allowed to have zero divisors, but constrains them to be strictly *nilpotent*. Moreover, both primary and prime ideals, in contradistinction to maximal ideals, are characterized by a covariance property under homomorphisms of rings, meaning that they have inverse images that are *still* primary or prime, hence, they behave well under base ring *change*.

6.2 SPECTRALITY: PRIME IDEALS AND GEOMETRIC PURE STATES

The duality between commutative algebras of observables and geometric state-spaces, is based primarily on the idea that evaluating an observable f at a state x is the *same* as evaluating x at the observable f:

$$x(f) = f(x)$$

More precisely, if we consider the simplest possible case, where, $\mathcal{A}$ is the $\mathcal{F}$-algebra of functions on a set X with values in a field F, any element $x \in X$ defines a morphism $\mathcal{A} \to \mathcal{F}$ by assigning to a function f its value at x. The kernel of this morphism is a maximal ideal μ in $\mathcal{A}$, suggesting in this way that it is possible to *recover* the set X as the set of maximal ideals in $\mathcal{A}$. In case that, $\mathcal{A} = \mathcal{C}^\infty(N)$, the maximal ideal μ_x is the ideal consisting of all smooth functions vanishing at the point x:

$$\mu_x = \{f \in \mathcal{C}^\infty(N) \,|\, f(x) = 0\}$$

Notice that the definition above, forces a decomposition of the algebra $\mathcal{C}^\infty(N)$ into a *direct sum* of linear vector spaces:

$$\mathcal{C}^\infty(N) = \mathcal{R} \oplus \mu_x$$
$$f = f(x) + (f - f(x))$$

and furthermore, the quotient ring $\mathcal{C}^\infty(N)/\mu_x$ is isomorphic to the field of real numbers $\mathcal{R}$.

We proceed by defining more concretely the notion of a *pure state* over a ring of measurement *scales*. Let us assume that $\mathcal{A}$ is a *commutative, unital* $\mathcal{F}$-algebra of observables, where $\mathcal{F}$ is an arbitrary field, and moreover, let $\mathcal{B} \supset \mathcal{F}$ be a ring *without* zero divisors (*integral domain*), interpreted as a ring of measurement scales.

We define a pure state of $\mathcal{A}$ over the ring $\mathcal{B}$ as a surjective morphism of $\mathcal{F}$-algebras $w : \mathcal{A} \to \mathcal{B}$. The notion of a pure state of $\mathcal{A}$ over $\mathcal{B}$ instantiates a schematizing, stable, and irreducible natural perspective, *encoded* geometrically, via the scales of $\mathcal{B}$, by means of evaluations of observables of $\mathcal{A}$ into $\mathcal{B}$, within the context of a corresponding measurement procedure.

Moreover, we say that two pure states, defined over the rings, $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, specified as above, are identical, if there exists an isomorphism of $\mathcal{F}$-algebras $\mathcal{B}_1 \cong \mathcal{B}_2$, effecting an isomorphism between the corresponding measurement scales. Hence, we may define a $\mathcal{B}$-pure state of an $\mathcal{F}$-algebra of observables $\mathcal{A}$, as an *equivalence class* of surjective morphisms of $\mathcal{F}$-algebras $w : \mathcal{A} \to \mathcal{B}$, where, the ring $\mathcal{B} \supset \mathcal{F}$ is called the *coordinatizing frame* of the pure state w.

Notice that according to the definition introduced, each pure state of an $\mathcal{F}$-algebra of observables $\mathcal{A}$ may have a *different* coordinatizing frame, depending upon the integral domain of *scales* employed for measurement. Thus, the *new* notion of a geometric state-space, being built from pure states, is a *multi-valued* one, in the sense that its generalized point-states may be coordinatized by means of different scales, namely scales belonging to different integral domains.

At this stage, it is essential to emphasize for reasons of clarity, that the identification of pure states with *generalized points* of a multi-valued geometric state space, corresponding to irreducible perspectives of observation or measurement, is precisely established by the existence of a *bijective* correspondence between $\mathcal{B}$-pure states of an $\mathcal{F}$-algebra of observables $\mathcal{A}$, where, $\mathcal{B} \supset \mathcal{F}$ is a coordinatizing ring without zero divisors, and the prime ideals of the algebra $\mathcal{A}$. The bijection can be established by defining the following assignment:

$$[w : \mathcal{A} \to \mathcal{B}] \mapsto Ker(w)$$

It is easy to verify, if we consider a prime ideal $\mathcal{I}$ of $\mathcal{A}$, that the quotient ring $\mathcal{A} / \mathcal{I}$ must be an integral domain, and that the canonical epimorphism $\iota : \mathcal{A} \to \mathcal{A} / \mathcal{I}$ is an $\mathcal{A} / \mathcal{I}$-pure state of the algebra $\mathcal{A}$, such that, $Ker(\iota) = \mathcal{I}$.

Thus, we identify the set of all pure states of an $\mathcal{F}$-algebra of observables $\mathcal{A}$ with the set of all prime ideals of $\mathcal{A}$, or equivalently, with the *prime spectrum* of $\mathcal{A}$, denoted by $PSpec\mathcal{A}$. This term denotes the set of all equivalence classes of stable and irreducible spectral schematization perspectives, coordinatized through the measurement scales of integral domains, by means of evaluations of observables of $\mathcal{A}$ into $\mathcal{B}$.

The definition of $PSpec\mathcal{A}$ allows us to think of an observable $f \in \mathcal{A}$, intuitively, as a *function* on $PSpec\mathcal{A}$, in the sense that we have values $f(\mathcal{I})$, defined for each state $\mathcal{I} \in PSpec\mathcal{A}$. However, these functions have the property that the space where their values live *varies* depending on the state where the evaluating is being performed.

Therefore, we may summarize briefly the conceptual shift involved in the definition of a pure state by comparison with the classical definition as follows: In the case of the algebra of observables $\mathcal{C}^{\infty}(N)$, viz., an $\mathcal{R}$-algebra of smooth real-valued functions on a compact real differential manifold N, the value of a function $f \in \mathcal{C}^{\infty}(N)$ at a state χ, corresponding to a maximal ideal μ_{χ} of $\mathcal{C}^{\infty}(N)$ lives in the quotient ring $\mathcal{C}^{\infty}(N)/\mu_{\chi}$, and all of these quotient rings can be canonically identified with the field of real numbers $\mathcal{R}$. In contradistinction, the notion of a generalized pure state introduced here stands opposed to the *absolute* representability principle of the classical theory over the coordinatizing field of real numbers. In its own terms, rather, it allows the geometric representation of stable and irreducible spectral perspectives, in terms of generalized points of a multi-valued geometric state space, and the evaluation of observables at these points. This is achieved by *relativizing* representability over a *multitude* of measurement scales, belonging to different coordinatizing rings without zero divisors, giving rise eventually to the above multi-valued geometric state space, constructed as the prime spectrum of the corresponding ring of observables.

6.3 INFINITESIMAL EXTENSION: NILPOTENT OBSERVABLES AND DIFFERENTIAL FORMS

It is particularly elucidating to clarify the peculiar nature of this multi-valued geometric state space. For this purpose, we consider an observable that vanishes at all states. We will show, by using a simple example, that such an observable is *not necessarily* zero. Consider the $\mathcal{F}$-algebra of *dual* numbers $\mathcal{F}[\varepsilon]/(\varepsilon)^2$ over $\mathcal{F}$.

Formally, the elements of the algebra $\mathcal{F}[\varepsilon]/(\varepsilon)^2$ are constituted by the linear combinations $a+b\varepsilon$, where a, $b \in \mathcal{F}$ and ε is a formal symbol, enunciated as follows: The addition is given by adding coefficient-wise, whereas the multiplication is given by applying $\varepsilon^2 = 0$, thus algebraically rendering ε an infinitesimal unit. Hence, we have that $\varepsilon \neq 0$, but $\varepsilon^2 = 0$. Then, the prime spectrum of this algebra, $PSpec(\mathcal{F}[\varepsilon]/(\varepsilon^2))$ consists of a *single* generalized point-state, corresponding to the unique prime ideal (ε).

The $\mathcal{F}$-algebra of dual numbers is the *smallest* ring with *nilpotent* elements. Hence, we may consider an observable ε on $PSpec(\mathcal{F}[\varepsilon]/(\varepsilon^2))$ that is nilpotent, namely the observable ε itself is *not* the zero function, but its square ε^2 *is* the zero function. Note that in this example, the fact that $\varepsilon^2 = 0$ means that the observable ε takes the value 0 at every (the) state of $PSpec(\mathcal{F}[\varepsilon]/(\varepsilon^2))$, but ε is not considered to be the zero function.

Generalizing, we state that any observable vanishing at all states might not be zero, but some power of it will be zero. In conclusion, observables on the multi-valued geometric space $PSpec\mathcal{A}$ will *not* be entirely determined by their values at states, and thus, we *no* longer distinguish observables based on their values at states. Hence, observables on $PSpec\mathcal{A}$ will have values at states, but are *not* going to be *determined* by those values, instead, they will be entirely determined by their germs. In order to explain the situation clearly, it is necessary to endow $PSpec\mathcal{A}$ with a *sheaf* structure, that intuitively constitutes an encoding of observables which is local and global in a compatible way, as we shall see as we go on.

At this point, we must study the generation of the $\mathcal{F}$-algebra of dual numbers $\mathcal{F}[\varepsilon]/(\varepsilon)^2$ from $\mathcal{F}$ in more detail. Since this extension pertains to any algebra of observables defined over $\mathcal{F}$, as an *infinitesimal extension*, it is important to examine the general case, where for simplicity the scalars can be identified with the real numbers, meaning that $\mathcal{F} \equiv \mathbb{R}$, in this case.

In general algebraic terms, the process of *extending* the observables of an $\mathbb{R}$-algebra A is described by means of a *fibration*, defined as an injective homomorphism of $\mathbb{R}$-algebras $\iota : A \infty B$. Thus, the $\mathbb{R}$-algebra B is considered as a *module* over the algebra A, analogously to a vector space over a field. A *section* of the fibration $\iota : A \infty B$, is represented by a homomorphism of $\mathbb{R}$-algebras $s : B \rightarrow A$ such that $\iota \circ s = id_B$.

The *fundamental* extension of observables of the $\mathbb{R}$-algebra A is obtained by *tensoring* A with itself over the subalgebra of the base field, that is $\iota : A \infty \ A \otimes_{\mathbb{R}} A$. Trivial cases of extensions, in fact *isomorphic* to A, induced by the fundamental one, are obtained by tensoring A with $\mathbb{R}$ from both sides, that is $\iota_1 : A \infty \ A \otimes_{\mathbb{R}} \mathbb{R}$, $\iota_2 : A \infty \ \mathbb{R} \otimes_{\mathbb{R}} A$.

The basic idea of *Leibniz* in conceptualizing *infinitesimal analysis* as a geometric calculus, which was further elaborated by *Riemann* in relation to the development of *differential geometry*, is that it should be thought of in terms of the extension from the *infinitesimal* to the *global*. For this purpose, we consider the extension of the algebra of observables A by infinitesimal quantities, defined as the fibration:

$$d_* : A \infty \ A \oplus M \cdot \varepsilon \quad (1)$$
$$f \mapsto f + d(f) \cdot \varepsilon \quad (2)$$

where $d(f) =: df$ is understood as the infinitesimal part of the extended observable, and ε denotes the infinitesimal unit obeying $\varepsilon^2 = 0$. The algebra of infinitesimally extended observables $A \oplus M \cdot \varepsilon$ is called the *algebra of dual numbers* over A with coefficients in the A-module M. It is immediately evident that the algebra $A \oplus M \cdot \varepsilon$, as an Abelian group is just the direct sum $A \oplus M$, whereas the multiplication is defined as follows:

$$(f + df \cdot \varepsilon) \bullet (f' + df' \cdot \varepsilon) = f \cdot f' + (f \cdot df' + f' \cdot df) \cdot \varepsilon \quad (3)$$

It is also required that the composition of the augmentation $A \oplus M \cdot \text{Ú} \to A$, with d_* is the identity.

Equivalently, the above fibration defined by the injective homomorphism of algebras $d_* : A \infty \ A \oplus M \cdot \varepsilon$, can be reformulated as a *derivation*, that is in terms of an additive $\mathbb{R}$-linear morphism:

$$d : A \to M, \quad (4)$$
$$f \mapsto df \quad (5)$$

that satisfies the *Leibniz rule*:

$$d(f \cdot g) = f \cdot dg + g \cdot df. \quad (6)$$

Seen in this light, the Leibniz rule *descends* from the *linearization* of the above fibration. Since the formal symbols of differentials $\{df, f \in A\}$, are reserved for the *universal* derivation, the A-module M is identified as the free A-module $\Omega := \Omega^1(A)$ of *differential* 1*-forms* generated by these formal symbols, modulo the Leibniz rule, where the *scalars* of the distinguished subalgebra $\mathbb{R}$, that is the real numbers, are treated as constants.

The fundamental insight of *Erich Kähler* in this algebraic setting consists in the realization that the free A-module Ω can be constructed explicitly through the tensor product *self-extension* of A, that is $\iota : A \infty \ A \otimes_{\mathbb{R}} A$ by considering the homomorphism:

$$\mu : A \otimes_{\mathbb{R}} A \to A \quad (7)$$

$$\sum_i f_i \otimes g_i \mapsto \sum_i f_i \cdot g_i \quad (8)$$

Then, by taking the *kernel* of this homomorphism of algebras, that is, the ideal:

$$I = Ker\mu = \{\theta \in A \otimes_{\mathbb{R}} A : \mu(\theta) = 0\} \subset A \otimes_{\mathbb{R}} A \quad (9)$$

we obtain the following: The homomorphism of A-modules

$$\Sigma : \Omega \to \frac{I}{I^2} \quad (10)$$

$$df \mapsto 1 \otimes f - f \otimes 1 \quad (11)$$

is an isomorphism.

Thus, the free A-module Ω of 1-forms is *isomorphic* with the free A-module $\frac{I}{I^2}$ of *Kähler differentials* of the algebra of observables A over $\mathbb{R}$, conceived as distinguished ideals within the algebra of infinitesimally extended scalars $A \oplus \Omega \cdot \varepsilon$, according to the following *split short exact sequence*:

$$\Omega \rightarrowtail A \oplus \Omega \cdot \varepsilon \twoheadrightarrow A, \quad (12)$$

equivalently formulated as:

$$0 \to \Omega \to A \otimes_{\mathbb{R}} A \to A. \quad (13)$$

By *dualizing*, we obtain the *dual* A *-module* of Ω, that is, $\Xi := Hom(\Omega, A)$. Consequently, we have at our disposal, expressed in terms of infinitesimal extension of the algebra A, semantically intertwined with the generation of geometry from the infinitesimal to the global new types of observables related with the incorporation of differentials and their duals, identified as vectors.

If the geometry is generated by a *metric*, then there is an associated *unique* dual to a vector, meaning an *isomorphism* $\tilde{g}$ between the A-module Ω and its dual A-module $\Xi = Hom(\Omega, A)$, that is:

$$\tilde{g} : \Omega \to \Xi, \quad (14)$$

such that:

$$\tilde{g} : \Omega \cong \Xi, \quad (15)$$
$$df \mapsto v_f := \tilde{g}(df). \quad (16)$$

Equivalently, a metric g stands for an $\mathbb{R}$-valued *symmetric bilinear form* on Ω, that is $g : \Omega \times \Omega \to \mathbb{R}$, yielding an *invertible* $\mathbb{R}$-linear morphism $\tilde{g} : \Omega \to \Xi$. Notice that for df, $dh \in \Omega$, a symmetric bilinear form g acts, via $\tilde{g}$, on df to give an element of the dual, $\tilde{g}(df) \in \Xi$, which then acts on dh to give $(\tilde{g}(df))(dh) = (\tilde{g}(dh))(df)$, or equivalently, $v_f(dh) = v_h(df) \in \mathbb{R}$. Also note that the invertibility of $\tilde{g}$ amounts to the property of *non-degeneracy* of g, meaning that for each $df \in \Omega$, there exists $dh \in \Omega$, such that $(\tilde{g}(df))(dh) = v_f(dh) \neq 0$.

6.4 LINEAL EXTENSION: MULTIPLICATION OF DIRECTED LINE SEGMENTS

The geometric essence of Leibniz's infinitesimal analysis, independently of any notion of absolute pre-existing space in the original sense of a geometric calculus, can be enunciated utilizing the metaphora involved in the development of *Hermann Grassmann's* theory of *lineal extension*. The latter emerged out of Grassmann's struggle to establish a *universal apparatus* for research in geometry by *algebraic* means, based on the notion of *extension*. The latter is implemented as a *lineal* process of unfolding geometric observables in a *series* of hierarchically organized, and nested *layers*. This process is expressed in terms of the conceptually inverse geometric actions of joining and separating, involving the ascent and descent respectively, from one layer to another layer. The crucial aspect of this theory is that the algebraic encapsulation of lineal extension necessarily requires the operation of *multiplication*, i.e. the product operation specifying the composition type of geometric observables from layer to layer.

In Grassmann's conception the process of extension is initiated at the first geometric layer, which is occupied by *directed line* segments, to be thought of in modern terms, as vectors. It is very important that the existence of a zero geometric layer, identified with the scalars, is also assumed, but only *implicitly*, and not explicitly, as is the case starting with the first layer. The significance of the above is that lineal extension theory actually constitutes *an indirectly self-referential* schema for the *articulation* of the notion of a *geometric point* through a many-layered metaphora conducted by means of *ascending* to higher and higher geometric layers, and then *descending* back. We argue that it is *precisely this* feature that sheds light on the geometric essence of calculus, in such a way that Grassmann's theory *cannot* be viewed independently from Leibniz's infinitesimal analysis.

The algebraic operation of multiplication that allows ascending is conceived *indirectly* in relation to the operation of addition. More specifically, if there is an operation identified as addition, which is operative within a single layer, then any operation satisfying the *distributive law* in terms of this addition is called multiplication. In this manner, Grassmann introduced the operation of *multiplication* of directed line segments, called the *exterior product*, as a means of their composition, according to the satisfaction of the distributive law with respect to the addition of directed line segments.

The exterior product is an associative, multiplicative operation described as follows, when applied at the first geometric layer: It takes two directed line segments A and B at the first layer, and produces a *directed area* or parallelogram at the second layer. This can be accomplished in *two* different ways that are *not* equivalent to each other. Either the parallelogram $A \vee B$ is the product of replicating B a total of A times along the linear extension of A, and then concatenating or adding these based replications, or the parallelogram $B \vee A$ is the product of replicating A a total of B times along the linear extension of B and then adding these based replications again.

Consequently, we obtain two *oppositely-oriented* extension processes from the linear layer to the bilinear or *area-bounding* layer. They mutually cancel each other in the precise sense that their product is oriented oppositely, meaning that $A \vee B = -B \vee A$. The fact that the multiplication product area of the linear extensions A and B is *directed* or *signed*, according to the above, means that this geometric product is *non-commutative*. Unfortunately in the literature, the "join-$\vee$" composition operation expressed by the exterior product is denoted by the "conjunction-$\wedge$" sign, whose symbolism is the exact opposite to the one intended by this operation. Keeping in mind this cautious remark, we may switch to the "wedge-notation" used in exterior algebra in order to conform with that convention in the literature, according to which $A \wedge B = -B \wedge A$.

We recapitulate that the multiplicative exterior product of two linear directed segments is non-commutative; it depends on the *order* of their composition, and this can take place in *two* possible, oppositely-oriented ways. Henceforth, the *orientation* of the produced signed area is dependent on the order of composition, and thus it is signed. If we identify the first geometric layer in which the independent linear extensions A and B are located as the *first exterior power* space, then the layer that the product signed area is located is identified as the *second exterior power* space. In a totally analogous fashion, the lineal extension process proceeds to higher exterior power spaces using the property of associativity of the exterior product and the property of distributivity with addition.

Analogously to the directed area element $A \wedge B$, the lineal extension process instantiates, at the *third* geometric layer, the *directed volume* element $A \wedge B \wedge C$ for three independent linear extensions A, B, C. Due to the associativity property, this should also be thought of in a *temporally ordered* way, by replicating the directed area of two of them along the linear extension of the third, and then concatenating to obtain the directed volume element. Note that, due to the *antisymmetric* property of the exterior product, if two elements at a layer have a common element of a lower layer, then their product is zero. This provides a conceptual understanding of the lineal extension process driven by the application of this multiplicative product. More precisely, it is named *exterior* because the non-nullity of the product of two extensive geometric magnitudes requires that each one of them is located geometrically to the exterior of the other making them independent.

6.5 EXTERIOR ALGEBRA: ASCENDING THE LADDER OF EXTERIOR POWER VECTOR SPACES

If we consider a finite dimensional vector space V of dimension ν over a field $\mathbb{K}$, then the first layer is identified with the first exterior power vector space $\bigwedge^1 V := V$, the second layer with the second exterior power vector space $\bigwedge^2 V$, and so on. The characteristic of this extension process is that in the μ exterior power vector space, where $\mu < \nu$, we have:

$$\ldots \wedge v_i \wedge \ldots \wedge v_j \wedge \ldots = 0 \quad \text{if} \quad v_i = v_j \quad \forall i \neq j$$

expressing the pertinent *alternating* property. In each exterior power space layer, addition of exterior products located at this layer is allowed, so the general elements of these power spaces are *linear superpositions* of products of elements *descending* to lower layers. Moreover, the vector v_1, v_2, v_3, $\ldots$ v_μ are *linearly independent* in V, where $\mu \leq \nu$, if and only if

$$v_1 \wedge v_2 \wedge \ldots \wedge v_\mu \neq 0 \quad \text{in} \quad \bigwedge^\mu V$$

The exterior product underlying the lineal extension process from layer to layer may be considered as a *bilinear* map from $\bigwedge^\lambda V \times \bigwedge^\mu V$, where $\lambda, \mu \leq \nu$, to $\bigwedge^{\lambda+\mu} V$, where $\lambda + \mu \leq n$. Hence, for $\circlearrowright \in \bigwedge^\lambda V$, and $\sigma \in \bigwedge^\mu V$, their exterior product is an element $\circlearrowright \wedge \sigma \in \bigwedge^{\lambda+\mu} V$, meaning that this is a bilinear operation on $\circlearrowright$, σ. As an immediate consequence of the alternating property of the exterior product, we obtain that

$$\circlearrowright \wedge \sigma = (-1)^{\lambda\mu} \sigma \wedge \circlearrowright$$

Being endowed with the exterior multiplication product, which is associative and distributive over addition, the *direct sum* of all exterior power vector spaces forms a non-commutative algebra over $\mathbb{K}$ (we exclude the case that $1 = -1$ in $\mathbb{K}$):

$$E(V) = \mathbb{K} + V + \bigwedge^2 V + \bigwedge^3 V + \ldots$$

Note that $dim \bigwedge^\mu V = \dfrac{\nu!}{\mu!(\nu - \mu)!}$, and the direct sum terminates at $\mu = \nu$. Consequently, the *top exterior power* is *one*-dimensional, and $dim E(V) = 2^\nu$.

For example, consider that V is a real vector space of dimension 3, with basis e_1, e_2, e_3. Then, we obtain that

$$\bigwedge^0 V = \mathbb{R}$$
$$\bigwedge^1 V = V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$$
$$\bigwedge^2 V = \mathbb{R}(e_1 \wedge e_2) \oplus \mathbb{R}(e_1 \wedge e_3) \oplus \mathbb{R}(e_2 \wedge e_3)$$
$$\bigwedge^3 V = \mathbb{R}(e_1 \wedge e_2 \wedge e_3)$$

The exterior algebra $E(V)$ is the direct sum of the above exterior power vector spaces and its dimension is $1+3+3+1=8=2^3$.

In the same manner that Grassmann's lineal extension process applies to vectors, it analogously applies to their dual objects, called *co-vectors* or *linear forms*. Moreover, it also applies to a *linear transformation* ϕ of a vector space, where in this case, the top exterior power of ϕ, i.e. $\bigwedge^v \phi : \bigwedge^v V \to \bigwedge^v V$, is scalar multiplication with the *determinant* of ϕ. Furthermore, the same extension process can be transferred to free modules, and finitely generated locally free modules, over a commutative algebra of observables.

We emphasize that the *total exterior algebra* of a vector space V is characterized universally by an intrinsic feature that is independent of its constituent layer-by-layer exterior power vector spaces. More precisely, if we consider any $\mathbb{K}$-algebra Q and *a* $\mathbb{K}$*-linear* map from V to Q, i.e. $l : V \to Q$, such that the square of the image elements of V in Q is zero, it can always be extended *uniquely*, up to a unique isomorphism, to an $\mathbb{K}$*-algebra* map from $E(V)$ to Q. This is a *universal* property, and thus characterizes $E(V)$ uniquely in the fashion of category theory. In other words, every linear map from V to Q having the square zero property in the image, *factorizes uniquely* through the exterior algebra $E(V)$. This intrinsic feature of $E(V)$ plays a decisive role in what is to follow.

Each exterior power vector space, considered *within* the exterior algebra $E(V)$, delineates a *homogenous part* of this total algebra, in the sense that its elements instantiate the homogenous elements of the *degree* specified by the specific exterior power. The homogenous elements of a degree injected into the total exterior algebra may be thought of as the *rays* corresponding to the pertinent exterior power in the lineal extension process forming this total graded algebra. Reciprocally, the whole process *descends* back to the level of directed linear extensions, where we should recall that the *scalars* occupying the zero layer are only *implicitly* assumed.

Conceptually, we may think of a directed linear extension in *temporal* terms, in the sense of a provisional order of actions referring to the *replication* of another independent linear extension along it, according to the exterior product composition we described previously. We may simply say that a directed linear extension assumes the role of a *temporal order* for another independent linear extension to be composed with it by means of the exterior product. Equivalently, the replication procedure itself may be thought of as an *one-parameter flow* over the considered temporal order. Notice also that the *origin* of this order does *not* have to be fixed *a priori*, since the replication flow of an independent directed linear extension along it does not depend on where the origin is positioned, or else, it is of an *affine* character.

If we take the universality property into account, we realize that the exterior algebra pertaining to a directed linear extension, as above, is *isomorphic* with the $\mathbb{K}$-algebra of dual numbers over $\mathbb{K}$. More precisely, we have the isomorphisms:

$$E(\mathbb{K}e) \cong \mathbb{K} \oplus \mathbb{K} \cdot \varepsilon \cong \frac{\mathbb{K}[\varepsilon]}{(\varepsilon^2)}$$

where the square of ε is zero, $\varepsilon^2 = 0$. The important fact is that $E(\mathbb{K}e)$ is a *commutative* algebra over $\mathbb{K}$. Note that the nullity of the exterior product operation of the single directed linear extension e with itself, i.e. $e \wedge e = 0$ implies that the directed area of e along itself is zero. Thus, the non-commutative exterior product has a commutative shadow in the exterior algebra $E(\mathbb{K}e)$ expressed by the unit ε whose square vanishes.

We identify this unit as an *infinitesimal unit* with respect to which the flow over the considered temporal order takes place. This flow is expressed in *commutative algebraic*, and thus, *spectrally geometric* terms by means of $E(\mathbb{K}e) \cong \mathbb{K} \oplus \mathbb{K} \cdot \varepsilon$. In turn, this is equivalent to a *first order* infinitesimal variation or *infinitesimal flow* along the $\mathbb{K}$-parameterized variable running along the directed linear extension of e, interpreted as a temporal order in this direction.

We conclude that the exterior product operation applied to a single directed linear extension e is reflected in the commutative algebraic shadow of this extension as an ordinary *multiplication* of its scalar *image* ε, and moreover, the *nullity* of the directed area of e along itself is reflected as the vanishing of ε^2, granting it the role of an infinitesimal unit with respect to which an one-parameter potential flow along the temporal order of e's directed extension can be initiated.

Equivalently phrased, and conclusively, directed linear extensions are *reflected* as *infinitesimals* in their respective *commutative spectral shadows*.

We stress that the non-commutativity of the exterior product operation applied to two independent directed linear extensions is due to the antisymmetry in the order of their composition, which gives rise to a specific orientation (clockwise or anticlockwise) of the completed parallelogram area. The crucial thing in this case is that each of these directed linear extensions serves as a temporal order for the transportation or flow of the other along its extension.

Therefore we have two *distinct* temporal orders with respect to which the parallelogram area can potentially be *completed*, but they are *not* equivalent, they differ in the way they induce an orientation on this area. Let us consider two independent directed linear extensions e_1 and e_2, then the ordered composition $e_1 \wedge e_2$ stipulates the replication of e_2 a total of $\| e_1 \|$ times along the linear extension of e_1, considered as a temporal order for this purpose. In the course of this replication procedure, whereas we may think of a transportation from a determinate point to another determinate point along the linear extension of e_1 in terms of a sharply defined scalar parameterizing variable, the directed segment e_2 has to be taken in its *totality*, in other words, as a potentially *simultaneous whole*. This means that it assumes *potentially all* the points of its linear extension in a kind of *"tautochrone"* superposition. In this sense, it is objectively *totally indeterminate* in the course of its replication along the determinate extension of e_1 from point to point. To be sure, the directed linear extension of a segment is *not* considered to consist of determinate spatial points, rather the latter are *implicitly instantiated* in the process of replicating another directed segment along the former's extension. Precisely analogous considerations pertain to the oppositely oriented case, referring to the ordered composition $e_2 \wedge e_1$, which requires the replication of e_1 a total of $\| e_2 \|$ times along the linear extension of e_2, considered as a temporal order in its turn.

The above procedure actually constitutes the cornerstone of the *partial derivation* method underlying geometric analysis, that is to say, the *two-variable* geometric calculus expressed in terms of *differential forms* and their *integration*, which is generalized to multiple variables by means of the rules of exterior products. In this way, the geometric manifestation of the exterior product in terms of an one-parameter flow along a directed linear extension is expressed *locally* by way of a partial derivation along this extension, i.e. a one-parameter partial differentiation procedure. The fact that the differentiation procedure can be enacted in two ways that differ by a sign is expressed by the Leibniz rule for differentiating products. The Leibniz rule of differentiation is a formal expression of the replication procedure we have already described.

Given that directed linear extensions are reflected as infinitesimals in their respective commutative spectral shadows, we need also to grasp in what way the *infinitesimal commutativity* underlying the *mixed* partial derivations with respect to the two independent temporal orders, in the case of two composed linear extensions, *reflects* the non-commutativity of the exterior product of these extensions, which finally is manifested as a *sign difference* in the *orientation* of the composed parallelogram area.

Up to present, we have already seen that the nullity of the exterior product of a single directed linear extension e with itself, $e \wedge e = 0$, has a *commutative shadow* in the exterior algebra $E(\mathbb{K}e)$ expressed by the unit ε whose square vanishes. We have identified this unit as an infinitesimal unit, with respect to which the flow over the considered temporal order takes place and expressed in commutative algebraic, and thus, spectrally geometric terms, by means of $E(\mathbb{K}e) \cong \mathbb{K} \oplus \mathbb{K} \cdot \varepsilon$. The pertinent question is *how* the commutative shadow *displays* in case of two independent linear extensions. An answer to that will allow an easy generalization in all cases involving any finite number of independent linear extensions.

In the context of the above, consider two independent linear extensions in a two-dimensional vector space V over the real numbers. Then, we may define a bilinear map $\rho : V \times V \to V \otimes V$, where $V \times V \cong V \oplus V$, such that

$$(e_1, e_2) \mapsto \rho(e_1, e_2) = e_1 \otimes e_2 - e_2 \otimes e_1$$

Clearly, we obtain that $\rho(e_1, e_1) = 0$, $\rho(e_2, e_2) = 0$, thus ρ is *alternating*. Therefore, a *unique* linear map exists in this case $\chi : \bigwedge^2 V \to V \otimes V$, such that:

$$e_1 \wedge e_2 \mapsto \chi(e_1 \wedge e_2) = e_1 \otimes e_2 - e_2 \otimes e_1$$

Next, we consider the same vector space V and its exterior algebra $E(V)$. The latter is an $\mathbb{R}$-algebra generated by the *images* of e_1 and e_2, denoted by $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$, subject to the relations $\hat{\varepsilon}_1{}^2 = 0$, $\hat{\varepsilon}_2{}^2 = 0$, and $\hat{\varepsilon}_1 \hat{\varepsilon}_2 = -\hat{\varepsilon}_2 \hat{\varepsilon}_1$. Note that $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ *anti-commute*, and thus $E(V)$ is a *non-*

commutative $\mathbb{R}$ -algebra in striking *contrast* to the case of a single directed linear extension.

Therefore, the exterior algebra $E(V)$ *cannot* be considered as the infinitesimally-generated commutative shadow we are looking for here. Note that, generally speaking, the commutative shadow always admits of a local description, since infinitesimal quantities are meaningful only within a *local* context. Globally only their *integrated products* are observable.

Analogously to the case of a single directed linear extension the sought-after commutative shadow should be thought of as a *commutative extension* with respect to *two* infinitesimal units, ε_1, ε_2, together with a *mixed* commutative term of the form $\varepsilon_1 \cdot \varepsilon_2 = \varepsilon_2 \cdot \varepsilon_1$ as follows:

$$\tilde{E}(V) \cong \mathbb{R} \oplus \mathbb{R} \cdot \varepsilon_1 \oplus \mathbb{R} \cdot \varepsilon_2 \oplus \mathbb{R} \cdot (\varepsilon_1 \cdot \varepsilon_2)$$

where the squares of both ε_1 and ε_2 are zero, i.e. $\varepsilon_1{}^2 = 0$, $\varepsilon_2{}^2 = 0$, and the two infinitesimal units *commute*, i.e. they are *scalars* in the commutative $\mathbb{R}$ -algebra $\tilde{E}(V)$. Note that instead of the field $\mathbb{R}$ considered here, we may similarly think of another field $\mathbb{K}$.

Note that V may written equivalently in the form $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, since we are dealing with two independent linear extensions. Then, $E(V)$ as a *vector space* over $\mathbb{R}$, denoted by $\bar{E}(V)$ to *distinguish* it from the same space equipped with the non-commutative algebra structure $E(V)$, takes the form:

$$\bar{E}(V) \cong \bar{E}(\mathbb{R}e_1 \oplus \mathbb{R}e_2) \cong E(\mathbb{R}\varepsilon_1) \otimes E(\mathbb{R}\varepsilon_2)$$

We notice that both $E(\mathbb{R}\varepsilon_1)$ and $E(\mathbb{R}\varepsilon_2)$ are not only vector spaces, but they are commutative algebras over $\mathbb{R}$. Thus, $\bar{E}(V)$ is *also* a commutative $\mathbb{R}$ -algebra with respect to the standard $\mathbb{R}$ -tensor product, *identical* with the algebra $\tilde{E}(V)$. Thus, the exterior algebra $E(V)$ has a *commutative spectral shadow* represented by the algebra $\tilde{E}(V)$.

The significance of the commutative spectral shadow $\tilde{E}(V)$ of $E(V)$ consists in the fact that the *unique linear map* from the top exterior power in this case, i.e. $\chi : \wedge^2 V \to V \otimes V$, such that:

$$e_1 \wedge e_2 \mapsto \chi(e_1 \wedge e_2) = e_1 \otimes e_2 - e_2 \otimes e_1$$

descends in the commutative shadow to the map

$$e_1 \wedge e_2 \mapsto \varepsilon_1 \cdot \varepsilon_2 - \varepsilon_2 \cdot \varepsilon_1$$

Therefore, $e_1 \wedge e_2$ belongs to the *kernel* of this unique linear map after its descent to the commutative shadow $\tilde{E}(V)$. Hence, the *commutativity* condition of the infinitesimals ε_1 and ε_2 is obtained, given as $\varepsilon_1 \cdot \varepsilon_2 = \varepsilon_2 \cdot \varepsilon_1$, which is nothing else but the standard *commutation rule* of the *mixed* partial derivatives.

There is an analogous relation here. *Just as* the non-commutative exterior algebra $E(V)$ is an algebraic *encoding* of Grassmann's lineal extension process from layer to layer, starting from directed linear extensions, where the scalars or points are only implicitly assumed, *so* the commutative spectral shadow $\tilde{E}(V)$ *affords* a conceptual *inversion*, in the form of a *decoding* of this process, which takes place in terms of commuting infinitesimal units, where each one corresponds to an infinitesimal directed extension, culminating in the *determination* of points according to distinct linear infinitesimal extensions that become *dependent* at that *point*.

The crucial aspect of this metaphora is that points are only implicit in the ordered extension process, in the sense that they are *instantiated* in the procedure of *replicating* a linear extension along another independent one. They become explicit *only* in the *inversion* of the extension in the commutative shadow, and only locally and infinitesimally, by means of making two infinitesimal flows in distinct directions dependent at that point. In this manner, a point becomes explicit as an element *bounded* by distinct temporal orders manifested as infinitesimal directional flows that *meet* or become dependent at that point. The necessary condition for the temporal explication of points in this way, according to two different or even a whole *multiplicity* of temporal determination modes, is that the units of these flows becoming dependent at a *point-instant* locally *commute* in the infinitesimal vicinity of that point.

Henceforth, the lineal extension process cannot be conceived independently of its inversion, taking place by means of the commutative spectral shadow of the exterior algebra. In point of fact, in an attempt to reach beyond the exterior multiplication product, called a *progressive product*, bearing the analogous role of the logical *join* operation, Grassmann strived for the definition of another type of product, called a *regressive product*, bearing a role *analogous* to the logical *meet* operation, and thus constituting the *inversion* of the former. There is a not a clear consensus in the literature regarding the definition of the regressive product. The proposed metaphora from the non-commutative domain to the commutative spectral shadow of this domain, instantiated as it is infinitesimally, shows that Grassmann's lineal extension *cannot* be understood independently of Leibniz's infinitesimal analysis.

6.6 COBOUNDARY LAW: DESCENDING AND THE DIFFERENTIAL RESOLUTION OF A POINT

It is worth emphasizing the potent, and non-contradictory, indirect self-referential strategy underlying the conception of points or scalars according to this metaphora. It initiates by means of an *ascent* from the layer of independent directed linear extensions, where points are *only implicitly* assumed, to the layer of oriented areas, volumes and so on, being followed by a *descent* back to lower and lower layers, and culminates in the *explication* of points as *bounds* of distinct temporal orders becoming *dependent infinitesimally* at that point. This type of indirect self-referentiality arising from the process of ascending and then descending back in order to explicate temporally and multi-linearly what a point is in the context of Grassmann's geometric lineal analysis, is not void. Quite the reverse, it contains the germ of *cohomological analysis* in general topological spaces or even categorical sites, arguably, one of the most powerful and fruitful methods of modern mathematics and theoretical physics.

There are three basic conceptual issues underlying the dramatic generalization of this metaphora. The first issue pertains to the generalization of Grassmann's exterior algebra from the context of vector spaces defined over a field, to the context of *locally free modules* defined over an *algebra* of observables (in particular, locally free sheaves of modules defined over a sheaf of algebras of observables). The second issue pertains to conceptualizing algebraically the previously described metaphora as a process of *spectral resolution* of the constant scalars. The third issue pertains to viewing this process of resolution of the constants in terms of the notion of an algebraic-topological *complex*. The definition of the latter is based on the action of an operator, called the *coboundary* or, *exterior derivation operator* $\mathbf{d}$, characterized by the property that $\mathbf{d}^2 = 0$, i.e. it is a *nilpotent* operator of the *second* degree.

The notion of the coboundary operator that effects *differentiation* in this setting, is based precisely on the articulation of a point as a bound of distinct temporal orders becoming dependent infinitesimally, i.e. in terms of *commuting* one-parameter infinitesimal flows, at that point. The specification of the coboundary operator can actually be extracted from the *constitution* of the commutative shadow $\tilde{E}(V)$ of the exterior algebra $E(V)$.

The idea is to unfold the infinitesimal irreducible parts of $\tilde{E}(V)$ as *modules*, or simply vectors spaces, of differentials of different orders. In particular, in the two-dimensional case we have examined in detail, we consider the vector space of differentials of the first order, which contains only the pure infinitesimal units ε_1, and ε_2, and then, the vector space of differentials of the second order, which contains the mixed terms. Since the latter are vector spaces of differentials of the first and second order correspondingly their elements, while infinitesimals, are directed geometric magnitudes. Then, we may symbolically consider

$$\mathbf{d} = \varepsilon_1\hat{\varepsilon}_1 + \varepsilon_2\hat{\varepsilon}_1$$

$$\mathbf{d}^2 = \mathbf{d}(\mathbf{d}) = \mathbf{d}(\varepsilon_1\hat{\varepsilon}_1 + \varepsilon_2\hat{\varepsilon}_1) = (\varepsilon_1 \cdot \varepsilon_2)\hat{\varepsilon}_1\hat{\varepsilon}_2 + (\varepsilon_2 \cdot \varepsilon_1)\hat{\varepsilon}_2\hat{\varepsilon}_1 = (\varepsilon_1 \cdot \varepsilon_2 - \varepsilon_2 \cdot \varepsilon_1)\hat{\varepsilon}_1\hat{\varepsilon}_2 = 0$$

which demonstrates the *"coboundary law"* of constitution of a *complex* of vector spaces in cohomology theory. Note that the coboundary operator is defined locally, whose purpose is to *resolve* the points, or else the constant scalars, by way of the concomitant *ascent-descent* process of Grassmann's lineal geometric analysis.
In the context of locally free modules defined over observable algebras the "coboundary law" expresses the law of *"inertial variation"* in physical terminology. The discrepancy from inertial variation in the transition from the local to the global conceived topologically, is measured by certain equivalence classes, called *cohomology classes*. These classes are defined precisely through the notion of coboundary, and they capture in infinitesimal commutative terms some *invariant global* aspect obtained by their *integration*.

6.7 COHOMOLOGY: GROUP SHEAVES AS COEFFICIENT SYSTEMS FOR CALCULUS

The far-reaching applicability of Grassmann's lineal geometric analysis can be adequately appreciated only if it can be *raised* appropriately from the level of a vector space to the level of a "locally free module", defined over a smooth manifold or, more generally, over an arbitrary topological space, called accordingly a *vector sheaf*. The essential aspect of this generalization marking the powerful unifying combination of methods and ideas from the fields of geometry, topology, analysis, and homological algebra, is the notion of *localization* of a mathematical form, which reaches its greatest heights with the invention of *sheaf theory* and its subsequent application in the machinery of cohomology theory.

Conceptually, the invariant *transference* of Grassmann's lineal extension framework from the level of vectors to the level of locally free modules, necessitates the consideration of *locally-definable* and *variable* mathematical entities. At the first stage, the natural generalization of the notion of a vector space defined over a field of scalars is carried over by the notion of a module defined over a commutative algebra of scalars in which the underlying field of definition of this algebra is injected as the constant scalars. The scalars of the algebra may be thought of in a physical sense as the *observable* quantities.

In this sense, they assume at least the role of *continuous* functions defined over a topological space. For instance, the algebra of scalars may be generally considered as the $\mathbb{R}$-algebra, or more generally $\mathbb{C}$-algebra, of continuous functions evaluated in this field. The notion of evaluation also bears a physical semantics, since it refers to the evaluation of the observables at the states of the base topological space, that is, the recording of events in terms of values in the field of definition. Since the vectorial quantities defined over the algebra of scalars should be in principle expressible in terms of a *basis*, as in the case of vector spaces, the modules are required to be *finitely* generated or *free*.

Notwithstanding this fact, there are two essential issues that have to be dealt with in this generalization:

The first pertains to the realization that the operative manifestation of Grassmann's lineal geometric analysis requires both the commutativity, and the infinitesimally-generated nature of the algebraic structural shadow of the extension process. In the context of the generalization we are dealing with the base field is substituted by the algebra of scalars, whereas the infinitesimals of all orders are included as *ideals* in this commutative shadow. In an analogous fashion, they give rise to modules of differentials, that is, continuous differential forms of various orders, such that the pertinent *coboundary* operator is defined in terms of them. In this way, the latter may be simply identified as an *exterior* derivative operator from the algebra of scalars to the modules of differentials of the first degree obeying the Leibniz rule, and subsequently generalized to all higher order by means of Grassmannian calculus, such that the coboundary law is satisfied. Technically, the base topological space is assumed to be paracompact and Hausdorff.

The conceptual problem arising in this setting is that continuous functions are defined by their *values* at the points of the base topological space. This is quite *at odds* with the unfolding of Grassmann's geometric analysis, since the points are *only implicitly* assumed in the initiation of the extension process, whereas they are actually explicated as *bounds* of criss-crossing temporal orders becoming dependent at a point, and most importantly, this is enacted only in terms of infinitesimal flows. The latter are only *locally* defined, they just require a local cover of this point.

Therefore, to introduce here, in the generalized scheme of geometric analysis scalars as continuous functions determined by their values at points of the assumed topological space would be to beg the question. There is a way around this serious issue that makes the generalization of Grassmanian analysis proceed in a meaningful way preserving the unfolding semantics we have explicated previously.

The essentially simple idea required for this purpose, notably also of a physical descent, is to consider the *localization* of the whole scheme. This is something already mandated from the origin of Grassmannian analysis, since the *commutative shadow* of the non-commutative exterior algebra, generated as it is *infinitesimally* can only be of a *local* character. This localization procedure of all scalar and vectorial quantities as well as geometric forms bears the name of *sheaf theory*.

Instead of thinking in terms of globally defined continuous functions or forms of any order, we think in terms of locally defined functions, called *sheaf sections*, and their corresponding *germs*, as detailed in the preceding. Accordingly, the base topological space is considered *only implicitly* in the beginning of geometric analysis, with *no* reference required at all on its *point* determination. Instead what matters is the *local covering* structure of this space, which consists of the *open* sets covering the implicitly assumed points, that become *explicable* only in the culmination of the analysis in terms of the commutative shadow, or equivalently the algebra of scalars together with the *cohomological* temporal unfolding of the coboundary operator to higher orders of infinitesimal germs to be integrated appropriately as *potential* bearers of global information. Thus, under this essential localization requirement, the generalization of geometric analysis takes place in terms of a sheaf of commutative algebras of scalars or observables and locally free sheaves of modules defined over it so that the notion of basis persists in this context as well.

The second issue pertaining to this generalization has to do with the *doubly* articulated semantics of the unfolding of geometric analysis given the necessary condition of topological or sheaf-theoretic localization, as established. Since, the point structure of the base topological space is only implicitly assumed what really matters is what plays the role of a topology, i.e. the role of a *local covering* structure of the points that allows the formulation of the concept of a continuous function with respect to these covers. Traditionally, this is implemented in terms of a collection of open covers of a topological space that is closed under arbitrary unions and finite intersections.

Grothendieck generalized the notion of a covering family consisting of open sets by the notion of a *covering sieve*, not necessarily consisting of injective open covers, but allowing the definition of continuity in these generalized terms. The idea of generalized covers is inspired essentially by *Riemann's covering principle* together with its re-conceptualization from the perspective of *Galois theory*.

In a *Grothendieck topology* the open covers of a space are maps into this space, where instead of their pairwise intersections we have to look at their *fibered products*, or *pullbacks*, while unions play no essential role. In this conception, the idea of a cover is thought of as an observation *horizon* of a point, represented as an opening in a sieve that *infiltrates* observable information compatibly at different *resolution* layers by refinement until its temporal completion, thanks to which the *essence* of a point as a bearer of some globally irreducible, quantifiable, and *invariant* information may be unfolded cohomologically. The defining requirements of these generalized covers are the following: Covers are *transitive* meaning that covers of covers are also covers; covers are stable under pullback operations conforming to the *stability* of the notion of a cover under *change* of base; and finally, *isomorphisms* are qualified as covers.

Note that the present type of cohomological unfolding is not typically the same as the one implemented by Grassmanian analysis in terms of differential forms, or more precisely, *de Rham cohomology*. It is a *differently* conceived kind of unfolding by means of another suitably defined coboundary operator, called *Čech cohomology*. Recalling that the notion of a coboundary operator is the encoding of a multiplicity of point-bounding temporal orders in infinitesimal terms, subject to the coboundary law, the conception of Čech cohomology is based on a different articulation of *what* a *temporal order* is, independently of the concept of a local linearly extended geometric magnitude or form as in the de Rham case.

The amazing fact that these two different types of cohomological unfolding can be unified, an insight that lies deeply at the heart of *local gauge invariance* that will be expanded later, is based on Grothendieck's *vision* that a locally variable *sheaf of coefficients* is actually the *natural* argument of all cohomological theories in this context.

In other words, the natural argument of a cohomology theory is *not* just a space, as it was initially thought of, but a space *together* with an observable sheaf of coefficients, such that the space constitutes the observed *spectrum* of the sheaf employed *cohomologically* for this purpose. The global nature of points of this spectrum is typically determined, as it is actually expected, by a *constant sheaf* of coefficients, namely, a sheaf of locally constant sections valued in the integers, or the reals, or the complex numbers. In this sense, a commutative observable sheaf of coefficients plays locally the equivalent role of a measurement procedure, or *apparatus* in physical terminology, that is capable of capturing some global invariant feature only by cohomological means.

In Grothendieck's phraseology, we read the following excerpt (in English translation) from his autobiographical reflections contained in *Récoltes et semailles: Réflexions et témoignages sur un passé de* *mathématicien*: "Consider the set formed by all sheaves over a (given) topological space or, if you like, the formidable arsenal of all the "rulers" that can be used in taking measurements on it. We will treat this "ensemble", or "arsenal" as one equipped with a structure that may be considered "self-evident", one that crops up "in front of one's nose": that is to say, a Categorical structure ... It functions as a kind of "superstructure of measurement", called the "Category of Sheaves" (over the given space), which henceforth shall be taken to incorporate all that is most essential about that space. This is in all respects a lawful procedure, (in terms of "mathematical common sense") because it turns out that one can "reconstitute" in all respects, the topological space by means of the associated "category of sheaves" (or "arsenal" of measuring instruments").

Having already described the process of continuous unfolding from the local to the global achieved by the theory of sheaves, it is worthwhile for our purposes to focus on the basic ideas of Čech cohomology theory, as a basic predecessor of the later developments. This is particularly interesting in relation to the notion of double articulation laid out briefly above, since it introduces a novel way of thinking about the *global* nature of the points of a topological space in terms of the idea of *coverings*, which for reasons of simplicity, we may currently identify with the typical *open* covers of a topological space, whose global aspects are only implicitly assumed *ab initio*.

The basic characteristic of these open coverings is that they are *partially ordered* by inclusion. Note that this is just a partial and *not* total order of open covers, meaning that it is capable of subsuming a *variety* of potential local directed total orders or *chains*. The crucial idea is to force an *inductive system* out of these open covers that is capable of *resolving* points in terms of a corresponding *dual projective system* of Abelian groups of locally defined function elements, more precisely sections of a sheaf, over these covers at varying resolution horizons. Note that a section is conceived extensively as a whole over its domain or locus of definition.

In the initial conception of this theory these function elements were considered *constants*, for instance constant real-valued functions, or the constant coefficient system $\mathbb{R}$. In this sense, an *extension* of function elements is initiated by *gluing* them together through a process of joining together open covers, which has to be compatible with the inverse process of *restriction* of these elements to smaller open covers. For this purpose, it is required that for any *pair* of them there exists an *infimum* expressed by their intersection, or meet, or more generally pull-back operation, such that their overlap is totally *contractible*. Clearly, for every sub-collection of these covers there also exists a *supremum* with respect to their partial order relation. This implicitly also leads us to posit a minimal cover to serve as the *inductive limit* of all pairwise intersections of all covers forming a chain.

Thinking in temporal terms, we seek an articulation of points through this extension-restriction process in terms of an infinitesimally, and thus locally, generated commutative shadow, which can be equivalently expressed by means of a locally variable sheaf of coefficients (which can obviously be the constants) together with an appropriately defined coboundary operator. The latter should act on *cochains* of function elements in such a way that all function elements in the image of its action should form *coboundaries*, that is, local infinitesimally-generated flows over the directed temporal orders represented by the respective dual chains of covers, called in turn *boundaries*.

The first thing to note is that although coboundaries are *locally* defined entities, boundaries are *globally* defined entities with respect to the implicit base topological space. Thus although the coboundary operator is a *locally* defined operator on *cochains*, its dual boundary operator is a *globally* defined operator on *chains*. This eventually leads to the marking *difference* between cohomological and homological entities providing the appropriate hint required to *pair* them together in a suitable manner. At this stage, there is a second important thing we should also take into account, namely that in order that coboundary infinitesimal flows over distinct directed temporal orders can bound points by means of crisscrossing, they should be clearly alternating. Thus, both chains and cochains should be *directed* in the order-theoretic sense and *alternating*; something that it is implicitly assumed from the beginning usually, but has to be stressed at this stage.

The most important idea now is that in a topological space, in sharp *distinction* to a linear space, this procedure *does not exhaust* the ontology of points. The reason is that this implicitly postulated global topological space is generally neither *contractible*, nor *simply connected*. Thus, there appears a *new kind* of ontology of points, which may be characterized in terms of a variety of *singular points*, for instance holes, singularities, poles, branching points, sources, residues, degeneracies, foams, and so on. All these types of singular points should be accounted for *indirectly*, through metaphora, in terms of some *global invariant* information they eventually give rise to with respect to the local commutative spectral shadow, instantiated by the locally variable sheaf of coefficients together with the locally defined coboundary operator. Since the global nature of these singular points *cannot* be probed and exhausted at the infinitesimal level, meaning that singular points cannot be bounded via crisscrossing infinitesimal flows locally, the type of cochains needed for their description cannot be coboundaries as in the former case, but cochains which are *annihilated* by the coboundary operator, or equivalently cochains that are vanishing over boundary chains, called in turn *cocycles*.

In this case, the crucial issue is that we require a *novel* notion of directed *temporal order* to be applied for distinguishing singular points, of a different quality from the one conceptualized in terms of boundary chains. It is intuitive enough to think that a singular point, since it *cannot* be bounded by infinitesimal flows meeting at this point, can only be amenable to a process of repeated circulations around it, where all these circulations are *not* equivalent, but *are* in principle *distinguishable* in terms of different *global* attributes.

Dually thinking, the type of global chains needed for this purpose, called *cycles*, are clearly *not* boundaries, and most significantly, they encapsulate a novel type of temporal order that it is qualitatively different from the former one. More precisely, the type of temporal order encapsulated by cycles is characterized in terms of the *periods* of the locally-generated process of cocyclic circumscription, to be thought of as a global *integration* procedure, and not in terms of instants as in the former case.

The only issue remaining after making this qualitative difference is to discern the internal relation that cocycles bear with coboundaries, or equivalently cycles with boundaries. We can immediately see that due to the validity of the coboundary law for alternating cochains, all coboundaries are trivially cocycles as well. Equivalently, all boundary chains are trivially cycles as well, showing that the spectral unveiling of singular points is deeper than that of standard points, since the type of temporal order needed to account for the latter is trivially subsumed by the former.

Hence, at each degree, we may consider the *quotient* of the space of all cochains that are cocycles by the space of all cochains that are coboundaries to obtain the notion of a corresponding *cohomology class*. From this viewpoint, we realize that coboundaries function as bridges in *statu-nascendi* for the transition from one cocycle to another of the same cohomology class. We conclude that it is the concept of a *cocycle* that epitomizes the notion of a *gauge* in a topological context, whence the notion of *gauge invariance* is expressed through the concept of a *cohomology class*, as we shall elaborate in more detail later.

We proceed with a brief technical exposition to accompany the above conceptual undertaking of *Eduard Čech theory*. We emphasize that this cohomology theory is based on the *intersection* properties of open covers. Let $\mathbb{U} = U_k, k \in K$, be a system of open covers of the implicitly assumed global topological space X. For k_1, k_2, …, k_n $\in$ K, we denote:

$$U_\kappa := U_{k_0 \dots k_n} := U_{k_0} \cap \dots \cap U_{k_n}$$

and define a degree n-cochain with real coefficients for the covering $\mathbb{U}$ as a *totally* ordered collection of function elements $\phi := \{f_\kappa\}$, where κ is totally ordered with $n+1$ elements, and each f_κ is constant, real-valued and alternating.

We denote the set of all degree n-cochains with *real* coefficients obtained from a covering $\mathbb{U}$ of X by $C^n(X;\mathbb{R};\mathbb{U})$. Then, pointwise addition over the real numbers makes $C^n(X;\mathbb{R};\mathbb{U})$ an Abelian group, and scalar multiplication provides it with the structure of a real vector space. According to the above definition, an element of $C^0(X;\mathbb{R},\mathbb{U})$ corresponds to the assignment of a constant real-valued function to each open cover in $\mathbb{U}$. We consider that the covering $\mathbb{U}$ is finite, say of cardinality ν, hence we obtain $C^0(X;\mathbb{R},\mathbb{U}) = \mathbb{R}^\nu$. Analogously, the elements in $C^1(X;\mathbb{R},\mathbb{U})$ correspond to *constant* real-valued functions defined on the overlap of two open covers, and so on.

The coboundary map, or Čech differential is a linear map

$$\delta : C^n(X;\mathbb{R},\mathbb{U}) \to C^{n+1}(X;\mathbb{R},\mathbb{U})$$

for every index $n>0$, which sends each n-cochain ϕ to a $n+1$-cochain $\delta\phi$, i.e. to a coboundary, that is a set of constant real-valued functions defined on intersections of $n+2$ open covers, each expressed as an *alternating* sum of restrictions of ϕ's to these covers. The coboundary operator by construction satisfies the *coboundary law*, $\delta^2 = 0$.

We say that a cochain ϕ in $C^n(X;\mathbb{R},\mathbb{U})$ is *closed*, or a *cocycle*, if $\delta\phi = 0$, whereas we say that a cochain φ in $C^n(X;\mathbb{R},\mathbb{U})$ is a *coboundary* if it is in the *image* of the coboundary operator δ, i.e. there exists a cochain χ in $C^{n-1}(X;\mathbb{R},\mathbb{U})$, such that $\delta\chi = \varphi$. Then, we define the n-cohomology Abelian group of equivalence classes of cocycles modulo coboundaries with coefficients in $\mathbb{R}$, as the *quotient*:

$$H^n = \frac{Ker(\delta : C^n \to C^{n+1})}{Im(\delta : C^{n-1} \to C^n)}$$

Superficially, it seems that the above cohomology groups depend on the covering $\mathbb{U}$ chosen. This is *not* true, but this wrong impression can be rectified by applying the *inductive limit* construction over a refinement of $\mathbb{U}$. Practically, it suffices to consider all open covers of $\mathbb{U}$ which are *contractible*, so that all their intersections are contractible as well.

It is important to notice that the structure of the real numbers is *not constraining* in any way for the establishment of Čech's theory. All the arguments can be carried out for constant functions in *any* Abelian group. It follows that, we obtain Abelian cohomology groups with values in G for *any* Abelian group of coefficients G. Most appropriately, because of the pertinent localization philosophy, we should focus on a locally variable Abelian group structure of coefficients, i.e. an Abelian group *sheaf* of coefficients, which is usually an Abelian group sheaf of *locally constant* functions.

After this clarification, we may examine in more detail the structure of the $0-th$ Abelian cohomology group $H^0(X;\mathbb{R},\mathbb{U})$. This group emanates purely from 0-cocycles, that is, from cochains $\phi := \{f_\kappa\}$, where κ contains only 1 element, and each f_κ is a locally constant, real-valued function. We have:

$$\delta\phi = 0 \Leftrightarrow f_a = f_b$$

on the intersection $U_a \cap U_b$ of the open covers U_a and U_b.

Therefore, $H^0(X;\mathbb{R},\mathbb{U})$ is the Abelian group of locally constant real-valued functions. In this way, the group $H^0(X;\mathbb{R},\mathbb{U})$ detects the *connected components* of the topological space X. If X is connected then $H^0(X;\mathbb{R},\mathbb{U}) = \mathbb{R}$. In this case, $H^0(X;\mathbb{R},\mathbb{U})$ is identified as the Abelian group of all globally defined constant functions.

Let us now examine the $1-st$ Abelian cohomology group $H^1(X;\mathbb{R},\mathbb{U})$. This group emanates from 1-cocycles, i.e. from cochains $\phi := \{f_\kappa\}$, where κ contains 2 elements, and each f_κ is a locally constant, real-valued function. Equivalently, we consider the family $\phi := \{f_{ab} : U_a \cap U_b \to \mathbb{R}\}$, such that $f_{ab} = -f_{ba}$, and the following *cocycle relation* is satisfied:

$$\delta\phi = 0 \Leftrightarrow f_{ac} = f_{ab} + f_{bc}$$

on the *triple intersection* of open covers $U_a \cap U_b \cap U_c$. These 1-cocycles ϕ yield cohomology classes in $H^1(X;\mathbb{R},\mathbb{U})$ by taking them up to the addition of a coboundary. Equivalently, for any two cocycles ϕ, ϕ', we have that:

$$[\phi]=[\phi'] \Leftrightarrow f'_{ab}=f_{ab}+f_a-f_b$$

for some 0-cochain $\{f_a : U_a \to \mathbb{R}\}$.

After our interlude among the workings of Čech cohomology theory, the essential thing to keep in mind is that the natural argument of a cohomology theory is not a global topological space or a manifold, which after all is initially only implicitly posited, but a space together with an observable sheaf of coefficients, such that the *point constitution* (both standard and singular) of the former arises as the *spectrum* of the sheaf cohomological *analysis*. It is an established fact that in the case of *paracompact* topological spaces the calculation of cohomology with coefficients into a typical sheaf of coefficients is *equivalent* to the calculation of Čech cohomology theory with values in the corresponding sheaf (complete presheaf).

In this context, the calculation of cohomology with values in a constant group sheaf is of particular significance; such a group sheaf consists of locally constant sections, as well as the interrelation among constant sheaves in their function as cohomology coefficients. The most important tool in this respect, is provided by the *exponential short exact sequence* of constant group sheaves. The validity of this exact sequence of group sheaves conceptually descents from Riemann's covering principle, that we have already seen in the context of complex function theory on Riemann surfaces in relation to the complex exponential function and its local inverse complex logarithm function.

In terms of constant Abelian group sheaves, we have locally a certain interrelation of coefficients, expressed via the exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{exp} \tilde{\mathbb{C}} \to 1, \quad (1)$$
$$Ker(exp)=Im(\iota)\cong \mathbb{Z}, \quad (2)$$

where $\mathbb{Z}$ is the constant additive Abelian group sheaf of the integers, $\mathbb{C}$ is the constant additive Abelian group sheaf of the complexes, and $\tilde{\mathbb{C}}$ is the constant multiplicative Abelian group sheaf of non-zero complexes.

The above exponential short exact sequence can be specialized further to the following short exact sequence of constant Abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{R} \xrightarrow{exp(2\pi i)} \mathbb{U}(1) \to 1 \quad (3)$$

where $\mathbb{R}$ is the constant additive Abelian group sheaf of the reals, and $\mathbb{U}(1)$ is the constant multiplicative Abelian group sheaf of unit modulus complexes (phases).

The applicability of Grassmann's lineal geometric analysis is finally generalized and empowered to its full strength under the program of *sheaf-theoretic localization*, that marks the essential semantic transition from the level of vector spaces to the level of vector sheaves. The important thing is that Grassmann's analysis permeates the level of vector sheaves, meaning that it is invariant under localization, or equivalently exhibits a *functorial* behavior that justifies its universal relevance and applicability. For reasons of completeness, it is worth including the basic definition of the notion of a vector sheaf that characterizes the correlative generalization.

We consider a pair $(X, \mathcal{A})$ consisting of a paracompact (Hausdorff) topological space X and a *soft sheaf* of commutative rings $\mathcal{A}$ localized over X. The above pair is considered as the *Gelfand spectrum* of a corresponding algebra of observables A. We remind that if $\mathbb{C}$ is the field of complex numbers, then an $\mathbb{C}$-algebra A is a ring A together with a morphism of rings $\mathbb{C} \to A$ (making A into a vector space over $\mathbb{C}$) such that, the morphism $A \to \mathbb{C}$ is a linear morphism of vector spaces. Notice that the same holds if we substitute the field $\mathbb{C}$ with any other field, for instance, the field of real numbers $\mathbb{R}$. We also assume that the *stalk* $\mathcal{A}_x$ of *germs* is a local commutative $\mathbb{C}$-algebra for any point $x \in X$. A typical example is the case, where X is a smooth manifold of and $\mathcal{A}$ is the $\mathbb{C}$-algebra sheaf of germs of smooth functions localized over X. Together with a $\mathbb{C}$-algebra sheaf $\mathcal{A}$ we also consider the Abelian group sheaf of invertible elements of $\mathcal{A}$, denoted by $\mathcal{A}^{\bullet} := \tilde{\mathcal{A}}$.

An immediate generalization of the exponential short exact sequence of *constant* Abelian group sheaves is provided by the following short exact sequence of *variable* Abelian group sheaves, which models sheaf-theoretically the process of *exponentiation* in terms of the variable sheaves of coefficients $\mathcal{A}$ and $\tilde{\mathcal{A}}$:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{A} \xrightarrow{e} \tilde{\mathcal{A}} \to 1, \quad (4)$$

where $\mathbb{Z}$ is the constant abelian group sheaf of integers (sheaf of locally constant sections valued in the group of integers), such that:

$$Ker(e) = Im(\iota) \cong \mathbb{Z}. \quad (5)$$

An $\mathcal{A}$-module $\mathcal{E}$ is called a *locally free* $\mathcal{A}$-module of *finite rank* m, or simply a *vector sheaf* of states, if for any point $x \in X$ there exists an open set U of X such that:

$$\mathcal{E}|_U \cong \bigoplus^{m}(\mathcal{A}|_U) := (\mathcal{A}|_U)^m, \quad (6)$$

where $(\mathcal{A}|_U)^m$ denotes the m-terms *direct sum* of the sheaf of $\mathbb{C}$-observable algebras $\mathcal{A}$ restricted to U, for some $m \in \mathbb{N}$. We call $(\mathcal{A}|_U)^m$ the local sectional frame of $\mathcal{E}$ associated via the open covering $\mathcal{U} = \{U\}$ of X.

In case that the rank is 1, the corresponding vector sheaf is called a *line sheaf* of states, that is locally for any point $x \in X$ there exists an open set U of X such that:

$$\mathcal{E}|_U \cong \mathcal{A}|_U. \quad (7)$$

Furthermore, if for any point $x \in X$ there exists an open set U of X such that:

$$\mathbb{S}|_U \cong \bigoplus^{m}(\mathbb{C}|_U) := (\mathbb{C}|_U)^m, \quad (8)$$

then we call any locally free $\mathbb{C}$-module $\mathbb{S}$ of finite rank m, for some $m \in \mathbb{N}$, a complex *linear local system* of rank m.

The notion of a vector sheaf of states generalizes the notion of a vector space in the sense that locally every section of a finite rank vector sheaf can be written as a finite linear combination, or *superposition* of a basis of sections with variable coefficients from the local observable algebra. For example, if X is a smooth manifold and $\mathcal{A}$ is the $\mathbb{C}$-algebra or $\mathbb{R}$-algebra sheaf of germs of smooth functions on X, then every section can be locally written as a finite superposition of a basis of sections with coefficient being real-valued smooth functions. We note that the set of sections of every *vector bundle* on a topological space (not necessarily a smooth manifold) forms a vector sheaf of sections localized over this space.

Given a vector sheaf of states $\mathcal{E}$, there is specified a Čech 1-cocycle with respect to a covering $\mathcal{U} = \{U\}$ of X, called a *coordinate* 1-

cocycle in $Z^1(\mathcal{U}, GL(m, \mathcal{A})$ (with values in the sheaf of germs of sections into the complex general linear group $GL(m, \mathbb{C})$), as follows:

$$\eta_\alpha : \mathcal{E}|_{U_\alpha} \cong (\mathcal{A}|_{U_\alpha})^m, \quad (9)$$
$$\eta_\beta : \mathcal{E}|_{U_\beta} \cong (\mathcal{A}|_{U_\beta})^m. \quad (10)$$

Thus, for every $x \in U_\alpha$ we have a stalk isomorphism:

$$\eta_\alpha(x) : \mathcal{E}_x \cong \mathcal{A}_x^{\ m}, \quad (11)$$

and similarly for every $x \in U_\beta$. If we consider that $x \in U_\alpha \bigcap U_\beta$, then we obtain the *isomorphism*:

$$g_{\alpha\beta}(x) = \eta_\alpha(x)^{-1} \circ \eta_\beta(x) : \mathcal{A}_x^{\ m} \cong \mathcal{A}_x^{\ m}. \quad (12)$$

The $g_{\alpha\beta}(x)$ is thought of as an *invertible matrix* of germs at x. Consequently, $g_{\alpha\beta}$ is an invertible matrix section in the sheaf of germs $GL(m, \mathcal{A})$ (taking values in the general linear group $GL(m, \mathbb{C})$). Moreover, $g_{\alpha\beta}$ satisfies the *cocycle conditions* $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ on *triple* intersections whenever they are defined.

It is clear in this way that we obtain a *vector bundle* with *typical fiber* $\mathbb{C}^m$, *structure group* $GL(m, \mathbb{C})$, whose *sections* form the vector sheaf of states we started with. In particular, for $m = 1$, we obtain a *line bundle* L with *fiber* $\mathbb{C}$, *structure group* $GL(1, \mathbb{C}) \cong \tilde{\mathbb{C}}$ (the non-zero complex numbers), whose *sections* form a line sheaf of states $\mathcal{L}$. Clearly, by imposing a *unitarity* condition the structure group is reduced to $U(1)$. Thus, particularly in the case of a line sheaf of states we have a bijective correspondence:

$$\mathcal{L} \leftrightarrow (g_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \quad (13)$$

where $GL(1, \mathcal{A}) \cong \mathcal{A}^\bullet := \tilde{\mathcal{A}}$ is the group sheaf of invertible elements of $\mathcal{A}$ (taking values in $\tilde{\mathbb{C}}$), and $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$ is the set of coordinate 1-cocycles. In physical terminology, a coordinate 1-cocycle effects a *local frame transformation*, or equivalently a *local gauge transformation* of a vector sheaf of states $\mathcal{E}$.

Every 1-*cocycle* can be *conjugated* with a 0-cochain (t_α) in the set $C^0(\mathcal{U},\tilde{\mathcal{A}})$ to obtain another *equivalent* 1-cocycle:

$$g'_{\alpha\beta}=t_\alpha\cdot g_{\alpha\beta}\cdot t_\beta^{-1} \quad (14)$$

If we consider the *coboundary operator*:

$$\Delta^0: C^0(\mathcal{U},\tilde{\mathcal{A}})\to C^1(\mathcal{U},\tilde{\mathcal{A}}), \quad (15)$$

then the *image* of Δ^0 gives the set of *1-coboundaries* of the form $\Delta^0(t_\alpha^{-1})$ in $B^1(\mathcal{U},\tilde{\mathcal{A}})$:

$$\Delta^0(t_\alpha^{-1}):=t_\alpha\cdot t_\beta^{-1} \quad (16)$$

Thus, we obtain that the 1-cocycle $g'_{\alpha\beta}$ is equivalent to the 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$ if and only if there exists a 0-cochain (t_α) in the set $C^0(\mathcal{U},\tilde{\mathcal{A}})$, such that:

$$g'_{\alpha\beta}\cdot g_{\alpha\beta}^{-1}=\Delta^0(t_\alpha^{-1}) \quad (17)$$

where, the multiplication above is meaningful in the Abelian group of 1-cocycles $Z^1(\mathcal{U},\tilde{\mathcal{A}})$.

Due to the bijective correspondence of line sheaves with coordinate 1-cocycles with respect to an open covering $\mathcal{U}$, we immediately obtain the following:

The set of *isomorphism classes* of *line sheaves* of states over the same topological space X, denoted by $Iso(\mathcal{L})(X)$, is in *bijective* correspondence with the set of *cohomology classes* $H^1(X,\tilde{\mathcal{A}})$:

$$Iso(\mathcal{L})(X)\cong H^1(X,\tilde{\mathcal{A}}) \quad (18)$$

Furthermore, each equivalence class $[\mathcal{L}]\equiv\mathcal{L}$ in $Iso(\mathcal{L})(X)$ has an *inverse*, defined by:

$$\mathcal{L}^{-1}:=Hom_{\mathcal{A}}(\mathcal{L},\mathcal{A}) \quad (19)$$

where, $Hom_{\mathcal{A}}(\mathcal{L},\mathcal{A}) := \mathcal{L}^*$ denotes the *dual line sheaf* of $\mathcal{L}$. This is actually deduced from the fact that we can define the *tensor product* of two equivalence classes of line sheaves over $\mathcal{A}$ so that:

$$\mathcal{L} \otimes_{\mathcal{A}} \mathcal{L}^* \cong Hom_{\mathcal{A}}(\mathcal{L},\mathcal{L}) \equiv \mathcal{E}nd_{\mathcal{A}}\mathcal{L} \cong \mathcal{A} \quad (20)$$

Hence, we conclude that the set of isomorphism classes of line sheaves of states over the same topological space X, $Iso(\mathcal{L})(X)$, has an Abelian group structure with respect to the tensor product over the observable algebra sheaf $\mathcal{A}$, and isomorphically the set of cohomology classes $H^1(X,\tilde{\mathcal{A}})$ is also an Abelian group, where the tensor product of two line sheaves of states corresponds to the product of their respective coordinate 1-cocycles.

6.8 SINGULAR DISCLOSURE: INTEGRATION OVER CYCLES AND INVARIANTS

In the context of Grassmann's lineal geometric analysis we reached the following conclusion: In the same way that the non-commutative exterior algebra $E(V)$ is an algebraic representation of Grassmann's lineal extension process from layer to layer starting from directed linear extensions, where the scalars or points are only implicitly assumed, the commutative shadow $\tilde{E}(V)$ provides an inversion of this process, in terms of commuting infinitesimal units, where each one of them corresponds to a directed extension, culminating in the determination of points according to distinct linear extensions becoming dependent at that point.

More precisely, the constitution of the commutative shadow $\tilde{E}(V)$ of the exterior algebra $E(V)$ is elucidated by means of the locally defined coboundary operator, based on the idea of semantically unfolding the infinitesimal irreducible parts of $\tilde{E}(V)$ as modules or simply vectors spaces of differentials of different orders. In this manner, the notion of the coboundary operator is actually based precisely on the articulation of a point as a bound of distinct temporal orders becoming dependent infinitesimally, that is, in terms of commuting one-parameter infinitesimal flows, at that point.

The coboundary operator gives rise to an exterior derivation operator d, characterized by the property that $d^2 = 0$, in other words, it is a nilpotent operator of the second degree. More concretely, the exterior derivation operator is thought of as acting on modules of differentials, called differential forms.

In order that Kähler's algebraic extension method becomes suitable as a generator of a *universal* mechanism of differential geometric analysis it should be susceptible to the process of *sheaf-theoretic localization* of an observable algebra. In turn, this would allow the complete disassociation of the differential mechanism from any underlying spatial substratum opening up the way for a *functorial* formulation of differential geometry.

The localization of Kähler's algebraic extension method in sheaf theoretic terms requires first of all the notion of a *universal derivation* of the observable algebra sheaf $\mathcal{A}$ considered as an algebra sheaf over the constant sheaf of the reals $\mathbb{R}$, or more generally, as an algebra sheaf over the constant sheaf of the complexes $\mathbb{C}$.

The universal $\mathbb{R}$-derivation of the observable algebra sheaf $\mathcal{A}$ to the universal $\mathcal{A}$-module sheaf $\Omega^1(\mathcal{A}) := \Omega^1$, called the $\mathcal{A}$*-module sheaf of differential* 1*-forms*, is the *universal* $\mathbb{R}$-linear sheaf morphism (natural transformation) $\partial := d^0$:

$$d^0 : \mathcal{A} \rightarrow \Omega^1 \quad (1)$$

such that the Leibniz condition is satisfied:

$$d^0(s \cdot t) = s \cdot d^0(t) + t \cdot d^0(s), \quad (2)$$

for any continuous local sections s, t belonging to $\mathcal{A}(U)$, with U an open set in X. Notice that all the above definitions are strictly local, which means they are considered *stalk-wise*. We may also use the following notational convention: $\mathcal{A} := \Omega^0$.

The major ingredient in setting up a universal mechanism of differential geometric analysis is the validity of the *Poincaré Lemma*:

$$Ker(d^0) = \mathbb{R}. \quad (3)$$

For each $n \in N$, $n \geq 2$, the n-fold exterior product is defined as follows: $\Omega^n(\mathcal{A}) = \wedge^n \Omega^1(\mathcal{A})$ where $\Omega(\mathcal{A}) := \Omega^1(\mathcal{A})$. We notice that there exists an $\mathbb{R}$-linear sheaf morphism:

$$d^n : \Omega^n(\mathcal{A}) \rightarrow \Omega^{n+1}(\mathcal{A}) \quad (4)$$

for all $n \geq 0$. Let $\omega \in \Omega^n(\mathcal{A})$, then ω has the form:

$$\omega = \sum f_i (dl_{i1} \wedge \ldots \wedge dl_{in}) \quad (5)$$

with f_i, l_{ij}, $\in$ $\mathcal{A}$ for all integers i, j. Further, we define:

$$d^n(\omega) = \sum df_i \wedge dl_{i1} \wedge \ldots \wedge dl_{in}. \quad (6)$$

From the above, we immediately obtain that the composition of two consecutive $\mathbb{R}$-linear sheaf morphisms vanishes, that is $d^{n+1} \circ d^n = 0$, abbreviated in the symbolic form $d^2 = 0$, expressing the *coboundary law* in this context.

The sequence of $\mathbb{R}$-linear sheaf morphisms:

$$\mathcal{A} \to \Omega^1(\mathcal{A}) \to \ldots \to \Omega^n(\mathcal{A}) \to \ldots \quad (7)$$

is a complex of $\mathbb{R}$-vector space sheaves, called the *sheaf-theoretic differential complex* of $\mathcal{A}$.

Moreover, given the validity of the Poincaré Lemma, $Ker(d^0) = \mathbb{R}$, and the fact that $\mathcal{A}$ is a soft observable algebra sheaf, we obtain:

The sequence of $\mathbb{R}$-vector space sheaves is *exact*:

$$\mathbf{0} \to \mathbb{R} \to \mathcal{A} \to \Omega^1(\mathcal{A}) \to \ldots \to \Omega^n(\mathcal{A}) \to \ldots \quad (8)$$

Thus, the sheaf-theoretic differential complex of the observable algebra sheaf $\mathcal{A}$ constitutes a *resolution of the constant sheaf* $\mathbb{R}$ by *soft* sheaves (which are *acyclic* with respect to the *global sections functor*).

The preceding observations elucidate the three basic conceptual issues underlying the generalization of Grassmann's lineal geometric analysis under the prism of sheaf-theoretic localization:

The first pertains to the generalization of Grassmann's exterior algebra from the context of vector spaces defined over a field to the context of vector sheaves defined over a sheaf of algebras of observables.

The second pertains to conceptualizing and expressing Grassmann's scheme algebraically as a process of resolution of the constants, represented by the field of definition, the coordinatizers of points.

The third pertains to viewing this process of resolution of the constants in terms of the notion of a differential complex, where the latter is based on the action of the exterior derivation operator d, characterized by the property that $d^2 = 0$, that is, the coboundary law.

Most important, in cases where the observable algebra sheaf is the sheaf of smooth functions defined on a differential manifold, the above sheaf-theoretic process of resolving the constants provides a clear manifestation of de *Rham's cohomological scheme*. In precise terms, the *sheaf* cohomology with coefficients in the *constant sheaf* $\mathbb{R}$, is isomorphic with the *de Rham* cohomology of X. We emphasize that $\mathbb{R}$ denotes the domain of *constants* (locally constant sections) of the observable algebra sheaf of smooth functions $\mathcal{A}$ defined on a smooth finite dimensional manifold X, identified as the *spectrum* of $\mathcal{A}$. The same is equally true of the Čech cohomology with values in the constant sheaf $\mathbb{R}$. In short, all the respective cohomology groups are isomorphic:

$$H^n(X;\mathbb{R}) \cong H_{dR}{}^n(X) \quad (9)$$

The above isomorphism, first proved by *André Weil*, establishes the sheaf-theoretic formulation of de Rham's theorem. In order to gain a deeper insight it is of value to describe de Rham's initial *motive* and follow the type of cohomological unfolding it implements in terms of differential forms.

The essential issue is the *disclosure* of singular points on a manifold, given that its point structure is only implicitly assumed initially, according to the preceding. This disclosure may be though of in terms of a coboundary operator giving rise to a differential complex, where it is considered as an exterior derivation operator acting on differential forms that obey the rules of Grassmann's lineal geometric analysis. The basic idea is that differential forms are objects which can be temporally *integrated* over *chains* in a way that is compatible with pull-back operations. If the result of this integration procedure is not trivial, then the obtained *residue* characterizes *invariantly* a singular point *encycled* by an appropriate chain in terms of *periods*.

According to de Rham, we consider differential forms of degree p, $0 \leq p \leq n$, on a finite dimensional smooth manifold of dimension n. These differential forms constitute a real vector space, denoted by Ω^p. The *exterior differential* operator acts on forms ω^p, such that $d\omega^p$ is a form in Ω^{p+1}, via the *linear* mapping of real vector spaces

$$d : \Omega^p \rightarrow \Omega^{p+1} \quad (10)$$

satisfying the condition that $d^2 = 0$.

The notion of the *differential complex* arising in this manner, called the *de Rham complex*, encapsulates the idea that the *image* $d\Omega^{p-1}$ of the real vector space Ω^{p-1} in Ω^p via d, lies in the *kernel* of the following linear mapping $d : \Omega^p \to \Omega^{p+1}$, denoted by $\mho^p$. Then, by means of quotienting the *de Rham cohomology groups*

$$H_{dR}{}^p(X) := \mho^p / d\Omega^{p-1} \quad (11)$$

are defined, and they are real vector spaces as well. A form ω^p that belongs to $\mho^p$, whereby $d\omega^p = 0$ is called a *closed differential form*, whereas a form in $\mho^p$ expressed as the differential of a form in Ω^{p-1} is called an *exact differential form*. Clearly, every exact form is also closed, but the *inverse* does *not* hold. Thus, the de Rham cohomology group of some degree structurally measures the *discrepancy from exactness* at this degree in terms of closed forms failing to be exact.

At a further stage, we consider that X is *subdivided* into *cells*, and let A_p be the vector space of p-chains. This is a real vector space of linear combinations of p-cells with real coefficients. The boundary $\partial \circlearrowright_p$ of a p-cell $\circlearrowright_p$ is a $(p-1)$-chain with coefficients ± 1. This extends by linearity to a *boundary* operator

$$\partial : A_p \to A_{p-1} \quad (12)$$

such that $\partial^2 = 0$.

Next, we consider the vector space of real linear forms defined on p-chains, that is the real vector space A^p of p-cochains, *dual* to A_p. For these, there is associated the *coboundary* operator

$$\delta : A^p \to A^{p+1} \quad (13)$$

satisfying the condition that $\delta^2 = 0$. More precisely, since the coboundary operator δ acting on cochains is dual to the boundary operator ∂ acting on chains, for a cochain ξ^p in A^p

$$\delta\xi^p(\rho_{p+1}) = \xi^p(\partial\rho_{p+1}) \quad (14)$$

where ρ_{p+1} is a chain $\in A_{p+1}$. We conclude that the image dA^{p-1} of the vector space A^{p-1} in A^p via δ, lies in the *kernel* of $\delta : A^p \to A^{p+1}$, denoted by $\forall^p$. Then, the cohomology groups

$$\bar{H}^p(X) := \forall^p / dA^{p-1} \quad (15)$$

are defined by means of quotienting, and they are real vector spaces.

In this context, de Rham's theorem not only proves the *isomorphism:*

$$\bar{H}^n(X) \cong H_{dR}{}^n(X) \quad (16)$$

but additionally, it shows explicitly that the above isomorphism is *implemented* and *realized* by the *integration* of differential forms. In particular, a p-cell $\circlearrowright_p$ is considered as smooth so as to serve as the domain of integration of a corresponding differential form. For instance, for an 1-differential form ω, the expression $\int_{\circlearrowright} \omega$ stands for the *line integral* of ω along the 1-cell $\circlearrowright$. In an analogous manner, the integral $\int_{\circlearrowright_p} \omega^p$ extends by linearity to all p-chains ρ_p, such that:

$$\delta\xi^p(\rho_p) = \int_{\rho_p} \omega^p \quad (17)$$

defines a linear form ξ^p identified as a cochain in A^p. Thus, for every degree p there emerge linear mappings

$$\Psi : \Omega^p \to A^p \quad (18)$$

such that we have

$$\Psi(d\omega^p) = \delta\Psi(\omega^p) \quad (19)$$

culminating into the theorem of *George Stokes*:

$$\int_{\rho_{p+1}} d\omega^p = \int_{\partial\rho_{p+1}} \omega^p = \xi^p(\partial\rho_{p+1}) = \delta\xi^p(\rho_{p+1}) \quad (20)$$

for all $\rho_{p+1} \in A_{p+1}$. Notice that the linear mappings Ψ map the groups $\mho^p$ and $d\Omega^{p-1}$ into the groups $\forall^p$ and dA^{p-1} correspondingly. We obtain thereby a linear mapping of the quotient groups:

$$H_{dR}{}^p(X) \to \bar{H}^p(X) \quad (21)$$

that *de Rham's theorem* asserts is an *isomorphism*.

De Rham's considerations additionally involve the *homology groups* $\bar{H}_p(X)$ defined as the quotient groups:

$$\bar{H}_p(X) := \forall_p / \partial A_{p+1} \quad (22)$$

where $\forall_p$ is the *kernel* of $\partial : A_p \to A_{p-1}$. These chains ρ_p in A_p satisfying the condition $\partial\rho_p = 0$ are called *cycles*, whereas those satisfying the condition $\rho_p = \partial\rho_{p+1}$ are called *boundaries*. In this manner, an element of the homology group $\bar{H}_p(X)$ denotes an equivalence class of p-cycles differing by a boundary. The connotation of a cycle is tantamount to a closed region of integration.

Consider a cochain ξ^p in $\forall^p$, i.e. $\delta\xi^p = 0$, called a p-cocycle. Then we obtain that

$$\delta\xi^p(\rho_{p+1}) = \xi^p(\partial\rho_{p+1}) = 0 \quad (23)$$

meaning that ξ^p is 0 on ∂A_{p+1}. In this way, we obtain an $\mathbb{R}$-valued linear form on $\bar{H}_p(X)$ that is identically 0 if ξ^p is a p-coboundary. In the inverse direction, all $\mathbb{R}$-valued linear forms on $\bar{H}_p(X)$ are produced likewise, meaning that

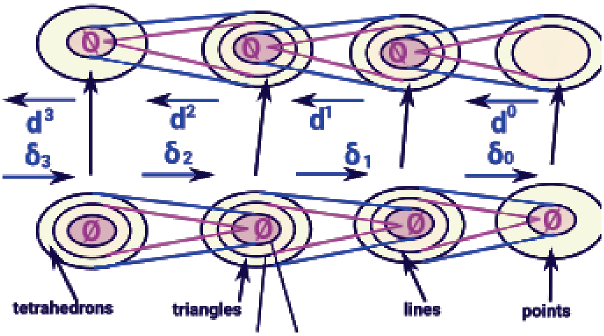
$$\bar{H}^p(X) = Hom(\bar{H}_p(X), \mathbb{R}) \quad (24)$$

In other words, $\bar{H}^p(X)$ is identified as the dual to $\bar{H}_p(X)$, i.e. the vector space of all $\mathbb{R}$-valued linear forms on $\bar{H}_p(X)$. Hence, the linear mappings Ψ can viewed equivalently as:

$$\Psi : H_{dR}{}^p(X) \to Hom(\bar{H}_p(X), \mathbb{R}) \quad (25)$$

which maps a closed differential form ω^p to an $\mathbb{R}$-valued linear form defined on the p-cycles, and is 0 on the p-boundaries. According to de Rham's theorem Ψ is an *isomorphism*, and the $\mathbb{R}$-values of this linear form are called the *periods* of the differential form ω^p. The injectivity of Ψ means that if all periods of the closed differential form ω^p are 0, then ω^p is an exact form, i.e. it is the differential of a form χ^{p-1}, equivalently $\omega^p = d\chi^{p-1}$. The surjectivity of Ψ means that given periods $\lambda \in Hom(\bar{H}_p(X), \mathbb{R})$, then there always exists a closed differential form ω^p, that it is associated with these periods.

Again the essential issue of de Rham cohomological analysis is the disclosure of singular points on a manifold in terms of invariant quantities obtained by the integration of differential forms. More precisely, closed differential forms are the natural integrands over cycles, i.e. they can be temporally integrated over closed chains encircling a singular point, such that the result of this integration procedure leaves a residue characterizing this singular point invariantly in terms of periods.



From the dual homology-theoretic viewpoint, the Abelian group $\bar{H}_0(X)$ measures the *number* of connected components of X. In turn, the Abelian group $\bar{H}_1(X)$ measures *how many independent* cycles, or *1-dimensional holes* X has, *modulo* the bounding cycles, meaning the *boundaries*. Since the cohomology groups are *dual* to their respective homology groups we have the following: The Abelian group ${H_{dR}}^0(X) \cong \bar{H}^0(X)$ measures the connected components of X *in terms* of real valued functions that are locally constant over these components. This is reflected on the dimension of the real vector space $\bar{H}^0(X)$. The Abelian group ${H_{dR}}^1(X) \cong \bar{H}^1(X)$ is constituted by *closed* differential 1-

forms that can be *integrated* over 1-*cycles* enclosing 1-dimensional holes or singular points, allowing them to be characterized in terms of *periods*.

The most characteristic example is provided by the *angular* differential 1-form ω, which is a closed form but *not* exact, since it cannot be expressed as an exact differential $\omega = d\theta$, where θ denotes the *angle* in polar coordinates of a 2-dimensional Euclidean space. Note that the angle is defined modulo $2\kappa\pi$, where $\kappa \in \mathbb{Z}$. In cartesian coordinates the angular differential 1-form ω is expressed as follows:

$$\omega = \frac{1}{2\pi}\frac{xdy - ydx}{x^2 + y^2} \quad (26)$$

It is defined everywhere on $\mathbb{R}^2$ except the point $(0,0)$, which is thus a singular point. In this sense, the integration of ω over oriented 1-cycles encircling the singularity $(0,0)$, i.e. the line integral of the closed differential form ω:

$$\oint_\gamma \omega \quad (27)$$

over any closed curve γ defined on $\mathbb{R}^2 - (0,0)$ measures its *period* or *winding number* around the singular point $(0,0)$, thus disclosing this singular point invariantly.

It is instructive to remember that differential forms ω^p constitute the natural entities to be integrated over cycles of the same dimension. Moreover, if we consider two forms ω^p in Ω^p, and $\phi^{(n-p)}$ in Ω^{n-p}, then their *exterior* product $\omega^p \wedge \phi^{(n-p)}$ is a form in Ω^n. This differential form can be integrated over X, considered as a closed, connected and orientable smooth manifold of finite dimension n:

$$\int_X \omega^p \wedge \phi^{(n-p)} \quad (28)$$

giving rise to a *bilinear pairing*:

$$\Omega^p \times \Omega^{n-p} \to \mathbb{R} \quad (29)$$

under which Ω^n and Ω^{n-p} become *dual* spaces. In turn, the above provides an equivalent expression of *Poincaré's duality*,

$$\bar{H}^p(X) \cong \bar{H}_{n-p}(X) \quad (30)$$

We close this circle of ideas related to de Rham cohomology by revisiting the *sheaf-theoretic* form of de Rham's theorem. We recall that the Čech cohomology with values in the constant sheaf $\mathbb{R}$, where $\mathbb{R}$ is the domain of constants (locally constant sections) of the observable algebra sheaf of smooth functions $\mathcal{A}$ on a smooth finite dimensional manifold X, conceived as the spectrum of $\mathcal{A}$, is isomorphic with the de Rham cohomology of X. This led to the conclusion that all the respective *cohomology groups* are *isomorphic*:

$$H^n(X;\mathbb{R}) \cong H_{dR}{}^n(X) \quad (31)$$

We recall that the $0-th$ abelian cohomology group $H^0(X;\mathbb{R},\mathbb{U})$ emanates purely from 0-cocycles, i.e. from cochains $\phi := \{f_\kappa\}$, where κ contains only one element, and each f_κ is a locally constant, real-valued function. We have:

$$\delta\phi = 0 \Leftrightarrow f_a = f_b$$

on the intersection $U_a \cap U_b$ of the open covers U_a and U_b. Therefore, $H^0(X;\mathbb{R},\mathbb{U})$ is the Abelian group of *locally constant* real-valued functions. Thus, $H^0(X;\mathbb{R},\mathbb{U})$ detects the connected components of X. More concretely, the dimension of the real vector space $H^0(X;\mathbb{R},\mathbb{U})$ provides the number of connected components of X. If X is connected then $H^0(X;\mathbb{R},\mathbb{U}) = \mathbb{R}$. In this case, $H^0(X;\mathbb{R},\mathbb{U})$ is the Abelian group of all globally defined constant functions.

For simplicity, we may consider an open cover V of an n-dimensional Euclidean space $\mathbb{R}^n$ and recall how we obtain $H^0(V;\mathbb{R})$. Then, the locally constant real-valued smooth functions in $\mathbb{R}^n$ over V constitute the Abelian group $H^0(V;\mathbb{R})$. These locally constant real-valued smooth functions f over V are exactly the *solutions* of the *differential equation*:

$$df = 0 \Leftrightarrow \sum_i \frac{\partial f}{\partial x_i} dx^i = 0 \quad (32)$$

Generalizing, for any smooth manifold the Abelian group $H^0(V;\mathbb{R})$, where V is an open cover, may be interpreted as the real vector space of *local* solutions on V of the differential equation $df = 0$, where f is qualified accordingly as a locally constant real-valued smooth function over V.

Most important, since $H^n(X;\mathbb{R}) \cong H_{dR}{}^n(X)$ at all orders, the constant sheaf of the reals $\mathbb{R}$, in its role as an Abelian coefficient sheaf of cohomology, should furnish the description of all higher cohomology groups in a way analogous to the above case. In this manner, $H^1(X;\mathbb{R})$ can be viewed as the real vector space of solutions of the differential equation $d\omega = 0$, where ω is a *closed* differential 1-form in Ω^1, *modulo* the *exact* differential forms, by which we mean these solutions $\omega = d\sigma$, where σ in Ω^0, considered trivial due to the validity of the coboundary law. Note that the difference between closed forms and exact forms is a *global* feature of X reflecting the topological complexity of singular points of X.

If X is a smooth manifold, we may express ω in the form $\omega = \Sigma_i \omega_i dx^i$, where the coordinate functions ω_i are smooth functions. Recalling that ω constitutes the natural entity to be integrated over cycles of the same dimension, we consider ω as a *function* on cycles:

$$o \mapsto \oint_o \omega \quad (33)$$

The important thing is that since the *evaluation* of this integral gives the *period* of ω with respect to an integration *cycle* o, the above function is a *locally constant* real-valued function on *homologous* cycles.

Conversely, considering a general differential 1-form ω, we think of it as a *function* on smooth paths γ:

$$\gamma \mapsto \int_\gamma \omega \quad (34)$$

and examine *when* the above function is a locally constant real-valued function on X. In other words, we are interested in *those* ω for which the line integral $\int_\gamma \omega$ remains *constant* under *perturbations* of γ, while keeping its endpoints fixed. This happens *only* if ω is a *closed* differential 1-form, i.e. $d\omega = 0$, meaning that the required condition is fulfilled by the solutions of the differential equation $d\omega = 0$ modulo the solutions $\omega = d\sigma$, where σ in Ω^0, satisfying trivially the local constancy condition.

Thus, $H^1(X;\mathbb{R})$ is a real vector space determined by the real-valued locally constant integrals of closed differential 1-forms modulo those that are trivially locally constant of exact differential 1-forms. The same prescription can be generalized to all higher order cohomology groups by the replacement of line integrals by their higher dimensional analogues. In conclusion, we realize that the *constant Abelian sheaf* of the reals $\mathbb{R}$ determines by the *localization* of the notion of *constancy* to local constancy *all* de Rham cohomology groups in its function as a coefficient sheaf of Čech cohomology.

7

CANONICS OF INVARIANCE: THE COUPLING OF MATTER WITH FIELDS

7.1 ARTICULATION OF MATTER: FROM LOCAL GAUGE SYMMETRY TO GAUGE POTENTIALS

Physical geometry may be thought of as the outcome of a measurement procedure, where the interwoven notion of a group action discloses a particular act of measurement. The general conceptual attitude towards physical geometry emanating from *Felix Klein's* Erlangen program requires that the *geometric configuration* of states of a physical system and the *symmetry group* of transformations of those states should be considered *equivalent* through the *free* and *transitive action* of the symmetry group on the space of states.

Gauge-invariant field theories of material interaction are modeled in terms of *fiber bundle* geometries endowed with a *connectivity* structure, which integrate Klein's *group* theoretical conception of geometry with Riemann's infinitesimal *metrical* viewpoint. These physical models are based on the fundamental notion of a *connection* on a fiber bundle, which generalizes *Weyl's* viewpoint of a *purely infinitesimally* geometry, generated by means of an affine connection that can be made metric-compatible. The notion of a connection, in relation to a fiber bundle geometry, is indispensable for the effectuation of a *covariant derivative* operator, which acts on the local sections-states of the bundle. This operator expresses analytically in infinitesimal terms the process of *parallel translation* along paths on the base space of the bundle induced by this connection.

In this manner, the infinitesimal expression of the connection is interpreted as a local *gauge potential* corresponding to a physical field. It is expressed by means of a *differential* 1*-form*, which takes values in the *Sophus Lie algebra* of the symmetry group. In turn, the observable, and thus measurable, effects of the potential are expressed invariantly via a *tensor*, which is called the *curvature* of the connection. The latter is physically identified locally with the observable *strength* of the corresponding physical field. In the setting of gauge theories the symmetry group is modeled locally on the fibers of a *principal* fiber bundle, which is defined over spacetime. Concomitantly, the state space of a gauge theory is identified with the *group sheaf* of sections of this bundle, or equivalently, with the *vector sheaf* of sections of its associated *vector* bundle. We note that the above setting of a gauge field theory constitutes a concrete expression of the principle of *sheaf-theoretic localization*, referring to both, the physical state space, and the pertinent free and transitive group action.

The genesis of gauge theory can be traced back to *Hermann Weyl's* conception of the interrelation between some type of *material source* and the corresponding *field* governing a physical interaction. According to Weyl, analogously to the fact that charge may be conceived as an electromagnetic effect, mass may be conceived as a gravitational effect. In particular, mass is the flux of the gravitational field through a surface enclosing a particle in the same sense that charge is the flux of the electric field. Thus, in the same way that it is impossible to introduce charge without the electromagnetic field, it is also impossible to introduce a non-vanishing mass without the gravitational field. In this respect, Weyl's fundamental idea concerning the notion of a local gauge interaction pertaining to a type of matter source with a corresponding physical field stems from the requirement of *invariance*.

More concretely, the crucial realization is that the localization of an *internal* symmetry in the physical description pertaining to a material form always gives rise to a *corresponding* gauge potential of a field, in order that the physical description remains *invariant*. Therefore, physical geometry in the setting of a gauge theory as a result of an interaction *always* proceeds from the *localization* of a *global* symmetry pertaining to the description of a material form. The demand of invariance under this process of localization requires the instantiation of a gauge field potential *transferring* an interaction, which thus couples the material form with the gauge field. Therefore, the localization of symmetry in relation to a material form constitutes the *necessary* condition for expressing both, the notion of an interaction field via its gauge potentials, and the minimal coupling of these potentials with the matter sources, under the demand of invariance.

The principle of sheaf-theoretic localization, which forces the functorial transference of Grassmann's lineal extension framework from the level of vector states to the level of vector sheaves of states, by which we mean locally free modules of states over a commutative algebra sheaf of observables, gives the possibility of a precise description of the physical geometry derived as a result of the interaction between a physical field and some type of a matter source, according to the preceding. For this purpose, the *base space* of the sheaf-theoretic localization procedure, pertaining to both, the symmetries, and the physical states, is identified with the spacetime manifold. Nevertheless, its construal as a base space is only implicit, in the sense that its point structure is not explicit *ab initio*, but has to be *articulated* by physical means.

The basic idea is that the information regarding the *point* constitution of this space, which points are *occupied* by some material form, or the space is *penetrated* by some current, can be expressed by means of some global invariant characterizing the physical interaction field whose *sources* are precisely these material forms. In the same context, the notion of physical geometry is always induced by the localization of some free and transitive group action, which expresses a fundamental internal symmetry in the description pertaining to a material form, in the sense that it is tantamount to a conservation law. The culmination of this conceptual setting bears the name derived by the principle of *gauge invariance* leading to the modeling of physical interaction geometries in terms of *gauge theories*.

The localization of the internal symmetry of a material form simply means that it may *vary independently* from point to point of the base space, within the range of *ambiguity* determined by the corresponding symmetry group. In this sense, a *copy* of the same symmetry group is *replicated* from point to point of the base space, in such a way that, at each of these points, a symmetry element is subsumed in total *independence* from all the others at *any* other point. It is precisely due to this freedom of independent symmetry variation from point to point that a proper means of following this variation from the infinitesimal to the global is required. These means establish the standards of *congruence* according to an imposed rule of *parallelism*, which is to say a rule of parallel transport along temporally parameterized paths on the base space. The latter derives from the notion of a connection on the corresponding principal group sheaf of symmetry coefficients, or equivalently, by a connection on the associated vector sheaf of states, where the latter assumes the role of the state representation space of the symmetry group.

In a standard way, a connection always gives rise to a covariant derivative operator, i.e. to a covariant means of *differentiating* the *sections* of this sheaf. The physical interpretation is that a field can be *covariantly* detected in terms of its differentiating effects on the section-states of the vector sheaf, and thus it should be *locally* characterized in terms of its gauge potentials. Henceforth, gauge field potentials are instantiated as necessary elements proceeding from the localization of the internal symmetry group of a material form, and being minimally coupled with the latter, such that the physical description remains always *invariant* with respect to the algebra sheaf of observables.

In the setting of a gauge theory the invariance in the description of physical geometry is always meaningful with respect to a *commutative algebra sheaf* of observables. It is this requisite invariance that demands our focus once again on the *commutative shadow* of the *exterior* algebra of the vector sheaf of states. According to the qualification of geometric calculus in relation to the sheaf-theoretic localization of Grassmann's lineal extension method, the commutative shadow is of a *cohomological* origin.

More concretely, it is enunciated by means of the locally defined coboundary, or exterior derivative, operator of the observable algebra sheaf. It is precisely based on the idea of unfolding and separating the infinitesimal irreducible parts of the commutative shadow, as modules of differentials of different orders. Recalling that the notion of the coboundary operator is based on the articulation of a point as a bound of distinct temporal orders becoming dependent infinitesimally, i.e. in terms of commuting one-parameter infinitesimal flows, at this point, it is necessary to account for the modification of these flows by the effect of localization of the internal symmetry of a material form occupying the point itself. In the descriptive capacity of the commutative shadow, this type of modification needs to be expressed in an invariant manner locally around this point by means of the associated gauge field potentials, which become minimally coupled with the material form.

Therefore, the coboundary operator is *covariantly adapted* to the additional requirement of localizing matter symmetry, and thus extended to act on the representation vector sheaf of states of the internal symmetry group, as a covariant derivative operator stemming from a connection on this sheaf. As a result, and in terms of the commutative shadow, the articulation of a point where some matter form is pertinent can be accounted for by the failure of this modified coboundary operator to extend to a differential de Rham complex as in the former case. This *obstruction* to *extendibility* of the *modified coboundary* operator to the next higher order is interpreted physically as the *encoding trace* of the field strength, associated with the curvature of the employed connection on the vector sheaf of states. The curvature bears the transformation properties of a tensor, and thus behaves covariantly with respect to the commutative algebra sheaf of observables.

7.2 COVARIANCE: GAUGE TRANSFORMATIONS OF LOCAL POTENTIALS' STRENGTHS

A connection $\mathcal{D}_{\mathcal{E}} := \nabla_{\mathcal{E}}$ on the vector sheaf of states $\mathcal{E}$ is a $\mathbb{C}$-*linear sheaf* morphism:

$$\nabla_{\mathcal{E}} : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}, \quad (1)$$

referring to $\mathbb{C}$-vector space sheaves, such that the following Leibniz type of condition is satisfied:

$$\nabla_{\mathcal{E}}(a \cdot s) = a \cdot \nabla_{\mathcal{E}}(s) + s \otimes d^0(a) \quad (2)$$

At the next stage, and since we are interested in the *local* form of a connection, we will show that *every* connection $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a finite rank-n vector sheaf of states on X, can be decomposed locally as follows:

$$\nabla_{\mathcal{E}} = d^0 + \omega. \quad (3)$$

where $\omega = \omega_{\alpha\beta}$ denotes an $n \times n$ matrix of sections of local 1-forms, called the *matrix potential* of $\nabla_{\mathcal{E}}$.

Moreover, under a change of local frame matrix $g = g_{\alpha\beta}$, we will demonstrate that the matrix potentials transform as follows:

$$\omega' = g^{-1}\omega g + g^{-1}d^0 g. \quad (4)$$

If we consider a coordinatizing *basis of sections* of the vector sheaf $\mathcal{E}$ of rank-n, defined over an open cover U of X, denoted by:

$$e^U \equiv \{U; (e_\alpha)_{1\leq\alpha\leq n}\} \quad (5)$$

called a *local sectional frame*, or equivalently, a *local gauge* of $\mathcal{E}$, then every continuous local section $s \in \mathcal{E}(U)$, where, $U \in \mathcal{U}$, can be expressed uniquely with respect to this local frame as the following *superposition*:

$$s = \sum_{\alpha=1}^{n} s_\alpha e_\alpha, \quad (6)$$

with coefficients s_α in $\mathcal{A}(U)$. The *action* of $\nabla_{\mathcal{E}}$ on these sections of $\mathcal{E}$ is expressed as follows:

$$\nabla_{\mathcal{E}}(s) = \sum_{\alpha=1}^{n}(s_\alpha \nabla_{\mathcal{E}}(e_\alpha) + e_\alpha \otimes d^0(s_\alpha)). \quad (7)$$

$$\nabla_{\mathcal{E}}(e_\alpha) = \sum_{\alpha=1}^{n} e_\alpha \otimes \omega_{\alpha\beta}, 1 \leq \alpha, \beta \leq n, \quad (8)$$

where $\omega = \omega_{\alpha\beta}$ denotes an $n \times n$ matrix of sections of local 1-forms. Consequently we have;

$$\nabla_{\mathcal{E}}(s) = \sum_{\alpha=1}^{n} e_\alpha \otimes (d^0(s_\alpha) + \sum_{\beta=1}^{n} s_\beta \omega_{\alpha\beta}) \equiv (d^0 + \omega)(s). \quad (9)$$

Thus, *every connection* $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a finite rank-n vector sheaf on X, can be *decomposed locally* as follows:

$$\nabla_{\mathcal{E}} = d^0 + \omega. \quad (10)$$

In this context, $\nabla_{\mathcal{E}}$ is identified as a covariant derivative operator acting on the sections of the vector sheaf of states $\mathcal{E}$. This operator can be decomposed locally as a sum consisting of a *flat or integrable part* identical with d^0, and a generally *non-integrable* part ω, called the *local frame (gauge) matrix potential* of the connection.

The behavior of the local potential ω of $\nabla_{\mathcal{E}}$ under local frame transformations constitutes the transformation law of local gauge potentials and is obtained as follows:

Let $e^U \equiv \{U; e_{\alpha=1\cdots n}\}$ and $h^V \equiv \{V; h_{\beta=1\cdots n}\}$ be two local frames of $\mathcal{E}$ over the open sets U and V of X, such that $U \bigcap V \neq \varnothing$. Let us denote by $g = g_{\alpha\beta}$ the following change of local frame matrix:

$$h_\beta = \sum_{\alpha=1}^{n} g_{\alpha\beta} e_\alpha. \quad (11)$$

Under such a local frame transformation $g_{\alpha\beta}$, we easily obtain that the local potential ω of $\nabla_{\mathcal{E}}$ transforms as follows in matrix form:

$$\omega' = g^{-1}\omega g + g^{-1} d^0 g. \quad (12)$$

The above is clearly a metaphora, which is expressed in terms of *conjugation* through the bridge g and its inverse:

$$\omega' = g^{-1}(d^0 + \omega) g. \quad (13)$$

Further, if we assume that the pair $(\mathcal{E}, \nabla_{\mathcal{E}})$ denotes a complex vector sheaf of states endowed with a *connection*, $\nabla_{\mathcal{E}}$, then $\nabla_{\mathcal{E}}$ induces a *sequence* of $\mathbb{C}$-linear morphisms:

$$\mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \to \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \quad (14)$$

or equivalently:

$$\mathcal{E} \to \Omega^1(\mathcal{E}) \to \ldots \to \Omega^n(\mathcal{E}) \to \ldots \quad (15)$$

where the morphism:

$$\nabla^n : \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^{n+1}(\mathcal{A}) \otimes_{\mathbf{A}} \mathcal{E}, \quad (16)$$

is given by the formula:

$$\nabla^n(\omega \otimes v) = d^n(\omega) \otimes v + (-1)^n \omega \wedge \nabla(v), \quad (17)$$

for all $\omega \in \Omega^n(\mathcal{A})$, $v \in \mathcal{E}$. Immediately it follows that $\nabla^0 = \nabla_{\mathcal{E}}$.

The *composition* of $\mathbb{C}$-linear morphisms $\nabla^1 \circ \nabla^0$ is called the *curvature* of the connection $\nabla_{\mathcal{E}}$:

$$\nabla^1 \circ \nabla^0 := R_\nabla : \mathcal{E} \to \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} = \Omega^2(\mathcal{E}). \quad (18)$$

Consequently, we derive that the curvature R_∇ of a connection $\nabla_{\mathcal{E}}$ on the vector sheaf of states $\mathcal{E}$ is an $\mathcal{A}$-*linear* sheaf morphism, that is, an $\mathcal{A}$-*covariant* morphism, or equivalently, an $\mathcal{A}$-*tensor*. The $\mathcal{A}$-covariant nature of the curvature R_∇ can significantly be contrasted with the connection $\nabla_{\mathcal{E}}$, which is *only* $\mathbb{C}$-covariant and *not* $\mathcal{A}$-covariant.

The *sequence* of $\mathbb{C}$-linear sheaf morphisms,

$$\mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \to \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \quad (19)$$

defines a *complex* of $\mathbb{C}$-vector space sheaves *if and only if* the following condition is satisfied:

$$R_\nabla = 0. \quad (20)$$

Thus, the curvature $\mathcal{A}$-covariant tensor R_∇ expresses the *obstacle*, or the *obstruction* for the above sequence to *qualify* as a complex. We say that the connection $\nabla_{\mathcal{E}}$ is an *integrable*, or equivalently, a *flat connection* if $R_\nabla = 0$. In this case, we refer to the above complex as the sheaf-theoretic *de Rham complex of the integrable connection* $\nabla_{\mathcal{E}}$ on the vector sheaf $\mathcal{E}$. Note that the universal $\mathbb{C}$-derivation d^0 on $\mathcal{A}$ always defines an integrable or flat connection.

A flat connection expresses a *maximally unobstructed* process of dynamical variation associated with the corresponding *field*. From a physical viewpoint, a flat connection sets up the standards of *congruence* under replication of the local internal symmetry group point by point. Thus, a non-vanishing curvature expresses covariantly the existence of a certain type of deviation from the maximally unobstructed form of this variation. Equivalently, curvature effects can be *cohomologically* identified as *obstructions* to deformation caused by the *matter* sources coupled to the field.

In case that, additionally, the representability principle over the field of the complex, or the real numbers is required at a point-event, the existence of *uniquely* defined *duals* is necessary. In this case, the physical field is identified with a *linear connection* on the $\mathcal{A}$-vector sheaf of states $\Xi = Hom(\Omega^1, \mathcal{A})$, which is cast *isomorphic* with Ω^1, by means of a bilinear form that plays the role of a *metric*:

$$g : \Omega^1 \simeq \Xi = \Omega^{1*}. \quad (21)$$

In this case, a physical observable geometry is considered with respect to a metric. Consequently, the physical field is properly expressed by *the pair* (Ξ, ∇_{Ξ}). The required *metric compatibility* of the connection is formulated as follows:

$$\nabla_{Hom_{\mathcal{A}}(\Xi,\Xi^*)}(g) = 0. \quad (22)$$

Taking into account the requirement of representability over the complex, or the real numbers, and thus considering the concomitant evaluation *trace* operator by means of the metric, we arrive at the *analogue* of *Albert Einstein's field equations*, which in the *absence* of matter sources with respect to $\mathcal{A}$, are expressed as follows:

$$\mathcal{R}(\nabla_{\Xi})(\Xi) = 0. \quad (23)$$

where $\mathcal{R}(\nabla_{\Xi})$ denotes the relevant *Ricci scalar curvature*. More precisely, we first define the *curvature endomorphism* $\mathfrak{R}_{\nabla} \in End(\Xi)$, called the *Ricci curvature operator*. Since the Ricci curvature $\mathfrak{R}_{\nabla}$ is locally matrix-valued, by taking its trace using the metric, that is, by considering its evaluation or contraction by means of the metric, we arrive at the definition of the Ricci scalar curvature $\mathcal{R}(\nabla_{\Xi})(\Xi)$ obeying the above equation.

Thus, the metric describing the *physical geometry*, as a *result* of field interactions, is dynamically determined as a *solution* of the above equation *in relation* to the metric compatible connection on the vector sheaf $Hom_{\mathcal{A}}(\Xi, \Xi^{*})$.

Finally, it is necessary to investigate the *local* form of the curvature R_{∇} of a connection $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a locally free finite rank-n sheaf of modules (vector sheaf) $\mathcal{E}$ on X, defined by the following $\mathcal{A}$-linear morphism of sheaves:

$$R_{\nabla} := \nabla^{1} \circ \nabla^{0} : \mathcal{E} \to \Omega^{2}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} := \Omega^{2}(\mathcal{E}). \quad (24)$$

Due to its property of $\mathcal{A}$-covariance, a non-vanishing curvature represents in this context, the $\mathcal{A}$-covariant, and thus, *geometrically observable deviation* from the inertial form of variation corresponding to an integrable connection.

Further, since the curvature R_{∇} is an $\mathcal{A}$-linear morphism of sheaves of $\mathcal{A}$-modules, that is an $\mathcal{A}$-tensor, R_{∇} may be thought of as an element of $End(\mathcal{E}) \otimes_{\mathcal{A}} \Omega^{2}(\mathcal{A}) := \Omega^{2}(End(\mathcal{E}))$, as follows:

$$R_{\nabla} \in \Omega^{2}(End(\mathcal{E})) \quad (25)$$

Hence, the local form of the curvature R_{∇} of a connection $\nabla_{\mathcal{E}}$, consists of local $n \times n$ matrices taking local 2-forms for entries. In particular, the local form of the curvature $R_{\nabla}|_{U}$, where U open in X, in terms of the local potentials ω is expressed by:

$$R_{\nabla}|_{U} = d^{1}\omega + \omega \wedge \omega, \quad (26)$$

as can easily be shown by substitution of the local potentials in the composition $\nabla^{1} \circ \nabla^{0}$. Furthermore, by application of the differential operator d^{2} on the above we obtain:

$$d^{2} R_{\nabla}|_{U} = R_{\nabla}|_{U} \wedge \omega - \omega \wedge R_{\nabla}|_{U}. \quad (27)$$

The behavior of the curvature R_∇ of a connection $\nabla_{\mathcal{E}}$ under local frame transformations constitutes the transformation law of the *gauge potentials' strength*. If we agree that $g = g_{\alpha\beta}$ denotes the change of local frame matrix that we considered in the discussion of the transformation law of the local gauge potentials previously, we derive the following local transformation law:

$$R_\nabla \overset{g}{\mapsto} R'_\nabla = g^{-1}(R_\nabla)g, \quad (28)$$

that is, the strength transforms covariantly by conjugation with respect to a local frame transformation.

Thus, we may summarize the preceding as follows:

i The local form of the curvature $R_\nabla|_U$, where U open in X, in terms of the local matrix potentials ω, is given by:

$$R_\nabla|_U = d^1\omega + \omega \wedge \omega \quad (29)$$

ii Under a change of local frame matrix $g = g_{\alpha\beta}$ the local form of the curvature transforms by conjugation with respect to *g*. Therefore, *g* and its inverse, are the bridges enunciating the metaphora of the field strength locally:

$$R_\nabla \overset{g}{\mapsto} R'_\nabla = g^{-1}(R_\nabla)g \quad (30)$$

We note that the above holds for *any* complex *vector* sheaf $\mathcal{E}$. Let us now specialize to the particular case of a *line* sheaf of states $\mathcal{L}$ endowed with a *connection*, denoted by the pair $(\mathcal{L}, \nabla)$. In this case, due to the isomorphism:

$$\mathcal{L} \otimes_{\mathcal{A}} \mathcal{L}^* \cong Hom_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \equiv End_{\mathcal{A}}\mathcal{L} \cong \mathcal{A}, \quad (31)$$

we obtain the following simplifications: the *local* form of a connection over an open set is *just* a *local* 1-*form* or a local potential, identified as a local continuous section of the sheaf Ω^1, whence the *local* form of the curvature of the connection over an open set is a *local* 2-*form*. The significant result obtained by the local transformation law in this case is that the curvature is actually a *local frame invariant*, i.e. it does not change under any local frame transformation.

$$R_{\nabla} \overset{g}{\mapsto} R_{\nabla}' = R_{\nabla} = d^{1}\omega \quad (32)$$

Thus, we obtain a global 2-form R_{∇} defined over X, which is also a closed 2-form since:

$$dR_{\nabla} = 0. \quad (33)$$

Therefore, we conclude that for a *line* sheaf of states $\mathcal{L}$ equipped with a *connection* ∇, denoted by the pair $(\mathcal{L}, \nabla)$, the *curvature* R_{∇} of the connection is a *global closed* 2-form.

7.3 INVARIANCE: GAUGE EQUIVALENT GEOMETRIC SPECTRA

The essence of a *geometric equivalence* problem in physical geometry refers to the determination of the conditions under which two *geometric spectra* of the same type are *equivalent* under an appropriate *group* of transformations. The most fruitful approach to geometric equivalence problems concerns the appropriate association of *invariants* with a type of geometric spectra, by which we mean attributes that *do not change* under an *isomorphism*. The idea is that invariants are capable of determining geometric spectra *uniquely* up to isomorphism.

In general, a *physical geometric spectral type* is expressed by an *equivalence* class of state spaces, each of which constitutes the geometric spectrum of a *commutative* algebra *sheaf* of observables. A state space incorporates the totality of *potential states* of a physical system, and is *vectorial* with respect to the algebra of observables of this system. In the case of a gauge theory, the notion of a state space arises from, and it is therefore equivalent, to the transitive group action space of an internal symmetry group, at least *locally*, in the sense that the symmetry group delineates locally the range of potential attributes that a physical geometric type can assume. Being a member of a geometric type, a physical matter entity can be in any of the permissible potential states locally.

This is the cornerstone of the local gauge freedom and constitutes a concrete physical manifestation of the *criterion* of geometric equivalence that a *local gauge group* furnishes in the case of gauge theories. The sheaf-theoretic formulation of gauge theories captures precisely the formation of physical geometric types under equivalence criteria constituted by the actions of local symmetry groups. In this state of affairs, a physical geometric spectral type is expressed by means of an *equivalence class* of *vector sheaves* endowed with a *connectivity* structure. The base manifold of a vector sheaf of states equipped with a connection is an integral part of the coupling between a gauge field and the matter sources occupying its points. For this reason, its point constitution is only implicitly posited, specified neither *ab initio* nor *a priori*. Rather, it is just the *base carrier* of the geometry by which matter is transformed, understood in this framework as a *structural quality* of the corresponding *physical field*.

The strategy to tackle the problem of equivalence of physical geometric spectra in the setting of gauge theories involves the determination of invariant *global characteristic classes* associated with vector sheaves of states endowed with a connection. A characteristic class is *represented* analytically in the pertinent geometric context in terms of an appropriate *differential form* stemming from the *connection*. For instance, in the case of a *line* sheaf, the global curvature differential form provides *the seed* for the articulation of the observable geometric spectrum, congredient with the field strength, via its *global de Rham cohomology class*.

We recall that the essential aspect of the de Rham cohomological analysis is precisely the *disclosure* of singular points on an implicitly assumed manifold, or more generally, a nice topological space, in terms of *invariant* quantities obtained by the *integration* of *differential forms*. In particular, closed differential forms are the natural integrands over cycles, meaning that they can be temporally integrated over closed chains encycling a singular point, such that the result of this integration procedure unveils a *residue* characterizing invariantly the *presence* of this singular point in terms of its *spectral periods*.

With this concept we reach the culmination of a key idea, an idea which addresses the specific problem of grasping the nature and essence of a *singular point* when some form of *matter* is in play. Since such a point *cannot* be bounded by infinitesimal flows converging on it, it can only be amenable to description by means of processes which *circulate* around it. These circulations are *not* equivalent, but they are in principle *distinguishable* in terms of some recognizable *global* attribute. Such an attribute obtained through integration is a spectral period, as above. Of especial significance is the resolving domain of spectral periods corresponding to the curvature form of the circulating connection. We stress the fact that cycles are not boundaries, since they generate homology classes encapsulating a *novel* type of temporal order with reference to singular points that is qualitatively different from boundaries. More concretely, the type of temporal order encapsulated by cycles is precisely characterized in terms of spectral periods via a global integration procedure, and not in terms of instants as in the former case.

In the context of the *equivalence* problem of *gauge geometric spectral types*, we consider two line sheaves of states which are equivalent due to an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$. Because of the *bijective* correspondence between line sheaves and coordinate 1-cocycles with respect to an open covering $\mathcal{U}$ of the base topological space X, which is considered to be paracompact, the set of *isomorphism classes* of line sheaves over X, denoted by $Iso(\mathcal{L})(X)$, is in bijective correspondence with the set of *cohomology classes* $H^1(X, \tilde{\mathcal{A}})$:

$$Iso(\mathcal{L})(X) \cong H^1(X, \tilde{\mathcal{A}}) \quad (1)$$

Moreover, the set of isomorphism classes of line sheaves of states over X, $Iso(\mathcal{L})(X)$, is an *Abelian group* with respect to the *tensor product* over the observable algebra sheaf $\mathcal{A}$. The tensor product of two line sheaves corresponds to the *product* of their respective coordinate 1-cocycles. Thus, $Iso(\mathcal{L})(X)$ is isomorphic with the Abelian group of cohomology classes $H^1(X, \tilde{\mathcal{A}})$. In this setting, $\mathcal{A}$ is considered as a soft sheaf, meaning that *every* section over some closed subset in X can be *extended* to a section over X.

Then, the process of *exponentiation*, in local sheaf-theoretic terminology, is expressed by the following *short exact sequence* of Abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{A} \xrightarrow{exp} \tilde{\mathcal{A}} \to 1, \quad (2)$$

where $\mathbb{Z}$ is the *constant* abelian group sheaf of the *integers*, namely the sheaf of *locally constant* sections valued in the group of integers, such that:

$$Ker(e) = Im(\iota) \cong \mathbb{Z}. \quad (3)$$

Note that all the elements in the above short exact sequence are Abelian group sheaves *generalizing* the corresponding short exact sequences of constant Abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{exp} \tilde{\mathbb{C}} \to 1, \quad (4)$$

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{R} \xrightarrow{exp(2\pi i)} \mathbb{U}(1) \to 1 \quad (5)$$

As an outcome, we obtain a *long exact sequence* in sheaf cohomology, which is reduced to a long exact sequence in Čech cohomology because of paracompactness of X:

$$\ldots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{A}) \to H^1(X,\tilde{\mathcal{A}}) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{A}) \to H^2(X,\tilde{\mathcal{A}}) \to \ldots \quad (6)$$

Furthermore, since $\mathcal{A}$ is a soft sheaf:

$$H^1(X,\mathcal{A}) = H^2(X,\mathcal{A}) = 0 \quad (7)$$

$$0 \to H^1(X,\tilde{\mathcal{A}}) \xrightarrow{\delta_c} H^2(X,\mathbb{Z}) \to 0. \quad (8)$$

Thus, we obtain the following isomorphism of Abelian groups, called the *Shiing-Shen Chern isomorphism*:

$$\delta_c : H^1(X,\tilde{\mathcal{A}}) \xrightarrow{\cong} H^2(X,\mathbb{Z}). \quad (9)$$

Finally, since $Iso(\mathcal{L})(X)$, is in *bijective* correspondence with the Abelian group of cohomology classes $H^1(X,\tilde{\mathcal{A}})$, we have:

$$Iso(\mathcal{L})(X) \cong H^2(X,\mathbb{Z}). \quad (10)$$

Therefore, the Chern isomorphism establishes that an *equivalence class* of line sheaves of states is in *bijective* correspondence with a *cohomology class* in the *integral* 2 *-dimensional cohomology group* of X, called a *Chern class* of X. Taking into account that a line sheaf of states is actually the representation vector sheaf of states of the internal symmetry group sheaf of some corresponding matter form occupying a singularity, the Chern isomorphism reveals that this *matter* form is *encoded* in terms of a *two dimensional integral cohomology class* of X.

In other words, if we integrate a representative two dimensional cocycle of this class over a two dimensional cycle in X enclosing it, we obtain a characteristic *spectral period*, qualified in terms of a $\mathbb{Z}$-invariant. The pertinent problem, in relation to the equivalence problem of gauge geometric spectral types, is whether such an integer cohomology class can be *expressed* in terms of the cohomology class of a two-dimensional differential form, namely a de Rham cohomological invariant. The underlying reason is that the decoding of an integral cohomology class *in dynamical terms*, that is, in terms of the gauge potentials of the field, and by extension, in terms of the potentials' *strength*, or curvature, as a *flow* induced by this matter form, can be expressed only by means of an appropriate corresponding *differential* de Rham cohomology class.

7.5 THE OBSTACLE OF A MATTER SOURCE: HARMONICS OF SPECTRA AND QUANTIZATION

Initially, we observe that there exists a natural injection $\mathbb{Z} \infty \mathbb{C}$:

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}), \quad (11)$$

where any Čech cohomology class belonging to the image of the above map is called an *integral* cohomology class.

Next, we have to take into account the compatibility of the above short exact sequence of Abelian group sheaves with the short exact sequence of $\mathbb{C}$-vector sheaves,

$$0 \to \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \xrightarrow{d^0} d^0\mathcal{A} \to 0, \quad (12)$$

arising as a fragment of the *de Rham resolution* of the constant sheaf $\mathbb{C}$, such that:

$$Ker(d^0) = Im(\varepsilon) \cong \mathbb{C}. \quad (13)$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "exp"] \arrow[d, "id"'] & \tilde{\mathcal{A}} \arrow[d, "\frac{1}{2\pi i}\tilde{d}^0"] \\
\mathcal{A} \arrow[r, "d^0"] & d^0\mathcal{A}
\end{tikzcd}$$

Thus we obtain the following key relation:

$$2\pi i \cdot d^0 = \tilde{d}^0 \circ exp. \quad (14)$$

The dynamical, gauge field-theoretic, decoding of an integral cohomology class, or Chern class, of X requires to extend the notion of equivalence of two line sheaves to the corresponding one of two line sheaves *equipped* with a connection, which is induced *locally* by gauge potentials, $(\mathcal{L},\nabla)$ and $(\mathcal{L}',\nabla')$.

If we consider an isomorphism $h:\mathcal{L}\xrightarrow{\cong}\mathcal{L}'$ of line sheaves of states, we say that ∇ is *frame or gauge equivalent* under *metaphora* to ∇' if they are *conjugate connections* under the action of the isomorphism h:

$$\nabla' = h\cdot\nabla\cdot h^{-1}. \quad (15)$$

Thus, we may consider the set of equivalence classes on *pairs* of the form $(\mathcal{L},\nabla)$ under an *isomorphism* h as previously, denoted by $Iso(\mathcal{L},\nabla)$. It is necessary to investigate the relation between $Iso(\mathcal{L},\nabla)$ and the Abelian group $Iso(\mathcal{L})$. For this purpose, we need to make use of the local form of the pair $(\mathcal{L},\nabla)$.

We call a line sheaf $\mathcal{L}$ endowed with a connection ∇ a *differential line sheaf*, and we denote it by the pair $(\mathcal{L},\nabla)$. Then, The *local* form of a differential line sheaf is given by:

$$(\mathcal{L},\nabla)\leftrightarrow(g_{\alpha\beta},\omega_\alpha)\in Z^1(\mathcal{U},\tilde{\mathcal{A}})\times C^0(\mathcal{U},\Omega^1) \quad (16)$$

Moreover, an arbitrary pair $(g_{\alpha\beta},\omega_\alpha)\in Z^1(\mathcal{U},\tilde{\mathcal{A}})\times C^0(\mathcal{U},\Omega^1)$ determines a differential line sheaf if the *transformation* law of local gauge potentials is valid:

$$\delta^0(\omega_\alpha)=\tilde{d}^0(g_{\alpha\beta}) \quad (17)$$

where,

$$\tilde{d}^0(g_{\alpha\beta})=g_{\alpha\beta}^{-1}\cdot d^0 g_{\alpha\beta}, \quad (18)$$

and δ^0 denotes the 0-th coboundary operator $\delta^0: C^0(\mathcal{U},\Omega^1)\to C^1(\mathcal{U},\Omega^1)$, such that:

$$\delta^0(\omega_\alpha)=\omega_\beta-\omega_\alpha. \quad (19)$$

In more detail, we note that a line sheaf is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$ associated with the covering $\mathcal{U}$. A connection ∇ is expressed bijectively in terms of a 0-cochain of gauge potential 1-forms, denoted by ω_α with respect to the covering $\mathcal{U}$ of X, that is $\omega_\alpha\in C^0(\mathcal{U},\Omega^1)$. Therefore, the local form of a differential line sheaf is the following:

$$(\mathcal{L},\nabla) \leftrightarrow (g_{\alpha\beta},\omega_\alpha) \in Z^1(\mathcal{U},\tilde{\mathcal{A}}) \times C^0(\mathcal{U},\Omega^1). \quad (20)$$

Conversely, an arbitrary pair $(g_{\alpha\beta},\omega_\alpha) \in Z^1(\mathcal{U},\tilde{\mathcal{A}}) \times C^0(\mathcal{U},\Omega^1)$ determines a differential line sheaf if the transformation law of local gauge potentials is satisfied by this pair, that is:

$$\omega_\beta = g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d^0 g_{\alpha\beta}, \quad (21)$$

$$\omega_\beta = \omega_\alpha + g_{\alpha\beta}^{-1} d^0 g_{\alpha\beta}. \quad (22)$$

Thus, given an open covering $\mathcal{U} = \{U_\alpha\}$, a 0-cochain (ω_α) valued in the sheaf Ω^1 determines the local form of a connection ∇ on the line sheaf $\mathcal{L}$, where the latter is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $(g_{\alpha\beta})$ valued in $\tilde{\mathcal{A}}$ with respect to $\mathcal{U}$, *if and only if* the corresponding local 1-forms ω_α of the 0-cochain with respect to $\mathcal{U}$ are pairwise inter-transformable, meaning *locally gauge-equivalent* on overlaps $U_{\alpha\beta}$ through the local gauge transition functions, namely the *local isomorphisms* $g_{\alpha\beta} \in \tilde{\mathcal{A}}(U_{\alpha\beta})$ according to the transformation law of local gauge potentials.

Next, we consider two line sheaves which are equivalent via an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$, such that their corresponding connections are *conjugate* under the action of h:

$$\nabla' = h \cdot \nabla \cdot h^{-1}. \quad (23)$$

Under these conditions the differential line sheaves $(\mathcal{L},\nabla)$ and $(\mathcal{L}',\nabla')$ are called *gauge or frame equivalent*. Thus, we may consider the set of gauge equivalence classes $[(\mathcal{L},\nabla)]$ of differential line sheaves as above, denoted by $Iso(\mathcal{L},\nabla)$. Then, it is easy to show that the set of gauge equivalence classes of differential line sheaves $Iso(\mathcal{L},\nabla)$, is indeed an *Abelian subgroup* of the abelian group $Iso(\mathcal{L})$.

If we consider the *local* form of the *tensor product* of two gauge equivalent differential line sheaves we have:

$$(\mathcal{L},\nabla) \otimes_{\mathcal{A}} (\mathcal{L}',\nabla') \leftrightarrow (g_{\alpha\beta} \cdot g'_{\alpha\beta}, \omega_\alpha + \omega'_\alpha), \quad (24)$$

which satisfies the transformation law of local gauge potentials:

$$\tilde{d}^0(g_{\alpha\beta}\cdot g'_{\alpha\beta}) = \tilde{d}^0(g_{\alpha\beta}) + \tilde{d}^0(g'_{\alpha\beta}) = \delta^0(\omega_\alpha) + \delta^0(\omega'_\alpha) = \delta^0(\omega_\alpha + \omega'_\alpha). \quad (25)$$

Moreover, the *inverse* of a pair $(g_{\alpha\beta}, \omega_\alpha)$ is given by $(g_{\alpha\beta}^{-1}, -\omega_\alpha)$, whereas the *neutral* element in the group $Iso(\mathcal{L}, \nabla)$ is given by $(id_{\alpha\beta}, 0)$, which corresponds to the trivial standard differential line sheaf $(\mathcal{A}, d^0)$.

The most important consequence of the above characterization of gauge equivalent differential line sheaves in *local* terms with respect to an open covering $\mathcal{U}$, is that *all* gauge *equivalent geometric spectral types* are characterized by the *same curvature*. We denote the curvature of a gauge equivalence class of differential line sheaves by R. The differential form R is a *global* 2-form on X since it is *invariant* under a gauge transformation. Moreover, R is also a *closed* global 2-form on X, because of the fact that:

$$d \circ d\omega_\alpha = dR = 0. \quad (26)$$

Thus, the global 2-form R, which belongs to $Ker(d): \Omega^2 \to \Omega^3$, called Ω^2_c, identified as a $\mathbb{C}$-vector sheaf *subspace* of Ω^2, determines a *global differential invariant* of gauge equivalent differential line sheaves. This is due to the fact that the global 2-form R determines a *2-dimensional complex-valued de Rham cohomology class* $[R]$. In turn, by virtue of the de Rham isomorphism, $[R]$ is *isomorphically* identified as a 2-dimensional complex Čech cohomology class in $H^2(X, \mathbb{C})$.

Most important, if we consider a differential line sheaf, the differential invariant de Rham cohomology class $[R]$ is *independent* of the *connection* used to *represent* R *locally*. Equivalently, a particular connection of a differential line sheaf provides *the means* to express this global differential invariant locally in terms of the corresponding *gauge potentials* of the field, whereas the latter is *independent* of the particular means used to represent it locally.

The fact that any two gauge equivalent differential sheaves have the *same* curvature, means that they are physically *indistinguishable*. Hence, the Abelian group $Iso(\mathcal{L}, \nabla)$ is spectrally *partitioned* into *orbits* over the image of $Iso(\mathcal{L}, \nabla)$ into Ω^2_c, where each orbit, or fiber of this partition, is labelled by a closed 2-form R of Ω^2_c:

$$Iso(\mathcal{L}, \nabla) = \sum_R Iso(\mathcal{L}, \nabla)_R. \quad (27)$$

Thus, the Abelian group of equivalence classes of differential line sheaves fibers over those elements of Ω_c^2, by which we mean that it fibers over those closed global 2-forms in Ω_c^2, which can be *identified* in terms of the *curvature* of the field.

Recapitulating the problem of gauge equivalent geometric spectra, the main issue is the *decoding* of a two-dimensional integral cohomology class, that is, the decoding of a *Chern class*, in *dynamical* terms expressed via the gauge field *potentials*. We bear in mind that a Chern class *encodes* the *singular* presence of a *matter* form via the Chern isomorphism.

This issue boils down to the idea of *decoding* a Chern class in terms of the curvature of the corresponding gauge field, which essentially amounts to decoding by means of the *flow* induced by the implicated singular matter form. According to this, the *Chern class* dynamical decoding takes place in an *invariant* manner through the *differential de Rham* cohomology class corresponding to the *curvature* of the field. In particular, since R is a global and closed 2-form on X it actually *determines* a 2-dimensional complex-valued de Rham cohomology class $[R]$, or equivalently a 2-dimensional complex Čech cohomology class in $H^2(X,\mathbb{C})$.

In the setting of gauge equivalent geometric spectra, the sheaf-theoretic formulation of the *Chern-Weil integrality theorem* states that the global closed 2-form is the curvature R of a differential line sheaf *if and only if* its 2-dimensional de Rham cohomology class is *integral*, more precisely, $[R] \in Im(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}))$.

This theorem establishes the *cohomological condition* for the consistency of the preceding encoding/decoding procedure, that is, the process of metaphora pertaining to gauge equivalent geometric spectra. More precisely, the singular presence of a matter form in gauge theory, encoded by means of a two-dimensional integral cohomology class according to the Chern isomorphism, is decoded dynamically by means of the two-dimensional cohomology class of the curvature of the gauge field, in such a manner that the latter is completely characterized intrinsically by the *integrality* cohomological condition.

It is precisely the above *integrality* condition that gives rise to *quantization* in the physical state of affairs. In other words, quantization is the condition that *resolves* the problem of *equivalence* of gauge geometric spectra. More precisely, a global closed 2-form is qualified as the curvature of a gauge field in its function to disclose the observable spectral periods of a singular matter form, *if and only if it is quantized*, meaning that the *integrality* condition is interpreted physically as a *quantization* condition, where the *intrinsic* characterization of the latter is purely of a *cohomological* nature.

The fact that the curvature cohomology class $[R]$ of any differential line sheaf in $H^2(X,\mathbb{C})$ is in the image of a cohomology class in the integral 2-dimensional cohomology group $H^2(X,\mathbb{Z})$ into $H^2(X,\mathbb{C})$ provides an intrinsic criterion for *recognition* of all those global closed 2-forms in Ω_c^2, which are instantiated as curvatures of gauge equivalence classes of differential line sheaves, according to the *fibration*:

$$Iso(\mathcal{L},\nabla) = \sum_R Iso(\mathcal{L},\nabla)_R. \quad (28)$$

The proof of this theorem is based on the following commutative diagram in our set-up:

$$\begin{tikzcd}
H^1(X,\tilde{A}) \arrow[r, "\delta_c"] \arrow[d, "\frac{1}{2\pi i}\tilde{d}^0"'] & H^2(X,\mathbb{Z}) \arrow[d, "\iota^*"] \\
H^1(X,d^0\mathcal{A}) \arrow[r, "\cong"] & H^2(X,\mathbb{C})
\end{tikzcd}$$

Note that the image of a cohomology class of $H^2(X,\mathbb{Z})$ into an integral cohomology class of $H^2(X,\mathbb{C})$ corresponds to the cohomology class specified by the image of the 1-cocycle $g_{\alpha\beta}$ into $H^1(X,d^0\mathcal{A})$, viz. by the 1-cocycle $\frac{1}{2\pi i}\tilde{d}^0(g_{\alpha\beta})$. Moreover, due to the exactness of the exponential sheaf sequence of Abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{A} \xrightarrow{exp} \tilde{\mathcal{A}} \to 1 \quad (29)$$

$$g_{\alpha\beta} = exp(w_{\alpha\beta}), \quad (30)$$

where $(w_{\alpha\beta}) \in C^1(\mathcal{U},\mathcal{A})$, and by means of the Chern isomorphism:

$$\delta_c(w_{\alpha\beta}) = (z_{\alpha\beta\gamma}) \in Z^2(\mathcal{U},\mathbb{Z}). \quad (31)$$

Explicitly, we may consider $w_{\alpha\beta} = ln(g_{\alpha\beta})$, so that:

$$\delta_c(w_{\alpha\beta}) = (z_{\alpha\beta\gamma}) := \frac{1}{2\pi i}(ln(g_{\alpha\beta}) + ln(g_{\beta\gamma}) - ln(g_{\alpha\gamma})) \in Z^2(\mathcal{U}, \mathbb{Z}). \quad (32)$$

The formulation of the *Chern-Weil* integrality theorem in *sheaf-theoretic* terms is the following:

$$[R] = 2\pi i \cdot [(z_{\alpha\beta\gamma})] \in H^2(X, \mathbb{C}), \quad (33)$$

$$[(g_{\alpha\beta})] = \frac{1}{2\pi i}[R] = [(z_{\alpha\beta\gamma})] \in H^2(X, \mathbb{Z}), \quad (34)$$

$$[R] \in Im(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})). \quad (35)$$

Thus, there exists an intrinsic *invariant* characterization of those global closed 2-forms in Ω_c^2, which are instantiated as curvatures of gauge equivalence classes of differential line sheaves, denoted by $\Omega_{c,\mathbb{Z}}^2$. Concretely, a global closed 2-form is the *curvature* R of a differential line sheaf if and only if it is *quantized*, i.e. its 2-dimensional de Rham cohomology class is *integral*, meaning that $[R] \in Im(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}))$. Therefore the *unveiling* of a gauge field's observable *geometric spectrum* as a means to qualify the implicit presence of a *singular matter* form with which the field interacts with entails *quantization*.

To conclude, the quantization condition expresses the *harmonics* of gauge equivalent geometric spectra. The harmonics in their manifestation in terms of *integral spectral periods* are obtained dynamically by the integration of a global closed differential 2-form qualified as the curvature, which represents the *potentials' strength* of the corresponding *gauge field*. In particular, the integration of such a closed differential 2-form over a two dimensional cycle in X enclosing a material form with which the field is coupled to, gives rise to a characteristic integral spectral period. This can be grasped equivalently as a harmonic serving as the invariant of all gauge equivalent differential line sheaves sharing this form as their curvature.

7.6 THE OBSTACLE OF INERTIA: INTEGRABLE CONNECTIONS AND POLARIZATION

Čech cohomology theory is characterized by the fact that the group coefficient structure of the real or the complex numbers does not constrain the validity of Čech's theoretical framework in any way. Hence, *all* the steps of the Čech construction can be carried out for constant functions in any Abelian group, giving rise to Abelian cohomology groups with values in G for any abelian group of coefficients G. In particular, we may also consider a *locally variable* Abelian group of coefficients, i.e. an Abelian group sheaf of coefficients.

Since in the case of a *paracompact* topological space the calculation of cohomology with coefficients into a typical sheaf of coefficients is *equivalent* to the calculation of *Čech cohomology* theory with *values* in the corresponding *sheaf*, the calculation of cohomology with values in an Abelian group sheaf of locally constant functions is of major importance. The basic idea is that the natural argument of a cohomology theory is not just a global topological space or a topological manifold, which it is invoked *only implicitly* in our considerations, but a space *together* with an observable algebra *sheaf of coefficients*, in such a way that the point constitution of this space, including both standard and singular points, is *unveiled* via the *spectrum* of our geometric calculus, by which we mean, in terms of the pertinent *cohomological analysis*.

Prior to the modern formulation of sheaf theory, the significance of the notion of *homology with local coefficients* has been treated in relation to the fundamental group of a connected topological space X, an approach pioneered by *Norman Steenrod*.

For any point x of X we consider the *fundamental group* F_x of X based at x. If Γ is a curve from x to y, the class of curves from x to y *homotopic* to Γ is denoted by γ_{xy}. Its inverse is denoted by $\gamma_{xy}{}^{-1} := \gamma_{yx}$. In this way, the elements of F_x are denoted by α_x, β_x, and so on. Moreover, the product $\alpha_x \cdot \beta_x$ denotes the element of F_x obtained by first traversing a curve of the class α_x, and then of the class β_x. The class γ_{xy} determines an *isomorphism* of groups $F_x \to F_y$, denoted by the same symbol, and defined by *conjugation*, i.e. $\gamma_{xy}(\alpha_x) = \gamma_{yx} \cdot \alpha_x \cdot \gamma_{xy}$. In this context, the combination of two isomorphisms of the above form is also an isomorphism.

Then, we may define a *system of local groups* in X in terms of the following three conditions:

First, for each point x of X, there is given a group G_x;

Second, for each class of paths γ_{xy}, there is given a group isomorphism $G_x \to G_y$, denoted by the same symbol;

Third, the result of the isomorphism γ_{xy} followed by γ_{yz} is the isomorphism corresponding to the path $\gamma_{xy} \cdot \gamma_{yz}$. Note that the identity path from x to x is the identity transformation in the group G_x.

We note that a closed path o_x of F_x determines an *automorphism* of G_x. From the third property, it follows that F_x is a group of automorphism of G_x. Now, the *invariant subgroup* of F_x acting as the identity on G_x is denoted by $F^1{}_x$.

Then, we define a system of *local groups* $\{G_x\}$ in X to be simple if every $F^1{}_x = F_x$. Moreover, if this happens for one x, it will be true for all x in X. If $\{G_x\}$ is a simple system of groups, then the isomorphism γ_{xy} is independent of the path from x to y. Choosing a fixed point z as *origin*, each G_x is uniquely isomorphic to G_z. Thus, the local system consists of *one* G_z and as many *copies* of G_z as there are points $x \neq 0$. In this context, the collection $\{F_x\}$ is a simple system of local groups if and only if it is *Abelian*. Furthermore, the *abelianization* of the fundamental group gives rise to a simple system of local groups consisting of isomorphic copies of the same abelian group, identified as the *first homology group* of X.

From the perspective of sheaf theory, the main objective consists in the operative role of a cohomology theory with values in a locally constant group sheaf of coefficients, namely a group sheaf on X for which every point x in X has an open cover U such that the restriction of this sheaf to U is a *constant* sheaf. Local coefficients may be extended to Čech cycles by constructing a system of local groups in the *simplicial nerve* of a finite open covering of X, and then, demonstrating the *isomorphism* of cohomology -taking values in a system of local groups- as above, with Čech cohomology.

If we consider vector sheaves of rank 1, i.e. line sheaves, we recall that locally, for any point $x \in X$, an open cover U of X exists such that: $\mathcal{L}|_U \cong \mathcal{A}|_U$. Furthermore, if for any point $x \in X$ an open cover U of X exists such that $\Psi|_U \cong (\mathbb{C}|_U)$, meaning that it is isomorphic to the *constant* sheaf $\mathbb{C}$, then the locally free $\mathbb{C}$-module Ψ of finite rank 1 is a complex *linear local system* of rank 1; a *line local system*.

A very important observation is that a *constant* 1 *-cocycle* $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathbb{C}})$ can be interpreted as the *coordinate* 1-cocycle of a *particular type* of a *line sheaf* with respect to an open covering $\mathcal{U}$. Since the coordinate 1 -cocycle $(\xi_{\alpha\beta})$ is constant, the *bijectively* corresponding to it line sheaf is a complex linear local system of rank 1, or *a line local system*. The natural question arising in this context is from where do line local systems *descend from* and what is their role in the *spectrum unveiling* conducted through cohomological analysis.

A first observation regarding line local systems is that if the underlying space X is simply-connected, then they are actually constant. More generally, if X is s path-connected and paracompact base topological space, then the pullback of a line local system on X to the *universal covering* space of X becomes a *constant sheaf*. In relation to the physical framework of gauge theory, the pertinent issue refers to the means of instantiation of a line local system in dynamical terms, that is, in terms of a differential line sheaf. It turns out that a *bijective* correspondence pertains between *differential line sheaves* equipped with an *integrable connection* and *line local systems*. Given that an integrable connection subsumes *the covariantly constant*, by which we mean the inertial structure of a gauge field, we derive that the *inertial structure* is *encoded cohomologically* in terms of a corresponding *line local system* in its function as a *coefficient sheaf* for cohomology.

For this purpose, let us consider any differential line sheaf $(\mathcal{L},\nabla)$, which lacks curvature and whose connection ∇ is thereby integrable. Then, the set of sections of $\mathcal{L}$, which reside in the *kernel* of the connection ∇, that is:

$$Ker(\nabla) := \{s \in \mathcal{L} : \nabla(s) = 0\} \quad (1)$$

forms a line local system. We call the sections of $Ker(\nabla)$ covariantly constant, or equivalently, inertial sections of $\mathcal{L}$ with respect to ∇. Inversely, given a line local system, denoted by Λ, we may define a differential line sheaf $(\mathcal{L},\nabla)$ by the prescription $\mathcal{L} := \mathcal{A} \otimes_{\mathbb{C}} \Lambda$, and for every pair of local sections $a \in \mathcal{A}(U)$, $s \in \Lambda(U)$, $\breve{\nabla}(a \otimes s) := da \otimes s$. The above defined connection is *integrable*, and therefore we conclude that a *bijective* correspondence exists between differential line sheaves with an integrable connection, denoted by $(\mathcal{L},\breve{\nabla})$ and line local systems. Most important, *every* line local system may be *identified* with the sheaf of *inertial sections* of an integrable differential line sheaf, meaning of a line sheaf $\mathcal{L}$ with respect to an integrable connection $\breve{\nabla}$ on $\mathcal{L}$, hence with $Ker(\breve{\nabla})_{\mathcal{L}}$.

Therefore, we derive the following equivalences:

$$Z^1(\mathcal{U},\tilde{\mathbb{C}}) \nsucceq (\xi_{\alpha\beta}) \cong (\mathcal{L},\breve{\nabla}) \cong Ker(\breve{\nabla})_{\mathcal{L}} \cong \Lambda, \quad (2)$$

$$H^1(X,\tilde{\mathbb{C}}) \cong [(\mathcal{L},\breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda]. \quad (3)$$

Since the underlying topological space X is locally path-connected, then the above series of equivalences is refined as follows:

$$Hom(\pi_1(X),\tilde{\mathbb{C}}) \cong H^1(X,\tilde{\mathbb{C}}) \cong [(\mathcal{L},\breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda], \quad (4)$$

where the first term denotes the set of *representations* of the fundamental group of the topological space X to $\tilde{\mathbb{C}}$.

All the previous considerations are immediately extended to the case of isomorphism classes of *Hermitian line sheaves*, that is, line sheaves equipped with a *Hermitian inner product* structure. More specifically, given a line sheaf $\mathcal{L}$ on X, an $\mathcal{A}$*-valued Hermitian inner product* on $\mathcal{L}$ is a skew-$\mathcal{A}$-bilinear sheaf morphism:

$$\circlearrowleft : \mathcal{L} \oplus \mathcal{L} \to \mathcal{A}, \quad (5)$$
$$\circlearrowleft(\alpha s, \beta t) = \alpha \cdot \bar{\beta} \cdot \circlearrowleft(s,t), \quad (6)$$

for any s, $t \in \mathcal{L}(U)$, α, $\beta \in \mathcal{A}(U)$, U open in X. Moreover, $\circlearrowleft(s,t)$ is skew-symmetric, viz. $\circlearrowleft(s,t) = \overline{\circlearrowleft(s,t)}$.

A line sheaf $\mathcal{L}$ on X, together with an $\mathcal{A}$-valued Hermitian inner product on $\mathcal{L}$ constitute a *Hermitian line sheaf*. A line sheaf is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$ associated with the open covering $\mathcal{U}$. A Čech coordinate 1-cocycle $g_{\alpha\beta}$ corresponding to a Hermitian line sheaf consists of local sections of $\mathcal{S}U(1,\mathcal{A})$, the *special unitary group sheaf* of $\mathcal{A}$ of order 1.

This is simply a coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$, such that the *unitarity* condition $|g_{\alpha\beta}|=1$ is satisfied. Clearly, in the case that a coordinate 1-cocycle $g_{\alpha\beta}$ is constant, we have $g_{\alpha\beta}$ in $Z^1(\mathcal{U},U(1))$, or equivalently $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\mathbb{S}^1)$. Next, a connection ∇ on $\mathcal{L}$ is called *Hermitian* if it is compatible with $\circlearrowleft$:

$$d^0 \circlearrowleft(s,t) = \circlearrowleft(\nabla s,t) + \circlearrowleft(s,\nabla t) \quad (7)$$

for any $s, t \in \mathcal{L}(U)$, U open in X. A *Hermitian differential line sheaf*, denoted by $(\mathcal{L}, \nabla, \circlearrowleft) := (\mathcal{L}, \nabla_{\circlearrowleft})$, is a Hermitian line sheaf equipped with a Hermitian connection.

The global 2-form R, which belongs to $Ker(d^2): \Omega^2 \to \Omega^3$, called Ω^2_c, identified as a $\mathbb{C}$-vector sheaf subspace of Ω^2, determines a global differential invariant of gauge equivalent Hermitian differential line sheaves, because the global 2-form R determines a 2-dimensional de Rham cohomology class $[R]$. The Hermitian connection of a Hermitian differential line sheaf provides *the means* to express this global differential invariant *locally*, whereas the latter is actually independent of the connection utilized to represent it locally.

From the curvature recognition integrality theorem, the abelian group $Iso(\mathcal{L}, \nabla)$ is partitioned into orbits over $\Omega^2_{c,\mathbb{Z}}$, where each orbit is labelled by an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$, providing the differential invariant $[R]$ of this orbit in de Rham cohomology:

$$Iso(\mathcal{L}, \nabla) = \sum_{R \in \Omega^2_{c,\mathbb{Z}}} Iso(\mathcal{L}, \nabla)_R. \quad (8)$$

If we *restrict* the Abelian group $Iso(\mathcal{L}, \nabla)$ to gauge equivalent Hermitian differential line sheaves we obtain an Abelian subgroup of the former, denoted by $Iso(\mathcal{L}, \nabla_{\circlearrowleft})$. It is clear that the latter Abelian group is also partitioned into orbits over $\Omega^2_{c,\mathbb{Z}}$, where each orbit is labelled by an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$, where R is the curvature of the corresponding gauge equivalence class of Hermitian differential line sheaves.

We call each Hermitian differential line sheaf $(\mathcal{L}, \nabla_{\circlearrowleft})$, which belongs to an equivalence class, that is to an *orbit* $Iso(\mathcal{L}, \nabla_{\circlearrowleft})_R$ a *unitary* R*-ray*. Concomitantly, we call the orbit itself a *spectral* $[R]$*-beam*, which is characterized by the *integral differential invariant* $[R]$. Each spectral $[R]$-beam consists of gauge equivalent unitary R-rays, which are *indistinguishable* from the perspective of their common curvature integral differential invariant $[R]$, physically meaning that they are characterized dynamically by the *same* field strength.

A natural question arising in the context of gauge equivalent quantum unitary R-rays is *how* they are related to each other. In other words, although all gauge equivalent unitary R-rays *cannot* be distinguished from the *perspective* of their *curvature* differential *invariant*, is there any other *intrinsic* way that we can distinguish among them? It is precisely at this point that the significance of the *global inertial structure* of a spectral $[R]$-beam manifests in unveiling the constitution of gauge equivalent geometric spectra.

Henceforth, the only intrinsic and invariant way of distinguishing among gauge equivalent unitary R-rays is through their *global inertial structure*, which is induced by the action of a line local system of the form Λ. Equivalently, there exists a free group action of the abelian group $H^1(X,\mathbb{S}^1)$ on the abelian group $Iso(\mathcal{L},\nabla_{\mathcal{O}})$, which is restricted to a free group action on each spectral $[R]$-beam.

First, we note that there exists a free group action of the Abelian group sheaf $\mathbb{S}^1 \infty\ \mathbb{C} \infty\ \mathcal{A}$ on the Abelian group sheaf $\tilde{\mathcal{A}}$ of invertible elements of $\mathcal{A}$, where $\mathbb{S}^1 \equiv \mathcal{U}(1) \equiv \mathcal{S}U(1,\mathbb{C})$:

$$\mathbb{S}^1 \times \tilde{\mathcal{A}}(U) \to \tilde{\mathcal{A}}(U), \quad (9)$$
$$(\xi, f) \mapsto \xi \cdot f. \quad (10)$$

with $\xi \in \mathbb{S}^1$ and $f \in \tilde{\mathcal{A}}(U)$ for any open U in X. This action is transferred naturally as a *free action* to the corresponding groups of coordinate 1-cocycles of the respective Abelian group sheaves:

$$Z^1(\mathcal{U},\mathbb{S}^1) \times Z^1(\mathcal{U},\tilde{\mathcal{A}}) \to Z^1(\mathcal{U},\tilde{\mathcal{A}}), \quad (11)$$
$$(\xi_{\alpha\beta}) \cdot (g_{\alpha\beta}) = (\xi_{\alpha\beta} \cdot g_{\alpha\beta}), \quad (12)$$

where $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U},\mathbb{S}^1)$, $(g_{\alpha\beta}) \in Z^1(\mathcal{U},\tilde{\mathcal{A}})$. This free action can be also *extended* to the corresponding *cohomology groups* still as a free action:

$$H^1(X,\mathbb{S}^1) \otimes H^1(X,\tilde{\mathcal{A}}) \to H^1(X,\tilde{\mathcal{A}}), \quad (13)$$
$$[(\xi_{\alpha\beta})] \cdot [(g_{\alpha\beta})] = [(\xi_{\alpha\beta} \cdot g_{\alpha\beta})], \quad (14)$$

where $[(\xi_{\alpha\beta})] \in H^1(X,\mathbb{S}^1)$, $[(g_{\alpha\beta})] \in H^1(X,\tilde{\mathcal{A}}) \cong Iso(\mathcal{L})$.

Next, we define a group action of $H^1(X,\mathbb{S}^1)$ on the Abelian group $Iso(\mathcal{L},\nabla)$ as follows: We consider $\xi \equiv [(\xi_{\alpha\beta})] \in H^1(X,\mathbb{S}^1)$, $[(\mathcal{L},\nabla)] \in Iso(\mathcal{L},\nabla)$, and we define the sought group action as follows:

$$\xi \cdot [(\mathcal{L},\nabla)] := [(\xi \cdot \mathcal{L},\nabla)] \equiv [(\mathcal{L}',\nabla)], \quad (15)$$

$$\mathcal{L}' = \xi \cdot \mathcal{L} \leftrightarrow (\xi_{\alpha\beta}) \cdot (g_{\alpha\beta}) = (\xi_{\alpha\beta} \cdot g_{\alpha\beta}). \quad (16)$$

We see easily that the pair $(\xi_{\alpha\beta} \cdot g_{\alpha\beta}, \omega_\alpha)$ satisfies the *transformation law* of local gauge potentials, i.e. $\delta(\omega_\alpha) = \tilde{d}^0(\xi_{\alpha\beta} \cdot g_{\alpha\beta})$. Given that $Ker(\tilde{d}^0) = \tilde{\mathbb{C}}$, as a consequence of the Poincaré Lemma, it follows directly that the above defined group action of $H^1(X, \mathbb{S}^1)$ on the abelian group $Iso(\mathcal{L}, \nabla)$ is actually *free*, where $[(\xi_{\alpha\beta})] = 1 \in H^1(X, \mathbb{S}^1)$.

Consequently, the free group action of $H^1(X, \mathbb{S}^1)$ on $Iso(\mathcal{L}, \nabla)$ is *restricted* to a free group action on its *Abelian subgroup* of *unitary rays* $Iso(\mathcal{L}, \nabla_{\circlearrowleft})$. Since the Abelian group $Iso(\mathcal{L}, \nabla_{\circlearrowleft})$ is *partitioned* into spectral $[R]$-beams over $\Omega^2_{c,\mathbb{Z}}$ constituting its *spectrum*, we derive that the above free group action is finally transferred as a *free group action* of $H^1(X, \mathbb{S}^1)$ on *each spectral* $[R]$*-beam*.

We call a cohomology class in the abelian group $H^1(X, \mathbb{S}^1)$ a *polarization phase germ* of a spectral $[R]$-beam. Our terminology derives from the fact that a cohomology class in the Abelian group $H^1(X, \mathbb{S}^1) \cong H^1(X, U(1))$ is evaluated through a representative *cocycle* at a homology *cycle* $\gamma \in H_1(X)$ by means of the *integration pairing*:

$$H_1(X) \times H^1(X, U(1)) \to U(1) \quad (17)$$

to obtain a *global* observable *gauge-invariant phase factor* in the Abelian group $U(1)$. Thus, gauge equivalent unitary R-rays are *intrinsically* and *invariantly* distinguished by means of a polarization phase germ, which is identified as a cohomology class in the group $H^1(X, \mathbb{S}^1)$.

The significant observation in this context is that a polarization phase germ of a spectral $[R]$-beam is *always realized* via a *representation* of the *fundamental group* of the topological space X to $\mathbb{S}^1$. As an immediate consequence of the *Hurewicz isomorphism*:

$$Hom(\pi_1(X), \tilde{\mathbb{C}}) \cong H^1(X, \tilde{\mathbb{C}}), \quad (18)$$

if we restrict to the unitary case we obtain:

$$Hom(\pi_1(X), \mathbb{S}^1) \cong H^1(X, \mathbb{S}^1). \quad (19)$$

Thus, a polarization phase germ, expressed in terms of a cohomology class in $H^1(X,\mathbb{S}^1)$, is realized by a representation of the fundamental group of the topological space X to $\mathbb{S}^1$.

We have demonstrated thus far that the action of the group $H^1(X,\mathbb{S}^1) \cong Hom(\pi_1(X),\mathbb{S}^1)$ on each spectral $[R]$-beam is a *free* group action. The *encoding* of the global inertial structure in group-theoretic terms requires the investigation of the conditions qualifying this free action as a *transitive* one as well in the context of gauge equivalent spectra of unitary rays.

Given the validity of the Poincaré Lemma, we consider the following sequence of abelian group sheaves:

$$1 \to \tilde{\mathbb{C}} \xrightarrow{\acute{\text{U}}} \tilde{\mathcal{A}} \xrightarrow{\tilde{d}^0} \Omega^1 \xrightarrow{d^1} d^1\Omega^1 \to 0. \quad (20)$$

The closed 1-forms θ_α of Ω^1, satisfying $Im(\tilde{d}^0) = Ker(d^1)$ are called *logarithmically exact closed* 1*-forms*.

The significance of logarithmically exact closed 1-forms in relation to the global inertial structure lies on the fact that the above *sequence* of Abelian group sheaves is an *exact sequence* if *restricted* to logarithmically exact closed 1-forms. In this case, we obtain a 0-cochain (θ_α) of logarithmically exact closed 1-forms:

$$(\theta_\alpha) \in C^0(\mathcal{U}, Ker(d^1)) = (\theta_\alpha) \in C^0(\mathcal{U}, Im(\tilde{d}^0)) = \tilde{d}^0(C^0(\mathcal{U},\tilde{\mathcal{A}})). \quad (21)$$

Hence, for a 0-cochain (θ_α) of logarithmically exact closed 1-forms θ_α, a 0-cochain t_α in $\tilde{\mathcal{A}}$ exists, such that $\theta_\alpha = \tilde{d}^0(t_\alpha)$. This 0-cochain (θ_α) may be thought of as the *representative* of an *integrable connection* $\breve{\nabla}$ of a differential line sheaf, whose coordinate 1-cocycle with respect to an open covering $\mathcal{U}$ is given by $\zeta_{\alpha\beta} = t_\beta^{-1} t_\alpha$, i.e. it is a *coboundary*, which *satisfies* the transformation law of local gauge potentials $\delta^0(\theta_\alpha) = \tilde{d}^0(\zeta_{\alpha\beta})$.

Next, we consider a spectral $[R]$-beam, namely the spectral partition class, $Iso(\mathcal{L},\nabla_{\circlearrowleft})_R$, consisting of gauge equivalent unitary R-rays, which are indistinguishable from the perspective of their common differential invariant $[R]$. We also consider those closed 1-forms θ_α of Ω^1, which qualify as logarithmically exact. If we take a pair of equivalent unitary $\mathcal{R}$ -rays, denoted by $(\mathcal{L},\nabla_{\circlearrowleft})$, $(\mathcal{L}',\nabla'_{\circlearrowleft})$ correspondingly, we obtain:

$$R=(d\omega_{\alpha})=(d\omega'_{\alpha}), \quad (22)$$

$$d(\omega_{\alpha}-\omega'_{\alpha})=0. \quad (23)$$

We conclude that $(\omega_{\alpha}-\omega'_{\alpha})$ is of the form θ_{α} of Ω^1, hence it is a logarithmically exact closed 1-form. Based on this fact, we derive that the *free* group action of $H^1(X,\mathbb{S}^1)$ on a spectral $[R]$-beam is *also transitive* with respect to logarithmically exact closed 1-forms.

In conclusion, a spectral $[R]$ -beam becomes a $H^1(X,\mathbb{S}^1)\cong Hom(\pi_1(X),\mathbb{S}^1)$-*affine space*, or equivalently, an affine space with *structure* group the *characters* of the fundamental group. The latter provides a complete characterization of the global inertial structure of a spectral $[R]$-beam in group-theoretic terms.

Each partition class of the spectrum, each orbit or fiber $Iso(\mathcal{L},\nabla_{\circlearrowleft})_R$, labelled by the curvature differential invariant $[R]$, which is to say each spectral $[R]$-beam is an affine space with structure group $H^1(X,\mathbb{S}^1)\cong Hom(\pi_1(X),\mathbb{S}^1)$. Thus, *any* two unitary $\mathcal{R}$-rays *differ* by an element of $H^1(X,\mathbb{S}^1)$, and conversely any two unitary rays which differ by an element of $H^1(X,\mathbb{S}^1)$ are characterized by the *same* differential invariant $[R]$; in short, they constitute $\mathcal{R}$-rays of the *same* spectral $[R]$-beam.

We conclude that, although all gauge equivalent unitary $\mathcal{R}$-rays *cannot* be distinguished from the perspective of their common curvature differential invariant, nevertheless a free and transitive action of the group $H^1(X,\mathbb{S}^1)\cong Hom(\pi_1(X),\mathbb{S}^1)$ exists, characterizing the global inertial structure of a spectral $[R]$-beam cohomologically in group-theoretic terms and corresponding *bijectively* to the respective line local system. Equivalently put, $\mathcal{R}$-rays of the same spectral $[R]$-beam can be *distinguished inertially* via the characters of the fundamental group of X.

Inversely, from any one unitary $\mathcal{R}$-ray of a spectral beam we can *generate* its *whole* orbit, identified with the *beam* itself, by means of the free and transitive action of the abelian group $H^1(X,\mathbb{S}^1)$ on the one depicted. Thus, whenever two unitary rays are characterized by the same differential invariant $[R]$, thereby belonging to the same orbit under the action of $H^1(X,\mathbb{S}^1)$ on $Iso(\mathcal{L},\nabla_{\circlearrowright})_R$, which is actually the *only* orbit due to transitivity of this action, identified as a spectral $[R]$-beam, then they differ by a *character* of the fundamental group of the carrier base topological space X.

7.7 INTERFERENCE: TOPOLOGICAL PHASE AND THE POLYDROMY OF A SPECTRAL BEAM

A polarization phase germ of a spectral $[R]$-beam, identified as a cohomology class in the Abelian group $H^1(X,\mathbb{S}^1)$, is realized by a representation of the fundamental group of the base connected topological space X to $\mathbb{S}^1$. We observe that the evaluation of a representative cocycle of $H^1(X,\mathbb{S}^1)$ at a homology cycle $\gamma \in H_1(X)$ by means of the integration pairing:

$$H_1(X)\times H^1(X,U(1))\to U(1) \quad (24)$$

gives a global observable gauge-invariant phase factor in the Abelian group $U(1)$. In this sense, gauge equivalent unitary spectral rays bearing the same strength are intrinsically distinguished by means of a polarization phase germ.

Consequently, the global polarization symmetry group of a spectral $[R]$-beam is *realized* in terms of a unitary line local system. Equivalently formulated, there exists a *bijective* correspondence between the polarization phase germs of a spectral $[R]$-beam and isomorphism classes of unitary line local systems, due to the following equivalences:

$$Z^1(\mathcal{U},\tilde{\mathbb{C}}) \nprec (\xi_{\alpha\beta}) \cong (\mathcal{L},\breve{\nabla}) \cong Ker(\breve{\nabla})_{\mathcal{L}} \cong \Lambda, \quad (25)$$

$$H^1(X,\tilde{\mathbb{C}}) \cong [(\mathcal{L},\breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda]. \quad (26)$$

The above equivalences read as follows in the case of Hermitian integrable differential line sheaves, or integrable unitary rays:

$$Z^1(\mathcal{U},\mathbb{S}^1) \nprec (\xi_{\alpha\beta}) \cong (\mathcal{L},\breve{\nabla}_{\circlearrowright}) \cong Ker(\breve{\nabla}_{\circlearrowright})_{\mathcal{L}} \cong \Lambda_{\circlearrowright}, \quad (27)$$

$$H^1(X,\mathbb{S}^1) \cong [(\mathcal{L},\breve{\nabla}_{\circlearrowright})] \cong [Ker(\breve{\nabla}_{\circlearrowright})_{\mathcal{L}}] \cong [\Lambda_{\circlearrowright}]. \quad (28)$$

Let us examine in more detail the realization of the polarization symmetry group of a spectral $[R]$-beam at a point of the base space X. At each point of the topological space X, the polarization symmetry group of a spectral $[R]$-beam is *realized* thanks to the *monodromy group* of a unitary line local system at the depicted point, or equivalently by the group of monodromies of the corresponding integrable unitary ray whose covariantly constant sections form this unitary line local system. This is established as follows:

We already know that a polarization phase germ of a spectral $[R]$-beam is realized by a representation of the fundamental group of X to $\mathbb{S}^1$. Moreover, $\tilde{\mathbb{C}}$ is identified locally with the group of automorphisms of a line local system, $GL(1,\mathbb{C}) \cong \tilde{\mathbb{C}}$. Thus, in the unitary case, $\mathbb{S}^1$ is *identified locally* with the group of *automorphisms* of a unitary line local system $\Lambda_{\mathcal{O}}$, which may be thought of as the *locally constant* sheaf of the inertial, or covariantly constant sections of a corresponding Hermitian integrable line sheaf (*integrable unitary ray*) $(\mathcal{L}, \breve{\nabla}_{\mathcal{O}})$.

We may fix a base point x_0 of X, and consider a path $\gamma:[0,1] \to X$, such that $\gamma(0) = x_0$, $\gamma(1) = x_1$. Thus, if Λ is a line local system on X, then it is *pulled back* to $[0,1]$ as a *constant* sheaf, denoted by $\gamma^{\leftrightarrows}(\Lambda)$. Hence, we obtain that:

$$(\gamma^{\leftrightarrows}(\Lambda))_0 \cong (\gamma^{\leftrightarrows}(\Lambda))_1. \quad (29)$$

Henceforth, because of the *isomorphisms* $(\gamma^{\leftrightarrows}(\Lambda))_0 \cong \Lambda_{\gamma(0)} := \Lambda_{x_0}$ and $(\gamma^{\leftrightarrows}(\Lambda))_1 \cong \Lambda_{\gamma(1)} := \Lambda_{x_1}$, we have at our disposal a $\mathbb{C}$-vector space *isomorphism* $\Lambda_{x_0} \cong \Lambda_{x_1}$, which depends only on the *homotopy class* of γ.

Furthermore, this isomorphism may be thought of as being induced by the *parallel transport condition* of the corresponding integrable connection of the Hermitian integrable line sheaf $(\mathcal{L}, \breve{\nabla}_{\mathcal{O}})$. If we consider a loop based at the point x_0 of X, we obtain an Abelian group homomorphism:

$$\mu : \pi_1(X, x_0) \to GL(\Lambda_{x_0}) \cong (GL(1,\mathbb{C}))_{x_0} \cong \tilde{\mathbb{C}}_{x_0}. \quad (30)$$

In the case of a unitary line local system, we correspondingly obtain the following Abelian group homomorphism:

$$\mu : \pi_1(X, x_0) \to \mathbb{S}^1_{x_0}. \quad (31)$$

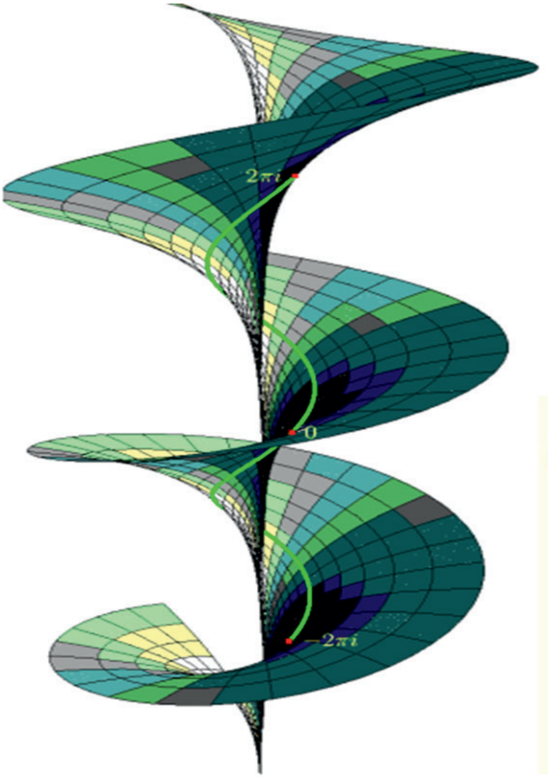
The *image* of μ in $\mathbb{S}^1_{x_0}$ is the *monodromy group* of the unitary local system $\Lambda_{\mathcal{O}}$. Thus, for each homotopy class of loops based at x_0, we obtain a topological integrable phase factor, identified with the monodromy of the unitary local system $\Lambda_{\mathcal{O}x_0}$. Equivalently, this is the same as the monodromy of the corresponding integrable quantum unitary ray $(\mathcal{L}, \breve{\nabla}_{\mathcal{O}})$, derived through the parallel transport along a loop based at x_0 and belonging to a homotopy class in $\pi_1(X, x_0)$.

The notion of *monodromy* originates from complex function theory on Riemann surfaces, and more specifically, from the theory of linear differential equations on Riemann surfaces. In general, a solution to a linear differential equation in the setting of analytic or holomorphic dynamics, is characterized as *polydromic* if the *circulation* around a loop enclosing a *singularity* produces a *different value* in comparison to the *initial* one.

By contrast, it is called *monodromic* if this phenomenon does not arise. In this manner, the crystallized term *"monodromy"* constitutes a *misnomer* compared to the meaning enclosed in this Greek term, i.e. it should be properly called *"polydromy"*. This should also be in accordance with Riemann's distinction between single and multi-valued magnitudes and the concomitant process of analytic continuation of a local solution along different paths or *"dromoi"*. Unfortunately, the term "monodromy" is currently established as such, so we will adopt the present usage modulo the present clarification.

As an example, we may consider the differential equation $zf'(z) = 1$, which is *singular* at the *origin* of the complex plane. As a consequence, the local solution $f(z) = \log(z)$, if circulated around the origin, for each closed path or *"dromos"*, its value is modified by an *integer multiple* of $2\pi i$. For this reason, we need to invoke the *universal covering* space defined by the complex exponential $exp(z): \mathbb{C} \to \mathbb{C}^*$.

Accordingly, a polydromic solution to the given differential equation, that is, a determination of the *many-valued* function $\log(z)$ takes place only in terms of *local sections* of the complex exponential covering projection map $exp(z)$. Each section defined on $\mathbb{C} - \{ray\}$ constitutes an *inversion* of $exp(z)$, but only *locally*, and as such, it provides to a local determination of the logarithm. In particular, each section bears the form $\log(z) + 2\kappa i\pi$, where κ is an integer. The compatible gluing of all these local determinations over their non-trivial overlaps gives rise to the *Riemann surface* of the many-valued function $\log(z)$, represented below in helicoidal form:



A concrete observable manifestation of a global integrable phase factor in the context of *quantum mechanics* is provided by the effect discovered by *Yakir Aharonov and David Bohm*. This effect demonstrates the significance of the *local* electromagnetic gauge *potentials* of a spectral beam whose strength is null. More specifically, what constitutes the *Aharonov-Bohm effect* is an experimentally verified and observed *global relative phase factor* whose origin is *topological*, and thus expressed via an integrable connection pertaining to the description of the electromagnetic field.

The setting involves a very long solenoid restricting the magnetic field *flux* within its borders, in consequence rendering the region it occupies *inaccessible* to a charged particle. In this sense, the base topological space of localization that carries the field is multiply-connected, bearing the homotopical symmetry of a *circle*. The evolving states are transported by an *integrable connection* because the propagation takes place in the field strength-free region outside the solenoid. The observed global phase factor measures the *monodromy* of this integrable connection due to the topological obstacle imposed by the inaccessible region, which is enclosed by the *boundary* of the solenoid, where the *flux* of the magnetic field takes place.

The Aharonov-Bohm effect constitutes a perfect demonstration of the nature and significance of global observable integrable phase factors arising from *topological obstacles* in quantum theory. In particular, it demonstrates the following: First, the local gauge freedom of the phase of a quantum state; Second, the mutually implicative roles of the local and global levels in the quantum field theoretic description pertaining to a line sheaf of states endowed with an integrable connection. This consists of an extensive integration process of the contributions of all local gauge potentials to monodromies from the local to the global, and inversely, of a differential localization process of the global topological phase invariant, characterizing all gauge equivalent rays, in terms of the whole multiplicity of local gauge potentials. Taken all together, these gauge potentials express the permissible contextual variability of the connection with respect to this invariant.

Let us examine more specifically the details pertaining to an *integrable topological phase factor* from our viewpoint. We start from the observation that the Abelian group of quantum unitary rays $Iso(\mathcal{L},\nabla_{\mathcal{O}})$ constitutes a *central extension* of the Abelian group of integral global closed 2-forms R of $\Omega^2_{c,\mathbb{Z}}$ by the Abelian group of polarization phase germs $H^1(X,\mathbb{S}^1)$. This is due to the exactness of the following sequence of Abelian groups:

$$1 \to H^1(X,\mathbb{S}^1) \xrightarrow{\sigma} Iso(\mathcal{L},\nabla_{\mathcal{O}}) \xrightarrow{\kappa} \Omega^2_{c,\mathbb{Z}} \to 0. \quad (32)$$

In particular, if we consider spectral $[R]$-beams, each corresponding *partition spectral block* of *equipotent rays* $Iso(\mathcal{L},\nabla_{\mathcal{O}})_R$ is an *affine* space with structure *group* $H^1(X,\mathbb{S}^1)$. The integrality condition in this context is tantamount to *Paul Dirac's quantization condition*, meaning cohomologically that $(2\pi i)^{-1}[R]$ is a 2-dimensional *integral* cohomology class of X. Moreover, a zero curvature spectral $[R]$-beam $Iso(\mathcal{L},\nabla_{\mathcal{O}})_0$ is isomorphic to the Abelian group of polarization phase germs $H^1(X,\mathbb{S}^1)$, according to the preceding.

We have that $Iso(\mathcal{L},\nabla_{\mathcal{O}})_R = \kappa^{-1}(R)$, where $R \in \Omega^2_{c,\mathbb{Z}}$. Therefore, due to the fact that $H^1(X,\mathbb{S}^1) \infty\ Iso(\mathcal{L},\nabla_{\mathcal{O}})$ we obtain:

$$\kappa^{-1}(0) = Iso(\mathcal{L},\nabla_{\mathcal{O}})_0 \cong H^1(X,\mathbb{S}^1). \quad (33)$$

Consequently, a spectral $[R]$-beam is an $Iso(\mathcal{L},\nabla_{\mathcal{O}})_0$*-torsor*, by which we mean an affine space with respect to *logarithmically exact closed* 1-*forms*.

The Aharonov-Bohm global phase factor refers to the realization of a zero curvature spectral $[R]$-beam $Iso(\mathcal{L},\nabla_{\mathcal{O}})_0$. Equivalently, it is the experimentally realized *global* gauge-invariant phase characteristic of a gauge equivalence class of integrable Hermitian differential line sheaves, or integrable quantum unitary rays. Thus, we derive the following:

$$Iso(\mathcal{L},\nabla_{\mathcal{O}})_0 \cong H^1(X,\mathbb{S}^1) \cong Hom(\pi_1(X),\mathbb{S}^1). \quad (34)$$

In particular, for each point x_0 of the base topological space X, we have:

$$\mu : \pi_1(X,x_0) \to \mathbb{S}^1{}_{x_0} \cong U(1). \quad (35)$$

Therefore, the *global phase factor* of the Aharonov-Bohm type is realized as the global *monodromy* group element $\mu(\gamma) \in U(1)$ for *each homotopy class* of loops γ based at x_0. Note that in the experimental setting the base topological space X is homotopically *contractible* to the circle, and hence, its second integer cohomology is trivial.

These features establish a gauge equivalence class of integrable Hermitian differential line sheaves, or equivalently a gauge equivalence class of line local systems on the *circle*, which form a zero curvature spectral beam. The global gauge-invariant phase factors by which this beam is realized is the monodromy group associated with it, which is identified with the image of the fundamental group of the circle; the integers $\mathbb{Z}$ into $U(1)$. The monodromy depends *only* on the *integer winding number* and is observed as a *shift* in the interference pattern of the beam. Physically, the integer winding numbers obtained topologically, descend from the *harmonics* of the spectrum of the beam, whereas *interference* constitutes its observable geometric manifestation.

We emphasize again that an Aharonov-Bohm type of phase for non-simply connected base space X, due to the presence of obstructions such as those in our experimental case, refers to the global gauge-invariant observable factor pertaining to a zero curvature beam, i.e. to a gauge equivalence class of zero curvature quantum unitary rays. From a *cohomological* perspective, the analysis of these types of phases proceeds as follows:

We notice that a zero curvature beam is actually *isomorphic* to the *Abelian group* $H^1(X,U(1))$. Hence, it can be grasped as follows:

$$[\Psi(\theta_\alpha,-)] = exp(\frac{ie}{\hbar c}\int_{[-]}\theta_\alpha), \quad (36)$$

where we have inserted the corresponding physical units, and we consider θ_α real-valued in the Lie algebra of $U(1)$.

We conclude that a *representative* of the *class* identified with a zero curvature beam $[\Psi(\theta_\alpha,-)]$ is an element of $H^1(X,U(1))$, which is *evaluated* at a homology cycle $\gamma \in H_1(X)$ by means of the *pairing* of groups:

$$H_1(X)\times H^1(X,U(1)) \to U(1), \quad (37)$$

$$(\gamma,[\Psi(\theta_\alpha,-)]) \mapsto [\Psi(\theta_\alpha,\gamma)] = exp(\frac{ie}{\hbar c}\int_\gamma \theta_\alpha), \quad (38)$$

where $[\Psi(\theta_\alpha,\gamma)]$ is identified as a global Aharonov-Bohm gauge-invariant topological phase factor of the beam in $U(1)$.

For reasons pertaining to the significance of the partition spectrum under investigation, it is instructive to point out that if we consider any unitary ray of this beam, then we only obtain a real-valued phase, defined by:

$$\Psi(\theta_\alpha,\gamma) = \frac{e}{\hbar c}\int_\gamma \theta_\alpha. \quad (39)$$

Due to the isomorphism of groups $\mathbb{R}/\mathbb{Z} \cong U(1)$, we have to take the quotient of the set of all $\Psi((\theta_\alpha)_i,\gamma)$ for all unitary rays by the equivalence relation: $\Psi((\theta_\alpha)_1,\gamma) \sim \Psi((\theta_\alpha)_2,\gamma)$ if $((\theta_\alpha)_1 - (\theta_\alpha)_2) \in \mathbb{Z}$. In physical terms, this means that the interference phase patterns of quantum unitary rays differing by an integer *cannot* be distinguished experimentally, and thus the physically meaningful global gauge-invariant information is only the topological phase factors of the form:

$$[\Psi(\theta_\alpha,\gamma)] = exp(\frac{ie}{\hbar c}\int_\gamma \theta_\alpha), \quad (40)$$

referring to the global realization of the beam, hence to the global realization of the *whole gauge equivalence class* of quantum unitary rays.

7.8 ANHOLONOMY: GEOMETRIC PHASE AND THE MEMORY OF A SPECTRAL BEAM

If the energy operator, the Hamiltonian of a quantum system, is functionally dependent on an underlying set of *control variables*, then a quantum state becomes *localized* parametrically on the base space constituted by these variables. The dynamical evolution of a quantum state is therefore driven by the *implicit temporal* dependence of the Hamiltonian through the control variables. Under the assumption that the set of these variables forms a smooth manifold, then the time dependence is depicted by means of *differentiable paths* on this space.

In the approach pioneered by *Michael Berry*, the *adiabatic cyclic* evolution of a quantum state if of a fundamental significance. The cyclic evolution signifies the *periodicity* property of a quantum state with respect to the control variables, whence the adiabatic hypothesis is equivalent to the specification of a *connection*, a parallel transport condition on the evolution of normalized state vectors, which in general is considered to be path-dependent. Due to this *path-dependency*, a quantum state upon completion of cyclic path acquires a *global non-integrable* geometric phase factor, called the *anholonomy* of the transport.

The fibers of the induced spectral line bundle stand for the eigenspaces of the energy operator. Thus, the adiabatic transportation rule amounts to a non-integrable connection, according to which, an eigenstate of the Hamiltonian is required to remain in the eigenspace of the same instantaneous eigenvalue during the adiabatic evolution. In turn, the non-integrable connection gives rise to a covariant derivative operator on the sections of the corresponding Hermitian line sheaf, constituted by the eigenstates of the Hamiltonian. The non-dynamical, by which we mean non-Hamiltonian dependent, global phase assembled during a cyclic evolution along a closed path on the base space is thought of as the *memory* of the evolution, since it *encodes* the global geometric features of the space of control variables in the algebraic, and more specifically, *group*-theoretic structure of the anholonomy of the connection.

The observable global phase factor is called *geometric* because it depends *solely* on the geometry of the base space *pathway* along which the quantum state is transported. If the eigenvalues of the Hamiltonian are degenerate or close to each other, then the adiabatic transportation constraint is not realistic and is substituted by another appropriate connection depending on the particular context. In this case, the gauge freedom of a state vector, localized at a fiber over an eigenspace of the Hamiltonian, is not an one-dimensional complex phase any more, but rather is an n-dimensional complex matrix of phases, called a *non-Abelian complex phase*. It turns out that even the adiabatic transportation rule is *not* necessary for the experimental detection of a global phase factor. This has been demonstrated convincingly through the *intrinsic line bundle* formulation of the complex Hilbert space of states over the *complex projective* Hilbert space. In this formulation, analogously to the preceding approach, the *one-dimensional projection* operators play the role of control variables. This line bundle, or equivalently, the line sheaf of its sections is endowed with a natural connection obtained by differentiation of the Hermitian inner product of the normalized sections, which are the quantum state vectors, and the adiabatic hypothesis is *not* involved at all. Then, the global phase factor is identified in terms of the global *anholonomy* of this connection with respect to a closed path on the complex projective space.

From the general perspective of our cohomological analysis pertaining to spectral beams, all observable geometric phase factors are actually *generated* by the *curvature* of a spectral $[R]$-beam. In particular, the *memory* of a spectral beam is the *global encoding* of the fact that $(2\pi i)^{-1}[R]$ is a 2-dimensional *integral* cohomology class of the base topological space of variables X. As already established, global observable topological phase factors can be completely understood in terms of the monodromies of zero curvature beams. Again, it is instructive to emphasize that an observable anholonomy pertains to a partition class of equipotent unitary R-rays; to a *spectral* $[R]$*-beam*, rather than to an individual unitary R-ray.

A global geometric phase factor arises cohomologically as follows: For any real valued form ϕ of degree k we define:

$$\hat{\phi}(\eta) := \Psi(\phi,\eta) = (\int_\eta \phi) + \mathbb{Z}, \quad (1)$$

where η is a k*-chain* of X and $\hat{\phi}$ is a k*-cochain* of X with values in $\mathbb{R}/\mathbb{Z}$. Next we consider the homomorphism of groups:

$$\Xi : Z_k(X) \to \mathbb{R}/\mathbb{Z}, \quad (2)$$

such that, a $k+1$-form τ exists, which satisfies $\Xi \circ \partial = \hat{\tau}$, or equivalently:

$$\Xi(\partial\zeta) = (\int_\zeta \tau) + \mathbb{Z} = \hat{\tau}(\zeta) \in \mathbb{R}/\mathbb{Z}, \quad (3)$$

for any smooth map $\zeta : \Delta_{k+1} \to X$. We note that Δ_n stands for the standard n-dimensional *simplex*, and the space of n-chains is generated by Δ_n.

In this framework, the following relation emerges:

$$d\left(\hat{\tau}\right) = d\tau = \Xi \circ \partial^2 = 0, \quad (4)$$

and therefore, τ is a *closed* $k+1$-form.

Next, we consider a unitary R-ray $(\mathcal{L}, \nabla_{\mathcal{O}})$ and apply the *quantization* condition, according to which, R is an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$. Similarly to the definition of gauge potentials in the case of Hermitian differential line sheaves, we consider R as a purely *imaginary closed* 2-form, such that $R = i \cdot \Theta$. Thus, according to the above, and since Θ is an *integral global* real-valued form of degree 2, we may instantiate the following group homorphism, which we call the *anholonomy homomorphism*:

$$\mathbb{H} : Z_1(X) \to \mathbb{R}/\mathbb{Z}, \quad (5)$$

$$\mathbb{H}(\partial S) = (\int_S \Theta) + \mathbb{Z} = \hat{\Theta}(S) \in \mathbb{R}/\mathbb{Z}, \quad (6)$$

for any smooth map $S : \Delta_2 \to X$. Equivalently, we have:

$$\mathbb{H} \circ \partial = \hat{\Theta}. \quad (7)$$

We observe that for a *fixed* curvature R of $\Omega^2_{c,\mathbb{Z}}$, the *same* anholonomy homorphism $\mathbb{H}$ is defined for any other unitary R-ray. Thus, it provides a characterization of the whole *gauge equivalence class* of unitary R-rays *classified* by the differential invariant $[R]$. Equivalently, it provides a characterization of a spectral $[R]$-beam, and therefore, we obtain an *anholonomy cohomology class* in $H^1(X, U(1))$, as follows:

$$Hol(\partial S) = exp(i\int_S \Theta). \quad (8)$$

From the above, we form the conclusion that taking into account the *Chern homomorphism*:

$$\delta_c : H^1(X, U(1)) \to H^2(X, \mathbb{Z}), \quad (9)$$

and the natural homomorphism:

$$\imath : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}), \quad (10)$$

the anholonomy cohomology class in $H^1(X, U(1))$ of a spectral beam for *fixed* Θ, is in the *inverse image* of the Chern characteristic class:

$$c_1 = -\frac{1}{2\pi i}[\Theta] \in H^2(X, \mathbb{Z}), \quad (11)$$

under δ_c, or equivalently, it is located in the inverse image of the cohomology class $[\Theta]$ in $H^2(X, \mathbb{R})$, such that:

$$(\imath \circ \delta_c)(Hol) = [\Theta]. \quad (12)$$

Thus, we conclude that an $\mathbb{R}/\mathbb{Z} \cong U(1)$-observable *anholonomy*, which formalizes the notion of a *non-integrable geometric phase factor*, is a global observable gauge-invariant characteristic of a spectral beam, qualified as the *memory* pertaining to the *whole* gauge equivalence class of quantum unitary rays having the *same curvature*, such that the above cohomological relation is satisfied.

ADJUNCTIONS: CATEGORY THEORY VIA NATURAL COMMUNICATION

8

8.1 CATEGORY THEORY FROM THE STANDPOINT OF NATURAL COMMUNICATION

The development of *category theory* is a natural and perhaps inevitable aspect of the late 20th century emphasis on the conceptual clarification of what specifies and characterizes an object of a mathematical inquiry pertaining to a universe of discourse as a hypostatic entity. The major precursor of this development can be located in the fields of algebraic topology and abstract algebra, where the specification of complex spaces was *not* based at all on their set-theoretic point-element constitution, but rather on certain algebraic symbolic *groups* incorporating some *invariant* characteristic in relation to this complex space.

This articulated a major change in the way of conceiving the conceptual *form* and *function* of mathematical objects, which *departed* from the axiomatic *set-theoretic* reductionist approach of analysis of objects in terms of pre-determined, or *a priori* distinguishable elements endowed with some particular *externally* imposed structure. The emphasis now has been put on the specification of objects in terms of the *communicative relations* they bear with other objects of the *same* species, where the notion of a *species of structure* is now derived from the whole homomorphic constitution of a certain mathematical universe of discourse, conceptualized in turn, by means of the notion of a *category* that respects or preserves this structural species.

In this respect, the central focus of the categorical way of rethinking the notion of a mathematical object algebraically and structurally can be described as a major transition in the conception and interpretation of what actually *characterizes* algebraic symbolic structures. More precisely, it represents a transition from a *substantial* to a *hypostatic* concept of structure. In the substantial constitutional set-theoretic mode of thinking, structures of any conceivable *morphe* are defined on a set-theoretic foundational basis, as sets of elements endowed with appropriate structural relations, like the ones characterizing a group structure.

In the hypostatic communicative category-theoretic mode, the emphasis is placed on the kind of *homomorphisms* among the objects of a category devised to capture and preserve a certain structure as its *instances*, by means of the pertinent structural constraints on these homomorphisms. In this sense, the notion of structure does *not* refer substantially to a fixed universe of sets of predetermined elements, but hypostatically acquires a *variable* reference to other generalized universes, called *topoi*.

In particular, the hypostatic structural elaboration of an object by way of its variable reference to a topos, to be thought of as a *scaffolding* entailed *contextually* by communication, and *not* as a foundation necessitated absolutely by constitutional needs, points to an indirect "obstacle-oriented" approach to the specification of an object as an algebraic structure arising from the category to which it pertains in as an instance. Most important, the specification of objects of a structural species can be even enunciated in terms of the *heteromorphic relations* they bear with objects of *another* structural species, which are thought of as partial resolving *probes* or *covers* of the former, under the proviso that these relations can be appropriately *internalized* within the former category via *adjunctions*.

The basic categorical principles can be expressed concisely as follows:

i To each kind of mathematical symbolic structure, there corresponds a *category* whose objects have that structure, and whose homomorphisms preserve it.

ii To any natural morphism on structures of one kind, yielding structures of another kind, there corresponds a *functor* from the category of the first kind to the category of the second. The implementation of this principle is associated with the fact that a morphism is not merely a function from objects of one kind to objects of another, but must *preserve* the essential *structural relationships* among objects.

iii To each natural translation between two functors having identical categorical domains and codomains there corresponds a natural transfiguration, called a *natural transformation* between these functors that can be restricted to a *natural isomorphism* of functors.

iv To any canonical bidirectional functorial correlation between two kinds of mathematical structures there corresponds an *adjunction*, expressed by a *pair* of *adjoint functors* between the corresponding categories. An adjunction is equivalent to a *categorical* process of *metaphora* effecting the *natural communication* between the correlated *hypostatic structural kinds.*

Therefore, if the standard framework of mathematical category theory is to be thought of as a conceptual pyramid based on the notion of a category and converging at the top on the notion of a categorical adjunction, up through the intermediate layers of functors and natural transformations, the emphasis on *natural communication* by contrast requires a *conceptual inversion* of this form. Precisely speaking, it is the need for expressing the conceptual norms of natural communication between two different hypostatic levels - the *norms* of the *metaphora* between *two different hypostases* conceived algebraically and structurally from a non-absolute elemental standpoint - that requires the notions of natural transformations and functors, and ultimately the notion of a category.

As we shall discuss later on, the inverted pyramid of category theory, emphasizing the natural communication between two different hypostatic structural kinds, is intimately related to an *"obstacle-oriented" approach* to the notion of *what* a *categorical object* is. More concretely, the enunciation of the categorical structure of an object of an unknown, novel, or even not directly accessible species, requires a potential *cyclically-embracing* process of *metaphora* through *another* comprehensible categorical species, together with the *canonics* of this heteromorphic metaphora, so illuminating the former structure by natural communication, or equivalently, *descending to* and *ascending from* another *covering* or *resolving* structure.

We outline below the standard basic definitions to be found in this order in standard formalist presentations of category theory, and further on, we are going to show how they actually arise *inversely* via the canonics and norms characterizing the natural-communication of non-absolute hypostatic structural kinds.

CATEGORIES:

A *category* $\mathcal{C}$ is a class of *objects* and *morphisms* of objects such that the following properties are satisfied:

i For any objects X, Y all morphisms $f: X \to Y$ form a set denoted $Hom_{\mathcal{C}}(X,Y)$;

ii For any object X an element $id_X \in Hom_{\mathcal{C}}(X,X)$ is distinguished; it is called the *identity* morphism;

iii For arbitrary objects X, Y, Z the set mapping is defined

$$Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) \to Hom_{\mathcal{C}}(X,Z)$$

For morphisms $g \in Hom_{\mathcal{C}}(X,Y)$, $h \in Hom_{\mathcal{C}}(Y,Z)$ the image of the pair (g,h) is called the *composition*; it is denoted $h \circ g$. The composition operation is *associative*.

iv For any $f \in Hom_{\mathcal{C}}(X,Y)$ we have $id_Y \circ f = f \circ id_X = f$.

For an arbitrary category $\mathcal{C}$ the *opposite* category $\mathcal{C}^{op}$ is defined in the following way: the objects are the same, but $Hom_{\mathcal{C}^{op}}(X,Y) = Hom_{\mathcal{C}}(Y,X)$, namely all arrows are *inverted.* A category $\mathcal{C}$ is called *small* if the classes of its objects and morphisms form genuine *sets* respectively.

FUNCTORS:

Let $\mathcal{C}$, $\mathcal{D}$ be categories; a *covariant functor* $\mathbf{F}:\mathcal{C}\to\mathcal{D}$ is a class mapping that transforms objects to objects and morphisms to morphisms *preserving* compositions and identity morphisms:

$$\mathbf{F}(id_X)=id_{\mathbf{F}(X)};\mathbf{F}(g\circ f)=\mathbf{F}(g)\circ\mathbf{F}(f)$$

A *contravariant functor* $\hat{\mathbf{F}}:\mathcal{C}\to\mathcal{D}$ is, by definition, a covariant functor $\mathbf{F}:\mathcal{C}^{op}\to\mathcal{D}$.

NATURAL TRANSFORMATIONS:

Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let further F, $\mathbf{G}$, be functors from the category $\mathcal{C}$ to the category $\mathcal{D}$. A *natural transformation* τ from F to $\mathbf{G}$ is a mapping assigning to each object A in $\mathcal{C}$ a morphism τ_A from $\mathbf{F}(A)$ to $\mathbf{G}(A)$ in $\mathcal{D}$, such that for every arrow $f:A\to B$ in $\mathcal{C}$ the following *diagram* in $\mathcal{D}$ *commutes*;

$$\begin{tikzcd}
\mathbf{F}(\mathcal{A}) \arrow[r, "\tau_{\mathcal{A}}"] \arrow[d, "\mathbf{F}(f)"'] & \mathbf{G}(\mathcal{A}) \arrow[d, "\mathbf{G}(f)"] \\
\mathbf{F}(\mathcal{B}) \arrow[r, "\tau_{\mathcal{B}}"] & \mathbf{G}(\mathcal{B})
\end{tikzcd}$$

That is, for every arrow $f:A\to B$ in $\mathcal{C}$ we have:

$$\mathbf{G}(f)\circ\tau_A=\tau_B\circ\mathbf{F}(f)$$

NATURAL ISOMORPHISMS:

A natural transformation $\tau:\mathbf{F}\to\mathbf{G}$ is called a *natural isomorphism* if every component τ_A is *invertible.*

ADJOINT FUNCTORS:

Let $\mathbf{F}:\mathcal{C}\to\mathcal{D}$ and $\mathbf{G}:\mathcal{D}\to\mathcal{C}$ be functors. We say that F is *left adjoint* to $\mathbf{G}$ (and correspondingly that $\mathbf{G}$ is *right adjoint* to F), if there exists a *bijective* correspondence between the arrows $\mathbf{F}(C)\to D$ in $\mathcal{D}$ and $C\to\mathbf{G}(D)$ in $\mathcal{C}$, which is *natural* in both C and D.

$$\mathbf{F}:\mathcal{C}\rightleftarrows\mathcal{D}:\mathbf{G}$$

Pictorially we have;

$$\begin{tikzcd} \mathbf{F}(C) \arrow[d] & C \arrow[d] \\ D & \mathbf{G}(D) \end{tikzcd}$$

where the left part is in $\mathcal{D}$ and the right in $\mathcal{C}$. Then, we say that the above pair of adjoint functors constitute a *categorical adjunction*.

DIAGRAMS:

A *diagram* $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$ is defined as an *indexed family* of objects $\{X_i\}_{i\in I}$ and a family of morphisms sets $\{F_{ij}\}_{i,j\in I} \subseteq Hom_{\mathcal{C}}(X_i, X_j)$.

COCONES:

A *cocone* of the diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$, consists of an object X in $\mathcal{C}$, and for every $i \in I$, a morphism $f_i : X_i \to X$, such that $f_i = f_j \circ f_{ij}$ for all $j \in I$, that is, such that for every i, $j \in I$, and for every $f_{ij} \in F_{ij}$ the diagram below commutes

$$\begin{tikzcd} & X & \\ X_i \arrow[ur, "f_i"] \arrow[rr, "f_{ij}"] & & X_j \arrow[ul, "f_j"'] \end{tikzcd}$$

COLIMITS:

A *colimit* of the diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ is a cocone with the property that for every other cocone given by morphisms $f_{i'} : X_i \to X'$, there exists exactly *one* morphism $f : X \to X'$, such that $f_{i'} = f \circ f_i$, for all $i \in I$ (*universality property*).

Reversing the arrows in the above definitions of cocone and colimit of a diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$, results in the dual notions called *cone* and *limit* of $\mathbf{X}$ respectively. Moreover, starting with a diagram $\mathbf{X} = (\{X_i\}_{i\in I}, \{F_{ij}\}_{i,j\in I})$ in a category $\mathcal{C}$, that consists only of the objects X_i, $i \in I$, as nodes but without morphisms, that is all $F_{ij} = \varnothing$, we obtain the notion of the categorical *coproduct*, $\coprod_{i\in I} X_i$ (as a special colimit) and *product*, $\prod_{i\in I} X_i$ (as a special limit) respectively. The morphisms f_i in the corresponding definitions are called *canonical injections* of the coproduct and *canonical projections* of the product, respectively. We emphasize that we can derive special notions of limits and colimits, corresponding to the *shape* of the base diagram $\mathbf{X}$. In this sense we obtain the following; an *initial* object is the colimit of the diagram consisting of the empty set. A *coequalizer* is the colimit of a diagram consisting of two parallel arrows $A \rightrightarrows B$. A *pushout* is the colimit of a diagram of the form:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

The dual notions are the following: a *terminal* object is the limit of the diagram consisting of the empty set. An *equalizer* is the limit of a diagram consisting of two parallel arrows $A \rightrightarrows B$. A *pullback* is the limit of a diagram of the form:

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"] & A \end{tikzcd}$$

8.2 ARTICULABILITY OF OBJECTS: HYPOSTATIC CATEGORICAL CANONICS

The natural communication between two different hypostatic structural kinds, conceptualized as different categories, is inextricably tied to an "obstacle-oriented" approach to the notion of what a categorical object is. It implies that the categorical structure of an object of a *non-directly* comprehensible or *novel* structural *species*, requires a potential *cyclically-embracing* process of *metaphora* through *another* categorical *species*. Since these species occupy *different hypostatic* structural levels the metaphora is inevitably of a *heteromorphic* nature. This means that a certain structural *canonics* is required for the accomplishment of the metaphora. The basic idea is that the metaphora bears the potential to illuminate the unknown or obstacle-laden structure by heteromorhically descending to and ascending from *another* comprehensible categorical hypostatic level.

The effectiveness of this metaphora is based on two factors: First, the directly comprehensible level should consist of categorical objects whose hypostasis should be somehow *illuminating* in relation to the ones of the obstacle-laden level. In this sense, ideally these categorical objects should manifest some *local* or *partial* structural spectral *invariance* characterizing the objects of the former species. In other words, their suitability should be based on their *capacity* to *encode* some local or partial structural *invariant* which is crucial for disclosing the structure of the former ones. It is precisely in this manner that they can be considered as structurally illuminating in terms of their categorical hypostasis. Secondly, since the metaphora is heteromorphic, a certain *canonics* must be in force that allows the natural communication between these levels. In other words, the objects of the comprehensible kind should be *adjoined* as probes or generalized pointers in the environment of the former ones not in a rigid, but in a *plastic* manner, so that *connectively*, and not only collectively, they can enforce a *homeotic cobounding* relation to an object of the unknown species that has the capacity to play the role of a *structural canon* for this object.

In turn, this qualification serves to facilitate the *inverse* canonization of the employed probes as *icons* for the deployment of the *ab initio* unknown object at least within the symbolic *algebraic milieu*. Note that in this categorical context of thinking the notion of an icon is *abstracted* from its direct visual connotation and is *elevated* at the symbolic algebraic level as a *local* or *partial invariant means* of internal illumination through its canonization. Therefore, the symbolic and structural specification of the unknown via metaphora through the connective network of icons cobounding it structurally serves to establish the structural *adaptability* of the icons to the species of the unknown as its partially *covering spectral* structural invariants. In this sense, the unknown object is illuminated connectively by this network of icons, while the latter are canonized reciprocally by their participation in the canon, which works metaphorically to unfold the structure of the former. Finally, and in terms of this canonization procedure, a *gnomon* emerges at the categorical level of the unknown structural species by the *adjunction* of the categorical level of its icons.

It is precisely these factors that we maintain fundamental for a careful re-evaluation of the novel mode of object-specification induced by the development of category theory in mathematical thinking in relation to the objective of natural communication. More precisely, the notion of a structural symbolic *icon* is conceived as the principal means of probing or resolving an object of an obstacle-laden species in a canonical manner, *independently* of any a priori requirement of analysis of this object in its set-theoretic elements. This is the case because the notion of a canonized icon is not subordinate to any analytic reduction, but on the contrary, subsumes a well-defined structural characterization derived from its canonical internalization within the category deciphering the species of the object under investigation. In other words, a probe is qualified as an icon if and only if it can be *canonized* and thus *internalized* within the category in question. Technically, this simply means that it can give rise to a structure-respecting homomorphism within this category targeting the object of inquiry. An immediate consequence of this characterization is that a canonized action is structurally *adaptable* by means of adjunction to the species of the investigated object.

The major issue arising in this state of affairs is the detection of the conditions that cast a probe into a canonized icon; these are the very conditions, conditions that render a probe canonically structurally adaptable to the object of inquiry, so that it can function as a source of local or partial spectral illumination on this object. Intuitively, since a probe should furnish a frame for *resolving* the unity of the investigated object, this frame can be structurally adapted to the species of the object, if and only if it *encodes* some structural *invariant* feature pertaining to the level of resolution or distinguishability of the investigated object with respect to the frame employed. The requirement of structural adaptability qualifies a probe as an illuminating symbolic icon, which plays the role of a local or partial structural frame for resolving or partitioning the investigated object under the action of this probe by virtue of the structurally *invariant context* it discloses. Concomitantly, the action of a canonized icon becomes *co-extensive* to a local or partial *cover* of the object of enquiry subordinate to the *aphairetic* filtering, or sieving, spectral invariant capacity of the icon. The covering action leads to a *localization* of the object of enquiry, thereby, it also gives rise to a topology on the object of enquiry *only under* the mild constraints of transitivity and compatibility of the covers under the operation of restriction to subcovers.

It has to be stressed that the implicated notion of localization with respect to an icon is derived internally and intrinsically only from the specific *aphairetic* structural invariant capacity of this icon in its function as a local cover, and *not* from any *spatial* embedding environment of any external kind. In this respect, an icon, although *incomplete* in its capacity to illuminate and resolve the investigated object *globally* or in its entirety, *shapes* this object locally or partially in a structurally adaptable manner. As a result, it can be inversely, internally extended beyond its compatible restriction as well, under the proviso that *overlaps compatibly* with some other icon deciphering another local cover of the investigated object.

A crucial feature of this local iconic schematism of an object in a category is that it does not assume or require the existence of an all-encompassing icon, meaning that any single particular icon should not be thought of as an independent part of an all-encompassing icon. In contradistinction, the iconic schematism of an object is based on the idea of a *multiplicity* of partially illuminating icons, covering the object connectively only under their joint canonic cobounding action. It is equally vital to point out that such a jointly covering multiplicity of icons is *not* merely a set of icons. It may be thought of as a colimiting, and hence, spectrally cobounding object, comprehending within itself the joint resolving spectral capacity of all contributing icons, participating in its integrity, coherently and compatibly synthesized in a way which does not require the treatment of those icons as independent parts.

Rather it may be visualized as a multi-layered *granulation weaving sieve*, whose variable concatenated openings comprise the resolving or illuminating power of the corresponding icon, and which becomes structurally adaptable to the categorial species of the investigated object. This colimiting, or cobounding, object arises out of heteromorphic canonics, by means of a categorical adjunction, in the sense that the *canon* pertaining to the *joint* spectral capacity of all the partially illuminating icons induces a *gnomon* for the object of enquiry under *homomorphic internalization within* its categorical species of structure.

In light of the above, it becomes possible to appreciate the significance of the central formal theorem of category theory - the *Yoneda-Grothendieck lemma* - for the proposed obstacle-oriented approach, referring to the means of *articulability* of an object in a category. More precisely, this theorem states formally that an object is *completely specified* by the network of all morphisms directed to this object by all other objects in the same category. In other words, an object in a category can be completely *resolved*, or *partitioned*, or even classified and retrieved, and thus become totally *illuminated*, by all internalized structure-respecting arrow-morphisms pointing towards it within the same category. The complete illumination promised by this theorem rests on the fact that the *whole network* of pointing morphisms specifies, and thus articulates, the object *uniquely* up to canonical isomorphism. Thus, it provides the *universal* means of articulation of the object in relation to its categorical structural species, in the sense that its specification as outlined is unique up to equivalence, established by an explicitly demonstrable *isomorphism*.

The ground-breaking consequence of this theorem is essentially that the object of interest can be legitimately *subjugated* or even conceptually substituted by the network of all internalized structure-respecting relations or morphisms targeting it within the same category. For this reason, the object constitutes a formal *symbolic representation* of the whole network of relations directed to it within its categorial species, and *inversely*, this network becomes uniquely *representable* symbolically up to equivalence by the targeted object. Note that the term *"representation"* is used in the formal sense, meaning that the object becomes the symbolic *placeholder* of the network of morphisms *pointing* to it. We would like to *enhance* this *non-spatial* notion of representation with a *temporal* connotation that emphasizes its *synthetic* functional role in this setting. In particular, representation concerns the *depth* of the *present* in its capacity to instantiate an object by *encapsulating* all the morphism pointing to it. In other words, the synthetic act of representability in category theory is enunciated temporally by means of the *capacity* of an *instantiated* object to encapsulate *all* possible *paths* of arrows pointing toward it, which in turn, constitute its *indexical iconization*. From now on, this is precisely the meaning that we attribute to the technical term "representation" in the context of category theory.

In practice, the specification of an object in a category by the network of all possible morphisms directed to it within the same category is *redundant*. It is precisely here that the notion of an internalized structurally adaptable, and thus structure-respecting, *probing relation*, emanating from *another* categorical *species* of structure, becomes significant. The underlying idea is that subject to theoretical, experimental, or computational reasons, a category of *probes* is always delineated in relation to an investigated object of some unknown, non-directly accessible, or obstacle-laden categorial species. This category of probes may have the status of a *subcategory* of the category in which the investigated object is structurally placed, but this is no general requirement. What is crucial, is that the *action* of a probe can be *structurally adapted* as an internalized, and therefore, structure-respecting directed morphism within the category of the investigated object. Henceforth, it is exactly this qualification of a probing relation that gives rise to the notion of an *icon* illuminating an investigated object of some category.

The natural issue arising in this setting is to leverage the redundancy in the specification of an object in a category through the network of *all* possible relations directed to it, by *restricting* this network to a *minimal* but *sufficient assemblage* of icons capable of *collectively* and *connectively* illuminating this object. Intuitively, the *sufficiency* condition pertains to the *joint covering* action of the object by the utilized assemblage of icons, as we have already seen. Under this restriction of the classifying network of internalized structure-respecting, directed morphisms on the investigated object in iconic schematic terms, the validity of the category-theoretic object-specification theorem remains intact. Namely, an object of a categorical structural species can be specified, classified, and retrieved by the assemblage of all partially illuminating icons upon it, uniquely up to equivalence. The objective is that this assemblage can be appropriately qualified as a *heteromorphic canon* capable of establishing *natural communication* between the categorical species of the object of enquiry and the species of the icons. The underlying idea again is that the heteromorphic canon naturally induces a gnomon for the investigated object under homomorphic internalization in its categorical species of structure.

An immediate consequence of the above is that for a fixed object under investigation, the network or assemblage of all icons directed towards it can be objectified category-theoretically into a *novel* object of a *higher* structural species, i.e. an *iconic schematism* functor, called simply a *representable functor*. The root of this terminology is that an iconic schematism functor becomes symbolically representable by the investigated object up to equivalence. An iconic schematism functor is a structure-respecting morphism from the category of the investigated object into the category of sets. That is to say, *for each* internalized source-probe it assigns a *set* of icons capable of partially illuminating the unknown object from the specified source. Clearly, the iconic schematism functor of the investigated object *varies* over *all possible* internalized source-probes and is also *evaluated* at each one of them by producing the set of icons corresponding to it. Then, the meaning of the previously expressed object-specification theorem boils down to the fact that the categorical species of the investigated object can be *fully* and *faithfully* embedded into the *category of functors* from this species to sets, where the *category of sets* is used as a *scaffolding* for this purpose. Henceforth, it can be identified by means of the *representable* functors, i.e. the respective iconic schematism functors, technically symbolized as Hom-functors. It is instructive again to think of an iconic schematism functor, representable by the investigated object up to equivalence, as a multi-layered *granulation sieve* whose variable concatenated openings comprise the resolving or illuminating *power* of the corresponding *icons*.

The heteromorphic natural communication emerges in categorical terms from the *conceptual inversion* of an iconic schematism functor. In general, if an iconic schematism functor is considered as an *encoding* functor of the unknown categorical species to the species of its icons, then its inversion would be a *decoding* functor in the opposite direction. Pairs of encoding/decoding functors give rise to categorical *adjunctions*, where these functors are called *adjoint* to each other. The notion of an adjunction as an expression of heteromorphic natural communication constitutes a far-reaching *generalization* of the algebraic notion of an *equation*. Precisely, the iconic schematism functor furnishes the algebraic *variables* of this equation with respect to the scaffolding of the category of sets, whereas its conceptual inverse decoding functor amounts to the *solution* of this equation, i.e. the specification of these variables.

More concretely, we have already concluded that an investigated object of some categorial species can be conceptually substituted due to the amenability of its classifying iconic schematism functor to symbolic representation by the former up to equivalence. The important point is that the iconic schematism functor and the representing object of this functor, are *located* within *different categorical species* of structure. Thus, the pertinent issue again is *how* the iconic schematism functor, or equivalently, a whole sieve of icons directed towards an object of inquiry, becomes structurally *adaptable* as a *totality* to the categorial species of this object. The structural adaptability of the iconic schematism functor, and thus, the internalization of the whole sieve of icons within the categorial species of the investigated object, is possible only as a *colimiting canon* pertaining to the *joint* spectral capacity of all the partially illuminating icons.

The pronounced inversion consists in the realization that the iconic schematism functor on an object of inquiry, after the selection of a certain category of probes - as structurally invariant iconic contexts of resolving and illuminating this object at various layers of distinguishability - is *sufficient* to depict, approximate, or even completely *disclose* the categorical species of the investigated object, up to equivalence. Note that in this setting, the categorical structural *species* of the investigated object, and *not merely* the investigated *object itself*, is considered an *unknown* variable. Thus, the proposed conceptual inversion can be utilized as a *recognition principle* of a *new* categorial *species* of structure. This becomes possible after the initial selection of a category of probes applied to a hypothetical object of a species unknowable *a priori*, which gives rise - modulo all the previously-stated requirements - to the iconic schematism functor of this object of inquiry.

This constitutes a novel approach to *revealing* new categorial species of structure by means of heteromorphic natural communication, through the employment of an encoding iconic schematism functor defined in terms of structurally invariant contexts for resolution of this unknown species, and then inverting or decoding it back. We may schematically assert that the categorical recognition principle formulated in terms of natural communication turns the formal object-specification theorem of category theory onto its head, in the sense that, considering all practical or theoretical applications, it is the *categorical species* of the investigated object that is the real *unknown* and not the object itself.

Thus, we finally arrive at the crucial point of formulating the proclaimed recognition principle of some new categorical species of structure given the set-up of a representable functor, i.e. of an iconic schematism functor, as previously. *A posteriori*, this constitutes an *interpretation* of the category-theoretic *syntax* from the standpoint of natural communication. More tellingly, we read from *Saunders MacLane*, one of the two inventors of the categorical syntax, the following cryptic remark: "But I emphasize that the notions category and functor were not formulated or put in print until the idea of a natural transformation was also at hand".

This is *at odds* with the usual axiomatic presentation of the subject, more accurately to be read as its *inversion*. Of course, for reasons of systematic presentation of the category-theoretic syntax, the definition of a natural transformation, being a morphism of functors, has to be preceded by the definitions of what a functor is and what a category is. Notwithstanding this fact, the idea of a natural transformation is solely based on the concept of covariance, meaning that the associated morphism of functors to which it refers, is *not subordinate* to any *ad hoc choice* of underlying object to which these functors can be applied and evaluated. In other words, a natural transformation between two functors *pertains* to the categorial *species* of the underlying objects and *not* to the objects themselves. This is the decisive *abstraction* for the formulation of the *recognition* principle of obstacle-laden categorical species of structure in natural communication terms.

Up to present, we have reached the conclusion that the recognition principle of some new categorical species of structure, should be properly expressed by means of a natural transformation of functors. It remains to examine the particularities of this natural transformation in detail, i.e. which *precise* morphism of functors is *pertinent* for the formulation of the recognition principle. What we have at our disposal is the set-up of an iconic schematism functor for each unknown hypothetical object of this new categorial species, after the selection of a category of probes, namely a category of structurally invariant contexts for resolution of these objects in terms of icons, according to the preceding argument. What we have also established is that given an iconic schematism functor on an object of inquiry, the associated *sieve* of all *icons* on this object becomes *internalized* as a totality in its categorial species by means of a novel *colimiting* or cobounding object, pertaining to the joint connective resolution capacity of all the involved icons illuminating this object. It is significant that this internalization process by means of a colimiting object is *universal*, in the sense that it is uniquely specified up to equivalence; it is unique up to a canonical isomorphism. In the terms of categorical natural communication, the universality property gives rise to a heteromorphic canon, which finally induces a homomorphic gnomon for the investigated object under internalization in its categorical species of structure.

Therefore, a colimiting object uniquely determines the *universal terminus* through which each one of the icons should factor in its function to probe and illuminate the species of the object of enquiry, *being* in this sense its *gnomon*. Clearly, the *instantiation* of such a *colimiting* object pertains to *any* iconic schematism functor associated with *any* object of the unknown categorial species. This prompts the realization that the iconic colimiting or cobounding process is actually a *functor* from the category of iconic schematism functors to the sought-after novel category. Most important, this colimiting functor conceptually *inverts* the iconic schematism functor on each object of the unknown species, because of the universality property characterizing the determination of the colimit object with respect to each object of the unknown species.

Put equivalently, for each object of the unknown species the set of all its partially illuminating icons factors uniquely through the *colimiting object* of the corresponding iconic schematism functor pointing to it. Henceforth, any object of the unknown species can be recognized *uniformly* and *universally* only through such a corresponding colimiting object *terminus* of an iconic schematism functor on it. In more philosophical terms, this means that the *colimiting functor* actually determines the *genus* of the whole new categorical species of structure *in relation* to the underlying category of icons, those structurally invariant contexts for resolution or illumination of the objects of this new species. Note that the notion of *genus* incorporates the *recognition principle* of the *whole* new categorical species of structure and not only of the objects of this species.

In a nutshell, for each object of the unknown categorical species of structure, after setting up an iconic schematism functor referring to it, we compute and apply the colimiting functor on the latter, thus, conceptually inverting it. This process is *sufficient* to determine the cobounding terminus uniquely up to equivalence, and thus, the universal homomorphic gnomon pertaining to the new categorical species through which the heteromorphic canonics of any icon has to factor in order to illuminate the object of inquiry. Since the same process applies uniformly and universally to *any* object of the unknown species, it is elevated to a process of articulation of the *genus* of the whole new categorical species. Henceforth, the unknown or obstacle-laden categorical species is recognized in iconic spectral terms by means of its *genus determination* through natural communication, where the iconic schematism and the colimiting functors play the role of the *encoding* and *decoding* bridges respectively.

We are now ready to express the recognition principle of some new categorical species of structure through natural communication, by means of a natural transformation of functors that has been our *main* objective. Conceptually, the leading idea is that object specification of an unknown categorical species in iconic terms always *follows from* genus determination of these species as a *whole*, as above. The unknown categorical species can equivalently be thought of, and objectified functorially, as the *identity functor* on itself. In this light, the recognition principle of this categorial species as a whole -irrespectively of any *ad hoc* choice of objects - should be expressed as a *natural transformation* of the identity functor induced by any appropriate selection of a category of probes to function as icons in relation to this categorical species. The suitability of a category of probes, furnishing structurally invariant contexts for resolving any object of the unknown species in iconic terms, is judged from its *capacity* to induce genus determination of this categorial species. This is possible if and only if a heteromorphic canon of natural communication is activated, which in turn, instantiates a universally cobounding gnomon of icons as a terminus of spectral recognition for each object of the sought-after species in a uniform manner. Equivalently, the heteromorphic canon is functorially tenable, if and only a colimiting functor exists whose *action* on the iconic schematism functor results in a natural transformation of the identity functor. Under these conditions, we say that the unknown categorical species is recognized by means of a natural communication *monad*, in the sense that the *unveiling* of these species emanates from the joint and interlinked illuminating capacity of its probes-icons.

Consequently, the recognition principle of some new categorical species of structure, initially taken to be unknown, is expressed as a natural transformation of the identity functor on this species by the *endo-functor* acting on the same species, which is obtained by *composing* the colimiting functor with the iconic schematism functor, *depending* on the appropriate selection of an *underlying* category of probes to act as icons. This natural transformation of the identity functor, determines the genus of the new categorical species in iconic terms, thereby establishing the sought-after *recognition* principle of this species. It is worth remarking that the above recognition-inducing natural transformation of the identity functor is equivalent to the conceptual inversion of the iconic schematism functor. More precisely, in technical terms, the colimiting functor that *inverts* the iconic schematism functor is called a *left adjoint* to the latter, since it acts on the left of it so that their composition provides the determining endofunctor on the unknown categorial species. It is precisely this determining endofunctor that plays the role of a heteromorphic canon of natural communication; it gives rise to a communicability monad between the species of the icons and the obstacle-laden species, which bears the capacity to unveil the latter in terms of the former.

We note that the notion of adjoint functors was first conceived, abstracted, and formulated more than a decade after the introduction of the category-theoretic framework in terms of categories, functors and natural transformations. Before the introduction of this notion category theory can be viewed as a useful tool for organizing, systematizing and classifying various species of already known mathematical objects. In other words, it served more as a theory of *taxonomy* than as a theory of invention and discovery. The notion of adjoint functors transformed its role completely and paved the way for forming bridges between seemingly unrelated mathematical disciplines; in short, it revealed its *natural communication* underpinnings. It is precisely this notion that discloses the philosophical significance of category theory, and at the same time, makes it suitable for solving difficult structural problems in the natural sciences.

The formulation of the theory of adjoint functors in category-theoretic terms has its roots in *homology theory* and *algebraic topology*. We arrived at this notion by following *another* more *philosophical* route involving the objective of heteromorphic natural communication, if expressed in terms of the category-theoretic *syntax*. Since the *canonics* characterizing this kind of natural communication, pertaining to categorical species of structure, is not to be taken for a merely taxonomic tool, but as a potent instrument of genuine *emergent novelty*, the basic problem posed is the *means* of recognition of an obstacle-laden categorical species in partial or local iconic terms. As we showed, this involves first, the setting-up of an iconic schematism functor, after the selection of an underlying category of probes in their function as icons, i.e. partially or locally congruent structural invariant contexts for illuminating and connectively resolving an object of the sought-after categorial species. The *solution* to this problem, which *a posteriori* characterizes the *suitability* of the selected category of probes and the *sufficiency* of the icons, requires the conceptual *inversion* of an iconic schematism functor. This conceptual inversion is equal in effect to finding a colimiting functor acting on a corresponding iconic schematism functor as a *left adjoint*, and thus, inverting it.

In turn, the above inversion *embraces* and *indirectly* solves the problem posed by means of the heteromorphic canonics establishing the natural communication between the categorical species of interest, that is, by means of a *pair of adjoint functors* between these species. In turn, the solution amounts to a natural transformation of the identity functor on the unknown categorical species, which is thus, recognized *gnomonically* by means of the unique homomorphic *termini* induced by the colimiting functor for *each* object of the sought-after species. Accordingly, the conceptual inversion of an iconic schematism functor leads to the genus determination of the initially unknown categorical species *by means of* the established recognition principle. Under certain conditions, the natural transformation of the identity functor solving the problem of recognition of a new categorical species of structure, can be *restricted* to a natural isomorphism. In this case, each object of this new species is specified uniquely up to equivalence by the corresponding colimiting or cobounding object emerging by the left adjoint to the iconic schematism functor referring to it. These conditions are usually implemented by imposing a particular *categorical topology* on the underlying category of probes, which amounts to *localizing* an assemblage of icons into a *sheaf of icons* for this topology.

8.3 MEASURABILITY: THE PHYSICAL ROOTS OF COORDINATIZATION

The modus-operandus of physics as a natural science is based on observation and measurement. In particular, measurement is the process corresponding to a well-defined *observational* procedure, according to which, various *attributes* or *magnitudes* are assigned *numbers*, or more generally, "number-like" quantities. In this sense, the general notion of number is understood physically as the *outcome* of a measurement corresponding to an observational procedure. The abstraction of the measurement process gives rise to the notion of *coordinatization* or *arithmetization*. The power of the measurement process to arithmetize phenomena according to a well-defined underlying observational procedure is in essence the *objectification* of magnitudes, in the sense that they can be *communicated* to other observers, and thus become amenable to comparison and transformation, again according to prescribed rules. Furthermore, the "number-like" quantities obtained by the coordinatization process can be subjected to *algebraic* operations, so that they can form suitable algebraic structures closed under the action of the corresponding operations.

Hence, from a physical perspective, algebraic structures of "number-like" quantities of any particular operational form, can be thought of as *solutions* to a physical measurement problem. Most commonly, the effectuation of the coordinatization process itself requires the conceptual extension of *what* can function as a "number-like" quantity. In turn, this is reflected in the algebraic process of *extension of scalarity*, which precisely extends the notion of a "number-like" quantity in a way that preserves the closure requirements under the application of algebraic operations characterizing some structure of already known "numbers". In categorical terms this means that the algebraic process of extension of scalars, solving a physical coordinatization problem, is a *functorial* process.

We recall that the father of the above described natural philosophy is Thales and his theory of measurement based on proportionality of magnitudes. The sole purpose of Thales' theory of proportions had been the measurement of a not directly accessible magnitude using a gnomon. This refers to the height x of an inaccessible pyramid, given the length c of its accessible shadow, as well as, the height a and shadow length

b of an accessible object, functioning as a measurement rod. The proportion between magnitudes resolving the Thalesian measurement problem reads as follows:

$$a \text{ to } b \text{ is as } x \text{ to } c$$

Symbolically, the above proportion is depicted by the equation $\frac{a}{b}=\frac{x}{c}$, from which the not directly accessible magnitude x can be obtained indirectly as $x=ac/b$. We remind that the geometric theory of proportions for the resolution of measurement problems of the Thalesian form, contains the seeds of conception of a group-theoretic structure, together with the concomitant formulation of an algebraic equation for the determination of unknowns.

In this mode of thinking, the geometric resolution of the Thalesian problem, in terms of proportionality, implicitly anticipates the discovery of the multiplicative monoid structure of positive integers, and subsequently, the multiplicative *group structure* of the rationals and the real numbers. The meaning of this assertion boils down to the realization that the determination of an unknown magnitude in the Thalesian setting, interpreted algebraically, requires the introduction of the multiplicative group structure of the rational or the reals (standing for magnitudes), in order to provide a solution to the associated equation expressing the corresponding proportion. Thus, the determination of unknown magnitudes by the method of proportion, algebraically entails the introduction of the group-theoretic closure structure on magnitudes, equipped with the operation of multiplication and possessing an inverse, which is division.

Up to now, we argued that algebraic structures of "number-like" quantities of any particular operational form, can be thought of as solutions to physical measurability problems. Until recently, these algebraic structures have been conceived as sets endowed with prescribed operations, like addition and multiplication, satisfying a closure condition with respect to the action of these operations on the elements of the underlying sets. Gradually the *primary* significance of considering the *homomorphisms* between algebraic structures of the same kind came to collective realization. In particular the existence of an *isomorphism* between two algebraic structures of the same kind essentially came to mean that these algebraic structures have exactly the *same* operational role.

Thus, the conception of algebraic structures of some kind should be considered primarily in terms of the *relations* between them, conceptualized in terms of *incoming* or *outgoing homomorphisms* (structure-preserving morphisms), and *not* in the restricted terms of their *elements*. The decisive fact in this conceptual re-articulation of algebraic structures has been the realization that their elements can be thought of as *special* homomorphisms from particular algebraic structures of the same kind.

Note that the notion of homomorphism *depends* on the *kind* of algebraic structure considered, since it reflects the preservation of the operations between structures of the same kind. Here arises the pure *algebraic* notion of a *"generalized element"* of an algebraic structure conceived as an *outgoing* homomorphism from this algebraic structure into some other of the *same* kind. Of course, simultaneously the *dual* notion of a "generalized element" applies, conceived as an *incoming* homomorphism from some other algebraic structure of the *same* kind. The artifact of this re-conceptualization requires an *interpretational* shift in the *semantics* of structure. More concretely, it is reasonable to coin the term *algebraic object* of some kind and reserve the *structural* qualification for the *environment* where these objects are situated and are related to each other by means of homomorphisms. This is precisely the semantic transition required for the category-theoretic re-interpretation of set-theoretic algebraic structures entailed by the primary role of homomorphisms (generalized elements and their duals) in the determination of algebraic objects of some kind.

The definition of a category of algebraic objects of some kind with arrows being homomorphisms between them, constitutes an *abstraction* of the behavior of *functions* closed under the associative operation of composition. More precisely, the notion of a function is generalized to the notion of a homomorphism, whereas the associative operation of composition becomes an operation on sets of homomorphisms between algebraic objects of the same kind satisfying the same properties that functions and compositions satisfy.

Note that the composition of two functions f, g, denoted as $f \circ g$ is defined *only* in case that the codomain of g is the domain of f. Moreover, the composition of a function f with the identity of either its domain or codomain gives f again. It is interesting then to notice that from a structural set-theoretic viewpoint, a category of algebraic objects of some kind, may be thought of as a *partial algebra* itself. More precisely, the elements of this algebra are the homomorphisms, and their composition is actually an associative binary operation which is only partial, being defined *only when* the composition of homomorphisms is meaningful. Then, the role of the objects is just that of *labeling* the homomorphisms for the determination of the domain of the binary operation of composition.

In a nutshell, the notion of a category of algebraic objects of some kind is a conception based on the behavior of functions closed under the associative operation of composition, abstracted in terms of homomorphisms, which in turn, have been idealized as algebraic "generalized elements" (or their duals) completely determining the algebraic objects themselves. Because an obvious duality obtains between incoming and outgoing homomorphisms with respect to an algebraic object, this is built into the definition of a category so that the operation of arrows-reversal leaves the concept of a category *invariant*, meaning that this operation gives again a category in *dual* or opposite relation in comparison to the given one.

A consequence of this fact is that *all* categorical constructions come in *dual pairs* corresponding to the dual viewpoints from which incoming or outgoing arrows with respect to a constituted algebraic object are seen. Apart from the duality property, we have seen above that the successful effectuation of the coordinatization process in more and more demanding physical measurement problems requires the conceptual extension of the meaning of a "number-like" quantity, described by the algebraic process of extension. For example, the resolution of the Thalesian measurement problem requires the *extension* of integer numbers into the rational numbers so that division becomes possible. Notably, the algebraic extension process provides algebraic objects of operationally extended "number-like" quantities including the initial ones. Note also that since we should always think in a dual way regarding the direction of arrows with respect to an algebraic object there is clearly an *inverse* algebraic process called *restriction*. The semantics of this inverse pair of algebraic processes will be explained in detail subsequently.

Now, the design of the categorical framework of reasoning with respect to the algebraic process of extension provides the conception of the algebraic object of extended "number-like" quantities, of the same kind as the initial one, uniquely up to isomorphism by means of a *universal* property. More precisely, each outgoing homomorphism $A \to B$ from the initial object A into a set B endowed with the *extra* structure, extends to an outgoing homomorphism $A \to B$ from the algebraic object A newly constructed by extension (solving a corresponding measurement problem) into B. Equivalently stated, every homomorphism $A \to B$ of the previous form *uniquely factors* through A.

8.4 NATURALITY: FUNCTORS AND COVARIANT TRANSFORMATIONS

We have seen that the idea of a category of algebraic objects of some kind for which arrows are homomorphisms incorporates from the start the basic idea of *duality* by arrow reversal, emanating from the fundamental role played by homomorphisms (algebraic "generalized elements") and their dual distinction into incoming and outgoing kinds with respect to an algebraic object completely determined by them. Moreover, understanding *algebraic objects* as providing *solutions* to physical *coordinatization* problems, the category-theoretic framework referring to algebraic objects of some kind, forces the conception of operationally extended objects of the same kind, *not* in terms of their set-theoretic *constitution* from given elements, but in terms of some *universal* property determining them up to isomorphism within the same category.

Now, the establishment of the notion of a category of algebraic objects naturally raises the difficulty of defining the notion of a *function* whose *domain* and *codomain* are *categories*. Obviously such a function should preserve the composition operation binding a category as an associatively closed universe of discourse. This is precisely the notion of *functor* between categories. From an equivalent viewpoint, since a category may be considered as a partial algebra with respect to the binary operation of composition, the notion of a functor corresponds to the notion of such a partial algebra homomorphism. A *covariant* functor is a functor which *preserves* the *directionality* of an arrow in the domain category, whereas a *contravariant* functor is a functor which *reverses* it.

Each object $\mathcal{A}$ of a category $\mathfrak{A}$ determines a covariant functor $\mathbf{y}_{\mathcal{A}} : \mathfrak{A} \to \mathbf{Sets}$, called the *covariant* $Hom_{\mathfrak{A}}$*-functor represented by* $\mathcal{A}$, defined as follows:

i For all objects $\mathcal{X}$ in $\mathfrak{A}$, $\mathbf{y}_{\mathcal{A}}(\mathcal{X}) := Hom_{\mathfrak{A}}(\mathcal{A}, \mathcal{X})$.

ii For all homomorphisms $f : \mathcal{X} \to \mathcal{Y}$ in $\mathfrak{A}$,

$$\mathbf{y}_{\mathcal{A}}(f) : Hom_{\mathfrak{A}}(\mathcal{A}, \mathcal{X}) \to Hom_{\mathfrak{A}}(\mathcal{A}, \mathcal{Y})$$

is defined as *post-composition* with f, viz., $\mathbf{y}_{\mathcal{A}}(f)(g) := f \circ g$.

The covariant representable functor $\mathbf{y}_{\mathcal{A}} : \mathfrak{A} \to \mathbf{Sets}$, can be thought of as constructing an image of $\mathfrak{A}$ in $Sets$ in a covariant way.

Now, let us consider the opposite category $\mathfrak{A}^{op}$, and let $\mathcal{A}$ be an object in this category. Then, the *contravariant* $Hom_{\mathfrak{A}}$ *-functor represented by* $\mathcal{A}$ is the contravariant functor $\mathbf{y}^{\mathcal{A}} : \mathfrak{A}^{op} \to \mathbf{Sets}$, defined as follows:

i For all objects $\mathcal{B}$ in $\mathfrak{A}^{op}$, $\mathbf{y}^{\mathcal{A}}(\mathcal{B}) := Hom_{\mathfrak{A}^{op}}(\mathcal{B}, \mathcal{A})$.

ii For all homomorphisms $f : \mathcal{C} \to \mathcal{B}$ in $\mathfrak{A}^{op}$,

$$\mathbf{y}^{\mathcal{A}}(f): Hom_{\mathfrak{A}^{op}}(\mathcal{B},\mathcal{A}) \to Hom_{\mathfrak{A}^{op}}(\mathcal{C},\mathcal{A})$$

is defined as *pre-composition* with f, viz., $\mathbf{y}^{\mathcal{A}}(f)(g) := g \circ f$.

The contravariant $Hom_{\mathfrak{A}}$-functor represented by $\mathcal{A}$, viz. $\mathbf{y}^{\mathcal{A}} : \mathcal{F}^{op} \to \mathbf{Sets}$, is called the *functor of generalized elements* (incoming homomorphisms) of $\mathcal{A}$. Moreover, the information contained in $\mathcal{A}$ is classified completely by its functor of generalized elements $\mathbf{y}^{\mathcal{A}}$. Dually the covariant $Hom_{\mathfrak{A}}$-functor represented by $\mathcal{A}$, viz. $\mathbf{y}_{\mathcal{A}} : \mathfrak{A} \to \mathbf{Sets}$ is called the *functor of generalized co-elements* (outgoing homomorphisms) of $\mathcal{A}$. Similarly, the information contained in $\mathcal{A}$ is classified completely by its functor of generalized co-elements $\mathbf{y}_{\mathcal{A}}$.

Now, given a *locally small* category $\mathfrak{A}$, that is, a category such that for all objects $\mathcal{B}$, $\mathcal{A}$, the Hom-class $Hom_{\mathfrak{A}}(\mathcal{B},\mathcal{A})$ is a set, we may consider the $Hom_{\mathfrak{A}}$*-bifunctor*:

$$\mathbf{Hom}_{\mathfrak{A}} := \mathbf{y}_{\mathcal{B}}^{\mathcal{A}} = \mathfrak{A}^{op} \times \mathfrak{A} \to \mathbf{Sets}$$

from the product category $\mathfrak{A}^{op} \times \mathfrak{A}$ to the category of sets, such that for objects $\mathcal{B}$, $\mathcal{A}$ of $\mathfrak{A}$, $\mathbf{y}_{\mathcal{B}} : \mathfrak{A} \to \mathbf{Sets}$ is the covariant representable functor (represented by $\mathcal{B}$), and $\mathbf{y}^{\mathcal{A}} : \mathfrak{A}^{op} \to \mathbf{Sets}$ is the contravariant representable functor (represented by $\mathcal{A}$).

Continuing in the same frame of thought, the next question arising is the following: Which is the proper notion of morphism to capture the notion of a transformation from some functor to another functor having both the same domain and the same codomain categories? Defining the proper notion of morphism between such functors is important because it would allow us to legitimate the notion of a functor category $\mathfrak{C}^{\mathfrak{A}}$, where the algebraic objects would be functors $\mathbf{F} : \mathfrak{A} \to \mathfrak{C}$ and the morphisms would be the sought transformations between such functors.

The leading idea has to do with the requirement that a transformation of the sought form should *compare* two functorial processes having the same domain and the same codomain in a way that is *not dependent* on the specific objects and arrows involved, that is it should relate the *processes* themselves without the intervention of *ad hoc* choices. This is precisely the notion required for the formalization of the concept of *naturality* referring to the relation or comparison of two functorial processes sharing the same source and the same target categories. Concomitantly the corresponding notion of morphism between functors of the above form is encapsulated in the notion of a *natural transformation*.

More precisely, if $\mathbf{F}$, $\mathbf{G}$, are functors from the category $\mathfrak{A}$ to the category $\mathfrak{C}$, a *natural transformation* τ from $\mathbf{F}$ to $\mathbf{G}$ is a function assigning to each object $\mathcal{A}$ in $\mathfrak{A}$ a morphism $\tau_{\mathcal{A}}$ from $\mathbf{F}(\mathcal{A})$ to $\mathbf{G}(\mathcal{A})$ in $\mathfrak{C}$, such that for every arrow $f : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{A}$ the following diagram in $\mathfrak{C}$ commutes:

$$\begin{tikzcd}
\mathbf{F}(\mathcal{A}) \arrow[r, "\tau_{\mathcal{A}}"] \arrow[d, "\mathbf{F}(f)"'] & \mathbf{G}(\mathcal{A}) \arrow[d, "\mathbf{G}(f)"] \\
\mathbf{F}(\mathcal{B}) \arrow[r, "\tau_{\mathcal{B}}"] & \mathbf{G}(\mathcal{B})
\end{tikzcd}$$

That is, for every arrow $f : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{A}$ we have:

$$\mathbf{G}(f) \circ \tau_{\mathcal{A}} = \tau_{\mathcal{B}} \circ \mathbf{F}(f)$$

A natural transformation $\tau : \mathbf{F} \to \mathbf{G}$ is called a *natural isomorphism* (or natural equivalence) if every component $\tau_{\mathcal{A}}$ is *invertible*. Obviously, a natural isomorphism is an invertible natural transformation in the functor category $\mathfrak{C}^{\mathfrak{A}}$. A natural isomorphism of functors defines precisely the *categorical* means of *metaphora*. Paraphrasing *Hermann Weyl*, we may say that a science can only determine its domain of investigation *up to* a natural isomorphism. In particular, it remains quite *indifferent* as to the *"essence"* of its objects. In this manner, the notion of a natural isomorphism demarcates the *insurmountable boundary* of cognition, in the sense that through the disclosure of natural isomorphisms of functors it is possible to *transfer* any insights gained in one domain to the *isomorphic* domain.

This is a *key* concept and justifies *a posteriori* the whole categorical framework, since it captures precisely the *criterion of naturality* referring to the comparison of any two *homoeoid functorial processes*. For this purpose, it is instructive to adopt the following terminology:

Suppose that $\mathbf{F}(\mathcal{R})$ is an expression with an argument $\mathcal{R}$, such that given an object $\mathcal{A}$ in $\mathfrak{A}$, the expression $\mathbf{F}(\mathcal{A})$ holds for an object in $\mathfrak{C}$, and given an arrow f in the set of incoming or outgoing arrows from $\mathcal{A}$ the expression $\mathbf{F}(f)$ holds for an arrow in $\mathfrak{C}$. We say that $\mathbf{F}(\mathcal{R})$ is *functorial* in $\mathcal{R}$, as $\mathcal{R}$ ranges over $\mathfrak{A}$, if this assignment yields *a functor* from $\mathfrak{A}$ to $\mathfrak{C}$. For example for a category $\mathfrak{A}$ the $Hom_{\mathfrak{A}}$-set expression $Hom_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$ is functorial in both $\mathcal{A}$ ranging over $\mathfrak{A}^{op}$ and $\mathcal{B}$ ranging over $\mathfrak{A}$. Thus, the $Hom_{\mathfrak{A}}$-set expression $Hom_{\mathfrak{A}}(\mathcal{A},\mathcal{B})$ is *bifunctorial* over the *product* category $\mathfrak{A}^{op}\times\mathfrak{A}$. Moreover, we say that the expressions $\mathbf{F}_1(\mathcal{R})$, $\mathbf{F}_2(\mathcal{R})$ both functorial in $\mathcal{R}$, as $\mathcal{R}$ ranges over $\mathfrak{A}$, are *naturally isomorphic* in $\mathcal{R}$, or equivalently, that there exists an isomorphism natural in $\mathcal{R}$, if and only if there is a *natural isomorphism* between the functors $\mathbf{F}_1$ and $\mathbf{F}_2$ from the category $\mathfrak{A}$ to the category $\mathfrak{C}$.

It is also useful to think of the category of *all* (locally small) categories and functors, denoted by $\mathbf{Cat}$. In $\mathbf{Cat}$ for any two categories $\mathfrak{A}$, $\mathfrak{C}$, we define the operation of *exponentiation* as follows:

$$\mathfrak{C}^{\mathfrak{A}} := \mathbf{Functors}(\mathfrak{A},\mathfrak{C})$$

where $\mathbf{Functors}(\mathfrak{A},\mathfrak{C})$ is the category of functors from $\mathfrak{A}$ to $\mathfrak{C}$ and natural transformations between them. From the perspective of $\mathbf{Cat}$ the notion of natural transformation between functors allows to transform the $Hom_{\mathbf{Cat}}$-set $Hom_{\mathbf{Cat}}(\mathfrak{A},\mathfrak{C})$ into a category with exponentials. In this setting we define the *evaluation functor*:

$$\mathbf{ev}:\mathfrak{C}^{\mathfrak{A}}\times\mathfrak{A}\to\mathfrak{C}$$

such that for any category $\mathfrak{D}$ and bifunctor $\mathbf{G}:\mathfrak{D}\times\mathfrak{A}\to\mathfrak{C}$ there is a functor $\tilde{\mathbf{G}}:\mathfrak{D}\to\mathfrak{C}^{\mathfrak{A}}$ defined by *transposition*, such that:

$$\mathbf{ev}\circ(\tilde{\mathbf{G}}\times\mathbf{1}_{\mathfrak{A}})=\mathbf{G}$$

This is clearly the case if we define the functor $\tilde{\mathbf{G}}$, given the bifunctor $\mathbf{G}:\mathfrak{D}\times\mathfrak{A}\to\mathfrak{C}$ as follows:

$$\tilde{\mathbf{G}}(\mathcal{D})(\mathcal{A})=\mathbf{G}(\mathcal{D},\mathcal{A})$$

As a simple application we may take the product category $\mathfrak{D}\times\mathfrak{A}$ and consider the first projection:

$$\mathfrak{D}\times\mathfrak{A}\to\mathfrak{D}$$

Then, if we transpose it we get a functor:

$$\Delta:\mathfrak{D}\to\mathfrak{D}^{\mathfrak{A}}$$

For an object $\mathcal{D}$ in $\mathfrak{D}$, the functor $\Delta(\mathcal{D})$ is the *constant* $\mathcal{D}$-valued functor, which for any object $\mathcal{A}$ of $\mathfrak{A}$ gives the constant value $\mathcal{D}$, and for any arrow f of $\mathfrak{A}$ gives the identity arrow on $\mathcal{D}$.

8.5 CATEGORICAL INVARIABILITY: FUNCTORIAL EQUIVALENCE AND DUALITY

Firstly, we introduce the notion of equivalence of two categories $\mathfrak{D}$ and $\mathfrak{A}$ as follows: An *equivalence* of two categories $\mathfrak{D}$ and $\mathfrak{A}$ is defined by means of a pair of oppositely directing functors;

$$\mathbf{F}:\mathfrak{D}\to\mathfrak{A}$$
$$\mathbf{G}:\mathfrak{A}\to\mathfrak{D}$$

and a pair of natural isomorphisms;

$$\tau:\mathbf{1}_{\mathfrak{D}}\to\mathbf{G}\circ\mathbf{F}$$
$$\circlearrowright:\mathbf{1}_{\mathfrak{A}}\to\mathbf{F}\circ\mathbf{G}$$

where τ is a natural isomorphism of the *identity* functor on $\mathfrak{D}$ in $\mathfrak{D}^{\mathfrak{D}}$ and $\circlearrowright$ is a natural isomorphism of the *identity functor* on $\mathfrak{A}$ in $\mathfrak{A}^{\mathfrak{A}}$. In this case, the functorial process $\mathbf{F}:\mathfrak{D}\to\mathfrak{A}$ is called *inverse* to the functorial process $\mathbf{G}:\mathfrak{A}\to\mathfrak{D}$.

The notion of equivalence of categories transcribes the concept of similarity, or *analogia*, in a categorical/functorial context. Furthermore, it makes precise the notion of *duality* between categories. More concretely, duality theorems can be expressed in the form $\mathfrak{A}\simeq\mathfrak{B}^{op}$, meaning that the category $\mathfrak{A}$ is equivalent to the opposite of the category $\mathfrak{B}$. The notion of duality between categories, formalized by means of an equivalence as above, is very important because it means that two categories related by a duality transformation are similar in the sense that they play the same functional role.

The interpretational consequences of this notion are far reaching and especially interesting from a physical viewpoint. We have argued previously that categories of algebraic objects can be considered as providing *solutions* to coordinatization processes (conceived within an associatively closed universe of discourse) arising from physical measurement problems. If a coordinatization process is modeled functorially as a functor $\mathbf{F}:\mathfrak{D}\to\mathfrak{A}$, which encodes the information of $\mathfrak{D}$ in terms of a category of algebraic objects $\mathfrak{A}$, then the existence of an inverse functorial process $\mathbf{G}:\mathfrak{A}\to\mathfrak{D}$ decodes the algebraic information of $\mathfrak{A}$ back into the terms of $\mathfrak{D}$, which in turn, is identified as the opposite or dual category of $\mathfrak{A}$.

In this way, a physical measurement problem presented in terms of a category of geometric objects $\mathfrak{D}$ can be *resolved* by finding a *duality*, that is by establishing a *natural equivalence* between a category of algebraic objects $\mathfrak{A}$ and the opposite of $\mathfrak{D}$. This frame of thought can be extrapolated in formal terms of duality between *geometric* objects and *algebraic* objects. Then, keeping in mind the *encoding* functionality of a functorial coordinatization process as well as the *decoding* functionality of its inverse (if it exists), we may interpret algebraic objects as *syntactical* objects and their dual geometric objects as *semantical* or *phenomenological* objects.

As a particular example we may cite the *Stone duality* established between the category of *Boolean algebras* (or equivalently Boolean unital rings) and the category of *compact totally disconnected Hausdorff topological spaces* (Stone spaces). In the finite case this duality restricts to an equivalence of the category of finite Boolean algebras and the opposite of the category of finite sets. In this way, each Boolean algebra B has an associated topological space $S(B)$, called its *Stone space*. The *points* of the Stone space $S(B)$ are the *ultrafilters* of the Boolean algebra B, or equivalently the homomorphisms from B to the 2-element Boolean algebra. A basis of the *topology* of $S(B)$ consists of sets of the form $\{x\in S(B)\,|\,b\in x\}$ where $b\in B$. Conversely, given any topological space X, the collection of subsets of X which are compact open or equivalently *clopen*, that is, both closed and open sets, is a Boolean algebra. Now, the assignment of a Boolean algebra to its Stone space is *functorial*, that is it corresponds to the object part of a *contravariant functor* from the category of Boolean algebras to the category of Stone spaces:

$$Hom_{\mathcal{B}}(-,2):\mathcal{B}^{op}\to\mathcal{T}$$

where $\mathcal{B}$, $\mathcal{T}$ are the categories of Boolean algebras and Stone spaces correspondingly. The functor $Hom_{\mathcal{B}}(-,2)$ sends a Boolean algebra B to the Stone space $S(B) := X_B$, and a Boolean algebra *homomorphism* $F : B \to A$ to the *continuous* function $Hom_{\mathcal{B}}(F,2)$, whose action is defined by precomposition with F, by which we mean that it sends $H : A \to 2$ of X_A to $F \circ H : B \to 2$ of X_B. Now, there exists an *inverse* functor:

$$Hom_{\mathcal{T}}(-,2) : \mathcal{T} \to \mathcal{B}^{op}$$

which sends a Stone space S to the set of *continuous* functions from S to the Stone space 2, which is endowed with the structure of a Boolean algebra, and further identified as the Boolean algebra of its clopen sets. Moreover, it sends a *continuous* function of Stone spaces $f : S_1 \to S_2$ to the *homomorphism* of Boolean algebras $Hom_{\mathcal{T}}(f,2)$, whose action is defined by postcomposition with f correspondingly.
Note that the *duality* between the category of Boolean algebras and the category of Stone spaces is based on a *dual* interpretation of the *two-element set* 2, or in other words; as the two-element trivial Boolean algebra, or again as a two-point Stone space. Essentially, the functorial process of *coordinatization* corresponding to a measurement problem of classical physics can be presented in the form of an information *encoding* functor $Hom_{\mathcal{T}}(-,2) : \mathcal{T} \to \mathcal{B}^{op}$, and is *resolved* by specifying the *inverse* information *decoding* functor $Hom_{\mathcal{B}}(-,2) : \mathcal{B}^{op} \to \mathcal{T}$ which establishes the *equivalence* of the geometric phenomenological category $\mathcal{T}$ with the dual of the algebraic category $\mathcal{B}$. Again from a physical viewpoint, the geometric category $\mathcal{T}$ is understood as a category of geometric spaces of physical states (*geometric state spaces*), whereas the algebraic category $\mathcal{B}$ providing an operational *solution* to the corresponding measurement problem is understood as a category of *algebras of observables*. Now, the *distinguished* two-element trivial Boolean algebra acts, from the perspective of the algebraic category, as an *evaluation* number-like object, that is, as a *binary* measurement device, for the evaluation of Boolean observables.
Concomitantly, the geometric state space *observed* by means of such a measurement procedure of observables is a Stone space, whose set of continuous functions into the Stone space 2 is identified as the Boolean algebra of its clopen sets. It is also important to notice that the above

duality *factors through* the associatively closed universe of *sets*, employed as a *scaffolding*. This is also clear from the dual interpretation that the two-element set 2 acquires by its conceptualization within the dual categories of Boolean algebras and Stone spaces. From this fact we can draw two conclusions: Firstly, the category of sets plays the role of *Aristotelian substance* in the categorical framework. The substance acquires *hypostatic* form only *in relation* to a specific category of algebraic objects or its dual category of geometric objects (if it exists) and most importantly, this process of form-acquisition is *functorial*. Then, its form is conceived in a dual sense, meaning that its algebraic and geometric manifestations are understood as *complementary* aspects related by means of the algebraic/geometric object duality. Secondly, the category of sets, understood as a category of Aristotelian substance, and due to its functionality for the enunciation of algebraic/geometric dualities, should play a fundamental role in the representable elaboration of more complicated functorial coordinatization processes. These *encoding* processes *may not* have an exact inverse *decoding* process, but still the latter can be approximated functorially as will be explained below.

The purpose of explaining in detail the case of *Stone duality*, being the *precursor* of other dualities, such as the case of *Gelfand duality*, or *Grothendieck duality*, has to do with the fact that it is also related to logic. Of course, from the perspective of *categorical logic*, algebra is connected with logic, in the sense that logical theories may be understood from the perspective of algebraic categories.

In particular, Boolean algebras correspond to *classical proposition* theories which describe the measurement procedures of classical physical theories. Thus, the category of Boolean algebras can be considered as the category of *logical* propositional theories of *classical* physics, whereas its dual category of Stone spaces can be seen as the category of *geometric* state-spaces *observed* by means of two-valued measurement procedures of *Boolean observables*. Hence, the set of geometric state models corresponding to a Boolean algebra in a classical measurement situation is obtained by taking *homomorphisms* from this algebra to the two-element Boolean algebra 2, that is the *functor*:

$$Hom_{\mathcal{B}}(-,2):\mathcal{B}^{op}\to\mathbf{Sets}$$

evaluated at B in $\mathcal{B}$, gives:

$$\mathbf{Models}_B \simeq Hom_{\mathcal{B}}(B,2)$$

Inversely, because of Stone duality, the definition of the appropriate *topology* on the set of geometric state models, gives a *topological* Stone space, by means of which we are able to retrieve the Boolean algebra B by taking morphisms in the category of Stone spaces from that space into the Stone space 2, that is:

$$B \cong Hom_{\mathcal{T}}(\mathbf{Models}_B, 2)$$

Again, it is instructive to observe the dual role that the two-element set 2 plays in the algebraic (logical)/geometric (phenomenological) Stone duality. By *analogia*, it is a natural question to ask if such a *dual* role can be played *not* from a *set*, like 2 in the case of Stone duality, but from a whole *category of sets* or even a *diagram in* $\mathbf{Sets}$.

For example, if we consider as a *dualizing object* the small category of sets $\mathbf{Sets}$, then a *logical theory* should correspond to a category $\mathfrak{L}$, whereas the category of logical theories would be an appropriate category of categories. Then, the category of $\mathbf{Sets}$-valued geometric state *models* of the logical theory $\mathfrak{L}$, would be a corresponding *category of functors* from $\mathfrak{L}$ to $\mathbf{Sets}$, that is:

$$\mathbf{Models}_{\mathfrak{L}} \simeq \mathbf{Functors}(\mathfrak{L}, \mathbf{Sets}) \simeq \mathbf{Sets}^{\mathfrak{L}}$$

where $\mathbf{Functors}(\mathfrak{L}, \mathbf{Sets})$ is the *category of functors* from $\mathfrak{L}$ to $\mathbf{Sets}$ and *natural transformations* between them. Then, the logical theory $\mathfrak{L}$ could be retrieved from the category of geometric state models $\mathbf{Models}_{\mathfrak{L}}$ as the category of suitable functors from $\mathbf{Models}_{\mathfrak{L}}$ to $\mathbf{Sets}$ preserving the required properties, so that:

$$\mathfrak{L} \simeq \mathbf{Functors}(\mathbf{Models}_{\mathfrak{L}}, \mathbf{Sets})$$

The exercise of treating a *diagram in* $\mathbf{Sets}$ as a dualizing object, and in particular, a diagram in the category of finite sets (being equivalent to the category of finite Boolean algebras) can be used as a natural starting point for the analysis of *quantum* logic from the perspective of *functorial duality*. In this way, the transition from classical measurement procedures to quantum measurement procedures can be understood operationally as a transition from a Boolean algebra to a *categorical diagram* of Boolean algebras. This physical viewpoint will be developed in detail as we progress.

The concept of algebraic/geometric duality also pertains to the notion of physical *coordinate systems*, or more generally, physical *reference frames*. More precisely a *global* coordinate system Σ can be understood as an *isomorphism* $\Sigma : M \to E$ between a *geometric object* M in some class and a *standard object* E in that class. A *local* coordinate system (such as the coordinate *charts* on a *manifold*) is an isomorphism between a local part of a geometric object in some class and a local part of a standard object in that class.

Thus, keeping in mind the distinction between local and global coordinate systems we may use as a standard object E a real *Euclidean* n-dimensional topological vector space. So if we identify the standard (local) geometric object as $E = \mathbb{R}^n$, then a (local) coordinate system is a (local) isomorphism $\Sigma : M \to \mathbb{R}^n$. We can think of this (local) isomorphism as related with an algebraic/geometric *duality*, where the algebraic object is identified as an algebra of observables (corresponding to a (local) measurement procedure) *evaluated* on a dualizing object, such that the (local) observable geometric state models of this algebra constitute the standard geometric object E.

In the particular case that $E = \mathbb{R}^n$, we can see immediately that $\mathbb{R}^n \cong Hom_{\mathcal{C}^\infty}(\mathcal{C}^\infty{}_{\mathbb{R}^n}, \mathbb{R})$. Thus, the *set* $\mathbb{R}$ plays the role of a *dualizing object*, interpreted algebraically as an $\mathbb{R}$-algebra (field), and geometrically as a Euclidean space. Hence, the set of geometric state models corresponding to the algebra of smooth functions $\mathcal{C}^\infty{}_{\mathbb{R}^n}$ in a classical smooth measurement situation is obtained by taking homomorphisms from this algebra to the field $\mathbb{R}$, that is:

$$\mathbf{Models}_{\mathcal{C}^\infty{}_{\mathbb{R}^n}} \simeq Hom_{\mathcal{C}^\infty}(\mathcal{C}^\infty{}_{\mathbb{R}^n}, R)$$

Inversely, the set of geometric state models viewed as the Euclidean space $E = \mathbb{R}^n$, allows us to retrieve $\mathcal{C}^\infty{}_{\mathbb{R}^n}$ by taking morphisms in the category of Euclidean spaces from that space into the Euclidean space $\mathbb{R}$, that is:

$$\mathcal{C}^\infty{}_{\mathbb{R}^n} \cong Hom_{\mathcal{E}}(\mathbf{Models}_{\mathcal{C}^\infty{}_{\mathbb{R}^n}}, \mathbb{R})$$

We have concluded above that the definition of a (local) coordinate system (for the example mentioned) is understood as a (local) isomorphism:

$$\Sigma : M \to \mathbf{Models}_{C^{\infty}_{\mathbb{R}^n}}$$

Thus, generally a (local) coordinate system identifies (in the sense of similarity induced by an isomorphism) a geometric object (locally) with a standard "number-like" object (like $\mathbb{R}^n$), which is understood as a (local) space of geometric state models observed by means of evaluating an algebra of observables on a dualizing object (like the real numbers $\mathbb{R}$) in the sense of functorial duality. Note that generally, the *identification* induced by a (local) coordinate system is *not natural.* This means that we may legitimately define a *whole class* of (local) coordinate systems depending on the operational means of observation. In this class, we may define an equivalence relation *partitioning* it into equivalence classes of (local) coordinate systems.

Thus, arises the necessity to adopt a *coordinate-invariant* point of view, meaning that every expression formed in the descriptive terms of a (local) coordinate system should remain *invariant* under a transformation of this coordinate system into another in the same equivalence class. Equally, given a (local) coordinate system $\Sigma : M \to \mathbf{Models}_A$, where A is an algebra of observables suited to a measurement procedure, and identifying by duality $Y \cong \mathbf{Models}_A$, then by an isomorphism $I : Y \to Y$ of the standard "number-like" object Y, we obtain a new (local) coordinate system of the geometric object M in the equivalence class of Y, that is, $I \circ \Sigma : M \to Y$ by *composition* of the two (local) isomorphisms. Inversely, every other (local) coordinate system of M, $\Theta : M \to Y$, is obtained in this way. Accordingly, we obtain a *group of isomorphisms* of Y, denoted as $G := Iso(Y)$, called the *gauge group* of the class of geometric objects.

Now, *instead* of taking a *single* geometric object, we may consider an *indexed family* of geometric objects. The indexing object, in the simplest case, is a set of parameters making a parameter space, or *base space*, over which the *variation* or *parametrization* of the family of geometric objects is conceived. This is equivalent to considering a *discrete diagram* of geometric objects in some appropriate category. Note that, the indexing object might be a category itself in the most general case. For example, if the indexing category is a set X which parameterizes geometric objects in the category of sets, then the *functor category* $\mathbf{Sets}^X$ is the category of X-indexed sets (category of *discrete diagrams of sets*). So the objects of $\mathbf{Sets}^X$ are X-indexed families of sets $(A_x)_{x \in X}$ and the morphisms are X-indexed families of functions:

$$(f_i : A_x \to B_x)_{x \in X} : (A_x)_{x \in X} \to (B_x)_{x \in X}$$

The functor category $\mathbf{Sets}^X$ can be described equivalently as the *"comma" , or slice category* of sets $\mathbf{Sets}$ over the indexing set X, denoted by $\mathbf{Sets}/X$. In this category, the objects are the *morphisms* $\alpha : A \to X$ and the arrows are the *commutative triangles* over the indexing set X. This is a very important example of an *equivalence* of categories, that is:

$$\mathbf{Sets}^X \simeq \mathbf{Sets}/X$$

The above functorial equivalence is described by means of the following inverse functors:

$$\mathbf{F} : \mathbf{Sets}^X \to \mathbf{Sets}/X$$
$$\mathbf{G} : \mathbf{Sets}/X \to \mathbf{Sets}^X$$

such that:

$$\mathbf{F}((A_x)_{x \in X}) = \pi : \sum_x A_x \to X$$

called the *indexing projection morphism*, and in the opposite direction:

$$\mathbf{G}(\alpha : A \to X) = (\alpha^{-1}\{x\})_{x \in X}$$

The advantage of the functorially equivalent slice category $\mathbf{Sets}/X$ where X is an indexing set (base space), is that it conforms with the intuition of a X *-indexed family* of objects of the category $\mathbf{Sets}$, or in short, an X-indexed family of sets (considered as geometric objects). Moreover, the slice category $\mathfrak{X}/X$ is meaningful for *any object* X in an *arbitrary* category (of geometric objects) $\mathfrak{X}$ conforming also with the intuition of an X-indexed family of objects of the category $\mathfrak{X}$. Now, if the category $\mathfrak{X}$ is *complete*, and in particular, it has *pullbacks*, then the process of *re-parameterization* along an arrow $h : X \to Y$ in $\mathfrak{X}$ is represented by the *pullback (limit) functor*:

$$h^* : \mathfrak{X}/Y \to \mathfrak{X}/X$$

In more familiar physical terms an X-indexed family of sets $(A_x)_{x \in X}$ (discrete diagram of sets) constitutes a *fiber bundle*, where the *fibers* A_x are sets and X is the *base* space of the bundle. Usually, the fibers have some *extra* structure, and topological or geometrical or differential *compatibility* conditions pertain between the fibers depending on the *corresponding* specification of the base space X. This is due to the natural requirement that the *kind* of variations in the *fiber* should conform to the *kind* of variations on the corresponding *base point.*

The crucial observation is that each fiber A_x in a fiber bundle of set-theoretic geometric objects, that is to say each object A_x in an X-indexed family of set-theoretic geometric objects $(A_x)_{x \in X}$, by virtue of belonging to a certain class of geometric objects, can be legitimately identified with a *standard "number-like"* object E in that class. This is possible by means of a parameterizing point x-based coordinate system, expressed in terms of a *point x-based* isomorphism $\Sigma_x : A_x \to E$.

In this way, we obtain a separate coordinate system for each base point x in the base space X. The *complete family* $(\Sigma_x)_{x \in X}$ (conforming with the corresponding topological, geometrical or differential compatibility conditions), if it exists, constitutes a *global gauge*, or global trivialization, for the fiber bundle $(A_x)_{x \in X}$. Correspondingly, a *local gauge* refers to a local trivialization for the fiber bundle $(A_x)_{x \in X}$ obtained in the same manner. Usually, a global gauge is *not* possible or available, but local gauges do exist, *trivializing* only *parts* of a fiber bundle in a consistent way. Note that the *trivial* bundle has the *same* fiber E for *each* point $x \in X$, denoted as $(E)_{x \in X}$. Then, a function $l : X \to E$ has a *graph* $\{(x, l(x)) : x \in X\} \subset X \times E$. The analogous notion to the graph of a function $l : X \to E$ referring to the trivial bundle $(E)_{x \in X}$ is the notion of *section* referring to an arbitrary fiber bundle.

In this sense, a *(local) section* of a fiber bundle may be thought of as a *collection* of *representative* elements $s = (s_x)_{x \in X}$ from the fibers of the fiber bundle $(A_x)_{x \in X}$. It is precisely *sections* of a fiber bundle that can be represented locally or globally in terms of "number-like" quantities by means of a local or global gauge correspondingly. Thus, a (local) gauge is a (local) fiber bundle *isomorphism* Σ from the bundle $(A_x)_{x \in X}$ to the trivial bundle $(E)_{x\in X}$. Now, analogously to the case of (local) coordinate systems of single geometric objects, every family of (local) isomorphisms $(I_x)_{x\in X}$ of $(E)_{x\in X}$, provides a new (local) gauge $(I_x \circ \Sigma_x)_{x\in X}$, thus transforming the (local) gauge $(\Sigma_x)_{x\in X}$ into the (local) gauge $(I_x \circ \Sigma_x)_{x\in X}$. In this case we say that $(I_x)_{x\in X}$ constitutes a *(local) gauge transformation*.

We may conclude the above discussion by noticing that the notion of a *gauge* corresponds to the notion of a *varying* coordinate system. In the example of fiber bundles presented above, the variation is *over* a parameter space (base space), such that the variation of coordinate system is dependent on the *location* of an observer occupying a point x of the base space. In the context of the same example, a gauge transformation is a *change* of coordinate system applied to *each* such location x of the base space X.

From a categorical perspective, this is an example of a *gauge theory* formulated within the category of *discrete diagrams of sets*, namely the *functor category* $\mathbf{Sets}^X$, or equivalently, the category of X*-indexed sets* $\mathbf{Sets}/X$ (or other set-theoretic geometric objects). The idea of gauge should *not* depend on the *indexing* object used for its manifestation, meaning that the indexing object might be a category itself. As an example, we may think of a *partially ordered* set, considered as a *category* with arrows being *inclusions*, corresponding to the partial order of the *open* sets on a topological space. Then, it becomes possible to formulate a gauge theory within the category of diagrams of sets indexed (or localized) by the category of open sets on a topological space. We will see later that this *functor category* is technically called the category of *(pre)sheaves (of sets)* on a topological space X, denoted by $Sets^{\mathcal{O}(X)^{op}}$.

The analogy with the previous case is going to become evident after we prove that we may formulate the theory *equivalently* within the category of (étale) topological bundles, which is the category of $\mathcal{O}(X)$-indexed topological spaces denoted by $\mathbf{Top}/X$. Therefore, when topology is involved it is appropriate to formulate the notion of a gauge theory within the category $Sets^{\mathcal{O}(X)^{op}}$, where $\mathcal{O}(X)$ is the *indexing (localizing)* category of open sets in the topology of X.

Hence, in the general case, where the *indexing* or *localization* is defined by a *category*, the notion of a gauge is that of a coordinate system (or more generally a reference frame) that varies depending on the *base object* of the indexing category over which it is defined. Then, a gauge transformation is a *change* of reference frame applied to *each* such base object of the indexing category, and a gauge theory is a theory describing the behavior of a physical system to which such gauge transformations can be applied. Concomitantly, the physical requirement of *gauge invariance* is a statement of the fact that all physical observable attributes should transform *naturally* (or remain invariant) under gauge transformations.

In order to understand the notion of a gauge within a general category of the form $Sets^{\mathfrak{A}^{op}}$, where $\mathfrak{A}$ is an indexing or localizing category, it is instructive to recall that the notion of a (local) gauge (coordinate system) for a set-theoretic geometric object in a class is understood as a (local) isomorphism with a standard geometric object in the same class:

$$\Sigma : G \to \mathbf{Models}_{\mathcal{A}}$$

where the set of geometric state models (geometric state-space representing the standard geometric object) corresponding to a coordinatizing algebra of observables in a measurement procedure, is obtained by taking homomorphisms from this algebra to some appropriate *dualizing* object in the same category $\mathfrak{A}$, denoted by $\mathcal{D}$ (like the real numbers $\mathbb{R}$), that is:

$$\mathbf{Models}_{\mathcal{A}} \simeq Hom_{\mathfrak{A}}(\mathcal{A}, \mathcal{D})$$

Thus, generally a (local) coordinate system identifies (in the sense of similarity induced by an isomorphism) a geometric object (locally) with a standard geometric object, which is understood as a (local) space of geometric state models observed by means of evaluating an algebra of observables on a dualizing object in the sense of *functorial duality*. Now, a natural question arising by *analogia* is the following: If the notion of a geometric object G in a class, is *substituted* by the notion of a *contravariant functor* $\mathbf{G} : \mathfrak{A}^{op} \to Sets$ in a functor category $Sets^{\mathfrak{A}^{op}}$, or by the notion of a *covariant functor* $\mathbf{G} : \mathfrak{A} \to Sets$ in a functor category $Sets^{\mathfrak{A}}$, where $\mathfrak{A}$ is an appropriate category of algebras of observables, then can we define a corresponding notion of a *functorial gauge* for the functor $\mathbf{G}$?

Intuitively, the notion of a functorial (local) gauge for $\mathbf{G}$ would be a (local) *natural isomorphism* with a *standard object* in the same category. Again, this standard functor would correspond to a functorial geometric state space, obtained as a category of functorial state models corresponding to a coordinatizing algebras of observables $\mathcal{A}$, that is:

$$\mathbf{FModels}_{\mathcal{A}} \simeq \mathbf{Hom}_{\mathfrak{A}}(\mathcal{A}, -)$$

or equivalently, in the dual formulation:

$$\mathbf{FModels}_{\mathcal{A}} \simeq \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A})$$

where $(-)$ is a varying algebraic object in $\mathfrak{A}$ (to be thought of as a varying dualizing object). Thus, intuitively a functorial (local) gauge of $\mathbf{G} : \mathfrak{A}^{op} \to Sets$ should be expressed as a (local) *natural isomorphism* (invertible natural transformation) of $Sets$-valued functors:

$$\mathbf{G} \cong \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A})$$

when $\mathbf{G}$ is a contravariant $Sets$-valued functor, and dually:

$$\mathbf{G} \cong \mathbf{Hom}_{\mathfrak{A}}(\mathcal{A}, -)$$

when $\mathbf{G}$ is a covariant $Sets$-valued functor. One might ask, why we have restricted our attention to the case of $Sets$-valued functors. The reason is that the case of $Sets$-valued functors is enough in order to perform the corresponding gauge formulation in the general case of functors valued in an arbitrary category, using the notion of parameterized functorial gauge. Conceptually, we have already seen the role that the category of $Sets$ plays in the case of functorial duality to enable factorization of duality through the scaffolding of the category of $Sets$, interpreted as a category of Aristotelian substance.

8.7 ICONICITY: GAUGE REPRESENTABILITY OF FUNCTORS

Before expanding on the notion of a functorial gauge for a functor of the form $\mathbf{G} : \mathfrak{A}^{op} \to Sets$ it is essential to provide a brief overview of the functor category $Sets^{\mathfrak{A}^{op}}$, where $\mathfrak{A}$ is an indexing or localizing category. The functor category $Sets^{\mathfrak{A}^{op}}$ may be thought of as the *category of diagrams* on the indexing category $\mathfrak{A}$. Technically, it is called the *category of presheaves* on $\mathfrak{A}$.

For a category $\mathfrak{A}$ the functor category of presheaves (of sets) $Sets^{\mathfrak{A}^{op}}$ is the category of all *contravariant functors* from $\mathfrak{A}$ to $Sets$ and all *natural transformations* between them. A functor $\mathbf{P}$ is a *structure-preserving* morphism of these categories, that is, it preserves composition and identities. A presheaf functor in the category $Sets^{\mathfrak{A}^{op}}$ can be thought of as constructing an *image* of $\mathfrak{A}$ in $Sets$ contravariantly, or as a contravariant *translation* of the language of $\mathfrak{A}$ into that of $Sets$.

Given another such translation (contravariant functor) $\mathbf{Q}$ of $\mathfrak{A}$ into $Sets$ we need to compare them. This can be done by giving, for each object $\mathcal{A}$ in $\mathfrak{A}$ a transformation $T_{\mathcal{A}} : \mathbf{P}(\mathcal{A}) \to \mathbf{Q}(\mathcal{A})$ which *compares* the two *images* of the object $\mathcal{A}$. Not any morphism will do, however, as we would like the construction to be *parametric* in $\mathcal{A}$, rather than *ad hoc*. Since $\mathcal{A}$ is an object in $\mathfrak{A}$ while $\mathbf{P}(\mathcal{A})$ is in $Sets$ we *cannot* link them by a morphism. Rather, the goal is that the transformation should *respect* the morphisms of $\mathfrak{A}$, or in other words, the interpretations of $v : \mathcal{C} \to \mathcal{A}$ by $\mathbf{P}$ and $\mathbf{Q}$ should be *compatible* with the transformation under T. Thus, T is a *natural transformation* in the category of presheaves $Sets^{\mathfrak{A}^{op}}$.

An object $\mathbf{P}$ of $Sets^{\mathfrak{A}^{op}}$ may be understood as a *left categorical action* of $\mathfrak{A}^{op}$ on a set which is *partitioned* into sorts *parameterized* by the objects of $\mathfrak{A}$, and such that whenever $v : \mathcal{C} \to \mathcal{A}$ is an arrow and p is an element of $\mathbf{P}$ of sort $\mathcal{A}$, then the pullback of p along $v : \mathcal{C} \to \mathcal{A}$, denoted by $p \cdot v$, is specified as an *element* of $\mathbf{P}$ of *sort* $\mathcal{C}$. Such a left action $\mathbf{P}$ is referred as a $\mathfrak{A}^{op}$*-variable set*.

The *category of elements* of a presheaf functor $\mathbf{P}$ (or *categorical diagram* of a presheaf functor $\mathbf{P}$), denoted by $\int(\mathbf{P}, \mathfrak{A})$, is defined as follows: Its objects are all *pairs* $(\mathcal{A}, p)$, and its morphisms $(\mathcal{A}', p') \to (\mathcal{A}, p)$ are *those* morphisms $u : \mathcal{A}' \to \mathcal{A}$ of $\mathfrak{A}$ for which $p \cdot u = p'$. *Projection* on the second coordinate of $\int(\mathbf{P}, \mathfrak{A})$, defines a *functor*:

$$\int_{\mathbf{P}} : \int(\mathbf{P}, \mathfrak{A}) \to \mathfrak{A}$$

$\int(\mathbf{P}, \mathfrak{A})$ together with the projection functor $\int_{\mathbf{P}}$ is called the *split discrete fibration* induced by $\mathbf{P}$, and $\mathfrak{A}$ is the *base indexing* or *localizing* category of the fibration. We note that the *fibers* are *categories* in which the only arrows are *identity* arrows. If $\mathcal{A}$ is an object of $\mathfrak{A}$, the inverse image under $\int_{\mathbf{P}}$ of $\mathcal{A}$ is simply the set $\mathbf{P}(\mathcal{A})$, although its elements are written as pairs so as to form a *disjoint union*.

Similarly, for a category $\mathfrak{A}$ the functor category of *copresheaves* (of sets) $Sets^{\mathfrak{A}}$ is defined as the category of all *covariant functors* from $\mathfrak{A}$ to $Sets$ and all *natural transformations* between them. Analogously, a copresheaf $\mathbf{P}$ may be understood as a *right action* of $\mathfrak{A}$ on a set which is partitioned into sorts parametrized by the objects of $\mathfrak{A}$ and such that whenever $v : \mathcal{C} \to \mathcal{A}$ is an arrow and p is an element of $\mathbf{P}$ of sort $\mathcal{C}$, then $v \cdot p$ is specified as an element of $\mathbf{P}$ of sort $\mathcal{A}$. Such a right action $\mathbf{P}$ is referred as a $\mathfrak{A}$-variable set. The category of elements of a copresheaf functor $\mathbf{P}$ (or diagram of a copresheaf functor $\mathbf{P}$), denoted by $\int(\mathbf{P}, \mathfrak{A})$ is defined as follows: Its objects are all pairs $(\mathcal{A}, p)$, and its morphisms $(\mathcal{A}', p') \to (\mathcal{A}, p)$ are those morphisms $u : \mathcal{A}' \to \mathcal{A}$ of $\mathfrak{A}$ for which $u \cdot p' = p$.

Notice that in both the functor categories of presheaves and copresheaves of sets standard objects are present. These are correspondingly the following:

i The contravariant $Hom_{\mathfrak{A}}$-functor represented by $\mathcal{A}$, that is:

$$\mathbf{y}^{\mathcal{A}} = \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A}) : \mathfrak{A}^{op} \to \mathbf{Sets}$$

called the *functor of generalized elements* (incoming morphisms) of $\mathcal{A}$, which classifies completely the information contained in $\mathcal{A}$.

ii The covariant $Hom_{\mathfrak{A}}$-functor represented by $\mathcal{A}$, that is:

$$\mathbf{y}_{\mathcal{A}} = \mathbf{Hom}_{\mathfrak{A}}(\mathcal{A}, -) : \mathfrak{A} \to \mathbf{Sets}$$

called the *functor of generalized co-elements* (outgoing morphisms) of $\mathcal{A}$. Similarly, the information contained in $\mathcal{A}$ is classified completely by its functor of generalized co-elements $\mathbf{y}_{\mathcal{A}}$.

In this way, we obtain *simultaneously both* a contravariant functor $\mathbf{y}^{(-)}$ and a covariant functor $\mathbf{y}_{(-)}$ valued on the category of copresheaves and presheaves (of sets) correspondingly as follows:

$$\mathbf{y}^{(-)} : \mathfrak{A}^{op} \to Sets^{\mathfrak{A}}$$
$$\mathbf{y}_{(-)} : \mathfrak{A} \to Sets^{\mathfrak{A}^{op}}$$

which constitute the *exponential transposes* of the $Hom_{\mathfrak{A}}$*-bifunctor*:

$$\mathbf{Hom}_{\mathfrak{A}} : \mathfrak{A}^{op} \times \mathfrak{A} \to \mathbf{Sets}$$

from the product category $\mathfrak{A}^{op} \times \mathfrak{A}$ to the category of $Sets$ with respect to each of its arguments.

The *covariant functor* $\mathbf{y}_{(-)} : \mathfrak{A} \to Sets^{\mathfrak{A}^{op}}$ realizes the *Nobuo Yoneda embedding* of the category $\mathfrak{A}$ into the category of presheaves $Sets^{\mathfrak{A}^{op}}$. The term embedding means that the functor $\mathbf{y}_{(-)}$ is *bijective* when restricted to each set of morphisms with a given domain and codomain, and that it is *injective on objects* as well. The Yoneda embedding constitutes a *representation* of the category $\mathfrak{A}$ within the functor category of presheaves (of sets) $Sets^{\mathfrak{A}^{op}}$. In detail we have:

$$\mathbf{y}_{(-)}(\mathcal{A}) = \mathbf{y}^{\mathcal{A}} = \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A}) : \mathfrak{A}^{op} \to \mathbf{Sets}$$
$$\mathbf{y}_{(-)}(f) = \mathbf{y}^{f} : \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A}) \to \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{C})$$

where $f : \mathcal{A} \to \mathcal{C}$ is a morphism in $\mathfrak{A}$.

The importance of the Yoneda embedding is manifested by the so called *Yoneda lemma*, according to which, for *any* object $\mathcal{A}$ in a (locally) small category $\mathfrak{A}$ and *any* presheaf functor $\mathbf{P}$ in $Sets^{\mathfrak{A}^{op}}$ there exists an *isomorphism*:

$$\mathbf{Hom}_{Sets^{\mathfrak{A}^{op}}}(\mathbf{y}^{\mathcal{A}}, \mathbf{P}) \cong \mathbf{P}(\mathcal{A})$$

written equivalently as:

$$\mathbf{Nat}(\mathbf{y}^{\mathcal{A}}, \mathbf{P}) \cong \mathbf{P}(\mathcal{A})$$

which is *natural* in both $\mathbf{P}$ and $\mathcal{A}$.

The main application of the Yoneda lemma is the following: Given objects $\mathcal{A}$ and $\mathcal{B}$ in a (locally) small category $\mathfrak{A}$, if $\mathbf{y}_{(-)}(\mathcal{A}) \cong \mathbf{y}_{(-)}(\mathcal{B})$, or equivalently, $\mathbf{y}^{\mathcal{A}} \cong \mathbf{y}^{\mathcal{B}}$, then $\mathcal{A} \cong \mathcal{B}$.

The advantage of working within the category of presheaves $Sets^{\mathfrak{A}^{op}}$ instead of the category $\mathfrak{A}$ is that it is both *complete* (has all small limits) and *cocomplete* (has all small colimits), and most importantly, this is so because the category of $Sets$ is both complete and cocomplete. Moreover, for every object $\mathcal{A}$ of $\mathfrak{A}$ the evaluation functor at $\mathcal{A}$, that is:

$$\mathbf{ev}_{\mathcal{A}} : Sets^{\mathfrak{A}^{op}} \rightarrow Sets$$

preserves both all limits and colimits.

Let us now return to the problem of understanding the notion of a functorial gauge (functorial reference frame) for a functor $\mathbf{G} : \mathfrak{A}^{op} \rightarrow Sets$, where $\mathfrak{A}$ is considered to be an appropriate category of coordinatizing algebras of observables. We have argued previously that a *functorial gauge* of $\mathbf{G} : \mathfrak{A}^{op} \rightarrow Sets$ should be expressed as a *natural isomorphism* (invertible natural transformation) of $Sets$ -valued functors:

$$\mathbf{G} \cong \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A})$$

where, the standard functor $\mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A})$ in the category of presheaves $Sets^{\mathfrak{A}^{op}}$ is physically equivalent to the functorial geometric state models of an algebra of observables $\mathcal{A}$, that is:

$$\mathbf{FModels}_{\mathcal{A}} \simeq \mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A})$$

In categorical terminology, we find the notion of *representability* of a (contravariant or covariant) $Sets$ -valued functor, defined as follows: A *representation* of a contravariant $Sets$ -valued functor of the form $\mathbf{G} : \mathfrak{A}^{op} \rightarrow Sets$, where $\mathfrak{A}$ is a (locally) small category, consists of an *object* $\mathcal{A}$ in $\mathfrak{A}$ and a *natural isomorphism*:

$$\mathbf{Hom}_{\mathfrak{A}^{op}}(-, \mathcal{A}) \cong \mathbf{G}$$

or equivalently:

$$\mathbf{y}^{\mathcal{A}} \cong \mathbf{G}$$

where $\mathcal{A}$ is called the *representing object* of the functor $\mathbf{G}:\mathfrak{A}^{op} \rightarrow Sets$. Thus, $\mathbf{G}$ is representable, *if and only if* such a representing object exists. Note that, representations of contravariant $Sets$-valued functors are *unique* up to a unique isomorphism. Evidently, a dual formulation obtains, referring to a representation of a *covariant* $Sets$-valued functor of the form $\mathbf{G}:\mathfrak{A} \rightarrow Sets$.
To be specific, the covariant functor $\mathbf{G}$ is *representable* if and only if there exists an object $\mathcal{A}$ in $\mathfrak{A}$ and a natural isomorphism:

$$\mathbf{Hom}_{\mathfrak{A}}(\mathcal{A},-) \cong \mathbf{G}$$

or equivalently:

$$\mathbf{y}_{\mathcal{A}} \cong \mathbf{G}$$

It follows immediately from the above that the *categorical* notion of *representability* of a $Sets$-valued *functor* is selfsame as the *physical* notion of a *functorial gauge* (functorial reference frame) of a $Sets$-valued functor in the intended interpretation.

Furthermore, if we use the Yoneda lemma, we have:

$$\mathbf{Nat}(\mathbf{y}^{\mathcal{A}},\mathbf{G}) \cong \mathbf{G}(\mathcal{A})$$

in the *contravariant* case, which is *natural* in both $\mathbf{G}$ and $\mathcal{A}$, and dually:

$$\mathbf{Nat}(\mathbf{y}_{\mathcal{A}},\mathbf{G}) \cong \mathbf{G}(\mathcal{A})$$

in the *covariant* case, which is *natural* in both $\mathbf{G}$ and $\mathcal{A}$.

If we consider, for instance, the *covariant* case in more detail, then if Υ is a natural transformation in $\mathbf{Nat}(\mathbf{y}_{\mathcal{A}},\mathbf{G})$, that is:

$$\Upsilon:\mathbf{y}_{\mathcal{A}} \rightarrow \mathbf{G}$$

the element y in $\mathbf{G}(\mathcal{A})$ corresponding to Υ is defined by:

$$y = \Upsilon_{\mathcal{A}}(id_{\mathcal{A}})$$

In the inverse direction, given any element y in $\mathbf{G}(\mathcal{A})$ we define a natural transformation $\Upsilon^y : \mathbf{y}_\mathcal{A} \to \mathbf{G}$ as follows:

$$\Upsilon^y{}_\mathcal{D}(h) = (\mathbf{G}(h))(y) = h \cdot y$$

where h is a generalized co-element of $\mathcal{A}$, which means h belongs to $Hom_\mathfrak{A}(\mathcal{A}, D)$. Thus, as a consequence of the Yoneda lemma, given any element y in $\mathbf{G}(\mathcal{A})$ we need to know when the induced natural transformation $\Upsilon^y : \mathbf{y}_\mathcal{A} \to \mathbf{G}$ is a *natural isomorphism*, because in this case the covariant $Sets$-valued functor $\mathbf{G}$ is representable, or equivalently, there exists a functorial gauge for $\mathbf{G}$.

This is actually the case if and only if there exists a *pair* $(\mathcal{A}, y)$ in $\int(\mathbf{P}, \mathfrak{A})$, where $\mathcal{A}$ in $\mathfrak{A}$ and y in $\mathbf{G}(\mathcal{A})$, such that, for every pair $(\mathcal{D}, z)$ with z in $\mathbf{G}(\mathcal{D})$, a unique morphism $h : \mathcal{A} \to \mathcal{D}$ pertains, such that $(\mathbf{G}(h))(y) = h \cdot y = z$. This is technically called a *universal element* of $\mathbf{G}$ and is interpreted as a *universal gauge* of $\mathbf{G}$. A universal gauge $(\mathcal{A}, y)$ of $\mathbf{G}$ actually acts as an *initial object* in $\int(\mathbf{G}, \mathfrak{A})$. Thus, we conclude that given any element y in $\mathbf{G}(\mathcal{A})$ the induced natural transformation $\Upsilon^y : \mathbf{y}_\mathcal{A} \to \mathbf{G}$ is a natural isomorphism, if and only if $(\mathcal{A}, y)$ is a *universal gauge* of $\mathbf{G}$. Hence, representations, or functorial gauges (functorial reference frames) of $\mathbf{G}$, that is, natural isomorphisms $\mathbf{y}_\mathcal{A} \cong \mathbf{G}$, are in *bijective* correspondence with universal gauges $(\mathcal{A}, y)$ of $\mathbf{G}$.

Precisely analogous conclusions hold in the dual case of a *contravariant* $Sets$-valued functor $\mathbf{G}$, that is, representations or functorial gauges (functorial reference frames) of $\mathbf{G}$, or natural isomorphisms $\mathbf{y}^\mathcal{A} \cong \mathbf{G}$, are in *bijective* correspondence with universal gauges $(\mathcal{A}, y)$ of $\mathbf{G}$, where a universal gauge $(\mathcal{A}, y)$ of $\mathbf{G}$ acts now as a *terminal object* in $\int(\mathbf{G}, \mathfrak{A})$. Note that in the intended physical interpretation, a universal gauge of $\mathbf{G}$ is provided by a pair $(\mathcal{A}, y)$, where $\mathcal{A}$ is a coordinatizing algebra in $\mathfrak{A}$, and y in $\mathbf{G}(\mathcal{A})$, which is acting as an initial (in the covariant case) or as a terminal (in the contravariant case) object in the corresponding category of elements $\int(\mathbf{G}, \mathfrak{A})$.

We have seen previously that the problem of finding a *functorial gauge* for (a covariant or contravariant) *Sets* -valued functor $\mathbf{G}$ is *resolved* by finding a corresponding *universal gauge* of $\mathbf{G}$. This in turn prompts the natural next step to find a concrete methodology with a sound physical interpretation by means of which we can define a functorial gauge for an *arbitrary* covariant functor $\mathbf{F}:\mathfrak{C}\to\mathfrak{L}$ or contravariant functor $\mathbf{H}:\mathfrak{C}^{op}\to\mathfrak{L}$.

A starting point to resolve this problem focusses on the observation that given any category $\mathfrak{A}$, the morphisms in $\mathfrak{A}$, or equivalently the morphisms in $\mathfrak{A}^{op}$, can be defined as the *elements* in the values of the $Hom_{\mathfrak{A}}$*-bifunctor*:

$$\mathbf{Hom}_{\mathfrak{A}}=\mathbf{y}(-,-)=\mathbf{y}^{(\mathcal{A})}_{(\mathcal{B})}=\mathfrak{A}^{op}\times\mathfrak{A}\to\mathbf{Sets}$$

from the product category $\mathfrak{A}^{op}\times\mathfrak{A}$ to the category of sets. This is because $\mathfrak{A}$ and $\mathfrak{A}^{op}$ are considered to have the same objects and reversed morphisms. The $Hom_{\mathfrak{A}}$-bifunctor operates in such a way that by *fixing* an object $\mathcal{B}$ in $\mathfrak{A}^{op}$:

$$\mathbf{y}_{\mathcal{B}}:\mathfrak{A}\to\mathbf{Sets}$$

is the *covariant representable* functor, or representable copresheaf of sets, represented by $\mathcal{B}$. Correspondingly by fixing an object $\mathcal{A}$ in $\mathfrak{A}$:

$$\mathbf{y}^{\mathcal{A}}:\mathfrak{A}^{op}\to\mathbf{Sets}$$

is the *contravariant representable* functor, or representable presheaf of sets, represented by $\mathcal{A}$. It is also instructive to keep in mind that $\mathbf{y}^{\mathcal{A}}$ defines a *representable left action* of $\mathfrak{A}^{op}$ on a set which is partitioned into sorts parameterized by the objects of $\mathfrak{A}^{op}$, whereas $\mathbf{y}_{\mathcal{B}}$ defines a *representable left action* of $\mathfrak{A}$ on a set which is partitioned into sorts parameterized by the objects of $\mathfrak{A}$.

Now we may consider oppositely directing functors:

$$\overrightarrow{\mathbf{T}}:\mathfrak{A}^{op}\to\mathfrak{A}$$
$$\overleftarrow{\mathbf{T}}:\mathfrak{A}\to\mathfrak{A}^{op}$$

which are *tautological* on objects and arrows-*reversing*. This being so, it is possible to rephrase the representability property of the $Hom_{\mathfrak{A}}$-bifunctor with respect to *fixing* one of its arguments as follows:

i The bifunctor $\mathbf{y}(-,-)$ is *representable* in $\mathfrak{A}$ because for each $\mathfrak{A}^{op}$-object $\mathcal{B}$ there exists a (tautological) $\mathfrak{A}$-object $\overrightarrow{\mathbf{T}}(\mathcal{B})$, such that $\mathbf{y}(\mathcal{B},-)$ is representable in $\mathfrak{A}$, that is:

$$\mathbf{Hom}_{\mathfrak{A}}(\overrightarrow{\mathbf{T}}(\mathcal{B}),\mathcal{A}) \cong \mathbf{y}(\mathcal{B},\mathcal{A})$$

which is *natural* in the argument $\mathcal{A}$ in $\mathfrak{A}$.

ii The bifunctor $\mathbf{y}(-,-)$ is *representable* in $\mathfrak{A}^{op}$ because for each $\mathfrak{A}$-object $\mathcal{A}$ there exists a (tautological) $\mathfrak{A}^{op}$-object $\overleftarrow{\mathbf{T}}(\mathcal{A})$, such that $\mathbf{y}(-,\mathcal{A})$ is representable in $\mathfrak{A}^{op}$, that is:

$$\mathbf{Hom}_{\mathfrak{A}^{op}}(\mathcal{B},\overleftarrow{\mathbf{T}}(\mathcal{A})) \cong \mathbf{y}(\mathcal{B},\mathcal{A})$$

which is *natural* in the argument $\mathcal{B}$ in $\mathfrak{A}^{op}$.

Thus, by combining the above, the representability of the bifunctor $\mathbf{y}(-,-)$ with respect to the categories $\mathfrak{A}$ and $\mathfrak{A}^{op}$, or equivalently the *birepresentability* of the bifunctor $\mathbf{y}(-,-)$ with respect to $\mathfrak{A}$ and $\mathfrak{A}^{op}$ is expressed as follows:

$$\mathbf{Hom}_{\mathfrak{A}}(\overrightarrow{\mathbf{T}}(\mathcal{B}),\mathcal{A}) \cong \mathbf{y}(\mathcal{B},\mathcal{A}) \cong \mathbf{Hom}_{\mathfrak{A}^{op}}(\mathcal{B},\overleftarrow{\mathbf{T}}(\mathcal{A}))$$

which is *natural* in both the arguments $\mathcal{B}$ in $\mathfrak{A}^{op}$ and $\mathcal{A}$ in $\mathfrak{A}$.

Although the above is based on tautological functors, it contains the seeds for a powerful generalization in the case of general functors between categories pointing in opposite directions. This generalization is exactly the crucial step needed in order to tackle the problem of defining a functorial gauge (functorial reference frame) for an arbitrary covariant functor $\mathbf{F}:\mathfrak{C} \to \mathfrak{L}$ or contravariant functor $\mathbf{H}:\mathfrak{C}^{op} \to \mathfrak{L}$. Since the distinction between the two cases can be handled trivially by means of tautological functors, from here on we formulate the solution to the problem posed above by referring to an arbitrary functor $\mathbf{F}:\mathfrak{C}^{op} \to \mathfrak{L}$.

Before proceeding it is instructive to observe that in the above series of *natural isomorphisms*, the first term refers to morphisms taken in $\mathfrak{A}$ and the last term refers to morphisms taken in $\mathfrak{A}^{op}$. Now, the middle term can refer either to morphisms in $\mathfrak{A}$ or to morphisms in $\mathfrak{A}^{op}$ because of the tautological nature of the objects in opposite categories. Beyond this, if we temporarily overlook this tautology, it is particularly interesting to consider an element of $\mathbf{y}(\mathcal{B},\mathcal{A})$ as a *signal-transmitter* from $\mathcal{B}$ in $\mathfrak{A}^{op}$ to $\mathcal{A}$ in $\mathfrak{A}$, or equivalently, as a *signal-receiver* in the opposite direction.

Note that, if we overlook again the object tautology of these categories, a signal-transmitter/receiver is *not* a morphism within any of the categories $\mathfrak{A}^{op}$, $\mathfrak{A}$, since it transfers a signal from some object in $\mathfrak{A}^{op}$ to some object in $\mathfrak{A}$. Notwithstanding this fact, a signal-transmitter/receiver between objects in *different* categories can be made into a morphism in a new category, namely in the product category $\mathfrak{A}^{op} \times \mathfrak{A}$ by considering the embeddings $\mathfrak{A}^{op} \infty \mathfrak{A}^{op} \times \mathfrak{A}$ and $\mathfrak{A} \infty \mathfrak{A}^{op} \times \mathfrak{A}$.

The important fact is that the set of signal-transmitter/receivers for any objects $\mathcal{B}$ in $\mathfrak{A}^{op}$ and $\mathcal{A}$ in $\mathfrak{A}$ can be represented as *morphisms within both* the categories in question, $\mathfrak{A}$ and $\mathfrak{A}^{op}$, using the oppositely directing functors $\vec{\mathrm{T}}$, $\overleftarrow{\mathrm{T}}$, by means of the *first* and the *last* term in the series of *isomorphisms*. Moreover, since these isomorphisms are *natural*, the translation process effectuating representability in both directions becomes possible because there exist *functorial gauges* for both $\mathbf{y}_{\mathcal{B}}$ and $\mathbf{y}^{\mathcal{A}}$, or equivalently, *universal gauges* $(\mathcal{B}, id_{\mathcal{B}})$, $(\mathcal{A}, id_{\mathcal{A}})$ of $\mathbf{y}_{\mathcal{B}}$, $\mathbf{y}^{\mathcal{A}}$ correspondingly.

By analogia, it is almost straightforward to proceed in the general case of a bifunctor:

$$\wedge(-,-) : \mathfrak{C}^{op} \times \mathfrak{L} \to \mathbf{Sets}$$

from the product category $\mathfrak{C}^{op} \times \mathfrak{L}$ to the category of sets. The elements in the values of $\wedge(-,-)$ are considered as *signal-transmitter/receivers* between objects of $\mathfrak{C}^{op}$, $\mathfrak{L}$ correspondingly. It is important to notice the directionality build into the process of signal transfer, by which we mean that it is understood as a *transfer* from an object of $\mathfrak{C}^{op}$ to an object of $\mathfrak{L}$. Again, a signal-transmitter/receiver between objects of $\mathfrak{C}^{op}$, $\mathfrak{L}$ correspondingly, can be made into a morphism in a new category, namely in the product category $\mathfrak{C}^{op} \times \mathfrak{L}$ by considering the embeddings $\mathfrak{C}^{op} \infty \mathfrak{C}^{op} \times \mathfrak{L}$ and $\mathfrak{L} \infty \mathfrak{C}^{op} \times \mathfrak{L}$.

We say that the bifunctor: $\wedge(-,-):\mathfrak{C}^{op}\times\mathfrak{L}\to\mathbf{Sets}$ is *birepresentable*, i.e. it is representable within both the categories $\mathfrak{C}^{op}$, $\mathfrak{L}$, if and only if there exist two functors pointing into opposite directions, that is:

$$\overrightarrow{\mathbf{A}}:\mathfrak{C}^{op}\to\mathfrak{L}$$
$$\overleftarrow{\mathbf{A}}:\mathfrak{L}\to\mathfrak{C}^{op}$$

and a series of isomorphisms:

$$\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(\mathcal{C}),\mathcal{L})\cong\wedge(\mathcal{C},\mathcal{L})\cong\mathbf{Hom}_{\mathfrak{C}^{op}}(\mathcal{C},\overleftarrow{\mathbf{A}}(\mathcal{L}))$$

which is *natural* in both the arguments $\mathcal{C}$ in $\mathfrak{C}^{op}$ and $\mathcal{L}$ in $\mathfrak{L}$.

Equivalently, we state that the bifunctor: $\wedge(-,-):\mathfrak{C}^{op}\times\mathfrak{L}\to\mathbf{Sets}$ is birepresentable, if and only if there exist two functors $\overrightarrow{\mathbf{A}}:\mathfrak{C}^{op}\to\mathfrak{L}$, $\overleftarrow{\mathbf{A}}:\mathfrak{L}\to\mathfrak{C}^{op}$ together with *natural isomorphisms* of bifunctors:

$$\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(-),-)\cong\wedge(-,-)\cong\mathbf{Hom}_{\mathfrak{C}^{op}}(-,\overleftarrow{\mathbf{A}}(-))$$

In this case, we say that the two oppositely pointing functors $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ form a *categorical adjunction*, which is induced by the requirement of birepresentability of the bifunctor: $\wedge(-,-):\mathfrak{C}^{op}\times\mathfrak{L}\to\mathbf{Sets}$, where $\overrightarrow{\mathbf{A}}:\mathfrak{C}^{op}\to\mathfrak{L}$ is the *left adjoint* functor of the adjunction, and symmetrically, $\overleftarrow{\mathbf{A}}:\mathfrak{L}\to\mathfrak{C}^{op}$ is the *right adjoint* functor of the adjunction. The notion of a categorical adjunction was conceived and formulated by *Daniel Kan*.
If we ignore the middle term in the above series, we obtain the *natural isomorphism* of the $\mathbf{Hom}$-bifunctors:

$$\Psi:\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(-),-)\cong\mathbf{Hom}_{\mathfrak{C}^{op}}(-,\overleftarrow{\mathbf{A}}(-))$$

where, the bifunctors $\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(-),-)$, $\mathbf{Hom}_{\mathfrak{C}^{op}}(-,\overleftarrow{\mathbf{A}}(-))$ are obtained as follows:

$$\mathfrak{C}^{op} \times \mathfrak{L} \xrightarrow{\overrightarrow{\mathbf{A}} \times \mathbf{id}_{\mathfrak{L}}} \mathfrak{L}^{op} \times \mathfrak{L} \xrightarrow{\mathbf{Hom}_{\mathfrak{L}}} \mathbf{Sets}$$

$$\mathfrak{C}^{op} \times \mathfrak{L} \xrightarrow{\mathbf{id}_{\mathfrak{C}^{op}} \times \overleftarrow{\mathbf{A}}} \mathfrak{C}^{op} \times \mathfrak{C} \xrightarrow{\mathbf{Hom}_{\mathfrak{C}^{op}}} \mathbf{Sets}$$

Equivalently, we obtain the isomorphism:

$$\Psi_{\mathcal{C},\mathcal{L}} : \mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(\mathcal{C}),\mathcal{L}) \cong \mathbf{Hom}_{\mathfrak{C}^{op}}(\mathcal{C},\overleftarrow{\mathbf{A}}(\mathcal{L}))$$

which is *natural* in both the arguments $\mathcal{C}$ in $\mathfrak{C}^{op}$ and $\mathcal{L}$ in $\mathfrak{L}$. The morphisms in $\mathfrak{L}$, $\mathfrak{C}^{op}$ related to each other by the above isomorphism are called *adjoint transposes*. Hence, if we consider a morphism $h:\overrightarrow{\mathbf{A}}(\mathcal{C}) \to \mathcal{L}$ in $\mathfrak{L}$, we find by virtue of the adjunction isomorphism, it has an adjoint transpose morphism in $\mathfrak{C}^{op}$, namely:

$$\Psi_{\mathcal{C},\mathcal{L}}(h) = h^* : \mathcal{C} \to \overleftarrow{\mathbf{A}}(\mathcal{L})$$

In dual correlation, if we consider a morphism $g:\mathcal{C} \to \overleftarrow{\mathbf{A}}(\mathcal{L})$ in $\mathfrak{C}^{op}$, we again find by virtue of the adjunction isomorphism, it has an adjoint transpose morphism in $\mathfrak{L}$, namely:

$$\Psi^{-1}{}_{\mathcal{C},\mathcal{L}}(g) = g^* : \overrightarrow{\mathbf{A}}(\mathcal{C}) \to \mathcal{L}$$

Now, we may consider the identity morphism at $\overrightarrow{\mathbf{A}}(\mathcal{C})$ in $\mathfrak{L}$, that is $id_{\overrightarrow{\mathbf{A}}(\mathcal{C})}$ belonging to the set $\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(\mathcal{C}),\overrightarrow{\mathbf{A}}(\mathcal{C}))$. The adjoint transpose of $id_{\overrightarrow{\mathbf{A}}(\mathcal{C})}$ is called the *unit morphism* at $\mathcal{C}$, that is:

$$\Psi_{\mathcal{C},\overrightarrow{\mathbf{A}}(\mathcal{C})}(id_{\overrightarrow{\mathbf{A}}(\mathcal{C})}) = (id_{\overrightarrow{\mathbf{A}}(\mathcal{C})})^* = \eta_{\mathcal{C}} : \mathcal{C} \to \overleftarrow{\mathbf{A}}(\overrightarrow{\mathbf{A}}(\mathcal{C}))$$

belonging to the set $\mathbf{Hom}_{\mathfrak{C}^{op}}(\mathcal{C},\overleftarrow{\mathbf{A}}(\overrightarrow{\mathbf{A}}(\mathcal{C})))$. Since the above is *natural* on the argument $\mathcal{C}$ in $\mathfrak{C}^{op}$, we obtain a *natural transformation* of the *identity functor* on $\mathfrak{C}^{op}$, called the *unit natural transformation* of the adjunction:

$$\eta : \mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}}$$

As a further dual correlation, we may consider the identity morphism at $\overleftarrow{\mathbf{A}}(\mathcal{L})$ in $\mathfrak{C}^{op}$, that is $id_{\overleftarrow{\mathbf{A}}(\mathcal{L})}$ belonging to the set $\mathbf{Hom}_{\mathfrak{C}^{op}}(\overleftarrow{\mathbf{A}}(\mathcal{L}), \overleftarrow{\mathbf{A}}(\mathcal{L}))$. The adjoint transpose of $id_{\overleftarrow{\mathbf{A}}(\mathcal{L})}$ is called the *counit morphism* at $\mathcal{L}$, that is:

$$\Psi^{-1}{}_{\overleftarrow{\mathbf{A}}(\mathcal{L}),\mathcal{L}}(id_{\overleftarrow{\mathbf{A}}(\mathcal{L})}) = (id_{\overleftarrow{\mathbf{A}}(\mathcal{L})})^* = \varepsilon_{\mathcal{L}} : \overrightarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}(\mathcal{L})) \to \mathcal{L}$$

belonging to the set $\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}(\mathcal{L})), \mathcal{L})$. Since the above is *natural* on the argument $\mathcal{L}$ in $\mathfrak{L}$, we obtain a natural transformation of the identity functor on $\mathfrak{L}$, called the *counit natural transformation* of the adjunction:

$$\varepsilon : \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}} \to \mathbf{id}_{\mathfrak{L}}$$

From the unit and counit natural transformations of the adjunction we obtain the following identities, called *triangular identities*:

$$\overrightarrow{\mathbf{A}} \xrightarrow{\overrightarrow{\mathbf{A}}\eta} \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}} \xrightarrow{\varepsilon\overrightarrow{\mathbf{A}}} \overrightarrow{\mathbf{A}}$$

$$\overleftarrow{\mathbf{A}} \xrightarrow{\eta\overleftarrow{\mathbf{A}}} \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}} \xrightarrow{\overleftarrow{\mathbf{A}}\varepsilon} \overleftarrow{\mathbf{A}}$$

The above identities may be written equivalently in equation form as follows:

$$id_{\overrightarrow{\mathbf{A}}} = \varepsilon\overrightarrow{\mathbf{A}} \circ \overrightarrow{\mathbf{A}}\eta$$
$$id_{\overleftarrow{\mathbf{A}}} = \overleftarrow{\mathbf{A}}\varepsilon \circ \eta\overleftarrow{\mathbf{A}}$$

where, $id_{\overrightarrow{\mathbf{A}}}$, $id_{\overleftarrow{\mathbf{A}}}$ denote the identity natural transformations on the functors $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ respectively. We conclude that the categorical adjunction being formed by $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ can be *represented equivalently* in terms of the *unit* and *counit* natural transformations *obeying* the triangular identities.

Next, we observe that if we consider a morphism $h : \overrightarrow{\mathbf{A}}(\mathcal{C}) \to \mathcal{L}$ in $\mathfrak{L}$, by virtue of the *unit* natural transformation, its *adjoint transpose* morphism in $\mathfrak{C}^{op}$, namely:

$$\Psi_{\mathcal{C},\mathcal{L}}(h) = h^* : \mathcal{C} \to \overleftarrow{\mathbf{A}}(\mathcal{L})$$

factors uniquely via the unit morphism at $\mathcal{C}$ as follows:

$$\mathcal{C} \xrightarrow{\eta_{\mathcal{C}}} \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}}(\mathcal{C}) \xrightarrow{\overleftarrow{\mathbf{A}}h} \overleftarrow{\mathbf{A}}(\mathcal{L})$$

or, more explicitly:

$$\Psi_{\mathcal{C},\mathcal{L}}(h) = h^* = \overleftarrow{\mathbf{A}}(h) \circ \eta_{\mathcal{C}} : \mathcal{C} \to \overleftarrow{\mathbf{A}}(\mathcal{L})$$

Thus, the pair $(\overrightarrow{\mathbf{A}}(\mathcal{C}), \eta_{\mathcal{C}})$ constitutes a *universal gauge* for the functor:

$$\mathbf{Hom}_{\mathfrak{C}^{op}}(\mathcal{C}, \overleftarrow{\mathbf{A}}(-)) = \mathbf{y}_{\mathcal{C}} \circ \overleftarrow{\mathbf{A}} : \mathfrak{L} \to \mathbf{Sets}$$

Equivalently stated, the pair $(\overrightarrow{\mathbf{A}}(\mathcal{C}), \eta_{\mathcal{C}})$ is an *initial object* in the category of elements of the *Sets* -valued functor $\mathbf{y}_{\mathcal{C}} \circ \overleftarrow{\mathbf{A}} : \mathfrak{L} \to \mathbf{Sets}$.

In dual correlation, if we consider a morphism $g : \mathcal{C} \to \overleftarrow{\mathbf{A}}(\mathcal{L})$ in $\mathfrak{C}^{op}$, by virtue of the *counit* natural transformation, its *adjoint transpose* morphism in $\mathfrak{L}$, namely:

$$\Psi^{-1}{}_{\mathcal{C},\mathcal{L}}(g) = g^* : \overrightarrow{\mathbf{A}}(\mathcal{C}) \to \mathcal{L}$$

factors uniquely via the counit morphism at $\mathcal{L}$ as follows:

$$\overrightarrow{\mathbf{A}}(\mathcal{C}) \xrightarrow{\overrightarrow{\mathbf{A}}g} \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}}(\mathcal{L}) \xrightarrow{\varepsilon_{\mathcal{L}}} \mathcal{L}$$

or, more explicitly:

$$\Psi^{-1}{}_{\mathcal{C},\mathcal{L}}(g) = g^* = \varepsilon_{\mathcal{L}} \circ \overrightarrow{\mathbf{A}}(g) : \overrightarrow{\mathbf{A}}(\mathcal{C}) \to \mathcal{L}$$

Thus, the pair $(\overleftarrow{\mathbf{A}}(\mathcal{L}), \varepsilon_{\mathcal{L}})$ constitutes a *universal gauge* for the functor:

$$\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(-), \mathcal{L}) = \mathbf{y}^{\mathcal{L}} \circ \overrightarrow{\mathbf{A}} : \mathfrak{C}^{op} \to \mathbf{Sets}$$

This is to say that the pair $(\overleftarrow{\mathbf{A}}(\mathcal{L}), \varepsilon_{\mathcal{L}})$ is a *terminal* object in the category of elements of the *Sets*-valued functor $\mathbf{y}^{\mathcal{L}} \circ \overrightarrow{\mathbf{A}} : \mathfrak{C}^{op} \to \mathbf{Sets}$.

Let us focus again on the bifunctor: $\wedge(-,-) : \mathfrak{C}^{op} \times \mathfrak{L} \to \mathbf{Sets}$. We see immediately that if the bifunctor $\wedge(-,-)$ is birepresentable, then the set of signal-transmitter/receivers for any objects $\mathcal{C}$ in $\mathfrak{C}^{op}$ and $\mathcal{L}$ in $\mathfrak{L}$ can be *represented* as morphisms within both the involved categories $\mathfrak{L}$, $\mathfrak{C}^{op}$, using the *left* and *right adjoint functors* $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ of the induced *adjunction*, by means of the first and the last term in the series of isomorphisms. Moreover, since these isomorphisms are natural, the translation process effectuating representability in both directions becomes possible, since there exist functorial gauges for both the functors:

$$\wedge(\mathcal{C},-) : \mathfrak{L} \to \mathbf{Sets}$$
$$\wedge(-,\mathcal{L}) : \mathfrak{C}^{op} \to \mathbf{Sets}$$

More concretely, for each $\mathcal{C}$ in $\mathfrak{C}^{op}$, there exists a representing element $\overrightarrow{\mathbf{A}}(\mathcal{C})$ in $\mathfrak{L}$, such that:

$$\Theta_{\mathcal{C},\mathcal{L}} : \mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(\mathcal{C}), \mathcal{L}) \cong \wedge(\mathcal{C}, \mathcal{L})$$

which is natural in $\mathcal{L}$. For $\mathcal{L} = \overrightarrow{\mathbf{A}}(\mathcal{C})$ we obtain:

$$\Theta_{\mathcal{C},\overrightarrow{\mathbf{A}}(\mathcal{C})} : \mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(\mathcal{C}), \overrightarrow{\mathbf{A}}(\mathcal{C})) \cong \wedge(\mathcal{C}, \overrightarrow{\mathbf{A}}(\mathcal{C}))$$

Now, we may consider the identity morphism at $\overrightarrow{\mathbf{A}}(\mathcal{C})$ in $\mathfrak{L}$, that is, $id_{\overrightarrow{\mathbf{A}}(\mathcal{C})}$ belonging to the set $\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(\mathcal{C}), \overrightarrow{\mathbf{A}}(\mathcal{C}))$. The image of $id_{\overrightarrow{\mathbf{A}}(\mathcal{C})}$ in $\wedge(\mathcal{C}, \overrightarrow{\mathbf{A}}(\mathcal{C}))$ is called the $\wedge$*-unit signal-transmitter/receiver at* $\mathcal{C}$, that is:

$$\Theta_{\mathcal{C},\overrightarrow{\mathbf{A}}(\mathcal{C})}(id_{\overrightarrow{\mathbf{A}}(\mathcal{C})}) = \delta_{\mathcal{C}} : \mathcal{C} \to \overrightarrow{\mathbf{A}}(\mathcal{C})$$

belonging to the set $\wedge(\mathcal{C},\overrightarrow{\mathbf{A}}(\mathcal{C}))$. Since the above is *natural* on the argument $\mathcal{C}$ in $\mathfrak{C}^{op}$, we obtain a natural transformation of the identity functor on $\mathfrak{C}^{op}$, called the $\wedge$*-unit natural signal-transmitter/receiver*:

$$\delta : \mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \overrightarrow{\mathbf{A}}$$

We conclude that the pair $(\overrightarrow{\mathbf{A}}(\mathcal{C}),\delta_{\mathcal{C}})$ constitutes a *universal gauge* for the functor:

$$\wedge(\mathcal{C},-) : \mathfrak{L} \to \mathbf{Sets}$$

Equivalently stated, the pair $(\overrightarrow{\mathbf{A}}(\mathcal{C}),\delta_{\mathcal{C}})$ is an *initial object* in the category of elements of the $Sets$-valued functor $\wedge(\mathcal{C},-)$.

Similarly, for each $\mathcal{L}$ in $\mathfrak{L}$, there exists a *representing* element $\overleftarrow{\mathbf{A}}(\mathcal{L})$ in $\mathfrak{C}^{op}$, such that:

$$\Phi_{\mathcal{C},\mathcal{L}} : \wedge(\mathcal{C},\mathcal{L}) \cong \mathbf{Hom}_{\mathfrak{C}^{op}}(\mathcal{C},\overleftarrow{\mathbf{A}}(\mathcal{L}))$$

which is *natural* in $\mathcal{C}$. For $\mathcal{C} = \overleftarrow{\mathbf{A}}(\mathcal{L})$ we obtain:

$$\Phi_{\overleftarrow{\mathbf{A}}(\mathcal{L}),\mathcal{L}} : \wedge(\overleftarrow{\mathbf{A}}(\mathcal{L}),\mathcal{L}) \cong \mathbf{Hom}_{\mathfrak{C}^{op}}(\overleftarrow{\mathbf{A}}(\mathcal{L}),\overleftarrow{\mathbf{A}}(\mathcal{L}))$$

Now, we may consider the identity morphism at $\overleftarrow{\mathbf{A}}(\mathcal{L})$ in $\mathfrak{C}^{op}$, that is $id_{\overleftarrow{\mathbf{A}}(\mathcal{L})}$ belonging to the set $\mathbf{Hom}_{\mathfrak{C}^{op}}(\overleftarrow{\mathbf{A}}(\mathcal{L}),\overleftarrow{\mathbf{A}}(\mathcal{L}))$. The inverse image of $id_{\overleftarrow{\mathbf{A}}(\mathcal{L})}$ in $\wedge(\overleftarrow{\mathbf{A}}(\mathcal{L}),\mathcal{L})$ is called the $\wedge$ *-counit signal transmitter/receiver at* $\mathcal{L}$, that is:

$$\Phi^{-1}_{\overleftarrow{\mathbf{A}}(\mathcal{L}),\mathcal{L}}(id_{\overleftarrow{\mathbf{A}}(\mathcal{L})}) = \text{Ú}_{\mathcal{L}} : \overleftarrow{\mathbf{A}}(\mathcal{L}) \to \mathcal{L}$$

belonging to the set $\wedge(\overleftarrow{\mathbf{A}}(\mathcal{L}),\mathcal{L})$. Since the above is natural on the argument $\mathcal{L}$ in $\mathfrak{L}$, we obtain a natural transformation of the identity functor on $\mathfrak{L}$, called the $\wedge$*-counit natural signal transmitter/receiver*:

$$\text{Ú} : \overleftarrow{\mathbf{A}} \to \mathbf{id}_{\mathfrak{L}}$$

We conclude that the pair $(\overleftarrow{\mathbf{A}}(\mathcal{L}), \acute{\mathrm{U}}_{\mathcal{L}})$ constitutes a *universal gauge* for the functor:

$$\wedge(-,\mathcal{L}):\mathfrak{C}^{op} \to \mathbf{Sets}$$

which again provides that the pair $(\overleftarrow{\mathbf{A}}(\mathcal{L}), \acute{\mathrm{U}}_{\mathcal{L}})$ is a *terminal* object in the category of elements of the *Sets* -valued functor $\wedge(-,\mathcal{L})$.

Therefore, if we consider a signal transmitter/receiver $\chi \in \wedge(\mathcal{C},\mathcal{L})$, the corresponding morphisms, induced by the birepresentability of the bifunctor:

$$\wedge(-,-):\mathfrak{C}^{op} \times \mathfrak{L} \to \mathbf{Sets}$$

within $\mathfrak{C}^{op}$, $\mathfrak{L}$ respectively, that is:

$$g(\chi):\mathcal{C} \to \overleftarrow{\mathbf{A}}(\mathcal{L})$$
$$h(\chi):\overrightarrow{\mathbf{A}}(\mathcal{C}) \to \mathcal{L}$$

play the role of *conjugates* with respect to the signal transmitter/receiver χ.

As a consequence, if χ is the $\wedge$ -unit signal-transmitter/receiver at $\mathcal{C}$, namely $\chi = \delta_{\mathcal{C}}:\mathcal{C} \to \overrightarrow{\mathbf{A}}(\mathcal{C})$, then:

$$g(\delta_{\mathcal{C}}) = \eta_{\mathcal{C}}:\mathcal{C} \to \overleftarrow{\mathbf{A}}(\overrightarrow{\mathbf{A}}(\mathcal{C}))$$
$$h(\delta_{\mathcal{C}}) = id_{\overrightarrow{\mathbf{A}}(\mathcal{C})}:\overrightarrow{\mathbf{A}}(\mathcal{C}) \to \overrightarrow{\mathbf{A}}(\mathcal{C})$$

are *conjugates* with respect to the $\wedge$ -unit signal-transmitter/receiver at $\mathcal{C}$. Since the above is natural on the argument $\mathcal{C}$ in $\mathfrak{C}^{op}$, we conclude that:

$$g(\delta) = \eta:\mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}}$$
$$h(\delta) = id_{\overrightarrow{\mathbf{A}}}:\overrightarrow{\mathbf{A}} \to \overrightarrow{\mathbf{A}}$$

Which is to say that the *unit* natural transformation of the adjunction η, and the $\overrightarrow{\mathbf{A}}$ *-identity* natural transformation $id_{\overrightarrow{\mathbf{A}}}$ are *functorial* *conjugates* with respect to the $\wedge$ *-unit* natural signal-transmitter/receiver $\delta : \mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \overrightarrow{\mathbf{A}}$. Similarly, we conclude that:

$$g(\acute{\text{U}}) = \varepsilon : \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}} \to \mathbf{id}_{\mathfrak{L}}$$
$$h(\acute{\text{U}}) = id_{\overleftarrow{\mathbf{A}}} : \overleftarrow{\mathbf{A}} \to \overleftarrow{\mathbf{A}}$$

meaning that, the *counit* natural transformation of the adjunction ε, and the $\overleftarrow{\mathbf{A}}$ *-identity* natural transformation $id_{\overleftarrow{\mathbf{A}}}$ are *functorial conjugates* with respect to the $\wedge$ *-counit* natural signal-transmitter/receiver $\acute{\text{U}} : \overleftarrow{\mathbf{A}} \to \mathbf{id}_{\mathfrak{L}}$.

Furthermore, if we consider an *arbitrary* signal transmitter/receiver $\chi \in \wedge(\mathcal{C},\mathcal{L})$, we obtain the following factorizations:

$$\chi = h(\chi) \circ \delta_{\mathcal{C}}$$
$$\chi = \acute{\text{U}}_{\mathcal{L}} \circ g(\chi)$$

presented equivalently, as follows:

$$h(\chi) \circ \delta_{\mathcal{C}} = \chi = \acute{\text{U}}_{\mathcal{L}} \circ g(\chi)$$

where, $h(\chi)$, $g(\chi)$ are *conjugates* with respect to the signal transmitter/receiver χ. Denoting the conjugate of $h(\chi)$, that is $g(\chi)$, by $g(\chi) = h^{*}(\chi)$, or equivalently, identifying the conjugate of $h(\chi)$ (with respect to the signal transmitter/receiver χ) with the *adjoint transpose* of $h(\chi)$, we obtain:

$$h(\chi) \circ \delta_{\mathcal{C}} = \chi = \acute{\text{U}}_{\mathcal{L}} \circ h^{*}(\chi)$$

Now, we are ready to tackle the problem of defining a functorial gauge (functorial reference frame) for an *arbitrary* functor. We say that a functor $\mathbf{F} : \mathfrak{C}^{op} \to \mathfrak{L}$ has a *functorial gauge relative to* $\mathfrak{L}$ if and only if the bifunctor: $\wedge(-,-) : \mathfrak{C}^{op} \times \mathfrak{L} \to \mathbf{Sets}$ is *representable within* $\mathfrak{L}$, that is we have a *natural isomorphism* of bifunctors:

$$\mathbf{Hom}_{\mathfrak{L}}(\mathbf{F}(-),-) \cong \wedge(-,-)$$

Equivalently, the functor $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$ has a *functorial gauge relative to* $\mathfrak{L}$ if and only if the functor:

$$\bigwedge(\mathcal{C},-):\mathfrak{L}\to\mathbf{Sets}$$

is *representable*, that is for each $\mathcal{C}$ in $\mathfrak{C}^{op}$, there exists a *representing element* $\mathbf{F}(\mathcal{C})$ in $\mathfrak{L}$, such that:

$$\Theta_{\mathcal{C},\mathcal{L}}:\mathbf{Hom}_{\mathfrak{L}}(\mathbf{F}(\mathcal{C}),\mathcal{L})\cong\bigwedge(\mathcal{C},\mathcal{L})$$

which is *natural* in $\mathcal{L}$.
Equivalently, the functor $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$ has a *functorial gauge relatively to* $\mathfrak{L}$ if and only if the *pair* $(\mathbf{F}(\mathcal{C}),\delta_{\mathcal{C}}:\mathcal{C}\to\mathbf{F}(\mathcal{C}))$ is a *universal gauge* for the functor $\bigwedge(\mathcal{C},-)$.

Symmetrically, we say that an oppositely pointing functor $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ has a *functorial gauge relative to* $\mathfrak{C}^{op}$ if and only if the bifunctor: $\bigwedge(-,-):\mathfrak{C}^{op}\times\mathfrak{L}\to\mathbf{Sets}$ is *representable within* $\mathfrak{C}^{op}$, that is, we have a natural isomorphism of bifunctors:

$$\bigwedge(-,-)\cong\mathbf{Hom}_{\mathfrak{C}^{op}}(-,\mathbf{G}(-))$$

Equivalently, the functor $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ has a *functorial gauge relative to* $\mathfrak{C}^{op}$ if and only if the functor:

$$\bigwedge(-,\mathcal{L}):\mathfrak{C}^{op}\to\mathbf{Sets}$$

is *representable*, that is for each $\mathcal{L}$ in $\mathfrak{L}$, there exists a *representing element* $\mathbf{G}(\mathcal{L})$ in $\mathfrak{C}^{op}$, such that:

$$\Phi_{\mathcal{C},\mathcal{L}}:\bigwedge(\mathcal{C},\mathcal{L})\cong\mathbf{Hom}_{\mathfrak{C}^{op}}(\mathcal{C},\mathbf{G}(\mathcal{L}))$$

which is *natural* in $\mathcal{C}$.
Equivalently, the functor $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ has a *functorial gauge relative to* $\mathfrak{C}^{op}$ if and only if the pair $(\mathbf{G}(\mathcal{L}),\acute{\mathrm{U}}_{\mathcal{L}}:\mathbf{G}(\mathcal{L})\to\mathcal{L})$ is a *universal gauge* for the functor $\bigwedge(-,\mathcal{L})$.

We say that an *arbitrary* functor of the form $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$ has a *functorial gauge* if and only if it has a *right adjoint* functor $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$.

Equivalently, the functor $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$ has a *functorial gauge* if and only if each functor $\mathbf{y}^{\mathcal{L}}\circ\mathbf{F}$ is *representable* for each object $\mathcal{L}$ in $\mathfrak{L}$. If this is the case, a functor $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ right adjoint to $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$ assigns to each $\mathcal{L}$ in $\mathfrak{L}$ an object $\mathbf{G}(\mathcal{L})$ in $\mathfrak{C}^{op}$ representing the functor:

$$\mathbf{Hom}_{\mathfrak{L}}(\mathbf{F}(-),\mathcal{L})=\mathbf{y}^{\mathcal{L}}\circ\mathbf{F}:\mathfrak{C}^{op}\to\mathbf{Sets}$$

Moreover, any other functor right adjoint to $\mathbf{F}$ is *naturally isomorphic* with $\mathbf{G}$.

Equivalently, the functor $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$ has a *functorial gauge* if and only if there exists for each object $\mathcal{L}$ in $\mathfrak{L}$ an object $\mathbf{G}(\mathcal{L})$ in $\mathfrak{C}^{op}$ and an arrow $\varepsilon_{\mathcal{L}}:\mathbf{F}(\mathbf{G}(\mathcal{L}))\to\mathcal{L}$, such that the *pair* $(\mathbf{G}(\mathcal{L}),\varepsilon_{\mathcal{L}})$ constitutes a *universal gauge* for the functor $\mathbf{y}^{\mathcal{L}}\circ\mathbf{F}$.

Symmetrically, we say that an arbitrary functor of the form $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ has a *functorial gauge* if and only if it has a *left adjoint* functor $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$.

Equivalently, the functor $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ has a *functorial gauge* if and only if each functor $\mathbf{y}_{\mathcal{C}}\circ\mathbf{G}$ is *representable* for each object $\mathcal{C}$ in $\mathfrak{C}^{op}$. If this is the case, a functor $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$ left adjoint to $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ assigns to each $\mathcal{C}$ in $\mathfrak{C}^{op}$ an object $\mathbf{F}(\mathcal{C})$ in $\mathfrak{L}$ representing the functor:

$$\mathbf{Hom}_{\mathfrak{C}^{op}}(\mathcal{C},\mathbf{G}(-))=\mathbf{y}_{\mathcal{C}}\circ\mathbf{G}:\mathfrak{L}\to\mathbf{Sets}$$

Moreover, any other functor left adjoint to $\mathbf{G}$ is *naturally isomorphic* with $\mathbf{F}$.

Equivalently, the functor $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$ has a *functorial gauge* if and only if there exists for each object $\mathcal{C}$ in $\mathfrak{C}^{op}$ an object $\mathbf{F}(\mathcal{C})$ in $\mathfrak{L}$ and an arrow $\eta_{\mathcal{C}}:\mathcal{C}\to\mathbf{G}(\mathbf{F}(\mathcal{C}))$, such that the *pair* $(\mathbf{F}(\mathcal{C}),\eta_{\mathcal{C}})$ constitutes a *universal gauge* for the functor $\mathbf{y}_{\mathcal{C}}\circ\mathbf{G}$.

Finally, we say that two oppositely pointing functors between the same categories, that is, $\mathbf{F}:\mathfrak{C}^{op}\to\mathfrak{L}$, $\mathbf{G}:\mathfrak{L}\to\mathfrak{C}^{op}$, have *mutual functorial gauges* with respect to each other if and only if the functor $\mathbf{F}$ is *left adjoint* to the functor $\mathbf{G}$ and the functor $\mathbf{G}$ is *right adjoint* to the functor $\mathbf{F}$, such that, the two oppositely pointing functors $\overrightarrow{\mathbf{A}}:=\mathbf{F}$,

$\overleftarrow{\mathbf{A}} := \mathbf{G}$ form a *categorical adjunction*, expressed by the *natural isomorphism* of the $\mathbf{Hom}$-bifunctors:

$$\Psi : \mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(-), -) \cong \mathbf{Hom}_{\mathfrak{C}^{op}}(-, \overleftarrow{\mathbf{A}}(-))$$

Equivalently, $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ have *mutual functorial gauges* with respect to each other if and only if the *bifunctor* $\wedge(-,-) : \mathfrak{C}^{op} \times \mathfrak{L} \to \mathbf{Sets}$ is *birepresentable*, that is, we have a series of *natural isomorphisms* of bifunctors:

$$\mathbf{Hom}_{\mathfrak{L}}(\overrightarrow{\mathbf{A}}(-), -) \cong \wedge(-,-) \cong \mathbf{Hom}_{\mathfrak{C}^{op}}(-, \overleftarrow{\mathbf{A}}(-))$$

When this is the case, the two oppositely pointing functors $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ form a *categorical adjunction*, induced by the requirement of birepresentability of the bifunctor $\wedge(-,-)$, where $\overrightarrow{\mathbf{A}}$ is the *left adjoint* functor of the adjunction, and symmetrically, $\overleftarrow{\mathbf{A}}$ is the *right adjoint* functor of the adjunction.

Equivalently, $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ have *mutual functorial gauges* with respect to each other if and only if there exist a natural transformation of the identity functor on $\mathfrak{C}^{op}$ (*unit natural transformation*):

$$\eta : \mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}}$$

as well as a natural transformation of the identity functor on $\mathfrak{L}$ (*counit natural transformation*):

$$\varepsilon : \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}} \to \mathbf{id}_{\mathfrak{L}}$$

obeying the following identities:

$$id_{\overrightarrow{\mathbf{A}}} = \varepsilon\overrightarrow{\mathbf{A}} \circ \overrightarrow{\mathbf{A}}\eta$$
$$id_{\overleftarrow{\mathbf{A}}} = \overleftarrow{\mathbf{A}}\varepsilon \circ \eta\overleftarrow{\mathbf{A}}$$

Equivalently, $\overrightarrow{\mathbf{A}} = \mathbf{F}$, $\overleftarrow{\mathbf{A}} = \mathbf{G}$ have *mutual functorial gauges* with respect to each other if and only if, each functor $\mathbf{y}^{\mathcal{L}} \circ \mathbf{F}$ is *representable* (via $\mathbf{G}$) for each object $\mathcal{L}$ in $\mathfrak{L}$, and each functor

$\mathbf{y}_{\mathcal{C}} \circ \mathbf{G}$ is *representable* (via $\mathbf{F}$) for each object $\mathcal{C}$ in $\mathfrak{C}^{op}$, that is, if and only if:

i There exists for each object $\mathcal{L}$ in $\mathfrak{L}$ an object $\mathbf{G}(\mathcal{L})$ in $\mathfrak{C}^{op}$ and an arrow $\varepsilon_{\mathcal{L}} : \mathbf{F}(\mathbf{G}(\mathcal{L})) \to \mathcal{L}$, such that the pair $(\mathbf{G}(\mathcal{L}), \varepsilon_{\mathcal{L}})$ constitutes a universal gauge for the functor $\mathbf{y}^{\mathcal{L}} \circ \mathbf{F}$.

ii There exists for each object $\mathcal{C}$ in $\mathfrak{C}^{op}$ an object $\mathbf{F}(\mathcal{C})$ in $\mathfrak{L}$ and an arrow $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbf{G}(\mathbf{F}(\mathcal{C}))$, such that the pair $(\mathbf{F}(\mathcal{C}), \eta_{\mathcal{C}})$ constitutes a *universal gauge* for the functor $\mathbf{y}_{\mathcal{C}} \circ \mathbf{G}$.

Equivalently, $\overrightarrow{\mathbf{A}} = \mathbf{F}$, $\overleftarrow{\mathbf{A}} = \mathbf{G}$ have mutual functorial gauges with respect to each other if and only if:

i The functor $\wedge(\mathcal{C}, -) : \mathfrak{L} \to \mathbf{Sets}$ is *representable*, such that the pair $(\mathbf{F}(\mathcal{C}), \delta_{\mathcal{C}})$ is a *universal gauge* for the functor $\wedge(\mathcal{C}, -)$.

ii The functor $\wedge(-, \mathcal{L}) : \mathfrak{C}^{op} \to \mathbf{Sets}$ is *representable*, such that the pair $(\mathbf{G}(\mathcal{L}), \acute{\mathrm{U}}_{\mathcal{L}})$ is a *universal* gauge for the functor $\wedge(-, \mathcal{L})$.

8.9 COMMUNICABILITY: MUTUAL FUNCTORIAL GAUGES AND MONADS

Let us now present some important consequences of the existence of mutual functorial gauges for two oppositely pointing functors between the same categories. As a simple illustration we may consider the *diagonal* functor:

$$\Delta : \mathfrak{K} \to \mathfrak{K}^{\mathfrak{A}}$$

For an object $\mathcal{K}$ in $\mathfrak{K}$, the functor $\Delta(\mathcal{K})$ is the *constant* $\mathcal{K}$-valued functor, which for any object $\mathcal{U}$ of a (locally) small category $\mathfrak{A}$ gives the constant value $\mathcal{K}$, and for any arrow f of $\mathfrak{A}$ gives the identity arrow on $\mathcal{K}$. Furthermore, if we consider that $\mathfrak{K}$ is the category of $Sets$, then $\mathbf{Sets}^{\mathfrak{A}}$ is the functor category of copresheaves (of sets), that is, the category of all covariant functors $\mathbf{P}$ from $\mathfrak{A}$ to $Sets$ and all natural transformations between them, where $\mathfrak{A}$ is an indexing (locally) small category.

$$\Delta : \mathbf{Sets} \to \mathbf{Sets}^{\mathfrak{A}}$$

$$\Delta(\mathcal{K}) : \mathfrak{A} \to \mathbf{Sets}$$

Thus, $\Delta(\mathcal{K})$ is the constant $\mathcal{K}$-valued functor taking each object $\mathcal{U}$ in $\mathfrak{A}$ to the set $\mathcal{K}$, and each arrow $f : \mathcal{U} \to \mathcal{V}$ in $\mathfrak{A}$ to the identity morphism $id_{\mathcal{K}}$. Now, if we consider the diagonal functor Δ as a functor of the above form $\mathbf{F}$, then it has a *functorial gauge* if and only if it has a *right adjoint* functor of the above form $\mathbf{G}$, denoted by:

$$\mathbf{O} : \mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$$

Moreover, the oppositely pointing functors Δ, $\mathbf{O}$ have *mutual functorial gauges* with respect to each other if and only if Δ is *left adjoint* to $\mathbf{O}$ and $\mathbf{O}$ is *right adjoint* to Δ, such that, the functors $\overrightarrow{\mathbf{A}} := \Delta$, $\overleftarrow{\mathbf{A}} := \mathbf{O}$ form a *categorical adjunction*, expressed by the *natural isomorphism* of the $\mathbf{Hom}$-bifunctors:

$$\Psi : \mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(\overrightarrow{\mathbf{A}}(-), -) \cong \mathbf{Hom}_{\mathbf{Sets}}(-, \overleftarrow{\mathbf{A}}(-))$$

Equivalently, $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$ have *mutual functorial gauges* with respect to each other if and only if the bifunctor $\wedge(-,-) : \mathbf{Sets} \times \mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$ is *birepresentable*, that is, we have a series of natural isomorphisms of bifunctors:

$$\mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(\overrightarrow{\mathbf{A}}(-), -) \cong \wedge(-,-) \cong \mathbf{Hom}_{\mathbf{Sets}}(-, \overleftarrow{\mathbf{A}}(-))$$

If we consider a signal transmitter/receiver $\chi \in \wedge(\mathcal{K}, \mathbf{P})$, then we realize that χ is actually a *cone*. Hence, the corresponding morphisms, induced by the birepresentability of the bifunctor $\wedge(-,-)$ within $\mathbf{Sets}$, $\mathbf{Sets}^{\mathfrak{A}}$ respectively, that is:

$$g(\chi) : \mathcal{K} \to \overleftarrow{\mathbf{A}}(\mathbf{P})$$
$$h(\chi) : \overrightarrow{\mathbf{A}}(\mathcal{K}) \to \mathbf{P}$$

are *conjugates* with respect to the *cone* signal transmitter/receiver χ. The morphism $h(\chi) : \Delta(\mathcal{K}) \to \mathbf{P}$ is a natural transformation in $\mathbf{Sets}^{\mathfrak{A}}$ from the constant $\mathcal{K}$-valued copresheaf $\Delta(\mathcal{K}) : \mathfrak{A} \to \mathbf{Sets}$ to the copresheaf $\mathbf{P} : \mathfrak{A} \to \mathbf{Sets}$. The conjugate morphism of $h(\chi)$ with respect to the cone χ, that is, $g(\chi):\mathcal{K}\to\overleftarrow{\mathbf{A}}(\mathbf{P})$ is a morphism in *Sets* (representing the cone χ) from $\mathcal{K}$ to the set $\overleftarrow{\mathbf{A}}(\mathbf{P})=\mathbf{O}(\mathbf{P})$ obtained by the application of the sought right adjoint $\mathbf{O}$ on a copresheaf $\mathbf{P}$. By further taking into account that the functor $\mathbf{O}$ (as a right adjoint to Δ) should assign to each $\mathbf{P}$ in $\mathbf{Sets}^{\mathfrak{A}}$ an object $\mathbf{O}(\mathbf{P})$ in *Sets* representing the functor:

$$\mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(\Delta(-),\mathbf{P})=\mathbf{y}^{\mathbf{P}}\circ\Delta:\mathbf{Sets}\to\mathbf{Sets}$$

or equivalently that the pair $(\mathbf{O}(\mathbf{P}),\varepsilon_{\mathbf{P}})$ should be a *universal gauge* for the functor $\mathbf{y}^{\mathbf{P}}\circ\Delta$, we conclude that:

$$\mathbf{O}(\mathbf{P})=\overleftarrow{\mathbf{A}}(\mathbf{P})=Lim(\mathbf{P})$$

where

$$\mathbf{O}=\overleftarrow{\mathbf{A}}=Lim:\mathbf{Sets}^{\mathfrak{A}}\to\mathbf{Sets}$$

is the sought *right adjoint* to Δ. Actually, the identification of the right adjoint to Δ with the functor $Lim:\mathbf{Sets}^{\mathfrak{A}}\to\mathbf{Sets}$ is immediate from the identification of a signal transmitter/receiver $\chi \in \bigwedge(\mathcal{K},\mathbf{P})$ with a cone and the requirement of universality of the gauge formed by $Lim(\mathbf{P})$ for each $\mathbf{P}$ in $\mathbf{Sets}^{\mathfrak{A}}$.

Dually, if we consider the diagonal functor Δ as a functor of the form $\mathbf{G}$, then it has a *functorial gauge* if and only if it has a *left adjoint* functor of the form $\mathbf{F}$, denoted by:

$$\Lambda:\mathbf{Sets}^{\mathfrak{A}}\to\mathbf{Sets}$$

Moreover, the oppositely pointing functors Δ, $\mathbf{O}$ have *mutual functorial gauges* with respect to each other if and only if Δ is *right adjoint* to Λ and Λ is *left adjoint* to Δ, such that the functors $\overrightarrow{\mathbf{A}}:=\Lambda$, $\overleftarrow{\mathbf{A}}:=\Delta$ form a *categorical adjunction*, expressed by the natural isomorphism of the $\mathbf{Hom}$-bifunctors:

$$\Xi:\mathbf{Hom}_{\mathbf{Sets}}(\overrightarrow{\mathbf{A}}(-),-)\cong\mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(-,\overleftarrow{\mathbf{A}}(-))$$

Equivalently, $\overrightarrow{\mathbf{A}} = \Lambda$, $\overleftarrow{\mathbf{A}} = \Delta$ have mutual functorial gauges with respect to each other if and only if the bifunctor $\curlywedge(-,-):\mathbf{Sets}^{\mathfrak{A}} \times \mathbf{Sets} \to \mathbf{Sets}$ is *birepresentable*, that is, we have a series of *natural isomorphisms* of bifunctors:

$$\mathbf{Hom}_{\mathbf{Sets}}(\overrightarrow{\mathbf{A}}(-),-) \cong \curlywedge(-,-) \cong \mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(-,\overleftarrow{\mathbf{A}}(-))$$

If we consider a signal transmitter/receiver $\psi \in \curlywedge(\mathbf{P},\mathcal{K})$, then we realize that ψ is actually a *cocone*. Hence, the corresponding morphisms, induced by the birepresentability of the bifunctor $\curlywedge(-,-)$ within $\mathbf{Sets}^{\mathfrak{A}}$, $\mathbf{Sets}$, respectively, that is:

$$f(\psi):\mathbf{P} \to \overleftarrow{\mathbf{A}}(\mathcal{K})$$
$$x(\psi):\overrightarrow{\mathbf{A}}(\mathbf{P}) \to \mathcal{K}$$

are *conjugates* with respect to the cocone signal transmitter/receiver ψ. The morphism $f(\psi):\mathbf{P} \to \overleftarrow{\mathbf{A}}(\mathcal{K})$ is a natural transformation in $\mathbf{Sets}^{\mathfrak{A}}$ from the copresheaf $\mathbf{P}:\mathfrak{A} \to \mathbf{Sets}$ to the constant $\mathcal{K}$ -valued copresheaf $\Delta(\mathcal{K}):\mathfrak{A} \to \mathbf{Sets}$. The conjugate morphism of $f(\psi)$ with respect to the cocone ψ, viz. $x(\psi):\overrightarrow{\mathbf{A}}(\mathbf{P}) \to \mathcal{K}$ is a morphism in *Sets* (representing the cocone ψ) from the set $\overrightarrow{\mathbf{A}}(\mathbf{P}) = \Lambda(\mathbf{P})$ obtained by the application of the sought left adjoint Λ on a copresheaf $\mathbf{P}$ to a set $\mathcal{K}$. By further taking into account that the functor Λ (as a left adjoint to Δ) should assign to each $\mathbf{P}$ in $\mathbf{Sets}^{\mathfrak{A}}$ an object $\Lambda(\mathbf{P})$ in $\mathbf{Sets}$ representing the functor:

$$\mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(\mathbf{P},\Delta(-)) = \mathbf{y}_{\mathbf{P}} \circ \Delta : Sets \to \mathbf{Sets}$$

or equivalently, that the pair $(\Lambda(\mathbf{P}),\eta_{\mathbf{P}})$ should be a *universal gauge* for the functor $\mathbf{y}_{\mathbf{P}} \circ \Delta$ we conclude that:

$$\Lambda(\mathbf{P}) = \overrightarrow{\mathbf{A}}(\mathbf{P}) = Colim(\mathbf{P})$$

where

$$\Lambda = \overrightarrow{\mathbf{A}} = Colim : \mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$$

is the sought *left adjoint* to Δ. Again, the identification of the left adjoint to Δ with the functor $Colim:\mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$ follows immediately from the identification of a signal transmitter/receiver $\psi \in \bigwedge(\mathbf{P},\mathcal{K})$ with a *cocone* and the requirement of universality of the gauge formed by $Colim(\mathbf{P})$ for each $\mathbf{P}$ in $\mathbf{Sets}^{\mathfrak{A}}$.

We conclude that the functors $Lim:\mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$, $Colim:\mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$ are *right* and *left* adjoints correspondingly of the diagonal functor Δ. More precisely:

i The functors $\Delta:\mathbf{Sets} \to \mathbf{Sets}^{\mathfrak{A}}$, $Lim:\mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$ have mutual functorial gauges with respect to each other forming a categorical adjunction, induced by the birepresentability of the bifunctor of cones $\bigwedge(-,-):\mathbf{Sets}\times\mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$, within $\mathbf{Sets}$, $\mathbf{Sets}^{\mathfrak{A}}$ respectively:

$$\mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(\Delta(-),-) \cong \bigwedge(-,-) \cong \mathbf{Hom}_{\mathbf{Sets}}(-,\mathbf{Lim}(-))$$

ii The functors $Colim:\mathbf{Sets}^{\mathfrak{A}} \to \mathbf{Sets}$, $\Delta:\mathbf{Sets} \to \mathbf{Sets}^{\mathfrak{A}}$ have mutual functorial gauges with respect to each other forming a categorical adjunction, induced by the birepresentability of the bifunctor of cocones $\bigwedge(-,-):\mathbf{Sets}^{\mathfrak{A}}\times\mathbf{Sets} \to \mathbf{Sets}$, within $\mathbf{Sets}^{\mathfrak{A}}$, $\mathbf{Sets}$ respectively:

$$\mathbf{Hom}_{\mathbf{Sets}}(\mathbf{Colim}(-),-) \cong \bigwedge(-,-) \cong \mathbf{Hom}_{\mathbf{Sets}^{\mathfrak{A}}}(-,\Delta(-))$$

Furthermore, we can show immediately that *right adjoint* functors *commute with limits*, while *left adjoint* functors *commute with colimits*. It is instructive to think of these properties of left/right adjoint functors as preservation properties induced by the requirement that functorial gauges should apply. Obviously, these properties can be used conversely in order to prove that a given functor *does not* have a functorial gauge by showing that it *does not preserve* limits or colimits. Finally, the above arguments remain valid if we replace the category of $Sets$ by any other complete category of algebras for the case [i], and by any other cocomplete category of algebras for the case [ii] respectively.

Along the same lines, we can also show that for any small category $\mathfrak{A}$, every *presheaf* $\mathbf{P}$ in the functor category of presheaves $Sets^{\mathfrak{A}^{op}}$, is a *colimit* of standard (representable) functors, by which we mean that it constitutes a *colimit of functorial gauges*. This is expressed as follows: We consider the category of elements of a presheaf $\mathbf{P}$ in $Sets^{\mathfrak{A}^{op}}$, that is the category $\int(\mathbf{P},\mathfrak{A})$. Projection on the second coordinate of $\int(\mathbf{P},\mathfrak{A})$, gives a functor:

$$\int_{\mathbf{P}} : \int(\mathbf{P},\mathfrak{A}) \to \mathfrak{A}$$

defining the split discrete fibration induced by $\mathbf{P}$, where $\mathfrak{A}$ is the base indexing or localizing category of the fibration. Then, as a consequence of the Yoneda lemma we obtain a *natural transformation*:

$$\blacktriangleright : \mathbf{Colim}(\mathbf{y} \circ \int_{\mathbf{P}}) \cong \mathbf{P}$$

which is a *natural isomorphism*. More precisely, we may consider a signal transmitter/receiver from $(\mathbf{y} \circ \int_{\mathbf{P}})$ to $\mathbf{P}$, or equivalently from $\mathbf{Hom}(\int_{\mathbf{P}}(-),-)$ to $\mathbf{P}$. This is actually a *cocone*, which for every $(p,\mathcal{A})$ in $\int(\mathbf{P},\mathfrak{A})$, provides a component $\mathbf{y}_{\mathcal{A}}$, and thus, becomes *representable* within $Sets^{\mathfrak{A}^{op}}$ in terms of the natural transformation $\mathbf{y}_{\mathcal{A}} \to \mathbf{P}$. Now, by the Yoneda lemma we have a *bijective* correspondence between natural transformation $\mathbf{y}_{\mathcal{A}} \to \mathbf{P}$ and elements p of $\mathbf{P}(\mathcal{A})$. Thus, each arrow in the cocone of the form $\mathbf{y}_{\mathcal{A}} \to \mathbf{P}$ for each $\mathcal{A}$ in $\mathfrak{A}$ is a corresponding element p of $\mathbf{P}(\mathcal{A})$. It is straightforward to see that this cocone in $Sets^{\mathfrak{A}^{op}}$ is a *universal* one, and thus the natural transformation $\mathbf{Colim}(\mathbf{y} \circ \int_{\mathbf{P}}) \to \mathbf{P}$ is a natural isomorphism.

Another important consequence of the existence of mutual functorial gauges for two oppositely pointing functors between the same categories is the following: We have seen that $\overrightarrow{\mathbf{A}} : \mathfrak{C}^{op} \to \mathfrak{L}$, $\overleftarrow{\mathbf{A}} : \mathfrak{L} \to \mathfrak{C}^{op}$ have mutual functorial gauges with respect to each other if and only if there exist a natural transformation of the identity functor on $\mathfrak{C}^{op}$ (unit natural transformation) $\eta : \mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}}$, as well as a natural transformation of the identity functor on $\mathfrak{L}$ (counit natural transformation) $\varepsilon : \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}} \to \mathbf{id}_{\mathfrak{L}}$ obeying the triangular identities: $id_{\overrightarrow{\mathbf{A}}} = \varepsilon\overrightarrow{\mathbf{A}} \circ \overrightarrow{\mathbf{A}}\eta$, $id_{\overleftarrow{\mathbf{A}}} = \overleftarrow{\mathbf{A}}\varepsilon \circ \eta\overleftarrow{\mathbf{A}}$. If *both* the *unit* and the *counit* natural transformations of the adjunction formed by the functors $\overrightarrow{\mathbf{A}}$, $\overleftarrow{\mathbf{A}}$, are *natural isomorphisms* then we obtain an *equivalence* of the categories $\mathfrak{C}^{op}$, $\mathfrak{L}$, that is, we obtain a *functorial duality*:

$$\mathfrak{C}^{op} \simeq \mathfrak{L}$$

Thus, the notion of functorial duality is a consequence of the existence of mutual functorial gauges for two oppositely pointing functors between the same categories when both the unit and counit natural transformations of the formed adjunction are natural isomorphisms.

We reached the conclusion that two oppositely pointing functors between the same categories, $\overrightarrow{\mathbf{A}}:\mathfrak{C}^{op} \to \mathfrak{L}$ and $\overleftarrow{\mathbf{A}}:\mathfrak{L} \to \mathfrak{C}^{op}$ form a categorical adjunction, induced by the requirement of birepresentability of the bifunctor $\wedge(-,-)$, where $\overrightarrow{\mathbf{A}}$ is the left adjoint functor of the adjunction and symmetrically $\overleftarrow{\mathbf{A}}$ is the right adjoint functor of the adjunction, or equivalently $\overrightarrow{\mathbf{A}}$ and $\overleftarrow{\mathbf{A}}$ have mutual functorial gauges with respect to each other, if and only if there exist a natural transformation of the identity functor on $\mathfrak{C}^{op}$ (unit natural transformation):

$$\eta : \mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}}$$

as well as a natural transformation of the identity functor on $\mathfrak{L}$ (counit natural transformation):

$$\varepsilon : \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}} \to \mathbf{id}_{\mathfrak{L}}$$

obeying the following identities:

$$id_{\overrightarrow{\mathbf{A}}} = \varepsilon\overrightarrow{\mathbf{A}} \circ \overrightarrow{\mathbf{A}}\eta$$
$$id_{\overleftarrow{\mathbf{A}}} = \overleftarrow{\mathbf{A}}\varepsilon \circ \eta\overleftarrow{\mathbf{A}}$$

The *composite endofunctor* $\mathbf{T} := \overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}} : \mathfrak{C}^{op} \to \mathfrak{C}^{op}$, together with the *natural transformations* $\omega : \mathbf{T} \circ \mathbf{T} \to \mathbf{T}$, called *multiplication*, and also, $\eta : \mathbf{id}_{\mathfrak{C}^{\mathbf{op}}} \to \mathbf{T}$, called *unit*, where $\mathbf{id}_{\mathfrak{C}^{\mathbf{op}}}$ is the identity functor on $\mathfrak{C}^{op}$, is defined as a *monad* $(\mathbf{T}, \omega, \eta)$ on the category $\mathfrak{C}^{op}$, provided that, the diagrams below commute for each object $\mathcal{C}$ of $\mathfrak{C}^{op}$;

$$
\begin{tikzcd}
\mathbf{T}\mathcal{C} & \mathbf{T}^2\mathcal{C} \arrow[l, "\omega_{\mathcal{C}}"'] \\
\mathbf{T}^2\mathcal{C} \arrow[u, "\omega_{\mathcal{C}}"] & \mathbf{T}^3\mathcal{C} \arrow[l, "\mathbf{T}\omega_{\mathcal{C}}"'] \arrow[u, "\omega_{\mathbf{T}\mathcal{C}}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathbf{T}\mathcal{C} \arrow[d, equal] & \\
\mathbf{T}^2\mathcal{C} \arrow[ur, "\mathbf{T}\eta_{\mathcal{C}}"] \arrow[r, "\omega_{\mathcal{C}}"] & \mathbf{T}\mathcal{C} & \mathbf{T}^2\mathcal{C} \arrow[ul, "\eta_{\mathbf{T}\mathcal{C}}"'] \arrow[l, "\omega_{\mathcal{C}}"']
\end{tikzcd}
$$

From the above, we conclude that a *monad* $(\mathbf{T},\omega,\eta)$ on the category $\mathfrak{C}^{op}$ can be understood as a *monoid* in the category of *endofunctors* of $\mathfrak{C}^{op}$ with the morphisms being the *natural transformations* between them.

We have another dual relation; the *composite endofunctor* $\mathbf{O} := \overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}} : \mathfrak{L} \to \mathfrak{L}$, together with the *natural transformations* $\delta : \mathbf{O} \to \mathbf{O} \circ \mathbf{O}$, called *comultiplication*, and also, $\varepsilon : \mathbf{O} \to \mathbf{id}_{\mathfrak{L}}$, called *counit*, where $\mathbf{id}_{\mathfrak{L}}$ is the identity functor on $\mathfrak{L}$, is defined as a *comonad* $(\mathbf{O},\delta,\varepsilon)$ on the category $\mathfrak{L}$, provided that the diagrams below commute for each object $\mathcal{L}$ of $\mathfrak{L}$;

$$
\begin{tikzcd}
\mathbf{O}\mathcal{L} \arrow[r, "\delta_{\mathcal{L}}"] \arrow[d, "\delta_{\mathcal{L}}"'] & \mathbf{O}^2\mathcal{L} \arrow[d, "\delta_{\mathbf{O}\mathcal{L}}"] \\
\mathbf{O}^2\mathcal{L} \arrow[r, "\mathbf{O}\delta_{\mathcal{L}}"] & \mathbf{O}^3\mathcal{L}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathbf{O}\mathcal{L} \arrow[dl, equal] \arrow[d, "\delta_{\mathcal{L}}"] \arrow[dr, equal] & \\
\mathbf{O}\mathcal{L} & \mathbf{O}^2\mathcal{L} \arrow[l, "\varepsilon_{\mathbf{G}\mathcal{L}}"'] \arrow[r, "\mathbf{O}\varepsilon_{\mathcal{L}}"] & \mathbf{O}\mathcal{L}
\end{tikzcd}
$$

For a *comonad* $(\mathbf{O},\delta,\varepsilon)$ on $\mathfrak{L}$, a $\mathbf{O}$ -*coalgebra* (comodule) is an object $\mathcal{L}$ of $\mathfrak{L}$, being equipped with a *structural map* $\kappa:\mathcal{L}\to\mathbf{O}\mathcal{L}$, such that the following conditions are satisfied:

$$1_{\mathcal{L}}=\varepsilon_{\mathcal{L}}\circ\kappa:\mathcal{L}\to\mathcal{L}$$
$$\mathbf{O}\kappa\circ\kappa=\delta_{\mathcal{L}}\circ\kappa:\mathcal{L}\to\mathbf{O}^2\mathcal{L}$$

With the above obvious notion of morphism, this gives a *category* $\mathfrak{L}_{\mathbf{O}}$ of all $\mathbf{O}$ -*coalgebras*.

By dual correlation, if $(\mathbf{T},\omega,\eta)$ is a *monad* on the category $\mathfrak{C}^{op}$, we define the category of $\mathfrak{C}^{op}{}_{\mathbf{T}}$ -*algebras* (modules) as follows: Its objects are pairs $(\mathcal{C},\mu_{\mathcal{C}})$, where, $\mathcal{C}$ in $\mathfrak{C}^{op}$, and, $\mu:\mathbf{T}(\mathcal{C})\to\mathcal{C}$ is a morphism in $\mathfrak{C}^{op}$, such that, the following conditions are satisfied:

$$1_{\mathcal{C}}=\mu\circ\eta_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}$$
$$\mu\circ\mathbf{T}\mu=\mu\circ\omega_{\mathcal{C}}:\mathbf{T}^2\mathcal{C}\to\mathcal{C}$$

Since an adjunction between two categories $\mathfrak{C}^{op}$, $\mathfrak{L}$, defined by a pair of adjoint functors $\overrightarrow{\mathbf{A}}:\mathfrak{C}^{op}\to\mathfrak{L}$ and $\overleftarrow{\mathbf{A}}:\mathfrak{L}\to\mathfrak{C}^{op}$ always gives rise to a monad on the category $\mathfrak{C}^{op}$, viz. $(\mathbf{T}=\overleftarrow{\mathbf{A}}\overrightarrow{\mathbf{A}},\omega=\overleftarrow{\mathbf{A}}\varepsilon\overrightarrow{\mathbf{A}},\eta)$, as well as to a comonad on the category $\mathfrak{L}$, viz. $(\mathbf{O}=\overrightarrow{\mathbf{A}}\overleftarrow{\mathbf{A}},\delta=\overrightarrow{\mathbf{A}}\eta\overleftarrow{\mathbf{A}},\varepsilon)$, we say that $\overrightarrow{\mathbf{A}}$ and $\overleftarrow{\mathbf{A}}$ have *mutual functorial gauges* with respect to each other, if and only if a *monad-comonad* pair exists as above.

CANONICS OF FUNCTORIAL RELATIONS: SIEVES AND REPRESENTABILITY

9

9.1 FUNCTORIAL COORDINATIZATION: CATEGORIES OF RINGS OF OBSERVABLES

The first fundamental notion, which is necessary for the intelligibility of the modeling process of any natural system's behavior is representation. *Representation* is expressed categorically as a canonical functorial relation for formulating the bidirectional process of categorical structural correspondence between *natural systems* and *formal symbolic systems*. The concept of a natural system refers to a deliberate abstraction that schematizes the external world in terms of a dyadic relationship between the former and its environment.

Hence, it is suitable to adopt a *flexible*, non-rigid notion of what a natural system actually is which, eventually, allows a *variability* in the determination of the separating *boundary* between what constitutes a natural system and what its environment. The essential aspect of this relation amounts to representing a natural system, conceived in the above sense, by means of an appropriate formal system, capable of being effectively used for providing predictions about the behavior of the former. The notion of a formal system is formulated in the algebraic terms of *rings* capturing the structure of attributes of natural systems, *encoded* by means of generalized number-like quantities, called *observables*, where, the notion of observable signifies a physical attribute that, in principle, can be measured.

The basic hypothesis underlying the *bidirectional* process of representation of natural systems by mathematical formal systems, is that the behavior of the former can be adequately understood by establishing an appropriate *functorial congruence* between the structures of phenomena corresponding to the behavior of natural systems and some suitable algebraic structures of generalized number-like quantities. The latter are enriched with the semantics of observable attributes of natural systems in various measurement situations. Hence, the essential aspect of representation amounts to a structure preserving process of *coordinatization* or *arithmetization* of attributes in terms of observables. The strategy of arithmetization will prove to be successful, if and only if, a formal system becomes capable of providing predictions about the behavior of the natural system which represents.

Of course, as a condition required to substantiate that possibility, observables must amount to more than mere amorphous collections of coordinates; they must also be amenable to collective manipulations under the fundamental arithmetic operations of addition and multiplication, such that, the collection remains *closed* under these operations, preserving the algebraic *morphe* of such a *structure* of observables. Most significantly, the form of coordinatization of phenomena in the above sense, should not depend structurally on any particular natural system depicted in the external world. This means, that the form of the arithmetization process, effectuated in terms of an appropriate associatively closed universe (category) of algebraic structures of observables, should be constructed *covariantly* with respect to a corresponding associatively closed universe (category) of natural systems abstracted from the external world.

Categorically speaking, we mean that *arithmetization* should be properly expressed as a *functorial* process. More specifically, arithmetization is defined as a *functor* from the abstract category of natural systems to a concrete category of formal systems, identified algebraically, with structures of observables closed under the operations of addition and multiplication (rings). Note that, the notion of an observable denotes a physical attribute that, in principle, can be measured. The quantities admissible as measured results should belong to a number field, or more generally, to a ring of scalar quantities to be interpreted as generalized numbers in some context of observation. The latter defines the *valuation* codomain of observables in terms of measurement *scales*, conforming to the algebraic specification of, *at least*, the *ring*-theoretic structure. Note also that, the designation of both, the observables and the measurement scales, should be *structurally* on an *equal* footing, meaning that they should both be algebraic structures (rings) in a *category* of objects of the same kind (category of rings).

In this sense, the process of measurement is precisely formalized by means of a *surjective* morphism of rings, within the corresponding category of formal systems. It is important to note that, from this perspective, an abstract ring of number-like elements (scalars) simultaneously incorporates a *dual role* within the categorical environment where it operates. More concretely, if it stands on the *domain* of a surjective morphism of rings it is interpreted as a closed algebraic structure of observables *measured* by means of a closed algebraic structure of the same form consisting of measurement scales or numbers in the codomain of this morphism, within that category.

Equally, if it stands on the *codomain* of the rings morphism, it is interpreted as an algebraic structure of number-like values, which *measure* the observables belonging in the domain-ring. The essence of this *duality* of roles amounts to a *symmetrization* of the notions of *observer* and *observed* in an algebraic categorical framework of reasoning based primarily on the concept of structure preserving transformation between objects.

From this perspective, the *same* algebraic object can serve simultaneously, as *both* an observer and as an observed, *depending on* the *context* of interpretation of morphisms in the algebraic category emulating formal systems. In more detail, the positioning of a ring of number-like elements in the *codomain* of a surjective morphism acquires the semantics of an *observer*, that coordinatizes the observables of the system it measures in terms of distinguishable measurement scales. On the other hand, the positioning of that ring in the *domain* of a surjective morphism in the algebraic category, acquires the semantics of an *observed*, measured by means of evaluation of its observable attributes at the measurement scales of the codomain-ring.

In this sense, we are able to capture the natural duality between the symmetric functional roles of observer-observed in a categorical context of interpretation of surjective algebraic morphisms of rings. According to this relational categorical framework, the description of attributes related to the behavior of a natural system should be *covariant* with respect to *all* rings of measurement scales implemented as observational means, or equivalently, covariant with respect to the base ring (of observer's measurement scales) *change*.

We conclude that, arithmetization or coordinatization constitutes a directed functorial process $\mathbf{E}:\mathcal{S}\to\mathcal{F}$ of *encoding* attributes related to an abstract category of natural systems $\mathcal{S}$, in terms of a concrete category of formal systems $\mathcal{F}$, constructed algebraically as a category of rings of scalar number-like quantities (coordinates), interpreted as observables of natural systems, which can be measured by means of surjective morphisms (evaluations) in the latter algebraic category. We say that the modeling of the abstract category of natural systems by the concrete category of formal systems is proper, if an *inverse* functorial process $\mathbf{D}:\mathcal{F}\to\mathcal{S}$ of *decoding* exists, which can be used for making predictions about the behavior of natural systems, such that, the pair of functorial processes $(\mathbf{E},\mathbf{D})$ constitutes a *categorical adjunction*:

$$\mathbf{E}:\mathcal{S}\rightleftarrows\mathcal{F}:\mathbf{D}$$

Thus, the bidirectional functorial process of representation of natural systems by formal systems is defined accurately as a proper categorical relation of *metaphora*, according to the previous adjunction. Finally, the encoding arithmetization functor $\mathbf{E}:\mathcal{S}\to\mathcal{F}$ is said to define an *equivalence* between the categories of natural systems and formal systems respectively, if a decoding functor $\mathbf{D}:\mathcal{F}\to\mathcal{S}$ exists, such that the composite functor $\mathbf{DE}$ is isomorphic to $\mathbf{Id}_S$ and the composite functor $\mathbf{ED}$ is isomorphic to $\mathbf{Id}_F$.

9.2 REPRESENTABLE FUNCTOR: MEASUREMENT AND COMMUNICATION

Let us now concentrate our attention on the category of formal systems $\mathcal{F}$, in an attempt to describe explicitly its representational functionality. The category $\mathcal{F}$ is defined as a category of unital rings of observables that coordinatize structurally the attributes of natural systems. The observables are measured by means of *evaluation* morphisms in appropriately specified rings of measurement *scales* in the same category $\mathcal{F}$. A ring in $\mathcal{F}$ is *commutative* if and only if the multiplication is commutative.

We will restrict ourselves intentionally to the case of commutative rings of scalars. This is due to the fact that the hypothesis of commutativity of the observable attributes of a natural system can be satisfied at least *locally* within the categorical environment for all natural systems, even if, globally their rings of observables are only *partially* commutative, or even *non-commutative*.

For the sake of completeness, we should recall that, a ring is a division ring, if and only if, every non-zero element has a multiplicative inverse, whereas, in case that it is also commutative, it is called a *field*. Moreover, a commutative ring without non-trivial divisors of zero is called an *integral domain*. According to our previous comments, the morphisms in $\mathcal{F}$ are considered to be the additive and multiplicative identity-preserving homomorphisms of commutative rings. The surjective morphisms in $\mathcal{F}$ formalize the structure preserving process of measurement of observables (contained in the domain of a rings-morphism) with respect to observers' measurement scales (contained in the codomain respectively), according to the preceding physical interpretation.

The essential aspect of casting the measurement process in a categorical form, as above, is twofold: Firstly, the categorical framework provides the means to state precisely a *criterion* of complete determination of a natural system's behavior, formulated in terms of its coordinatization ring of observables, by way of measurements in variable rings of measurement scales, according to the Yoneda-Grothendieck philosophy. Secondly, the same framework, seen from a *dual* categorical perspective, permits the *geometric* encoding of the information acquired by evaluating a physical attribute at a measurement scale, in terms of the notion of *state*. Thus, it becomes possible to conceptually and technically unify, through duality transformations in the category of formal systems, the *algebraic information* acquisition by measuring observables at scales, together with, the *geometric representation* of that information in terms of state-spaces.

The covariant description of the measurement process of natural systems in relation to the above two-fold interpretational schema renders a functorial formulation of that process. In turn, this is to be achieved using the machinery of *representable* functors. More precisely, we consider the category of formal systems restricted to surjective morphisms of unital commutative rings of observables, and let $\mathcal{A}$ be a ring of observables of a corresponding natural system. Then, the functor *represented* by $\mathcal{A}$ is the *covariant* functor $\mathbf{y}_{\mathcal{A}} : \mathcal{F} \to \mathbf{Sets}$, defined as follows:

i For all rings $\mathcal{X}$ in $\mathcal{F}$, $\mathbf{y}_{\mathcal{A}}(\mathcal{X}) := Hom_{\mathcal{F}}(\mathcal{A}, \mathcal{X})$.

ii For all rings-morphisms $f : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{F}$,

$$\mathbf{y}_{\mathcal{A}}(f) : Hom_{\mathcal{F}}(\mathcal{A}, \mathcal{X}) \to Hom_{\mathcal{F}}(\mathcal{A}, \mathcal{Y})$$

is defined as post-composition with f, viz., $\mathbf{y}_{\mathcal{A}}(f)(g) := f \circ g$.

The covariant representable functor $\mathbf{y}_{\mathcal{A}} : \mathcal{F} \to \mathbf{Sets}$, can be thought of as constructing an image of $\mathcal{F}$ in $Sets$ covariantly, or equivalently, as a covariant translation of the information induced by measurement procedures in the category $\mathcal{F}$ into that of $Sets$. The important thing to notice is that, if we set up some measurement procedure of the observables of a natural system, represented formally as a surjective morphism of rings $f : \mathcal{A} \to \mathcal{B}$, then we get an *induced* morphism of representable functors (*natural transformation*) $\mathbf{y}_f : \mathbf{y}_{\mathcal{B}} \to \mathbf{y}_{\mathcal{A}}$ by pre-composing with f.

Subsequently, the implementation of the Yoneda lemma in the current setting, gives that, if $\eta : \mathbf{y}_{\mathcal{B}} \to \mathbf{y}_{\mathcal{A}}$ is *any* natural transformation of covariant representable functors, then, there is a measurement procedure, a morphism of rings $f : \mathcal{A} \to \mathcal{B}$, such that, $\eta = \mathbf{y}_f$. This implies, in particular, that if two rings of observables represent *isomorphic functors*, then the corresponding formal systems themselves are *isomorphic*.

The importance of this fact, constituting the cornerstone of the Yoneda-Grothendieck philosophy, is that a formal system conceived within the associatively closed structure of a category of objects and structure-preserving morphisms of the same form $\mathcal{F}$, encoded algebraically as a ring of observables of a natural system, can be *classified* completely up to *unique* isomorphism, by analyzing the set-valued functor that it *represents*. More precisely, the information encoded in the ring of observables of a natural system, can be *recovered* completely by means of *all* measurement procedures applied to that ring, incorporated in the definition of the corresponding representable functor as ring-morphisms from that ring of observables towards variable rings of measurement scales of observers within the same category.

Consequently, this realization means that the behavior of a corresponding natural system within $\mathcal{S}$, can be reproduced in its entirety, by studying the totality of *variable* observational perspectives imposed upon it within $\mathcal{S}$, or equivalently, the totality of its interactions with all other natural systems within $\mathcal{S}$. Note that, the notion of variable observational perspective on a natural system in $\mathcal{S}$, is operationally realized by the *instantiation* of measurement procedures of the coordinatization ring of observables $\mathcal{A}$ of that natural system with respect to variable rings of measurement scales in $\mathcal{F}$, and consequently, modeled by means of the corresponding *covariant representable* functor $\mathbf{y}_{\mathcal{A}} : \mathcal{F} \to \mathbf{Sets}$. In the same vein of ideas, we may say conclusively that, the abstraction related to the notion of a natural system, within a closed categorical environment $\mathcal{S}$, is completely understood by the system of all *relational* referential viewpoints on it (variable observational perspectives) instantiated within $\mathcal{S}$, encoded as variable measurement procedures of its coordinatization ring of observables with respect to variable rings of measurement scales within the modeling category of formal systems $\mathcal{F}$. This is finally translated covariantly in $Sets$ via the corresponding representable functor of that ring of observables.

9.3 FUNCTORIAL SPECTRUM: SIEVES AND MULTI-LAYERED RESOLUTION

At a further stage of development, the operational role of the covariant representable functor of a ring of observables, classifying it completely up to unique isomorphism, is equivalent to the functorial process of translating geometrically the information collected by all evaluations of that ring at all rings of measurement scales, in sum, the information collected by the totality of variable measurement procedures as above. In this manner, the *geometric encoding* of the information related to the behavior of a natural system is being generated *functorially* through the functioning of the covariant representable functor of the ring of observables arithmetizing the former.

The key concept that explicates the geometric representation of natural systems in appropriate functorial terms is the notion of state. Subsequently, the geometric representation of natural systems in terms of state-spaces is generated functorially. Generally speaking, a *state* of a ring of observables $\mathcal{A}$ *over* a ring of measurement scales $\mathcal{B}$ (called a $\mathcal{B}$-state of $\mathcal{A}$) signifies the geometric encoding of the information acquired by evaluating the physical attributes contained in the former ring at the measurement scales of the latter. Thus, any $\mathcal{B}$-state of $\mathcal{A}$ is a geometric representation of a morphism of rings $f : \mathcal{A} \to \mathcal{B}$ in the intended semantic interpretation.

Consequently, the *set* of *all* morphisms of rings $f:\mathcal{A}\to\mathcal{B}$, where the domain ring is considered as a ring of observables and the codomain ring as a ring of measurement scales, should be tautosemous with the set of all $\mathcal{B}$-states of $\mathcal{A}$. This set is called the $\mathcal{B}$*-spectrum* of the unital ring of observables $\mathcal{A}$, where the ring $\mathcal{B}$ is called the *coordinatizing frame* of that state. The geometric semantics of this connotation denotes the set of elements which can be $\mathcal{B}$-observed by a measurement procedure on $\mathcal{A}$. Eventually, that set of elements, constituting the $\mathcal{B}$-spectrum of $\mathcal{A}$, are properly identified with the $\mathcal{B}$*-coordinatized points* of a geometric state-space that can be observed by means of the ring $\mathcal{A}$. Obviously, the above conception of a geometric state-space of a natural system is *functorial,* since it admits a *covariant* description with respect to variation of the evaluation ring of measurement scales $\mathcal{B}$ in the category $\mathcal{F}$. This fact effectively means that, the geometric state space of a natural system is identified with the covariant representable functor $\mathbf{y}_{\mathcal{A}}:\mathcal{F}\to\mathbf{Sets}$ of the coordinatization ring of observables of that natural system.

For this reason, $\mathbf{y}_{\mathcal{A}}$ is called the *functorial spectrum* of $\mathcal{A}$, denoted by $\mathrm{S}pec_{\mathcal{A}}$, which gives rise to a *spectral sieve* of $\mathcal{A}$. It describes functorially the *multi-layered* geometric state-space related to the behavior of a natural system under *variable* observational perspectives, where its evaluation at a layer $\mathcal{B}$, constituting the $\mathcal{B}$-spectrum of the ring of observables $\mathcal{A}$, gives the set of all $\mathcal{B}$-states of $\mathcal{A}$. In this vein of ideas, each ring of measurement scales, where the evaluation of a ring of observables takes place, by the effectuation of a corresponding measurement procedure, is the *spectral carrier* of a specific geometric *layer of spatiality*, corresponding to the ontological observational perspective of *point-schematization* dictated by the nature of measurement scales contained in that ring.

9.4 RELATIVIZATION: MODULES AND REPRESENTABILITY OF COMPOSITION

A natural question arising in this categorical setting is the following: Is it possible to express the notion of a *module* of a commutative unital ring of observables $\mathcal{A}$ in the category $\mathcal{F}$, *intrinsically* with respect to the information contained in the category $\mathcal{F}$? This can be accomplished by using the method of *categorical relativization*, which is based on the passage to the *slice* category $\mathcal{F}/\mathcal{A}$. More concretely, the basic problem has to do with the possibility of representing the information contained in an $\mathcal{A}$-module, where $\mathcal{A}$ is a commutative unital ring of observables in $\mathcal{F}$, by means of a suitable object of the relativization of $\mathcal{F}$ with respect to $\mathcal{A}$, namely with an object of the slice, or comma category $\mathcal{F}/\mathcal{A}$.

For this purpose, we define the *split extension* of the commutative ring $\mathcal{A}$ by an $\mathcal{A}$-module M, denoted by $\mathcal{A}\bigoplus M$, as follows: The underlying set of $\mathcal{A}\bigoplus M$ is the cartesian product $\mathcal{A}\times M$, where the group and ring theoretic operations are defined respectively as;

$$(a,m)+(b,n):=(a+b,m+n)$$
$$(a,m)\bullet(b,n):=(ab,a\cdot n+b\cdot m)$$

Note that the identity element of $\mathcal{A}\bigoplus M$ is $(1_{\mathcal{A}},0_M)$, and also that, the split extension $\mathcal{A}\bigoplus M$ contains an *ideal* $0_{\mathcal{A}}\times M:=\langle M\rangle$, that corresponds *naturally* to the $\mathcal{A}$-module M. Thus, given a commutative ring $\mathcal{A}$ in $\mathcal{F}$, the information of an $\mathcal{A}$-module M, consists of an object $\langle M\rangle$ (*ideal* in $\mathcal{A}\bigoplus M$), together with a *split short exact sequence* in $\mathcal{F}$;

$$\langle M\rangle \rightarrowtail \mathcal{A}\oplus M \twoheadrightarrow \mathcal{A}$$

We infer that the ideal $\langle M\rangle$ is identified with the *kernel* of the epimorphism $\mathcal{A}\oplus M \twoheadrightarrow \mathcal{A}$:

$$\langle M\rangle = Ker(\mathcal{A}\oplus M \twoheadrightarrow \mathcal{A})$$

From now on we focus our attention to the comma category $\mathcal{F}/\mathcal{A}$, noticing that $id_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}$ is the *terminal* object in this category. If we consider the split extension of the commutative ring of observables $\mathcal{A}$, by an $\mathcal{A}$-module M, that is $\mathcal{A}\bigoplus M$, then the morphism:

$$\lambda:\mathcal{A}\bigoplus M\to\mathcal{A}$$
$$(a,m)\mapsto a$$

is obviously an object of $\mathcal{F}/\mathcal{A}$. Moreover, it easy to show that it is actually an *Abelian group* object in the comma category $\mathcal{F}/\mathcal{A}$. This equivalently means that for every object ξ in $\mathcal{F}/\mathcal{A}$ the set of morphisms $Hom_{\mathcal{F}/\mathcal{A}}(\xi,\lambda)$ is an Abelian group in **Sets**. Moreover, the arrow $\gamma:\kappa\to\lambda$ is a morphism of Abelian groups in $\mathcal{F}/\mathcal{A}$, if and only if for every ξ in $\mathcal{F}/\mathcal{A}$ the morphism:

$$\hat{\gamma}_{\xi}:Hom_{\mathcal{F}/\mathcal{A}}(\xi,\kappa)\to Hom_{\mathcal{F}/\mathcal{A}}(\xi,\lambda)$$

is a morphism of Abelian groups in **Sets**. We denote the *category of Abelian group objects* in $\mathcal{F}/\mathcal{A}$ by the suggestive symbol $[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}}$. Based on our previous remarks, it is straightforward to show that the category of Abelian group objects in $\mathcal{F}/\mathcal{A}$ is *equivalent* with the *category* of $\mathcal{A}$*-modules*:

$$[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}}\cong\mathcal{M}^{(\mathcal{A})}$$

Thus, we have managed to characterize $\mathcal{A}$-modules intrinsically as Abelian group objects in the *relativization* of the category of commutative unital rings of observables $\mathcal{F}$ with respect to $\mathcal{A}$, and moreover, we have concretely identified them as kernels of split extensions of $\mathcal{A}$.

Let us now consider two $\mathcal{A}$-modules M, N. The *tensor product* of M and N over $\mathcal{A}$, denoted by $M\bigotimes_{\mathcal{A}}N$ is the *unique* object in $[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}}\cong\mathcal{M}^{(\mathcal{A})}$, which satisfies the following *universal* property: There exists a *bilinear* morphism $\tau:M\times N\to M\bigotimes_{\mathcal{A}}N$ such that given *any* $\mathcal{A}$*-module* S and *any bilinear* morphism $f:M\times N\to S$ there exists a *unique* $\mathcal{A}$-modules (*linear*) morphism $g:M\bigotimes_{\mathcal{A}}N\to S$ such that the bilinear morphism admits the *factorization* $f=g\circ\tau$. Consequently, factorization via the tensor product of two $\mathcal{A}$-modules M, N, that is, $M\bigotimes_{\mathcal{A}}N$, should be understood as the *universal* way to *linearize* any bilinear morphism form $M\times N$ to any other $\mathcal{A}$-module S. Note that, since the tensor product construction is defined by a universal mapping property it is *unique* up to unique isomorphism, as a consequence of Yoneda's lemma.

The tensor product of two $\mathcal{A}$-modules M, N, viz. $M\bigotimes_{\mathcal{A}}N$, is *generated* by elements of the form $m\otimes n$. This means that every element of $M\bigotimes_{\mathcal{A}}N$ is of the form $\sum_{i=1}^{k}m_i\otimes n_i$, where $m_i\in M$, $n_i\in N$ for $1\leq i\leq k$. Moreover, if M, N are *free* $\mathcal{A}$-modules of *finite* rank m,

n correspondingly, then their tensor product over $\mathcal{A}$, $M \bigotimes_{\mathcal{A}} N$ is a *free* $\mathcal{A}$-module of finite rank mn.

The tensor product in the category $\mathcal{M}^{(\mathcal{A})}$ is an *associative*, *commutative* and *unital* operation, that is:

i If M, N, P are $\mathcal{A}$-modules, then:

$$(M \bigotimes_{\mathcal{A}} N) \bigotimes_{\mathcal{A}} P = M \bigotimes_{\mathcal{A}} (N \bigotimes_{\mathcal{A}} P)$$

ii If M, N are $\mathcal{A}$-modules, then:

$$M \bigotimes_{\mathcal{A}} N = N \bigotimes_{\mathcal{A}} M$$

iii The commutative ring $\mathcal{A}$ is a *unit* for the tensor product, that is, for any $\mathcal{A}$-modules M:

$$\mathcal{A} \bigotimes_{\mathcal{A}} M = M \bigotimes_{\mathcal{A}} \mathcal{A} = M$$

Most importantly, the tensor product is a *functorial* operation, meaning that if M is an $\mathcal{A}$-module, then:

$$M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$$

is a *functor*, defined on objects as:

$$(M \bigotimes_{\mathcal{A}} -)(N) = M \bigotimes_{\mathcal{A}} N$$

and on morphisms $f : N \to L$ in $\mathcal{M}^{(\mathcal{A})}$ as follows:

$$(M \bigotimes_{\mathcal{A}} -)(f) = 1 \otimes f : M \bigotimes_{\mathcal{A}} N \to M \bigotimes_{\mathcal{A}} L$$

It is also defined on generators by:

$$(1 \otimes f)(m \otimes n) = m \otimes f(n)$$

The functor $M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$ is *right exact*. More precisely, if we consider a *short exact sequence* of $\mathcal{A}$-modules:

$$0 \to N_1 \to N \to N_2 \to 0$$

then, *tensoring* by M over $\mathcal{A}$, we get the *right exact sequence* of $\mathcal{A}$-modules:

$$M \bigotimes_{\mathcal{A}} N_1 \to M \bigotimes_{\mathcal{A}} N \to M \bigotimes_{\mathcal{A}} N_2 \to 0$$

Note that, in general, the above sequence of $\mathcal{A}$-modules is *not left exact* as well. In case that the tensor product functor $M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$ is *exact* (meaning that it is left exact as well) then the $\mathcal{A}$-module M is called a *flat* module.

The conceptual meaning of the tensor product functor $M \bigotimes_{\mathcal{A}} - : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$ and consequently of the tensor product operation on the category of $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})}$ arises if we consider the *covariant representable functor* in the category $[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}} \cong \mathcal{M}^{(\mathcal{A})}$ valued in the same category, that is the covariant functor:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, -) : \mathcal{M}^{(\mathcal{A})} \to \mathcal{M}^{(\mathcal{A})}$$

represented by the $\mathcal{A}$-module M, defined as follows:

i For all $\mathcal{A}$-modules N in $\mathcal{M}^{(\mathcal{A})}$, the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor maps N to the $\mathcal{A}$-module $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, N)$.

ii For all $\mathcal{A}$-modules $f : N \to P$ in $\mathcal{M}^{(\mathcal{A})}$,

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(f) : \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, N) \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, P)$$

is defined as post-composition with f, viz., $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(f)(g) := f \circ g$.

The advantage of considering the covariant representable functor in the category $\mathcal{M}^{(\mathcal{A})}$, that is, $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, -)$, is that, by virtue of the categorical equivalence $[\mathcal{F}/\mathcal{A}]_{\mathbf{Ab}} \cong \mathcal{M}^{(\mathcal{A})}$, we *retain* the physical interpretation of the $\mathbf{Hom}$-functor (enriched in the category of $\mathcal{A}$-modules) in terms of measurement procedures relativized with respect to a fixed ring of observables $\mathcal{A}$, according to the Yoneda-Grothendieck philosophy explained previously.

The covariant representable functor in the category $\mathcal{M}^{(\mathcal{A})}$, viz. $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$ is a *left exact* functor. More precisely, if we consider a short exact sequence of $\mathcal{A}$-modules:

$$0 \to N_1 \to N \to N_2 \to 0$$

then, applying the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor, we get the *left exact sequence* of $\mathcal{A}$-modules:

$$0 \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,N_1) \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,N) \to \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,N_2)$$

Note that, in general, the above sequence of $\mathcal{A}$-modules is *not* right exact as well. In case that the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor is exact (meaning that it is right exact as well) then the $\mathcal{A}$-module M is called a *projective* module. Analogously we may consider the contravariant representable functor in the category $\mathcal{M}^{(\mathcal{A})}$, that is, $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(,M)$. If the latter functor is *exact* then the $\mathcal{A}$-module M is correspondingly called an *injective* module. Another immediate observation is that:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(\mathcal{A},N) = N$$

The above properties of the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor, $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$, and the corresponding properties of the tensor product functor $(M \bigotimes_{\mathcal{A}} -)$ imply that they stand for *inverse functorial processes* in the category of $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})}$. This reveals the presence of an *adjunction*, that is, $(M \bigotimes_{\mathcal{A}} -)$ and $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$ are *adjoint functors*, described by the *natural isomorphism*:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M \bigotimes_{\mathcal{A}} N,P) \cong \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(N,P))$$

for any $\mathcal{A}$-modules M, N and P, where the tensor product functor is the *left adjoint* and the covariant $\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}$-functor is the *right adjoint* of the $\left[\otimes \rightleftarrows \mathbf{Hom}\right]_{\mathcal{M}(\mathcal{A})}$ adjunction.

Consequently, the physical meaning of the tensor product operation in the category of $\mathcal{A}$-modules $\mathcal{M}^{(\mathcal{A})} \cong [\mathcal{F} / \mathcal{A}]_{\mathbf{Ab}}$ is obtained as follows: We consider the interpretation of the covariant **Hom**-functors $\mathbf{Z}_M := \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M,-)$ and $\mathbf{Z}_N := \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(N,-)$ (enriched in the category of $\mathcal{A}$-modules) in terms of corresponding measurement procedures on the rings $\mathcal{A} \bigoplus M$ and $\mathcal{A} \bigoplus N$ relativized with respect to the fixed ring of observables $\mathcal{A}$.

Then, the *composition* of the covariant **Hom**-functors $\mathbf{Z}_M$ and $\mathbf{Z}_N$, by which we mean the composition of the corresponding measurement procedures with respect to the fixed ring $\mathcal{A}$, becomes *representable* in the category of $\mathcal{A}$-modules by the tensor product *entanglement* of M and N, that is we have the *natural isomorphism*:

$$\mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M \bigotimes_{\mathcal{A}} N, P) \cong \mathbf{Z}_M \circ \mathbf{Z}_N(P) = \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(M, \mathbf{Hom}_{\mathcal{M}^{(\mathcal{A})}}(N, P))$$

9.5 COMPARISON OF THE FUNCTORIAL WITH THE CLASSICAL REPRESENTABILITY

At this stage, a brief discussion of the formal model of a natural system idealized by classical theories will serve to throw light on the connection with the categorical generalization implied by the previous analysis. The basic postulate of classical theories stipulates in advance that the form of observation be globally expressible by *real number representability*, and subsequently, observables are modeled by *real-valued* functions corresponding to measuring devices calibrated to register real numbers.

At a further stage of development of this idea, two further assumptions are imposed on the structure of observables: the first of them specifies the *algebraic* nature of the set of all observables used for the description of a natural system, by assuming the structure of a commutative unital ring, which is, a *commutative unital algebra* $\mathcal{A}$ over the *real* numbers. The second assumption restricts the content of the set of real-valued functions corresponding to physical observables to those that admit a mathematical characterization as *measurable*, *continuous* or *smooth*.

Thus, depending on the means of description of a physical system, observables are modeled by $\mathbb{R}$-algebras of measurable, continuous or smooth functions corresponding to suitably specifiable measurement environments in each case. Usually the assumption of *smoothness* is postulated because it is desirable to consider *derivatives* of observables and effectively set-up a *dynamical* framework of description in terms of *differential equations*. Moreover, since we have initially assumed that real-number representability constitutes the prefixed form of observation in terms of the readings of measuring devices, the set of all $\mathbb{R}$ -algebra unital morphisms $\mathcal{A} \to \mathbb{R}$, assigning to each observable in $\mathcal{A}$, the reading of a measuring device in $\mathbb{R}$, encapsulates all the *states-related* information collected about a system in measurement situations in terms of algebras of real-valued observables.

Mathematically, the set of all $\mathbb{R}$ -algebra morphisms $\mathcal{A} \to \mathbb{R}$ is identified as the $\mathbb{R}$ -*spectrum* of the unital commutative algebra of observables $\mathcal{A}$. The physical semantics of this connotation denotes the set that can be $\mathbb{R}$ -*observed* by means of this algebra. It is well known that, in case $\mathcal{A}$ stands for a smooth algebra of real-valued observables, $\mathbb{R}$ -algebra morphisms $\mathcal{A} \to \mathbb{R}$ can be legitimately identified (thanks to the *Gelfand duality*) with the $\mathbb{R}$ -*coordinatized points* of a space, which can be *observed* by means of $\mathcal{A}$. These are the points of a compact real differential *manifold* that, in turn, denote the states of the observed system. From this perspective, geometric state spaces in classical theories are compact real differential manifolds N consisting of sets of points being $\mathbb{R}$ -observed by means of unital $\mathbb{R}$ -algebras of smooth real-valued functions, denoted by $\mathcal{C}^{\infty}(N)$.

If we attempt a comparison of the functorial with the classical case, outlined above, we notice that, according to the generalized definition of a geometric state introduced previously, each state of a ring of observables $\mathcal{A}$ may have a *different coordinatizing frame*, depending upon the ring of scales employed for measurement. Thus, the *new* notion of a geometric state-space, is a *multi-valued* one, in the sense that its generalized points may be coordinatized by means of *different scales*, namely scales belonging to different rings. Thus, in contradistinction to the classical conception, the notion of generalized state introduced, *rejects* the *absolute representability* principle of the classical theory over the coordinatizing field of real numbers, rather to allow, in this sense, the *geometric* representation of states, in terms of *generalized* points of a *multi-valued* geometric state space, as well as, the *evaluation* of observable attributes at those points. This is achieved by the *relativization* of *representability* over a multitude of measurement scales, belonging to different coordinatizing rings, giving eventually rise to the above multi-valued geometric state space.

9.6 THE GEOMETRIC QUALIFICATION OF THE SPECTRUM FUNCTOR

We have already concluded that, in the context of the functorial approach to modeling a natural system's behavior, the geometric state space of a natural system is identified with the *covariant representable functor* $\mathbf{y}_A : \mathcal{F} \to \mathbf{Sets}$ of the coordinatization ring of observables of that natural system. For this reason, the functor $\mathbf{y}_A$ is called the *spectrum* of $\mathcal{A}$, and denoted as, $Spec_A$.

It is essential to examine the geometric semantics of the Spectrum functor in some detail. If we consider the opposite of the category of rings of observables, that is, the category with the same objects but with arrows *reversed* $\mathcal{F}^{op}$, each object in the context of this category can be thought of as the *locus of states* of a ring of observables, or else it carries the connotation of *space*. The crucial observation is that, any such space is determined, up to canonical isomorphism, if we know *all* morphisms into this locus from *any* other locus in that category. For instance, the set of morphisms from the one-point state locus to the locus A in the categorial context of $\mathcal{F}^{op}$ determines the set of point-states of the locus A. The philosophy behind this approach amounts to treating *any* morphism in $\mathcal{F}^{op}$ with the locus A for target as a *generalized point state* of A.

Let us consider the category of loci of states $\mathcal{F}^{op}$, and let A be an object in this category. Then, the functor *represented* by A is the *contravariant* functor $\mathbf{y}^A : \mathcal{F}^{op} \to \mathbf{Sets}$, defined as follows:

i For all loci of states B in $\mathcal{F}^{op}$, $\mathbf{y}^A(B) := Hom_{\mathcal{F}^{op}}(B, A)$.

ii For all loci-morphisms $f : C \to B$ in $\mathcal{F}^{op}$,

$$\mathbf{y}^A(f) : Hom_{\mathcal{F}^{op}}(B, A) \to Hom_{\mathcal{F}^{op}}(C, A)$$

is defined as pre-composition with f, viz., $\mathbf{y}^A(f)(g) := g \circ f$.

The functor $\mathbf{y}^A : \mathcal{F}^{op} \to \mathbf{Sets}$ being represented by the locus of states A, is called the *functor of generalized point states* of A. Moreover, the information contained in the locus of states A is classified completely by its functor of generalized point states $\mathbf{y}^A$. Hence, the functor $\mathbf{y}^A$ gives a *geometric* form to the abstract extension of the spatial locus of states A in the environment of $\mathcal{F}^{op}$. From the above, the direct conclusion is that the *Spectrum functor* can be specified *equivalently* in a dual manner: it can be specified by means of the *contravariant* representable functor from the category of loci of states to the category of sets, or equivalently, via the *covariant* representable functor from the category of rings of observables to the category of sets. Thus, it admits a well-defined operational determination in terms of measurement procedures referring to a coordinatization ring of observables, according to our preceding remarks.

9.7 QUANTUM SYSTEMS: FUNCTORIAL GEOMETRIC SPECTRA

At this stage there is every need to distinguish between classical and quantum systems. A category of classical natural systems, according to the above general description, admits a representation in terms of a category of formal systems (commutative, unital $\mathbb{R}$ -algebras of observable attributes) whose qualitative features are well understood and can be simultaneously determined with precision by means of valuations to the field of real numbers.

In this sense, a category of systems is characterized as a category of *quantum* systems in relation to the global complexity of the corresponding representing *formal* systems, with respect to those used to model the behavior of *classical* systems. This aspect of global complexity is, according to quantum theory, due to the *inability* of *simultaneous precise* measurement of all the attributes of the corresponding natural system within the same *local* measurement *context*.

More concretely, *Heisenberg's uncertainty* principle sets the *limits* of simultaneous precise measurement of *incompatible* observables by means of valuations in the field of real numbers, like position and momentum, within the same measurement context. On the other hand, *maximal families* of compatible observables can be *simultaneously* measured *precisely* within some appropriate measurement context, but these families stand in a *complementary* relation to each other with respect to the property of *incompatibility* of their observables. Thus, the behavior of a quantum system can be *approximated* in terms of maximal compatible families of simultaneously measurable observables within an appropriate local commutative measurement context, considered together with the notion of complementarity between such families.

In this frame of reasoning, the functorial conception of natural systems proves to be particularly relevant for the representation of quantum systems. More specifically, in striking *contrast* to the *global* classical conception, the notion of *generalized state* of a natural system, *rejects* the *absolute* global and simultaneous representability of all observables of the classical theory over the coordinatizing field of real numbers, rather it allows, in this sense, the geometric representation of states, in terms of *generalized* points of a *multi-valued* geometric state space, as well as, the *evaluation* of observable attributes at those points. This is achieved by a process of *local relativization* of physical *representability* with respect to a *multitude* of measurement scales, belonging to *different local commutative* coordinatizing rings, leading eventually to the above multi-valued geometric state space.

In relation to quantum theory, if we further assume that *local* commutative rings of measurement scales, corresponding to maximal families of compatible observables, can *co-finally* be translated simultaneously into the field of real numbers by means of suitable event-registering measuring devices, then the representation of quantum systems can be made possible by means of *diagrams* of simultaneously determinable *commutative* and *unital local* $\mathbb{R}$ -algebras of compatible observables.

This essentially means that the transition from the classical to the quantum representation of natural systems, is effectuated by the *replacement* of a *category* of commutative unital $\mathbb{R}$ -algebras of observables evaluated globally and simultaneously to the field of real numbers, by a *category of appropriate diagrams* of local commutative unital $\mathbb{R}$ -algebras of compatible observables, amenable to a local simultaneous translation into the field of real numbers. Most importantly, such a semantic transition becomes possible only via the *functorial* representational framework of natural systems, which entails the relativization of physical representability of observable attributes of natural systems with respect to varying local commutative coordinatizing frames of scale coefficients.

The force of this present line of argument will be missed without a precise indication of the sense in which the qualifications of global and local nature are used here. We have seen that in the environment of the functorial representation of natural systems, a representing formal system stands for a structure specified concretely as a coordinatizing unital ring of observables of some corresponding natural system. Thus, the *distinction* between classical and quantum systems should also be reflected in the appropriate *qualification* of their corresponding *rings* of observables.

The crucial distinguishing requirement with respect to the coordinatizing rings of observables has to do with the property of *global* commutativity. In this line of thought, a classical system is being globally represented by a commutative ring ($\mathbb{R}$ -algebra) of observable attributes, whereas, a quantum one is represented by a ring of observables attributes which is only *partially commutative*, and thus, *globally non-commutative*. The conceptual underpinning of this distinction, referring to the property of commutativity of observable coordinates, has to do with the fact that, a globally non-commutative or partially commutative ring of observables *determines* an underlying *diagram* of *commutative* rings. Then, each commutative ring can be *locally* identified with a ring of commutative scalar coordinates. Thus, there exists the possibility that the information contained in a ring of a quantum system's observables may be approximated or recovered by a *sheaf-theoretic pasting* construction referring to *diagrams* of commutative subrings, identified *locally* with *commutative* rings of compatible observables.

The implications of these considerations are the following: Firstly, we claim that observational *complexity* is a property of a system's behavior that is conceived *topologically* as a global attribute, admitting an *algebraic* description in terms (globally non-commutative but locally commutative) of its *observables*. Secondly, the behavior of a quantum system can be modeled in terms of well-defined families of local commutative algebras of compatible observables, such that the globally complex aspect of its behavior is due to the *non-trivial interlocking* of its local manifesting *icons* in commutative localization measurement environments. Thirdly, a suitable representational framework of quantum systems, taking into account the functorial interpretation of the measurement processes of natural systems, should *not* be a category of non-commutative rings of observables, but a category of appropriate *diagrams* of local commutative unital $\mathbb{R}$ -algebras of compatible observables. This means that the behavior of a quantum system in relation to the measurement processes is not captured by a rigid non-commutative $\mathbb{R}$ -algebra of observables, but by diagrams of local commutative unital $\mathbb{R}$ -algebras of compatible observables. In turn, this is equivalent to the preconditional requirement for *local variation* of the *coordinatizing frame-ring* of coefficients to enable the capture of quantum phenomena, together with the requirement of *amalgamation* of locally compatible information into diagrams of such a form. Fourthly, according to the above, physical *representability* of observables of quantum systems should be *relativized* or *localized* with respect to local commutative reference frames of measurement, since *simultaneous global* determination of their attributes into the field of real numbers is *impossible*. We shall see later that the mathematical transcription of these ideas requires the explicit adoption of a *topos-theoretic* model of the *physical continuum* for the description of quantum phenomena, realized as a category of *(pre)-sheaves* over a base *localizing* category of commutative measurement contexts.

By taking into account the previous distinction, we represent a category of quantum systems by means of a cocomplete category of formal systems $\mathcal{Z}$, such that: Its objects (called *quantum objects*), Z, are quantum information structures, identified as partially commutative unital rings ($\mathbb{R}$ -algebras) of observables, whereas, its *arrows* are the structure-preserving morphisms between them.

The basic idea is that the behavior of a quantum system can be comprehended in terms of a *local to global* contravariant functorial construction referring to its *Spectrum functor*. This can be realized for each quantum structure Z in $\mathcal{Z}$, as an *interlocking* family of incoming morphisms from the loci of intentionally depicted commutative measurement structures, which, characterize the behavior of simple, sufficiently understood *local* systems. From this perspective, we consider a locally small category $\mathcal{Y}$, whose objects, Y, are intentionally selected commutative unital rings ($\mathbb{R}$ -algebras) of scalar coordinates, called *partial* or *local information carriers*, whereas its *arrows* are structure-preserving maps of these carriers. Their role is inextricably connected with the philosophy illustrating their attachment to a *quantum* information structure as *localization* devices, or information *filters* or even as modes of perception.

The epistemological purpose of their introduction is, eventually, the construction of a *covering* system of a quantum information structure with respect to commutative domains of measurement. The notion of a covering system signifies an intentional structured decomposition of a globally non-commutative information structure in terms of partial or local commutative carriers, such that the functioning of the former can be *approximated*, or completely *recovered*, by the *interconnecting* machinery governing the organization of the covering system. Evidently, each local or partial information carrier, includes the amount of information related to a *filtering* process, objectified by a specified context, or a localization environment, and thus, it represents the abstractions associated with the intentional aspect of its use. A further claim, necessary for the development of the proposed functorial model, has to do with the technical requirement that the category of quantum information structures must meet a condition, phrased in category- theoretic language, as *cocompleteness*. This condition means that the category of quantum information structures has arbitrary *small colimits*. The existence of colimits expresses the basic intuition that a quantum object may be conceived as arising from the structured *interconnection* of partially or locally defined information carriers in a specified covering system.

We recall that the formal system corresponding to a quantum system is completely determined by its Spectrum functor in the corresponding representing categorical environment $\mathcal{Z}$. From the preceding discussion we have concluded that the Spectrum functor can be specified equivalently, either by, the contravariant representable functor from the category of partially commutative loci of states to the category of sets, or by means of, the covariant representable functor from the category of partially commutative rings of observables to the category of sets.

The crucial thing to remember is that any locus of states of a quantum system is determined, up to canonical isomorphism, if we know all morphisms into this locus from any other locus in the same category.

For our purposes we consider the description of a locus Z in terms of *all* possible morphisms from all other objects of $\mathcal{Z}^{op}$ as *redundant*. For this reason, we may restrict the generalized point states of Z to all those morphisms in $\mathcal{Z}^{op}$ having as domains loci corresponding to commutative subrings of the globally non-commutative ring of observables of a quantum system. Variation of generalized point states over all loci of the subcategory of $\mathcal{Z}^{op}$, consisting of commutative measurement loci, identified with $\mathcal{Y}^{op}$, produces the *Spectrum functor* of Z restricted to the subcategory of commutative loci.

The Spectrum functor of Z, specified as above, stands as an object in the category of *presheaves* (variable sets) $Sets^{\mathcal{Y}^{op}}$, representing a quantum object in the variable environment of the *topos* of presheaves over the category of its commutative subobjects. This methodology proves to be successful by the establishment of an *isomorphic* representation of Z in terms of its generalized point states $Y_i \to Z$, considered as morphisms in the same category, and further, amalgamated by sheaf-theoretic means.

COMMUNICATION TOPOI: QUALIFYING THE PHYSICAL "CONTINUUM"

10

The codomain of evaluation of physical attributes with respect to some measurement scale is usually identified with the concept of the "*physical continuum*". In standard approaches, the model used to represent these values is the real line $\mathbb{R}$ and its powers, specified as a set- theoretic structure of points, to be identified under instantiation as events, which are independent and distinguishable with precision.

The standard model of the "physical continuum" identifying events globally with the point-elements of the real line faces serious shortcomings when cases of *subjective uncertainty*, as in standard probability theory, or cases of *objective uncertainty* and *indistinguishability*, as in quantum mechanics, have to be taken into account. In these case, the notion of the physical "continuum" does not rely on an assumed preexisting set-theoretic structure of points on the real line. Rather, the evaluation of observables require the prior instantiation of well-defined *local* measurement *frames*, or even *local contexts* of observation, that depend on the prior *infiltration* or *percolation* of events through these pertinent local frames of measurement, or spectral observation. It is precisely in the extensive correlations among these local frames that continuity of observables can be assigned and part-whole or local-global relations can be meaningfully formulated.

In this sense, particular attention is needed in the clarification of what is meant by *localization*, and concomitantly, how it affects our model of the "physical continuum". The basic premise is that only through a consistent localization process does it become possible to *discern* observable events and assign an *individuality* to them. Generally, such a process should *not* depend on the existence of points, and moreover the standard notions of space and time should be *derivative* from localization, rather than the other way round.

Thinking in physical measurement terms, localization is tantamount to a process of *filtering* or *percolating* observables through an appropriate category of frames, such that the ordered structure of events emerging by their evaluation, *fibers over* the underlying local frames including their extensive correlations. In this sense, and from the reciprocal viewpoint, an *event* bears the *depth* of a *sieve* of local spectral frames through which it *percolates*, that resolve it compatibly at various frames of resolution through local observables. In turn, the latter defines a *homologous* physical procedure of observation or measurement.

Since it is assumed that frames together with their structural morphisms give rise to a *category*, the localization process should be understood in terms of an *action* of the category of frames on the global structure of observed events, according to the above. Then, the event structure is qualified in terms of a *partition spectrum*; it is partitioned into sorts parameterized by the objects of the category of frames. Thus, localization can be represented by means of a fibered structure, understood geometrically as a variable set over the base category of frames. The *fibers* are qualified, analogously to the case of the action of a group on a set of points, as the *generalized orbits* of the action of the category of frames. The notion of functional dependence incorporated in this action, forces the *ordered* structure of physical events to *fiber* over the base category of frames.

The partition spectrum emerging out of this action is characterized by *uniformity*. More precisely, for any two events observed over the same frame, the structure of all frames that relate to the first *cannot* be distinguished in any possible way from the structure of frames relating to the second. Given this uniformity, the *ordering* relation between events should be *induced* from the *base* category of frames, that is, by *lifting* relations between frames at the base to the fibers.

10.2 FUNCTORIAL LOCALIZATION: SHEAVES OF GERMS OF OBSERVABLES

In order to clarify the functioning of a localization process we will describe in detail the important case of *localization* of a commutative, unital $\mathbb{R}$-algebra of observables of a natural system over a base localizing category $\mathcal{O}(X)$, consisting of *open loci* U of a topological space X, the arrows between them being *inclusions*. In this case, the frames are defined in terms of the open loci U of X, *partially ordered* by inclusion.

Since observables are conceived as *global functions* on the $\mathbb{R}$-coordinatized state-space of this system, the process of localization forces the *replacement* of the algebra of observables $\mathcal{A}$ by an algebraic structure which will give us all *local* and *global* functional information together. All these functional elements should *interlock* compatibly together in an appropriate manner, which serves to respect the *extension* from local to global, as well as the *restriction* from global to local implied by the localization process. The structure at issue is precisely formalized by the notion of a *sheaf of germs* of a commutative, unital $\mathbb{R}$-algebra of observables, denoted by $\mathbf{A}$, which, incorporates all compatible local and global information together. Let us first introduce precisely the categorical notion of a sheaf on an abstract topological space, and then, examine its applicability in the current situation.

For this purpose, we consider the *category* of open sets $\mathcal{O}(X)$ in an abstract topological space, partially ordered by inclusion. If $\mathcal{O}(X)^{op}$ is the *opposite* category of $\mathcal{O}(X)$, and $Sets$ denotes the scaffolding provided by the category of sets, we define:

A *presheaf* of sets on $\mathcal{O}(X)$ is a *contravariant* set-valued functor on $\mathcal{O}(X)$, denoted by $\mathbf{P}:\mathcal{O}(X)^{op} \to Sets$. For each base open set U of $\mathcal{O}(X)$, $\mathbf{P}(U)$ is a set, and for each arrow $F:V \to U$, $\mathbf{P}(F):\mathbf{P}(U) \to \mathbf{P}(V)$ is a set-function. If $\mathbf{P}$ is a presheaf on $\mathcal{O}(X)$ and $p \in \mathbf{P}(U)$, the *value* $\mathbf{P}(F)(p)$ for an arrow $F:V \to U$ in $\mathcal{O}(X)$ is called the *restriction* of p *along* F and is denoted by $\mathbf{P}(F)(p) := p \cdot F$. A presheaf $\mathbf{P}$ may be understood as a *right action* of $\mathcal{O}(X)$ on a set. This set is *partitioned* into sorts parameterized by the objects of $\mathcal{O}(X)$, and has the following property: If $F:V \to U$ is an *inclusion* arrow in $\mathcal{O}(X)$ and p is an element of $\mathbf{P}$ of sort U, then $p \cdot F$ is specified as an element of $\mathbf{P}$ of sort V. Such an action $\mathbf{P}$ is referred as an $\mathcal{O}(X)$-variable set. A variable set of this form is entirely determined by its category of elements.

The *category of elements* of a presheaf $\mathbf{P}$, denoted by $\int(\mathbf{P},\mathcal{O}(X))$, is described as follows: The objects of $\int(\mathbf{P},\mathcal{O}(X))$ are all *pairs* (U,p), with U in $\mathcal{O}(X)$ and $p \in \mathbf{P}(U)$. The arrows of $\int(\mathbf{P},\mathcal{O}(X))$, that is, $(U',p') \to (U,p)$, are those morphisms $Z:U' \to U$ in $\mathcal{O}(X)$, such that $p' = \mathbf{P}(Z)(p) := p \cdot Z$. Notice that the arrows in $\int(\mathbf{P},\mathcal{O}(X))$ are those morphisms $Z:U' \to U$ in the base category $\mathcal{O}(X)$, that *pull* a chosen element $p \in \mathbf{P}(U)$ *back* into $p' \in \mathbf{P}(U')$.

The category of elements $\int(\mathbf{P},\mathcal{O}(X))$ of a presheaf $\mathbf{P}$, together with, the projection functor $\int_{\mathbf{P}} : \int(\mathbf{P},\mathcal{O}(X)) \to \mathcal{O}(X)$ defines the *split discrete fibration* induced by $\mathbf{P}$, where $\mathcal{O}(X)$ is the *base* category of the fibration. We note that the fibers are categories in which the only arrows are identity arrows. If U is an open reference locus of $\mathcal{O}(X)$, the inverse image of U under $\int_{\mathbf{P}}$, is simply the set $\mathbf{P}(U)$, although its elements are written as pairs so as to form a disjoint union.

From a physical viewpoint, the purpose of introducing the notion of a presheaf $\mathbf{P}$ on $\mathcal{O}(X)$, is the following: We identify an *element* of $\mathbf{P}$ of *sort* U, that is $p \in \mathbf{P}(U)$, with a *local* observable, which, can be observed by means of a measurement procedure over the reference locus U, being an open set of a topological space X. This identification forces the interrelations of local observables, over all reference loci of the base category $\mathcal{O}(X)$, to fulfill the requirements of a *uniform* and *homologous* fibered categorical structure. We recall that the latter is understood according to the following requirements:

a The reference loci used for observational purposes, together with, their structural morphisms, should form a mathematical *category*.

b For any two local observables, both amenable to a measurement procedure, over the same open domain of measurement U, the structure of all reference loci that relate to the first *cannot* be distinguished, in any possible way, from the structure of loci relating to the second. According to this, all the localized observables, within any particular reference locus, should be *uniformly equivalent* to each other.

The split discrete fibration induced by $\mathbf{P}$, where $\mathcal{O}(X)$ is the base category of the fibration, provides a well-defined notion of a uniform homologous fibered structure of local observables in the following sense: Firstly, by the arrows specification defined in the category of elements of $\mathbf{P}$, any local observable p, determined over the reference locus U, is *homologously* related with any other local observable p' over the reference locus U', and so on, by variation over all the reference loci of the base category. Secondly, all the local observables p of $\mathbf{P}$, of the same sort U, determined over the same reference locus U, are uniformly equivalent to each other, since all the arrows in $\int(\mathbf{P},\mathcal{O}(X))$ are induced by lifting arrows from the base category $\mathcal{O}(X)$, formed by partially ordering the reference loci. We conclude that the topological localization process is consistent with the physical requirement of uniformity.

The next crucial step of the construction, aims at the satisfaction of the following physical requirement: Since, we have assumed the existence of reference contexts (open observational domains) locally, according to the operational requirements of a corresponding physical procedure of measurement, the information gathered about local observables in different measurement situations should be *collated* by appropriate means. Mathematically, this requirement is implemented by the methodology of *completion* of the *presheaf* $\mathbf{P}$, or equivalently, *sheafification* of $\mathbf{P}$.

A *sheaf* is characterized as a presheaf $\mathbf{P}$ that satisfies the following condition: If $U=\bigcup_a U_a$, U_a in $\mathcal{O}(X)$, and elements $p_a \in \mathbf{P}(U_a)$, $a \in I$:index set, are such that for arbitrary a, $b \in I$, it holds:

$$p_a | U_{ab} = p_b | U_{ab}$$

where, $U_{ab} := U_a \bigcap U_b$, and the symbol $|$ denotes the operation of *restriction* on the corresponding open domain, then there exists a *unique* element $p \in \mathbf{P}(U)$, such that $p | U_a = p_a$ for each a in I. Then, an element of $\mathbf{P}(U)$ is called a *section* of the sheaf $\mathbf{P}$ over the open locus U. The sheaf condition means that sections can be *glued* together *uniquely* over the reference loci of the base category $\mathcal{O}(X)$. In particular, the sheaf-theoretic qualification of a uniform and homologous fibered structure of observables, as above, makes the latter also *coherent*, in terms of local-global compatibility of the information content it carries, under the operations of restriction and collation.

Thus, we form the following conclusion: The structure of a sheaf arises by imposing on the uniform and homologous fibered structure of elements of the corresponding presheaf the following two requirements:

i Compatibility of observable information under *restriction* from the *global to the local* level, and

ii Compatibility of observable information under *extension* from the *local to the global* level.

According to the first of the above requirements, a sheaf constitutes a *separated presheaf* (monopresheaf) of local observables over a global topological space, meaning that two observables are identical globally, if and only if, they are identical locally. In turn, according to the second requirement, locally compatible observables can be collated together in some global observable, which, is also *uniquely* defined because of the first requirement.

Furthermore, it is obvious that each set of sort U, $\mathbf{P}(U)$, can be endowed with the structure of an $\mathcal{R}$-algebra under pointwise sum, product, and scalar multiplication, denoted correspondingly by $\mathbf{A}(U)$; in that case, the morphisms $\mathbf{A}(U) \to \mathbf{A}(V)$ stand for $\mathcal{R}$-linear morphisms of $\mathcal{R}$-algebras. In this algebraic setting, the sheaf condition means that the following *sequence* of $\mathcal{R}$-algebras of local observables is *left exact*;

$$0 \to \mathbf{A}(U) \to \prod_a \mathbf{A}(U_a) \to \prod_{a,b} \mathbf{A}(U_{ab})$$

As an important example of the above, if $\mathbf{A}$ is the contravariant functor that assigns to each open locus $U \subset X$, the set of all real-valued continuous functions on U, then we will show that $\mathbf{A}$ is actually a sheaf.

Finally, it is important to explain the construction of the *inductive limit* (colimit) of sets (or rings, or $\mathcal{R}$-algebras) $\mathbf{A}(U)$, denoted by $Colim[\mathbf{A}(U)]$, in order to explicate the physically important notions of stalks and germs of a sheaf. For this purpose, let us consider that x is a point of the topological measurement space X. Moreover, let K be a set consisting of open subsets of X, containing x, such that the following condition holds: For any two open reference domains U, V, containing x, an open set $W \in K$ exists, contained in the *intersection* domain $U \bigcap V$. We may say that K constitutes a *basis* for the system of open reference domains around x. We form the *disjoint union* of all $\mathbf{A}(U)$, denoted by;

$$\mathbf{D}(x) := \coprod_{U \in K} \mathbf{A}(U)$$

Then, we can define an *equivalence relation* in $\mathbf{D}(x)$, by requiring that $p \sim q$, for $p \in \mathbf{A}(U)$, $q \in \mathbf{A}(V)$, provided that, they have the *same restriction* to a smaller open set contained in K. Then we define;

$$Colim_K[\mathbf{A}(U)] := \mathbf{D}(x)/ \sim_K$$

Note that, if we denote, the inclusion mapping of V into U by;

$$i_{V,U} : V \infty U$$

and also, the restriction morphism of sets from U to V by;

$$\rho_{U,V} : \mathbf{A}(U) \to \mathbf{A}(V)$$

we can introduce well-defined notions of addition and scalar multiplication on the set $Colim_K[\mathbf{A}(U)]$, making it into an $\mathcal{R}$-module, or even, an $\mathcal{R}$-algebra, as follows:

$$[p_U] + [q_V] := [\rho_{U,W}(p_U) + \rho_{V,W}(q_V)]$$
$$\mu[q_V] := [\mu q_V]$$

where, p_U and q_V are elements in $\mathbf{A}(U)$ and $\mathbf{A}(V)$, and $\mu \in \mathcal{R}$.

Now, if we consider that K and Λ are two bases for the system of open sets domains around $x \in X$, we can show that there are canonical isomorphisms between $Colim_K[\mathbf{A}(U)]$ and $Colim_\Lambda[\mathbf{A}(U)]$. In particular, we may take all the open subsets of X containing x: Indeed, we consider first the case when K is arbitrary and Λ is the set of all open subsets containing x. Then $\Lambda \supset K$ induces a morphism;

$$Colim_K[\mathbf{A}(U)] \to Colim_\Lambda[\mathbf{A}(U)]$$

which is an isomorphism, since whenever V is an open subset containing x, there exists an open subset U in K contained in V. Since we can repeat that procedure for all bases of the system of open sets domains around $x \in X$, the initial claim follows immediately.

Then, the *stalk* of $\mathbf{A}$ at the point $x \in X$, denoted by $\mathbf{A}_x$, is precisely the inductive limit of sets $\mathbf{A}(U)$:

$$Colim_K[\mathbf{A}(U)] := \coprod_{U \in K} \mathbf{A}(U) / \sim_K$$

where K is a basis for the system of open reference domains around x, and $\sim_K$ denotes the equivalence relation of restriction within an open set in K. Note that the definition is *independent* of the chosen basis K. For an open reference domain W containing the point x, we obtain an morphism of $\mathbf{A}(W)$ into the stalk at the point x:

$$i_{W,x} : \mathbf{A}(W) \to \mathbf{A}_x$$

For an element $p \in \mathbf{A}(W)$ its image:

$$i_{W,x}(p) := p_x = germ_x p$$

is called the *germ* of p at the point x.

The fibered structure that corresponds to a sheaf of sets $\mathbf{A}$ is a *topological bundle* defined by the continuous mapping $\varphi : A \to X$, where;

$$A = \coprod_{x \in X} \mathbf{A}_x$$
$$\varphi^{-1}(x) = \mathbf{A}_x = Colim_{\{x \in U\}}[\mathbf{A}(U)]$$

The mapping φ is *locally* a *homeomorphism* of topological spaces. The topology in A is defined as follows: for each $p \in \mathbf{A}(U)$, the set $\{p_x, x \in U\}$ is open, and moreover, an arbitrary open set is a union of sets of this form. Obviously, the same arguments hold in the case of a sheaf of sets $\mathbf{A}$ endowed with some algebraic structure, for example rings or $\mathcal{F}$-algebras (where $\mathcal{F}$ is a field).

With respect to the physical interpretation, we remind that we have identified an *element* of $\mathbf{A}$ of sort U, that is a *local section* of $\mathbf{A}$, with a local observable p, which can be observed via a measurement procedure over the reference locus U. Then the equivalence relation, used in the definition of the stalk $\mathbf{A}_x$ at the point $x \in X$ signifies the following:

Two local observables $p \in \mathbf{A}(U)$, $q \in \mathbf{A}(V)$, induce the *same contextual information* at x in X, provided that they have the same restriction to a smaller open locus contained in the basis K. Then, the stalk $\mathbf{A}_x$ is the set containing all contextual information at x, that is, the set of all equivalence classes.

Moreover, the image of a local observable $p \in \mathbf{A}(U)$ at the stalk $\mathbf{A}_x$, that is, the equivalence class of this local observable p, is precisely the *germ* of p at the point x. Next, if we consider a local observable $p \in \mathbf{A}(U)$, it determines a function:

$$\dot{p} : x \mapsto germ_x p$$

whose domain is the open locus U and its codomain is the stalk $\mathbf{A}_x$, for each $x \in U$.

We may consider instead, the disjoint union $A = \coprod_{x \in X} \mathbf{A}_x$ as the codomain of the function $\dot{p}$. From this perspective, every local observable $p \in \mathbf{A}(U)$, gives rise to some *partial function*:

$$\dot{p} : U \to A$$

which, is defined on the open locus $U \subset X$. Hence, all local observables $p \in \mathbf{A}(U)$, admit a *functional representation*, established by means of the following correspondence:

$$\Delta(U) : p \mapsto \dot{p}$$

Stated equivalently, each local observable $p \in \mathbf{A}(U)$ can be legitimately considered a partial function:

$$\dot{p} : U \to A$$

defined over the open reference locus U, the value of which, at a point $x \in U$, that is, $germ_x p$, is the *contextual* observable information induced at x by the local observable p. Furthermore, such a partial function $\dot{p} : U \to A$ is identified with a *cross section* of the topological bundle of germs, defined by the *continuous* mapping $\varphi : A \to X$, such that,

$$\varphi^{-1}(x) = \mathbf{A}_x = Colim_{\{x \in U\}}[\mathbf{A}(U)]$$

Note that the mapping φ is locally a homeomorphism of topological spaces, and thus, the bundle is *étale*.

The previous discussion can be formalized categorically in terms of an *adjunctive correspondence*, defined fundamentally, between the category of *presheaves* of sets $Sets^{\mathcal{O}(X)^{op}}$ on the category of open loci $\mathcal{O}(X)$ of a topological space X, and the category of topological *bundles* $\mathcal{B}(X)$ over X, as follows:

$$\Gamma : \mathcal{B}(X) \rightleftarrows Sets^{\mathcal{O}(X)^{op}} : \Lambda$$

where, in the above adjunction, the functor $\Gamma : \mathcal{B}(X) \to Sets^{\mathcal{O}(X)^{op}}$, called *the cross sections-functor*, assigns to each bundle $\varphi : A \to X$ the sheaf of all cross-sections of A, while its left adjoint functor $\Lambda : Sets^{\mathcal{O}(X)^{op}} \to \mathcal{B}(X)$, called the *germs-functor*, assigns to each presheaf $\mathbf{A}$ the bundle of germs of $\mathbf{A}$. The adjunction is characterized completely by the *unit* and *counit* natural transformations, defined respectively as follows:

$$\eta_{\mathbf{A}} : \mathbf{A} \to \Gamma\Lambda\mathbf{A}$$
$$\acute{\mathrm{U}}_A : \Lambda\Gamma A \to A$$

Moreover, if $\mathbf{A}$ is a *sheaf*, then, the unit $\eta_{\mathbf{A}}$ is an isomorphism, while, if A is *étale*, then, the counit $\acute{\mathrm{U}}_A$ is an isomorphism. For these reasons, the above adjunction is restricted to a *natural equivalence* between the categories of *sheaves* $\mathbf{Sh}(X)$ on $\mathcal{O}(X)$, and the category of *étale topological bundles* $\mathcal{ET}(X)$ over X, as follows:

$$\Gamma : \mathcal{ET}\left(X\right) \rightleftarrows \mathbf{Sh}\left(X\right) : \Lambda$$

Note that the above adjunction (natural equivalence) is still valid if we consider instead of presheaves (sheaves) of sets, presheaves (sheaves) of *rings*, or $\mathbb{R}$ *-algebras*. Moreover, as a consequence of the categorical equivalence between sheaves on a topological space X and étale topological bundles over X, *every* sheaf can be considered as a sheaf of *cross-sections*.

It is also instructive to notice that the previous arguments can help us to understand the process of *completion* (or sheafification, or germification) of a presheaf. For this purpose, we realize that the notions of *germ*, *stalk* and *étale* bundle make sense for a general presheaf. More precisely, the germ at a point stands for an equivalence class of elements of the presheaf corresponding to open loci around that point, under the equivalence relation which follows from having the same germ. The stalk over this point is the set of all germs at this point. The étale bundle is the disjoint union of all stalks. The first crucial observation is that by the definition of a topology on the étale bundle, as described previously, it is legitimate to consider continuous sections of the *étale* bundle.

Stated equivalently, this procedure amounts to *transforming* the *elements* of the presheaf into partial continuous functions (*continuous sections*) *valued* into the *étale* space. Hence, we manage to *functionalize* the initial presheaf, by defining a *new* presheaf, called the *presheaf of sections* of the initial presheaf as follows: It is the presheaf, which associates to each open locus of the base topological space the set of continuous sections from that open locus into the *étale* space. Now, there is an obvious morphism from the initial presheaf to its presheaf of sections, which maps each element of the category of elements of the initial presheaf to the continuous section, which sends *each point* in an open locus of the base space to the *germ* of this element at that point.

The second crucial observation is that the associated functionalized presheaf of sections of a presheaf is actually also *localized* (locally determined), meaning that it is a sheaf, identified as the *sheaf of cross-sections* of the corresponding *étale* topological bundle. Thus, the latter sheaf is called the sheaf *associated* to the initial presheaf. Moreover, the process of completion of a presheaf into the sheaf of cross-sections of the corresponding *étale* topological bundle is *functorial*, meaning that for each presheaf there is a functor sending it to its completion, that is, to its associated sheaf of sections, called the *sheafification functor*. As a corollary, we conclude that a presheaf is a sheaf, which means a *complete* presheaf, if and only if the morphism to its associated functional presheaf of sections is an *isomorphism*.

Thus, the process of *completion* of a presheaf is equivalent to the combined processes of *functionalization* and *localization* of its elements. Consequently, the associated sheaf of sections of a corresponding presheaf, contains by its construction, the totality of local contextual information compatible with the one available from the initial presheaf due to its restriction property, and in this sense, it constitutes its completion.

Now, let us consider a sheaf of $\mathcal{R}$-algebras of local observables, identified as a sheaf of real-valued continuous *cross-sections* of the corresponding *étale* bundle. Then, the set of germs of all these sections at a point, the stalk at this point, is also an $\mathcal{R}$-algebra. Most importantly, the *stalk* at this point is a *local* $\mathcal{R}$-algebra, meaning that it has a *unique maximal ideal.* In turn, this maximal ideal consists of all germs vanishing at the point in question. The *quotient* of the stalk by this maximal ideal is *isomorphic* to the field of *real* numbers. Equivalently, this means that the morphism evaluating a germ of the stalk at a point to the real numbers, which provides a real value at the corresponding non-vanishing equivalence class of sections at the base point of interest, is a *surjective* morphism of $\mathcal{R}$-algebras taking as *kernel* the *maximal ideal* of the stalk at this point:

$$\mathbf{ev}_x : \mathbf{A}_x \to \mathbf{A}_x / \mu_x \cong \mathcal{R}$$
$$germ_x p \mapsto \mathbf{ev}_x(germ_x p) = p(x)$$

Thus the evaluation morphism of a germ of the stalk at a point of the base space is an $\mathcal{R}$-valued measurement of this observable germ, interpreted as an *observed event* of the corresponding natural system, and subsequently encoded by means of an $\mathcal{R}$*-state* of its topological state-space.

At a next stage of development of these ideas, the sheaf of germs of real-valued continuous functions on a topological space X is an object in the functor category of sheaves $\mathbf{Sh}(X)$ on varying reference loci U, being open sets of X, partially ordered by inclusion. The morphisms in $\mathbf{Sh}(X)$ are all *natural transformations* between sheaves. It is instructive to notice that a sheaf makes sense only if the base category of reference loci is specified, which is equivalent in our context to the determination of a topology on the space X. The *functor category of sheaves* $\mathbf{Sh}(X)$, provides an exemplary case of a construct known as *topos.*

A *topos* can be conceived as a *local* mathematical framework corresponding to a *generalized model* of set theory, or as a *generalized algebraic space*, corresponding to a categorical universe of variable observable information sets over the multiplicity of the reference loci of the base category. We recall that, formally a topos is a category, which has a terminal object, pullbacks, exponentials, and a subobject classifier, which in turn is understood as an object of *generalized truth values*. The particular significance of the sheaf of real-valued continuous functions on X, is due to the following *isomorphism*: The *sheaf* of germs of continuous real-valued functions on X, is isomorphic to the object of *Dedekind real* numbers in the topos of sheaves $\mathbf{Sh}(X)$. The aforementioned isomorphism validates the physical intuition which reads a local observable as a continuously variable real number over its locus of definition.

10.3 TOPOS-THEORETIC RELATIVIZATION OF REPRESENTABILITY

The transition in the semantics of the *physical continuum* from the topos of $\mathbf{Sets}$ to the topos of sheaves $\mathbf{Sh}(X)$ is an instance of the principle of *topos-theoretic relativization* of physical *representability* referring to the interpretation of observed events. We initially notice that, in the former case, observables are identified with (continuous) functions determined *completely* by their values at *points*. In the latter case, observables are identified with local continuous sections of the *étale* space determined *completely* by their *germs*.

In order to analyze in more detail the *transition* in the *semantics*, we note that, in the former case, a continuous function from a base topological space X to the topological space $\mathcal{R}$ can be considered as a *continuous section* from X to the product space $X \times \mathcal{R}$. This product space is set-theoretically isomorphic to a space containing a *copy* of the coordinatizing frame-field $\mathcal{R}$ at each *point*, being the *inverse image* of the *projection* from $X \times \mathcal{R}$ to the base X. The value that is taken at a point is the value taken by the function. Thus, this type of modeling the notion of an *observable* is only appropriate in capturing its *point-properties*.

In contrast, in the latter case we obtain *local properties* of observables. This is due to the fact that in the sheaf-theoretic local environment, we associate *not* with the *value* that a section takes at a point of the base space, but its *germ*. In this sense, *instead* of the product total space $X \times \mathcal{R}$, we have the *étale* topological space, such that the *inverse image* of each point of the base space is *not* a copy of the coordinatizing frame-field $\mathcal{R}$, but the *stalk* at that point.

This essentially means that the transition of semantics from the topos of sets **Sets** to the topos of sheaves $\mathbf{Sh}(X)$ amounts to *shifting* the focus *from point-wise* behavior of observables *to local* behavior of observables. Obviously the *étale* topological space is a much *richer* and *bigger* space than the rigid space $X \times \mathcal{R}$, since the étale space provides information about the *local* behavior of observables around each point of the base space in terms of *germs*, instead of merely point-wise behavior of observables in terms of their values in the real numbers.

Thus, *observed events* are *not* determined by the *values* of continuous functions at points of the base space, but by the *evaluation* morphisms of *germs* at those points, according to our previous remarks. We conclude that the meaning of the principle of topos-theoretic relativization of physical representability as effectuated by the transition from the topos of sets **Sets** to the topos of sheaves $\mathbf{Sh}(X)$ amounts to a *relativization* with respect to the *local* behavior of physical observables as *opposed to* their *point* behavior.

It is worth explaining in some further detail the important notion of relativization of physical representability by shifting the semantics of observables from the topos of sets **Sets** to the topos of sheaves $\mathbf{Sh}(X)$. The *absolute representability* principle is based on the *set-theoretic* conception of the *real line*, as a set of infinitely distinguished points coordinatized by means of the field of real numbers. Expressed categorically, this is equivalent to the interpretation of the algebraic structure of the reals inside the *absolute* universe of **Sets**, or more precisely inside the *topos of constant* **Sets**.

It is also well known that algebraic structures and mechanisms can admit a *variable* reference, formulated in category-theoretic jargon in terms of arrows-only specifications, *inside any* suitable *topos* of discourse. The *relativization* of physical representability with respect to the topos of sheaves $\mathbf{Shv}(X)$, amounts to the relativization of both the *notion* and the algebraic *structure* of the *real* numbers *inside* this *topos*. Regarding the notion of real numbers inside the topos $\mathbf{Shv}(X)$, this is equivalent to the notion of *continuously variable* real numbers over the open reference domains of X, or else, equivalent to the notion of real-valued continuous functions on X, when interpreted respectively inside the topos of **Sets**.

Equivalently stated, the internal object of *Dedekind reals* constructed within the *logic* of the topos $\mathbf{Shv}(X)$ is isomorphic to the *sheaf of germs* of continuous real-valued functions on the space X. Regarding the algebraic structure of the reals inside the topos $\mathbf{Shv}(X)$, they form *only* an algebra in this topos, which is identified with the *sheaf of commutative* $\mathbb{R}$-algebras of *germs* of continuous real-valued functions on X, where $\mathbb{R}$ corresponds in that case to the *locally constant* sheaf of germs of real numbers over X.

From a physical perspective, internally in the topos $\mathbf{Shv}(X)$ the *valuation* algebra of real numbers is *relativized* with respect to the base category of *open* sets of a topological space X. As a consequence it admits a description as a sheaf of of germs of continuous real-valued functions on X. In particular, for each open reference context U of X, we obtain a unital commutative algebra of continuous real-valued local sections. In this way, the semantics of localization of observables is transformed from a set-theoretic to a sheaf-theoretic one. More concretely, it is obvious that inside the topos $\mathbf{Sets}$ the *unique* localization measure of observables is a *point* of the $\mathbb{R}$-spectrum of the corresponding algebra of scalars, which is assigned a numerical identity.

By contrast, inside the topos $\mathbf{Shv}(X)$, the former is *substituted* by a variety of *localization measures*, dependent *only* upon the *open* sets in the topology of X. In the latter context, a point-localization measure, is identified precisely with the *ultrafilter* of all opens containing the point. This identification permits the conception of other *filters* owing their formation to admissible operations between opens as generalized measures of localization of observables. In a wider context, the relativization of representability effected in $\mathbf{Shv}(X)$ is physically significant, because the operational specification of measurement environments exists *only locally* and the underlying assumption is that the information gathered about local observables in different measurement situations can be collated together by appropriate means; a process that is precisely formalized by the notion of sheaf.

Conclusively, we assert that localization schemes referring to observables *may not depend* exclusively on the existence of *points*, and thus should *not* be tautosemous with the practice of conferring a numerical identity to them. Therefore, the relativization of representability with respect to the internal reals of the topos of sheaves $\mathbf{Shv}(X)$, amounts to the *substitution* of point-localization measures, represented numerically, with localization measures *fibering* over the base category of open reference loci, represented respectively by *local sections* in the sheaf of internal reals.

The *transition* in the semantics of physical *representability* under *relativization* from the topos $\mathbf{Sets}$ to the topos $\mathbf{Shv}(Y)$ can be formalized via the concept of an *admissible transformation* between *topoi*, called a *functorial geometric transformation*, or simply a *geometric morphism*. More concretely, a functorial geometric transformation from the topos $\mathbf{Sets}$ to the topos $\mathbf{Shv}(Y)$ is defined as a *pair* of *adjoint functors*:

$$\mathbf{F}^{\leftleftarrows}\colon \mathbf{Sets} \to \mathbf{Shv}(Y)$$
$$\mathbf{F}_{\leftleftarrows}\colon \mathbf{Shv}(Y) \to \mathbf{Sets}$$

where the functor $\mathbf{F}_{\leftleftarrows}$ is *right adjoint* to the functor $\mathbf{F}^{\leftleftarrows}$, which in turn is *left exact*. Then, the functor $\mathbf{F}_{\leftleftarrows}$ is called the *direct image* part of the functorial geometric transformation, while the functor $\mathbf{F}^{\leftleftarrows}$ is called the *inverse image* part.

The terminology arises from the simple realization that a *continuous* morphism between *topological spaces* X and Y, denoted by $h: X \to Y$, induces a *functorial geometric transformation* between the *categories of their sheaves* as follows:

$$\mathbf{H}^{\leftleftarrows}\colon \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$$
$$\mathbf{H}_{\leftleftarrows}\colon \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$$

such that:

$$\mathbf{Shv}\left(X\right) \rightleftarrows \mathbf{Shv}\left(Y\right)$$

form an *adjoint pair* of functors, where, if $\mathbf{A}$ is a *sheaf* on X, and U is an open locus in Y, then:

$$\mathbf{H}_{\leftleftarrows}(\mathbf{A})(U) = \mathbf{A}((h^{-1})(U))$$

called the *direct image* of the sheaf $\mathbf{A}$ under the morphism h. On the opposite side, the *inverse image* of a sheaf $\mathbf{B}$ under the morphism h, denoted by $\mathbf{H}^{\leftleftarrows}(\mathbf{B})$ is defined as the sheaf on X such that the stalk at any point $x \in X$ is the stalk at $h(x)$.

Most importantly, *any* functorial geometric transformation between the topos of sheaves on X and the topos of sheaves on Y *necessarily* arises from a *unique continuous* function between these spaces. The above is particularly useful, and can be demonstrated by a simple example as follows: the topos $\mathbf{Sets}$ can be considered as the topos of sheaves over the 1-point topological space, that is, $\mathbf{Shv}(1)$. Thus, a *point* y of a topological space, i.e. a continuous morphism $y: 1 \to Y$, gives rise to a *geometric morphism* described by:

$$\mathbf{y}^{\leftleftarrows}: \mathbf{Shv}(Y) \to \mathbf{Shv}(1)$$
$$\mathbf{y}_{\leftleftarrows}: \mathbf{Shv}(1) \to \mathbf{Shv}(Y)$$

such that:

$$\mathbf{Shv}(1) \rightleftarrows \mathbf{Shv}(Y)$$

form an *adjoint pair* of functors, where if $\mathbf{A}$ is a *sheaf* over the one-point topological space 1, which is just a set, and U is an open locus in Y, then:

$$\mathbf{y}_{\leftleftarrows}(\mathbf{A})(U) = \mathbf{A}((y^{-1})(U))$$

that is, the *direct image* of the set $\mathbf{A}$ under y, gives the value $\mathbf{A}$ if $y \in U$ and the value 1 otherwise. The sheaf $\mathbf{y}_{\leftleftarrows}(\mathbf{A})$ on Y is the *skyscraper sheaf* at the point y, which is a *totally discontinuous* sheaf on Y. Thus, if $\mathbf{A}$ is the sheaf of internal reals on 1, that is, the set of real numbers, then the skyscraper sheaf $\mathbf{y}_{\leftleftarrows}(\mathbf{A})$ on Y consists of a copy of the real numbers at the point y and is 1 at all other points. On the opposite side, the inverse image of a sheaf $\mathbf{B}$ under the morphism y, denoted by $\mathbf{y}^{\leftleftarrows}(\mathbf{B})$, is precisely the stalk of $\mathbf{B}$ at the point y. Thus, if $\mathbf{B}$ is the sheaf of internal real numbers in the topos $\mathbf{Shv}(Y)$, then its inverse image under y is the stalk of the internal real numbers at y, that is, the set (local ring) of germs of continuous real-valued sections at y. This means in turn that the notion of a continuously variable real number over Y, which is a real number from the perspective of $\mathbf{Shv}(Y)$, is transformed via the inverse image functor $\mathbf{y}^{\leftleftarrows}$ corresponding to $y: 1 \to Y$, into the notion of a germ of continuous real-valued sections at y from the perspective of $\mathbf{Shv}(1)$.

This simple example illustrates the *first* fundamental aspect of the principle of relativization of physical representability with respect to the internal reals of a topos of the form $\mathbf{Shv}(Y)$, where Y is a topological space. More precisely, since the sheaf of internal real numbers in the topos $\mathbf{Shv}(Y)$ is perceived via $\mathbf{y}^{\leftleftarrows}$ by the set (local ring) of germs of continuous real-valued sections at y in Y, this means that the transition in the semantics of observable representability at a point of a base topological space, as reflected within the topos of sets, amounts to the *substitution* of a *point-localization* measure of an observable, that is its real value at that point, *encoding point-wise information*, by its *germ* at the same point, *encoding local information.*

The *second* fundamental aspect of the principle of relativization of physical representability with respect to the internal real numbers of a topos of the form $\mathbf{Shv}(Y)$, where Y is a topological space, is implemented by means of the following *functorial geometric transformation*:

$$\mathbf{Shv}(Y) \rightleftarrows \mathbf{Sets}$$
$$\mathbf{F}^{\leftleftarrows}: \mathbf{Sets} \to \mathbf{Shv}(Y)$$
$$\mathbf{F}_{\leftleftarrows}: \mathbf{Shv}(Y) \to \mathbf{Sets}$$

where,

$$\mathbf{F}_{\leftleftarrows} := \Gamma : \mathbf{Shv}(Y) \to \mathbf{Sets}$$

is the *global sections functor*, which assigns to a sheaf $\mathbf{A}$ in $\mathbf{Shv}(Y)$ its set of global sections (*global elements*) $\Gamma(\mathbf{A}) = \mathbf{Nat}(\mathbf{1}, \mathbf{A})$, where $\mathbf{1}$ is the *terminal* object in $\mathbf{Shv}(Y)$.

In particular, if $\mathbf{A}$ is the sheaf of internal real numbers in the topos $\mathbf{Shv}(Y)$, then its set of global sections is the set of real-valued continuous functions on Y. This means that the notion of a continuously variable real number over Y, that is, a real number from the perspective of $\mathbf{Shv}(Y)$, is transformed by the inverse image functor $\mathbf{F}_{\leftleftarrows} := \Gamma$ to the notion of a real-valued continuous function on Y from the perspective of $\mathbf{Sets}$.

Consequently, the semantics of observable representability *globally*, as reflected within the topos of sets, remains *invariant* under the *relativization* with respect to the internal reals of a topos of the form $\mathbf{Shv}(Y)$. Hence, the relativization of physical representability as above, forces the encoding of local observable information in terms of germs, and thus transforms the semantics of observables from the *point* level to the *local* level, while it leaves *invariant* their *global* interpretation.

In the opposite direction, the functor:

$$\mathbf{F}^{\leftleftarrows} := \Delta : \mathbf{Sets} \to \mathbf{Shv}(Y)$$

assigns to each set S the corresponding *constant* presheaf $\Delta(S) := \mathbf{\Delta S}$. This constant presheaf sends each open set U of Y to the same set S. In particular, if S is the set or real numbers $\mathbb{R}$, then the constant presheaf sends each open set U of Y to the set of real numbers $\mathbb{R}$. Thus, the corresponding *étale* space of the constant presheaf $\Delta\mathbb{R}$ is the *projection* $Y \times \mathbb{R} \to Y$. Therefore, for each open locus U of Y, $\Delta\mathbb{R}(U)$ is the set of continuous functions from U to the *discrete* space $\mathbb{R}$. This is exactly the set of *locally constant* functions from U to $\mathbb{R}$. In this sense, the sheaf $\Delta\mathbb{R}$ is called the *constant sheaf* corresponding to the set of real numbers.

A particularly interesting application of the above arises even in the case that the base topological space Y is *discrete*, considered as an *infinite* set. We may consider the *counit* natural transformation of the corresponding pair of adjoint functors:

$$\text{Ú} : \mathbf{F}_{\rightleftarrows}\mathbf{F}^{\leftleftarrows} \to \mathbf{Id}_{\mathbf{Sets}}$$

$$\text{Ú}_S : \mathbf{F}_{\rightleftarrows}\mathbf{F}^{\leftleftarrows} S \to S$$

If S is the set or real numbers $\mathbb{R}$, then we obtain:

$$\text{Ú}_{\mathbb{R}} : \mathbf{F}_{\rightleftarrows}\mathbf{F}^{\leftleftarrows} \mathbb{R} \to \mathbb{R}$$

Note that the domain of the counit, is the set of global sections of the constant sheaf $\Delta\mathbb{R}$ in $\mathbf{Sets}$ as previously. This set is identified as the set of sections of the projection morphism from the cartesian product $Y \times \mathbb{R}$ (viewed as a set) to Y. It is denoted by $\mathbb{R}^Y$, while its elements $\chi \in \mathbb{R}^Y$ are mappings $\chi : Y \to \mathbb{R}$. Therefore, we obtain:

$$\text{Ú}_{\mathbb{R}} : \mathbb{R}^Y \to \mathbb{R}$$

This is precisely the *evaluation* morphism of the set $\mathbb{R}^Y$ to the set of real numbers, that is, the morphism evaluating the set of global sections $\mathbb{R}^Y$ of $\Delta\mathbb{R}$ at a point y of the base space Y to the set of real numbers. The result of evaluating the set $\mathbb{R}^Y$ at a point y of Y is equivalent to the process of *identification* of functions $\chi : Y \to \mathbb{R}$ in $\mathbb{R}^Y$ under the condition that their values at y are the *same*; in short, we may define an equivalence relation on the set $\mathbb{R}^Y$ as follows:

$$\chi \sim_y \psi$$

if and only if:

$$\chi(y) = \psi(y)$$

This means in turn that the set,

$$\{y_i \in Y : \chi(y_i) = \psi(y_i)\}$$

belongs to the *principal ultrafilter* of y, that is, to the set:

$$V_y = \{U \subseteq Y : y \in U\}$$

Now, if we *identify* the point y with its principal ultrafilter, we may reformulate the equivalence relation as follows:

$$\chi \sim_{V_y} \psi$$

if and only if:

$$\{y_i \in Y : \chi(y_i) = \psi(y_i)\} \in V_y$$

Consequently, the result of evaluating all the elements of the set $\mathbb{R}^Y$ at the point y, or equivalently, at the principal ultrafilter V_y, is the set of equivalence classes of $\mathbb{R}^Y$ modulo the equivalence relation $\sim_{V_y}$. This set is obviously isomorphic to the set of real numbers, that is:

$$\mathbb{R}^Y / \sim_{V_y} \simeq \mathbb{R}$$

and the evaluation morphism is actually the following:

$$\acute{U}_{\mathbb{R}} : \mathbb{R}^Y \to \mathbb{R}^Y / \sim_{V_y} \simeq \mathbb{R}$$

The above procedure also give us the possibility of evaluating the set of global sections $\mathbb{R}^Y$ of $\Delta\mathbb{R}$ at an *arbitrary ultrafilter* V of the base space Y, thought of as a *virtual point* of Y. More specifically, an ultrafilter V is a point of the *compactification* of Y. In this way, we may define the following equivalence relation on the set $\mathbb{R}^Y$:

$$\chi \sim_V \psi$$

if and only if:

$$\{y_i \in Y : \chi(y_i) = \psi(y_i)\} \in V$$

Similarly, the result of evaluating all the elements of the set $\mathbb{R}^Y$ at the *ultrafilter* (virtual point) V, is the set of equivalence classes of $\mathbb{R}^Y$ modulo the equivalence relation $\sim_V$. This set is *not* isomorphic to the set of real numbers, and is called the *ultrapower* of $\mathbb{R}^Y$ with respect to the *ultrafilter* V. Moreover, the set of real numbers can be naturally *embedded* in the ultrapower of $\mathbb{R}^Y$ with respect to the ultrafilter V. In this sense, the ultrapower of $\mathbb{R}^Y$ with respect to V, denoted by Υ_V, contains the *real* numbers and *additionally* contains *new* generalized *elements*.

Consequently we can imagine the elements of the ultrapower Υ_V as real numbers surrounded by a *cloud* of objective thickness. Of course, this procedure provides the possibility of a generalized interpretation of measurement states of the corresponding ring of global observables $\mathbb{R}^Y$, by means of the surjective morphism of rings:

$$\mathbb{R}^Y \to \Upsilon_V$$

Which is to say the legitimate consideration of Υ_V-states of $\mathbb{R}^Y$. Essentially, this means that the ring Υ_V can act as a *ring* of measurement *scales* for the evaluation of the observables in $\mathbb{R}^Y$. This is another indication of the fact that *absolute* representability with respect to $\mathbb{R}$-measurement scales should be *abandoned*, and instead a *covariance* principle referring to *all legitimate* rings of measurement scales should be *substituted* in its *place* for the evaluation of observables.

Based on the above conclusion, we may extend these ideas by taking into account the following: Firstly, the set of *global sections* $\mathbb{R}^Y$ of $\Delta\mathbb{R}$ is actually a *commutative* $\mathbb{R}$-*algebra* as can easily be verified. Secondly, an injective correspondence exists between the *proper ideals* of the $\mathbb{R}$-algebra $\mathbb{R}^Y$ and the *filters* of the discrete space Y

(considered as an infinite set). More concretely, if we recall that the elements $\chi \in \mathbb{R}^Y$ are mappings $\chi : Y \to \mathbb{R}$, then we can define the *zero set* of an element χ as follows:

$$Z(\chi) = \{y \in Y \mid \chi(y) = 0\}$$

Furthermore, if we denote by $\mathcal{J}$ an *ideal* in the $\mathbb{R}$-algebra $\mathbb{R}^Y$, and by F a *filter* on the infinite set Y we obtain the *injective* correspondences:

$$\mathcal{J} \mapsto F_{\mathcal{J}} = \{Z(\chi) \mid \chi \in \mathcal{J}\}$$
$$F \mapsto \mathcal{J}_F = \{\chi \in \mathbb{R}^Y \mid Z(\chi) \in F\}$$

The above correspondences are *order-preserving* and *idempotent* under iteration. It follows that every *reduced power* algebra $\mathbb{R}^Y / \mathcal{J}$, where $\mathcal{J}$ is an ideal in the $\mathbb{R}$-algebra $\mathbb{R}^Y$ is of the quotient form:

$$\mathcal{A}_F = \mathbb{R}^Y / \mathcal{J}_F$$

for a unique *generating filter* F on the index set Y.

Next, we note that reduced power algebras of the above form can be related to each other as follows: For two filters F, G on Y, such that $F \subseteq G$, we obtain the surjective morphism of $\mathbb{R}$-algebras

$$\mathcal{A}_F \twoheadrightarrow \mathcal{A}_G$$
$$\chi + \mathcal{J}_F \mapsto \chi + \mathcal{J}_G$$

from which we conclude that the algebras $\mathcal{A}_G$ and $\mathcal{A}_F / (\mathcal{J}_G / \mathcal{J}_F)$ are *isomorphic*. A degenerate case refers to the power algebras obtained when a filter F on Y is generated by a non-empty subset Σ of Y, that is, in case that ${}_\Sigma F = \{K \subseteq Y \mid K \supseteq \Sigma\}$. Then, we obtain the power algebras of the form:

$$\mathcal{A}_{{}_\Sigma F} = \mathbb{R}^\Sigma$$

Further, if Σ is a *finite* subset of Y having $n \geq 1$ elements we obtain:

$$\mathcal{A}_{\Sigma F} = \mathbb{R}^n$$

Consequently, the n-dimensional *Euclidean spaces* are *power algebras* of $\mathbb{R}^Y$ of the form $\mathbb{R}^\Sigma$, where Σ is a finite subset of Y having n-elements. The exclusion of all the degenerate cases, leading to the formation of power algebras, amounts to *restricting* the generating filters to those that are supersets of the *Maurice Frechet filter* on Y. The *Frechet* filter on Y, denoted by Fr is a *cofinite* filter on Y, where a cofinite filter on Y consists of all subsets of Y having finite complement in Y. Thus, non-degenerate reduced power algebras of $\mathbb{R}^Y$ are of the form:

$$\mathcal{A}_F = \mathbb{R}^Y / \mathcal{J}_F$$

for a *unique* generating filter F on the index set Y, such that $F \supseteq Fr$. Moreover, because of the relation $Fr \subseteq F$, every non-degenerate reduced power algebra $\mathcal{A}_F$ of $\mathbb{R}^Y$ is the *surjective image* of the reduced power algebra $\mathcal{A}_{Fr}$ corresponding to the *Frechet* filter on Y, and moreover, it is *isomorphic* to $\mathcal{A}_{Fr} / (\mathcal{J}_F / \mathcal{J}_{Fr})$, that is:

$$\mathcal{A}_F \simeq \mathcal{A}_{Fr} / (\mathcal{J}_F / \mathcal{J}_{Fr})$$

The two most important properties of all the non-degenerate reduced power algebras of $\mathbb{R}^Y$ are that they have *zero divisors*, unless the dividing ideal is prime, and that they are *non-Archimedean*. Still, they can legitimately act as rings of measurement scales for the evaluation of the observables in $\mathbb{R}^Y$. Thus, once again the absolute representability with respect to $\mathbb{R}$-measurement scales should be abandoned in favour of a covariance principle with respect to all legitimate rings of measurement scales for the evaluation of observables.

10.4 DIFFERENTIAL RINGED SPACES OF STATES

The transition in the semantics of the physical continuum from the topos of **Sets** to the topos of sheaves $\mathbf{Sh}(X)$ as an instance of the principle of topos-theoretic relativization of physical representability, entails a transition in the *semantics of observables* from (continuous) functions determined completely by their values at points, to local continuous sections of the *étale* space of a sheaf determined completely by their germs. Thus, in the latter case of localization within the environment of the topos of sheaves $\mathbf{Sh}(X)$ we are naturally led to the introduction of the notion of a *commutative ringed space* of states. We note that the notion of ringed spaces is used extensively in Algebraic Geometry, in the theory of Abstract Differential Geometry, and in the theory of $\mathcal{C}^{\infty}$-differentiable spaces.

A *commutative ringed space* of states is a pair $(X,\mathbf{R})$ consisting of a topological space X and a *sheaf of commutative rings* of observables $\mathbf{R}$ on X. The space X is called the underlying *state-space* of the ringed space, while the sheaf $\mathbf{R}$ is called the *structure sheaf* of observables. Now, for any open locus $U \subset X$, the pair $(U,\mathbf{R}|_U)$ is also a ringed space, called an open subspace of states of $(X,\mathbf{R})$. A morphism of ringed spaces $H=(h,\phi)$ from $(X,\mathbf{R})$ to $(Y,\mathbf{Q})$ consists of a continuous morphism of topological spaces $h: X \to Y$ and also a morphism of sheaves of rings $\phi: \mathbf{H}^{\leftarrow}(\mathbf{Q}) \to \mathbf{R}$.

We recall that if $\mathbb{R}$ is the field of real numbers, then an $\mathbb{R}$-algebra of observables $\mathcal{A}$ is a ring $\mathcal{A}$ together with a morphism of rings $\mathbb{R} \to \mathcal{A}$ (making $\mathcal{A}$ into a vector space over $\mathbb{R}$) such that, the morphism $\mathcal{A} \to \mathbb{R}$ is a linear morphism of vector spaces. Notice that the same holds if we substitute the field $\mathbb{R}$ with any other field, for instance, the field of complex numbers $\mathbb{C}$.

Next, we introduce the notion of a commutative (locally) $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states as a pair $(X,\mathbf{A})$ consisting of a topological space X and a sheaf of commutative $\mathbb{R}$-algebras of observables $\mathbf{A}$ on X, *such that* the *stalk* $\mathbf{A}_x$ of *germs* is a *(local) commutative* $\mathbb{R}$-algebra for *any point* $x \in X$. A morphism of $\mathbb{R}$-ringed spaces $H=(h,\phi)$ from $(X,\mathbf{A})$ to $(Y,\mathbf{B})$ consists of a continuous morphism of topological spaces $h: X \to Y$ and also a morphism of sheaves of $\mathbb{R}$-algebras $\phi: \mathbf{H}^{\leftarrow}(\mathbf{B}) \to \mathbf{A}$, such that for every $x \in X$ the induced morphism of stalks at x, that is, $\phi_x: \mathbf{B}_{h(x)} \to \mathbf{A}_x$ is a morphism of $\mathbb{R}$-algebras. We denote the category of commutative (locally) $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) spaces of states by $\mathcal{A}_{\mathbb{R}}$.

At this stage, we are able to introduce the notion of a *category of models* as a subcategory of the category $\mathcal{A}_{\mathbb{R}}$, denoted by $\mathcal{M}_{\mathbb{R}}$, which satisfies the following conditions:

i The base locus $\mathbb{U}$ of an object $(\mathbb{U}, \mathbf{A})$ of $\mathcal{M}_{\mathbb{R}}$ is some *model* topological space;

ii If $(\mathbb{U}, \mathbf{A})$ is an object of $\mathcal{M}_{\mathbb{R}}$, and $\mathbb{V} \subseteq \mathbb{U}$ an open locus, then $(\mathbb{V}, \mathbf{A} \mid \mathbb{V})$ is *also* an object of $\mathcal{M}_{\mathbb{R}}$, and the *injection* $(\mathbb{V}, \mathbf{A} \mid \mathbb{V}) \infty (\mathbb{U}, \mathbf{A})$ is a morphism in $\mathcal{M}_{\mathbb{R}}$.

We say that given a category of models $\mathcal{M}_{\mathbb{R}}$, an $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states $(Y, \mathbf{B})$ is an $\mathcal{M}_{\mathbb{R}}$-*manifold* if the following conditions are satisfied:

i Every point $y \in Y$ has an open locus U together with an *isomorphism* of $\mathbb{R}$-ringed spaces, that is:

$$H = (h, \phi) : (U, \mathbf{B} \mid U) \to (\mathbb{U}, \mathbf{A}) \in \mathcal{M}_{\mathbb{R}}$$

We call the above isomorphism an $\mathcal{M}_{\mathbb{R}}$-*coordinate chart*, or *reference frame* of $(Y, \mathbf{B})$ with respect to the category of models $\mathcal{M}_{\mathbb{R}}$;

ii For any pair of $\mathcal{M}_{\mathbb{R}}$-coordinate charts:

$$H_a = (h_a, \phi_a) : (U_a, \mathbf{B} \mid U_a) \to (\mathbb{U}_a, \mathbf{A}_a)$$
$$H_b = (h_b, \phi_b) : (U_b, \mathbf{B} \mid U_b) \to (\mathbb{U}_b, \mathbf{A}_b)$$

with $U_a \cap U_b \neq \varnothing$, the induced isomorphism:

$$H_{ab} := H_a \circ H_b^{-1} : (\mathbb{U}_{ba}, \mathbf{A}_b \mid \mathbb{U}_{ba}) \to (\mathbb{U}_{ab}, \mathbf{A}_a \mid \mathbb{U}_{ab})$$

is a *morphism* in the category of models $\mathcal{M}_{\mathbb{R}}$, where:

$$\mathbb{U}_{ba} := h_b(U_a \cap U_b)$$
$$\mathbb{U}_{ab} := h_a(U_a \cap U_b)$$

The isomorphism H_{ab} in the category of models $\mathcal{M}_{\mathbb{R}}$ is called a *gluing datum* between overlapping $\mathcal{M}_{\mathbb{R}}$-coordinate charts of $(Y,\mathbf{B})$ with respect to the category of models $\mathcal{M}_{\mathbb{R}}$;

iii The following *cocycle relations* are satisfied whenever they are defined:

1 $H_{aa}=1$

2 $H_{ab}=H_{ba}^{-1}$

3 $H_{ab}\circ H_{bc}=H_{ac}$

From the above, we conclude the following: Given a *gluing datum* between overlapping $\mathcal{M}_{\mathbb{R}}$-coordinate charts of $(Y,\mathbf{B})$ with respect to the category of models $\mathcal{M}_{\mathbb{R}}$, we consider the disjoint union:

$$\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)$$

with its natural structure as an $\mathcal{M}_{\mathbb{R}}$ *-manifold*, where I is a corresponding indexing set. Then, we introduce on $\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)$ a relation defined as follows:

$$(\mathbb{U}_a,\mathbf{A}_a)\not\ni(y_a,s_a)\sim(y_b,s_b)\in(\mathbb{U}_b,\mathbf{A}_b)$$

if and only if:

i $(y_a,s_a)\in\mathbb{U}_{ab}\subset\mathbb{U}_a$

ii $(y_b,s_b)\in\mathbb{U}_{ba}\subset\mathbb{U}_b$

iii $(y_a,s_a)=H_{ab}((y_b,s_b))$

Then, according to the *cocycle relations* of the given gluing datum we obtain that: Because of (1) the relation is *reflexive*, because of (2) the relation is *symmetric*, and because of (3) the relation is *transitive*. Hence the relation $\sim$ defined on $\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)$ is an *equivalence relation*, giving rise to a groupoid. The *quotient* space

$$\coprod_{a\in I}(\mathbb{U}_a,\mathbf{A}_a)/\sim$$

with respect to this equivalence relation has the induced structure of an $\mathcal{M}_{\mathbb{R}}$-*manifold*, such that the natural projection

$$p : \coprod_{a \in I} (\mathbb{U}_a, \mathbf{A}_a) \to \coprod_{a \in I} (\mathbb{U}_a, \mathbf{A}_a)/\sim$$

is a morphism of $\mathcal{M}_{\mathbb{R}}$-manifolds. Moreover, the set $\{p(\mathbb{U}_a, \mathbf{A}_a)\}_{a \in I}$ is defined as an $\mathcal{M}_{\mathbb{R}}$-*coordinate atlas* on $\coprod_{a \in I} (\mathbb{U}_a, \mathbf{A}_a)/\sim$.

An immediately obvious application of the notion of an $\mathcal{M}_{\mathbb{R}}$-manifold becomes clear if we consider as a category of models $\mathcal{M}_{\mathbb{R}}$ the smooth category of models, denoted by $\mathcal{M}_{\mathbb{R}}{}^{sm}$. The category $\mathcal{M}_{\mathbb{R}}{}^{sm}$ has as objects pairs of the form $(\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$, where $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ denotes the sheaf of real-valued smooth functions of class $\mathcal{C}^{\infty}$ on $\mathbb{R}^n$. We notice that in this case morphisms $(\mathbb{R}^m, \mathcal{C}^{\infty}{}_{\mathbb{R}^m}) \to (\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$ are just smooth maps $\mathbb{R}^m \to \mathbb{R}^n$.

Then, given the category of smooth models $\mathcal{M}_{\mathbb{R}}{}^{sm}$, an $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of observables $(Y, \mathbf{B})$ is called a *smooth* $\mathbb{R}$-*manifold* if it satisfies the conditions [I], [II], [III] given previously. We also notice that the structure of a smooth $\mathbb{R}$-manifold is obtained by the *equivalence relation* on *the disjoint union* of its *coordinate charts* induced by the corresponding *gluing datum* with respect to the category of smooth models. We denote a smooth $\mathbb{R}$-manifold by the pair $(Y, \mathbf{O}_Y)$, where, for every open subset U of Y, $(\mathbf{O}_Y)(U) = \mathcal{C}^{\infty}{}_Y(U)$ is the ring of smooth functions on U.

A morphism between smooth $\mathbb{R}$-manifolds is called a *diffeomorphism* when it is an *isomorphism* of the corresponding $\mathbb{R}$-*ringed spaces*. In this sense, some smooth functions $u_1, \ldots u_n \in \mathcal{C}^{\infty}{}_Y(U)$ define a smooth coordinate system on U if the corresponding morphism $(u_1, \ldots, u_n) : U \to \mathbb{R}^n$ induces a diffeomorphism of U onto an open locus of the model smooth topological $\mathbb{R}^n$ for some appropriate $n \in \mathbb{N}$.

An interesting observation has to do with the fact that the definition of a smooth $\mathbb{R}$-manifold as an $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states $(Y, \mathbf{O}_Y)$ obtained by means of a gluing datum with respect to the category of smooth models $\mathcal{M}_{\mathbb{R}}{}^{sm}$, takes into account the principle of *relativization* of physical *representability* with respect to the internal reals of a topos of the form $\mathbf{Shv}(Y)$, where Y is a topological space. We emphasize the significance of this principle concerning its dual aspect referring to both the local and the global behavior of observables.

From the other side, we know that a smooth $\mathbb{R}$-manifold, is completely determined by the ring ($\mathbb{R}$-algebra) of all global smooth functions (assuming that it is a Hausdorff topological space with a countable basis). This means that the semantics of a smooth $\mathbb{R}$-manifold is completely determined by the information encoded in the global observables only, or equivalently, its semantics is completely understood with respect to the constant topos of sets $\mathbf{Sets}$.

We consider this fact as a serious *drawback* which affects the interpretation of our current physical theories to a very significant degree, on which we shall expand later. Most importantly, the *localization* properties of observables expressed in terms of *observable germs* are actually overlooked. In this sense, it is necessary to understand the procedure by means of which the $\mathbb{R}$-ringed (or $\mathbb{R}$-algebraized) space of states $(Y, \mathbf{O}_Y)$ can be *reconstituted* from the ring ($\mathbb{R}$-algebra) $\mathcal{C}^{\infty}{}_{Y}$ of global real-valued smooth functions on Y.

Firstly, the *set* Y is recovered as the $\mathbb{R}$-*spectrum* of *global* observables (global real-valued smooth functions) $\mathcal{C}^{\infty}{}_{Y}$, that is the set of all surjective morphisms of $\mathbb{R}$-algebras:

$$\gamma : \mathcal{C}^{\infty}{}_{Y} \to \mathbb{R}$$

Thus, we have that, set-theoretically:

$$Y = \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}) := \mathbf{Hom}_{\mathbb{R}-alg}(\mathcal{C}^{\infty}{}_{Y}, \mathbb{R})$$

We recall that the $\mathbb{R}$-algebra (field) $\mathbb{R}$ is called the *coordinatizing frame* of each state of the formed state-space $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$. The geometric semantics of this connotation denotes the set of elements which can be $\mathbb{R}$-observed by a measurement procedure on the ring of observables $\mathcal{C}^{\infty}{}_{Y}$. Eventually, that set of elements, constituting the $\mathbb{R}$-spectrum of $\mathcal{C}^{\infty}{}_{Y}$, are identified with the $\mathbb{R}$-coordinatized *points* of a geometric state-space that can be observed by means of the ring of global real-valued functions $\mathcal{C}^{\infty}{}_{Y}$. The next task is to endow the set (state-space) $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$ with an appropriate topology, so that it bears the structure of a topological space. We consider the *Israel Gelfand topology* on $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$, which is defined by the requirement that it is the smallest topology such that:

$$\hat{f} : \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}) \to \mathbb{R}$$
$$\hat{f}(\gamma) := \gamma(f) \in \mathbb{R}$$

is *continuous* for any $f \in \mathcal{C}^{\infty}{}_{Y}$. Thus, we get a morphism of $\mathbb{R}$-algebras:

$$\mathcal{C}^{\infty}{}_{Y} \to \mathbf{Hom}_{cstates}(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbb{R})$$
$$f \mapsto \hat{f}$$
$$\hat{f}(\gamma) := \gamma(f) \in \mathbb{R}$$

where the $\mathbf{Hom}$-set contains morphisms between continuous state spaces. Now, according to the *Gelfand representation theorem*, if Y is a separated smooth $\mathbb{R}$-manifold whose topology has a countable basis, then the above morphism of $\mathbb{R}$-algebras is an *isomorphism*, and the *morphism*:

$$\lambda : Y \to \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$$

is a *homeomorphism* of topological state-spaces. Note that if Y is a *smooth* $\mathbb{R}$*-manifold* each point $y \in Y$ defines a morphism of of $\mathbb{R}$-algebras ($\mathbb{R}$-evaluation):

$$\lambda_y : \mathcal{C}^{\infty}{}_{Y} \to \mathbb{R}$$
$$\lambda_y(f) = f(y)$$

Equivalently stated, each point $y \in Y$, defines a *maximal ideal* μ_y of $\mathcal{C}^{\infty}{}_{Y}$, that is:

$$\mu_y := Ker\lambda_y = \{f \in \mathcal{C}^{\infty}{}_{Y} : f(y) = 0\}$$

Consequently, we obtain a natural morphism

$$\lambda : Y \to \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$$
$$\lambda(y) = \lambda_y \simeq \mu_y$$

which is a *homeomorphism* of topological spaces. Thus, identifying the topological space Y with the topological space $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$ via λ, the topology of Y coincides with the *Gelfand topology*, the morphism of $\mathbb{R}$-algebras $\mathcal{C}^{\infty}{}_{Y} \to \mathbf{Hom}_{states}(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbb{R})$, where $f \mapsto \hat{f}$, such that, $\hat{f}(\gamma) := \gamma(f) \in \mathbb{R}$ is an *isomorphism*, and the *evaluation* morphism λ_y is the same as γ, being identified with the *maximal ideal* μ_y for each $\mathbb{R}$-state y in Y.

Furthermore, if U is any open set in $Y = \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$, then we require that $\mathbf{O}_Y(U) = \mathcal{C}^{\infty}(U)$, where:

$$\mathcal{C}^{\infty}(U) = \mathcal{C}^{\infty}{}_{Y} \,|\, U$$

is the *algebraic localization* of $\mathcal{C}^{\infty}{}_{Y}$, or *ring of fractions* of $\mathcal{C}^{\infty}{}_{Y}$ with respect to the *multiplicative set* of all global real-valued smooth functions without zeros in U, that is:

$$\mathcal{C}^{\infty}(U) = \mathcal{C}^{\infty}{}_{Y} \,|\, U = \mathcal{C}^{\infty}(Y) \,|\, U := \{f / g : f, g \in \mathcal{C}^{\infty}{}_{Y} \,|\, g(y) \neq 0, \forall y \in U\}$$

The localization of $\mathcal{C}^{\infty}{}_{Y}$ with respect to the multiplicative set of all global real-valued smooth functions without zeros in U corresponds to the *localization* of $\mathcal{C}^{\infty}{}_{Y}$ at the *ideal* μ_K of all real-valued smooth functions vanishing at a *closed subset* K of Y (*zero-set* of the ideal μ_K), *complementary* to U, that is:

$$K = \{y \in Y : g(y) = 0, \forall g \in \mu_K\}$$

In this way, given the ring ($\mathbb{R}$-algebra) $\mathcal{C}^{\infty}{}_{Y}$ of global real-valued smooth functions on Y, we finally obtain a *topological space* $\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$ *identified* with Y, viz. the $\mathbb{R}$-*spectrum* of $\mathcal{C}^{\infty}{}_{Y}$, together with a *structure sheaf* of rings ($\mathbb{R}$-algebras) $\mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})}$, identified with the *completion* (sheafification) of the presheaf of rings ($\mathbb{R}$ -algebras) $U \to \mathcal{C}^{\infty}U(U)$, denoted by $\mathbf{C}^{\infty}{}_{Y} = \mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})}$, such that:

$$\mathbf{\Gamma}\mathbf{C}^{\infty}{}_{Y} = \mathbf{\Gamma}\mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})} = \mathcal{C}^{\infty}{}_{Y}$$

Consequently, we have constructed the $\mathbb{R}$ -*ringed* (or $\mathbb{R}$ -algebraized) space of observables

$$(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbf{O}_{\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})}) := (\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbf{C}^{\infty}{}_{Y})$$

from the *ring* ($\mathbb{R}$ -algebra) $\mathcal{C}^{\infty}{}_{Y}$ of *global* real-valued smooth functions on Y, where $Y = \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y})$. If we apply this procedure to the ring ($\mathbb{R}$ -algebra) $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$, we obtain the *local smooth model* $\mathbb{R}$ -ringed space $(\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$, as an object in the smooth model category $\mathcal{M}_{\mathbb{R}}{}^{sm}$, by means of which we have defined the notion of a smooth $\mathbb{R}$ -manifold previously within the topos of sheaves.

An extremely significant observation has to do with the fact that the *procedure* of *reconstitution* of the $\mathbb{R}$ -algebraized space of states $(\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{Y}), \mathbf{C}^{\infty}{}_{Y})$ from the ring ($\mathbb{R}$ -algebra) of observables $\mathcal{C}^{\infty}{}_{Y}$, where Y is a smooth $\mathbb{R}$ -manifold, can be applied for an *arbitrary* $\mathbb{R}$ -algebra of observables $\mathcal{A}$. Of course, in this case the $\mathbb{R}$ -spectrum of $\mathcal{A}$, that is, $\mathbb{R}\mathbf{S}pec\mathcal{A}$ is not a smooth $\mathbb{R}$ -manifold any more, meanong that it is *not* modeled on the local smooth model $\mathbb{R}$ -ringed space $(\mathbb{R}^n, \mathcal{C}^{\infty}{}_{\mathbb{R}^n})$. Still, the *associated* $\mathbb{R}$ -*ringed* space $(\mathbb{R}\mathbf{S}pec\mathcal{A}, \mathbf{A})$ *can be* an $\mathcal{M}_{\mathbb{R}}$-manifold for an appropriate *choice* of a category of models $\mathcal{M}_{\mathbb{R}}$.

A natural issue that arises in this setting is the conceptualization of an $\mathbb{R}$ -ringed space $(\mathbb{R}\mathbf{S}pec\mathcal{A}, \mathbf{A})$ as an $\mathcal{M}_{\mathbb{R}}$-manifold, where the category of models has objects (local models of an $\mathcal{M}_{\mathbb{R}}$-manifold) of the form $(\mathbb{R}\mathbf{S}pec(\mathcal{D}), \mathbf{D})$, where $\mathcal{D}$ is a *quotient* of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ by an *ideal* ξ of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$, that is:

$$\mathcal{D} := \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$$

for some natural number n. Obviously, $\mathbf{D}$ refers to the *completion of the presheaf* U Œ $\mathcal{D}(U)$ for U open locus in $\mathbb{R}\mathbf{S}pec(\mathcal{D})$ obtained by the procedure of *algebraic localization* of the $\mathbb{R}$-algebra $\mathcal{D}$. We think of the ideal ξ as the ideal of all elements of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ vanishing at a closed subset $[\xi]_0$ of $\mathbb{R}^n$, that is:

$$[\xi]_0 = \{x \in \mathbb{R}^n : f(x) = 0, \forall f \in \xi\}$$

Then, evidently $\mathbb{R}\mathbf{S}pec(\mathcal{D})$ is identified with the *zero-set* of the ideal ξ, that is:

$$\mathbb{R}\mathbf{S}pec(\mathcal{D}) = \mathbb{R}\mathbf{S}pec((\mathcal{C}^{\infty}{}_{\mathbb{R}^n})/\xi) = [\xi]_0 = \{x \in \mathbb{R}^n : f(x) = 0, \forall f \in \xi\}$$

We notice that, by analogia to the local smooth model case, we may call $(\mathbb{R}\mathbf{S}pec(\mathcal{D}), \mathbf{D})$ a *local differential model*. This is due to the fact that, although $\mathbb{R}\mathbf{S}pec(\mathcal{D}) = [\xi]_0$ is *not* a smooth manifold, it *can be* interpreted as a *differential space*, whose global algebra of differentiable functions is $\mathcal{D} := \mathcal{C}^{\infty}{}_{\mathbb{R}^n}/\xi$. In this sense, differentiable functions on $\mathbb{R}\mathbf{S}pec(\mathcal{D}) = [\xi]_0$ are thought of as *restrictions* of smooth functions on $\mathbb{R}^n$. This is due to the existence of the surjective morphism of $\mathbb{R}$-algebras:

$$\circlearrowleft : \mathcal{C}^{\infty}{}_{\mathbb{R}^n} \leq \mathcal{C}^{\infty}{}_{\mathbb{R}^n}/\xi$$

which is interpreted as a *restriction* morphism, that is for any $f \in \mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ the equivalence class $[f] \in \mathcal{C}^{\infty}{}_{\mathbb{R}^n}/\xi$ is said to be the restriction of the smooth function f to the differential function $[f]$, or else, the restriction of f to the *differential subspace* $[\xi]_0$ of the smooth space $\mathbb{R}^n$. In this manner, the notion of *differentiability* induced by local differential models of the form considered, and subsequently, the notion of an $\mathcal{M}_{\mathbb{R}}$-differential manifold obtained, is much more *general* than the notion of *smoothness* induced by local smooth models and the associated smooth manifolds. Moreover, the notion of differentiability *supersedes* the notion of smoothness, which in turn, is obtained as a *special* case of the former.

Motivated by the previous observation, we say that an *ideal* of a differentiable algebra is *closed* if and only if its *quotient* by this ideal is *also* a differentiable algebra. In particular, since $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ is a differentiable algebra, and the quotient algebra $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$ is also a differentiable algebra by *restriction*, the ideal ξ is *closed*. We also say that an $\mathbb{R}$-algebra $\mathcal{Z}$ is *simple* if 0 is the *unique* element of $\mathcal{Z}$ *vanishing* at any point of $\mathbb{R}\mathbf{S}pec(\mathcal{Z})$. This is equivalent to saying that the morphism of $\mathbb{R}$-algebras:

$$\mathcal{Z} \to \mathbf{Hom}_{cstates}(\mathbb{R}\mathbf{S}pec(\mathcal{Z}), \mathbb{R})$$
$$f \mapsto \hat{f}$$

is *injective*, so that any simple $\mathbb{R}$-algebra is *isomorphic* to an algebra of real-valued continuous functions on the topological space $\mathbb{R}\mathbf{S}pec(\mathcal{Z})$. It is clear that if $\mathcal{Z}$ is of the form $\mathcal{Z} = \mathcal{D} = \mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$ where ξ denotes the *closed ideal* of all elements of $\mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ vanishing at a closed subset $[\xi]_0$ of $\mathbb{R}^n$, then $\mathcal{Z}$ is *simple*. Now, we are ready to consider an *extension* of differentiable algebras to *non-simple* ones.

Intuitively, these algebras are *still* of the form $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \xi$, where ξ is a *closed ideal*, but now they contain *nilpotent* elements. This equivalently means that the ideal is of the form $\xi = \zeta^r$ for some *power* r. For example, if μ_x is the (maximal) ideal of all smooth functions of $\mathcal{C}^{\infty}_{\mathbb{R}^n}$ vanishing at x in $\mathbb{R}^n$, then $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x{}^2$ is considered a differential algebra containing *nilpotent* elements. Note that:

$$\mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x{}^2) = \mathbb{R}\mathbf{S}pec(\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x) = \{x\}$$

but $\mathcal{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x{}^2$ contains nilpotent elements. In this sense, we can consider *local differential model spaces* of the form

$$(\{x\}, \mathbf{C}^{\infty}{}_{\mathbb{R}^n} / \mu_x{}^{(r+1)})$$

interpreted as the [$r-th$ *infinitesimal region of* x] differential space, being a differential subspace of $(\mathbb{R}^n, \mathbf{C}^{\infty}{}_{\mathbb{R}^n})$. These differential spaces have an interesting physical interpretation, since the restriction of a smooth function $f \in \mathcal{C}^{\infty}{}_{\mathbb{R}^n}$ to a differential space of the above form is equivalent to the $r-th$ *Taylor expansion* of f at x. This is called the $r-th$-*jet* of f at x and we obtain:

$$j_x^r = [f] \in \mathcal{C}^\infty_{\mathbb{R}^n} / \mu_x^{(r+1)}$$

According to the preceding analysis we can now conceptualize an $\mathbb{R}$-ringed space $(\mathbb{R}\mathbf{S}pec\mathcal{A}, \mathbf{A})$ as an $\mathcal{M}_{\mathbb{R}}$-*manifold*, called a *differential* $\mathbb{R}$-*ringed space*, where the category of models has objects (local models of an $\mathcal{M}_{\mathbb{R}}$-manifold) of the form $(\mathbb{R}\mathbf{S}pec(\mathcal{D}), \mathbf{D})$.

In this context, $\mathcal{D}$ stands for a differentiable algebra, since it is a quotient of $\mathcal{C}^\infty_{\mathbb{R}^n}$ by a closed ideal ξ of $\mathcal{C}^\infty_{\mathbb{R}^n}$, that is:

$$\mathcal{D} := \mathcal{C}^\infty_{\mathbb{R}^n} / \xi$$

for some natural number n. In order to achieve the greatest generality we give the following preliminary definition:

A (locally) $\mathbb{R}$-ringed space $(Y, \mathbf{O}_Y)$ is called an *affine differential* $\mathbb{R}$-*ringed space* of states if it is *isomorphic* to a differential $\mathbb{R}$-ringed space of the form $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$, where $\mathcal{D}$ is a differentiable algebra of observables. Then, we consider as a category of models $\mathcal{M}_{\mathbb{R}}^{diff}$ the category of affine differential $\mathbb{R}$-ringed spaces.

Then, a differential $\mathbb{R}$-ringed space of states $(X, \mathbf{O}_X)$ is defined as an $\mathcal{M}_{\mathbb{R}}^{diff}$-*manifold*. This means that any point $x \in X$ has an open neighbourhood U in X, called an *affine open locus*, such that $(U, \mathbf{O}_X | U)$ is an affine differential $\mathbb{R}$-ringed space of states. Such open loci of X are called *affine open sets* and they naturally define a *basis* for the topology of X. The sections of $\mathbf{O}_X$ on an open locus U in X are said to be *differentiable* (observable) functions on U. The value of a differentiable function at a point $x \in U$ is identified with the *residue class* of its germ f_x at x, that is, with $f(x) \in \mathcal{O}_{X,x} / \mu_x = \mathbb{R}$. Then, any differentiable function $f \in \mathbf{O}_X(U)$ determines a *continuous* morphism $\breve{f} : U \to \mathbb{R}$, where $\breve{f}(x) = f(x)$, although f is *not* determined by $\breve{f}$.

An immediate consequence is the following: If ζ is a *closed ideal* of a differentiable algebra $\mathcal{D}$, then $(\mathbb{R}\mathbf{S}pec(\mathcal{D}/\zeta), \mathbf{D}/\zeta)$ is a *closed differential subspace* of $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$. Inversely, any *closed differential subspace* of $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$ is defined by a *unique closed ideal* of $\mathcal{D}$.

For brevity of notation, if a differentiable algebra is denoted by $\mathcal{D}$, then we denote the corresponding affine differential $\mathbb{R}$-ringed space by $\hat{\mathcal{D}}$. Now, there exists a categorical *natural equivalence* (duality) between the category of (commutative and unital) differentiable $\mathbb{R}$-algebras of observables $\mathcal{D}$, and the category of affine differential (commutative) $\mathbb{R}$-ringed spaces of states $\hat{\mathcal{D}}$, which is defined functorially, as follows:

$$\mathbb{R}\mathbf{Spec} : \mathcal{D} \rightleftarrows \hat{\mathcal{D}} : \Gamma$$

where, $\mathbb{R}\mathbf{Spec} : \mathcal{D} \to \hat{\mathcal{D}}$ is the *real spectrum* functor, which assigns to a commutative and unital differentiable $\mathbb{R}$-algebra of observables $\mathcal{D}$ its dual *affine differential* $\mathbb{R}$ *-ringed space* of states $(\mathbb{R}Spec\mathcal{D}, \mathbf{D})$, and, $\mathbf{\Gamma} : \hat{\mathcal{D}} \to \mathcal{D}$ is the *global sections* functor in the opposite direction, which assigns to the *structure sheaf* $\mathbf{D}$ over the real spectrum topological space $\mathbb{R}Spec\mathcal{D}$, its $\mathbb{R}$-algebra of *global sections* $\mathbf{\Gamma}(\mathbb{R}Spec\mathcal{D}, \mathbf{D}) := \mathbf{D}(\mathbb{R}Spec\mathcal{D})$, identified with the *differentiable* $\mathbb{R}$ *-algebra* of differentiable functions $\mathcal{D}$.

The conceptual importance of this natural equivalence lies in the fact that the notion of a *differential model* object, incorporates a fundamental *categorical duality*, which, when interpreted physically, *unifies* the *algebraic encoding* of differentiable observable information expressed in terms of (commutative and unital) differentiable $\mathbb{R}$-algebras of observables, together with, the *geometric-topological representation* of these model objects in terms of affine differential (commutative) $\mathbb{R}$-ringed spaces of states. Moreover, both, the algebraic, and, the equivalent dual geometric representation of differential model objects, implement and reciprocally respect faithfully the bidirectional *localization-globalization* process of observation within the topos of sheaves.

10.5 DIFFERENTIAL OBSERVABILITY AND LOCALLY FREE MODULES

In order to clarify the meaning of the observation process within the localization environment of a topos of sheaves it is instructive to use again the principle of *relativization* of physical *representability*. Let us consider a commutative and unital coordinatization ring, thought of as a $\mathbb{R}$-algebra of observables, whose representability is relativized with respect to the topos of sheaves $\mathbf{Shv}(Y)$, Then, the open sets of the global topological space Y, play the role of a category of extensional reference contexts of observation, partially ordered by inclusion. Consequently, the right action of this category on the ring of observables, partitions it into sorts parameterized by the base local contexts, inducing the uniform homologous fibered information structure of a presheaf of local observables over the base category of localizing contexts. In this sense, the *geometric classification* of the total information content included in the *representation* of the corresponding natural system by a formal system of the above form, through observation, which in addition respects the requirements of a coherent localization-globalization process, can be performed in a *dually equivalent* manner.

More precisely, observable information organized in the form of a *presheaf* can firstly be *contextualized* at base points by application of the *germs*-functor, and then, *glued* together (by taking the disjoint union of the fibers and topologizing the formed space as an *étale* bundle), such that, the set of cross sections constitutes a *sheaf* obtained by the subsequent application of the *cross-sections* functor. Alternatively, observable information organized in the form of a *presheaf* can firstly be made compatible with respect to *restriction* from the global to the local, as well as, *extension* from the local to the global, by application of the *sheafification* functor, and then, be *contextualized* at the base points of the equivalent *étale* bundle, by subsequent application of the *germs*-functor on the sheaf of local observables formed from the *sheafification* of the presheaf we started with.

Thus, observation of natural systems preserving the requirements of a coherent *localization-globalization* process, can be implemented equivalently, given the initial *organization* of observable information in the form of a presheaf of rings of local observables, either, by firstly *contextualizing* information at base points of a topological space in terms of observable germs, and then, *gluing* appropriately, or, by firstly making locally observable information *compatible* with respect to *restriction* and *extension*, and then, *contextualizing* at those perspectives. Again, there is a *key concept* that *unlocks* the meaning of this *equivalence*, referring to the sequence of the operations needed, in order to make observation of natural systems preserving the requirements of a coherent localization-globalization observation process; It is that, *any sheaf* of local observables can be conceived as a *sheaf of cross-sections* of its corresponding equivalent by duality *étale* bundle.

The formalization of the above takes place, when we first recollect that there is a categorical natural equivalence between the categories of *sheaves* $\mathbf{Sh}(Y)$ on $\mathcal{O}(Y)$, and the category of *étale topological bundles* $\mathcal{ET}(Y)$ over Y, as follows:

$$\Lambda : \mathbf{Sh}\big(Y\big) \rightleftarrows \mathcal{ET}\big(Y\big) : \Gamma$$

Now, the crucial fact that a sheaf of local observables is identified as a sheaf of cross-sections of the corresponding étale bundle, is expressed *functorially* by the requirement that the *unit* natural transformation of this adjunction (natural equivalence) is an *isomorphism* $\eta_{\mathbf{A}} : \mathbf{A} \to \Gamma\Lambda\mathbf{A}$. Moreover, in the case at issue, the *counit* Ú$_A : \Lambda\Gamma A \to A$ is also an *isomorphism*. Thus, the *cross sections-functor* $\Gamma : \mathcal{ET}(Y) \to \mathbf{Sh}(Y)$ and the *germs-functor* $\Lambda : \mathbf{Sh}(Y) \to \mathcal{ET}(Y)$ can be conceived as inverses to each other.

Let us now extend this line of thought, by considering the above $\mathbb{R}$ -algebra of observables as a *differentiable* $\mathbb{R}$ -algebra $\mathcal{D}$ corresponding to a *model differential space* of states $\hat{\mathcal{D}}$. We intend to conceptualize the procedure of model differentiable observable information acquisition constituting a corresponding model differential observation process. The reason that we focus our attention on the notion of a model differential observation process is that we can extend it for the case of an $\mathcal{M}_{\mathbb{R}}^{diff}$ -*manifold* by means of a *gluing datum*.

We claim that a model differential observation process should be conceived categorically as a *natural transformation of the identity* of $\mathcal{D}$, enacted through the multitude of *all variable* localization domains of its *real spectrum* topological space, which classifies its total information content. This natural transformation of the identity of $\mathcal{D}$ is expressed concretely, by the *counit* of the *adjunction* between the category of (commutative and unital) differentiable $\mathbb{R}$ -algebras of observables $\mathcal{D}$, and the category of affine differential (commutative) $\mathbb{R}$ -ringed spaces of states $\hat{\mathcal{D}}$.

Moreover, since the *adjunction* is actually a categorical *natural equivalence* (duality), the *counit* natural transformation is an *isomorphism*. In this sense, a model differential observation process, enacted through the multitude of all variable localization domains of its real spectrum topological space classifying its total information content, according to the localization-globalization process, is conceived as the *operational implementation* of the counit isomorphism:

$$Ú_{\mathcal{D}} : \Gamma\mathbb{R}\mathbf{Spec}\mathcal{D} \to \mathcal{D}$$

In this sense, the composite endofunctor:

$$\mathbf{\Pi} := \mathbf{\Gamma}\mathbb{R}\mathbf{Spec} : \mathcal{D} \to \mathcal{D}$$

may be called a model differential observation. Note that *any* differentiable $\mathbb{R}$ -algebra of observables can be considered as a *fixed point* of a model differential observation functor $\mathbf{\Pi}$.

Next, let us consider a commutative (locally) $\mathbb{R}$ -ringed (or $\mathbb{R}$ -algebraized) space of states denoted by the pair $(X, \mathbf{A})$. An $\mathbf{A}$ *-module* E is called a *locally free* $\mathbf{A}$ -module of finite rank m, if for any point $x \in X$ there exists an open locus U of X such that:

$$E|U = (\mathbf{A}|U)^m$$

where $(\mathbf{A}|U)^m$ denotes the $m-th$ *-direct sum* of the sheaf of $\mathbb{R}$ -algebras of observables $\mathbf{A}$ restricted to U, for some $m \in \mathbb{N}$. Furthermore, if $\mathbf{A}$ is a constant sheaf of $\mathbb{R}$ -algebras, then any locally free $\mathbf{A}$ -module of finite rank m, for some $n \in \mathbb{N}$, stands for a *local system* of coefficients.

In case that we view a smooth $\mathbb{R}$ *-manifold* as an $\mathbb{R}$ -ringed (or $\mathbb{R}$ -algebraized) space of states $(Y, \mathbf{O}_Y = \mathbf{C}^{\infty}{}_Y)$ obtained by means of a *gluing datum* with respect to the category of smooth models $\mathcal{M}_{\mathbb{R}}{}^{sm}$, then a (smooth) *locally free* $\mathbf{C}^{\infty}{}_Y$-module of *finite* rank corresponds *bijectively* to a (smooth) *vector bundle* on Y.

It is worth explaining the above stated bijective correspondence between locally free $\mathbf{C}^{\infty}{}_Y$-modules of finite rank and vector bundles on Y as follows:

Firstly, we notice that since sections of $\mathbf{C}^{\infty}{}_Y$ are smooth functions on Y, we may think of the corresponding sections of the $\mathbf{C}^{\infty}{}_Y$-module $(\mathbf{C}^{\infty}{}_Y)^m$ as *vector-valued* functions.

Secondly, we notice that if we have a general locally free $\mathbf{C}^{\infty}{}_Y$-module E of finite rank, then by definition, it is *locally isomorphic* to $(\mathbf{C}^{\infty}{}_Y)^m$ for some $m \in \mathbb{N}$. We bear in mind that, if y is a point of Y, and μ_Y is *the ideal* of $(\mathbf{C}^{\infty}{}_Y)_y$ consisting of *germs* of smooth functions at y, vanishing at y, we may consider for any $f \in (\mathbf{C}^{\infty}{}_Y)_y$ its image in

$(\mathbf{C}^{\infty}{}_{Y})_y / \mu_y$. Then, the *evaluation* morphism $(\mathbf{C}^{\infty}{}_{Y})_y \to \mathbb{R}$, which takes any smooth function f to its real value $f(y)$, provides an *isomorphism* of $(\mathbf{C}^{\infty}{}_{Y})_y / \mu_y$ with $\mathbb{R}$. Hence, the *evaluation* morphism of a smooth function f is *equivalent* to considering the *image* of f in $(\mathbf{C}^{\infty}{}_{Y})_y / \mu_y$. This observation leads us to the conclusion that, by analogia, the $\mathbb{R}$-vector space $\varepsilon_y = E_y / \mu_y E_y$ is the *vector space* in which the sections of E take values at the point $y \in Y$.

Thirdly, we notice that the essential difference between the case referring to $\mathbf{C}^{\infty}{}_{Y}$ and the case referring to a locally free sheaf E lies in the fact that in the latter case the vector space $\varepsilon_y = E_y / \mu_y E_y$ *associated with* $y \in Y$ *depends on the point* y in comparison to the former case where such a dependence is not existent. Put differently, there is *no natural isomorphism* of these $\mathbb{R}$-vector spaces at *two different points* y_1 and y_2 of Y.

Of course, we may consider the set-theoretic union of all ε_y, viz., $\varepsilon = \bigcup_y \varepsilon_y$. We notice that ε has a *natural projection* morphism π into Y, such that $\pi^{-1}(y) = E_y$. Moreover, for any $y \in Y$ there exists an open locus U such that $E \mid U$ is isomorphic to $(\mathbf{C}^{\infty}{}_{Y})^m$, so that $\pi^{-1}(U)$ may be identified with $U \times \mathbb{R}^m$, just as we may identify the *morphism* $\pi^{-1}(U) \to U$ with the *projection* $U \times \mathbb{R}^m \to U$. In particular, the set $\pi^{-1}(U)$ can be endowed with the smooth structure of the *product space* $U \times \mathbb{R}^m$. Then it is straightforward to construct an appropriate *gluing datum* and thus obtain a corresponding $\mathcal{M}_{\mathbb{R}}$-*manifold* if we consider as a category of models $\mathcal{M}_{\mathbb{R}}$ the *smooth* category of *vector models*, denoted by $\mathcal{M}_{\mathbb{R}}{}^{smv}$. The category $\mathcal{M}_{\mathbb{R}}{}^{smv}$ has as objects pairs of the form:

$$(\mathbb{U} \times \mathbb{R}^m \subset \mathbb{R}^{n+m}, \mathbf{C}^{\infty}{}_{\mathbb{R}^n}(\mathbb{U}) \bigotimes \mathbf{C}^{\infty}{}_{\mathbb{R}^m})$$

Then, given the category of smooth vector models $\mathcal{M}_{\mathbb{R}}{}^{smv}$, we may construct a *smooth* $\mathcal{M}_{\mathbb{R}}{}^{smv}$-*manifold* satifying the conditions [I], [II], [III] given previously. Again, we note that the *structure* of a smooth $\mathcal{M}_{\mathbb{R}}{}^{smv}$-manifold is obtained by the *equivalence relation* on the disjoint union of its coordinate charts induced by the corresponding gluing datum with respect to the category of smooth vector models.

Now, the datum (ε, Y, π), consisting of a *smooth* $\mathcal{M}_{\mathbb{R}}^{smv}$*-manifold* ε (which is an $n+m$ -dimensional $\mathbb{R}$ -manifold), a smooth n-dimensional $\mathbb{R}$ -manifold Y and a *smooth projection* morphism $\pi : \varepsilon \to Y$ with the above described properties instantiates a *smooth vector bundle over* Y of rank m.

Thus, for any *locally free* $\mathbf{C}^{\infty}{}_{Y}$-module E of *finite* rank we can construct an *associated smooth vector bundle* (ε, Y, π) of the same rank. Inversely, the *sheaf of sections* of any (smooth) vector bundle on Y is a *locally free* $\mathbf{C}^{\infty}{}_{Y}$*-module* of finite rank. Thus, the correspondence is *bijective*, and consequently, we can identify locally free sheaves of $\mathbf{C}^{\infty}{}_{Y}$-modules with sheaves of (smooth) sections of their associated vector bundles on Y.

This *bijective* correspondence between locally free sheaves of $\mathbf{C}^{\infty}{}_{Y}$-modules with sheaves of (smooth) sections of their associated vector bundles on Y, where $(Y, \mathbf{O}_Y = \mathbf{C}^{\infty}{}_{Y})$ is a smooth $\mathbb{R}$ -manifold, has received such close attention in order to provide the necessary geometric intuition underlying the general notion of a locally free sheaf of $\mathbf{A}$ -modules of finite rank.

Note that the definition of a locally free sheaf of $\mathbf{A}$ -modules (of finite rank) holds with respect to an arbitrary commutative (locally) $\mathbb{R}$ -ringed (or $\mathbb{R}$ -algebraized) space of states denoted by the pair $(X, \mathbf{A})$. Thus, it can be applied for example to the case of an affine differential $\mathbb{R}$ -ringed space of states, which is by definition, isomorphic to a differential $\mathbb{R}$ -ringed space of the form $(\mathbb{R}\mathbf{S}pec\mathcal{D}, \mathbf{D})$, where $\mathcal{D}$ is a differentiable algebra of observables, or even to the case of a general differential $\mathbb{R}$ -ringed space of states $(X, \mathbf{O}_X)$, since it is definition an $\mathcal{M}_{\mathbb{R}}^{diff}$-manifold, but one which equally retains the analogous geometric intuition by means of associated differential vector bundles.

Next, let us consider the *category* of *differential* $\mathbb{R}$ *-ringed spaces* of states $(X, \mathbf{O}_X)$, and let $\hat{X}$ be an object in this category. Then, the functor represented by $\hat{X}$ is the *contravariant functor* $\mathbf{y}^{\hat{X}} : \hat{\mathcal{X}} \to \mathbf{Sets}$, defined as follows:

i For all differential spaces $\hat{Y}$ in $\hat{\mathcal{X}}$, $\mathbf{y}^{\hat{X}}(\hat{Y}) := Hom_{\hat{\mathcal{X}}}(\hat{Y}, \hat{X})$.

ii For all differential spaces-morphisms $f:\hat{Z}\rightarrow\hat{Y}$ in $\hat{\mathcal{X}}$,

$$\mathbf{y}^{\hat{X}}(f):Hom_{\hat{\mathcal{X}}}(\hat{Y},\hat{X})\rightarrow Hom_{\hat{\mathcal{X}}}(\hat{Z},\hat{X})$$

is defined as pre-composition with f, that is, $\mathbf{y}^{\hat{X}}(f)(g):=g\circ f$.

The functor $\mathbf{y}^{\hat{X}}:\hat{\mathcal{X}}\rightarrow\mathbf{Sets}$ represented by the differential spaces of states $\hat{X}$, is called the *Grothendieck functor* of generalized points (states) of $\hat{X}$. Moreover, the information contained in the differential space of states $\hat{X}$ is classified completely by its functor of generalized points (states) $\mathbf{y}^{\hat{X}}$.

In this sense, it is possible to extract geometric information without knowing whether there is actually a differential space in possession of a functor of the above form as its functor of generalized points. Pursuing the functorial approach one step further, we notice that the functor of generalized points of an $\mathbb{R}$-ringed differential space of states is completely determined by its *restriction to the subcategory* of $\mathbb{R}$*-ringed affine differential spaces* of states, together with, the *gluing datum* between any two $\mathbb{R}$-ringed affine differential spaces belonging to the covering family of that differential space.

In this manner, it is specified by means of the contravariant representable functor from the category of $\mathbb{R}$-ringed affine differential spaces to the category of sets, or equivalently, via the covariant representable functor from the category of differentiable $\mathbb{R}$-algebras of observables to the category of sets (modulo the compatibility conditions), thus admitting a well-defined operational determination in terms of *model differential observation* processes, as previously outlined. Furthermore, the appropriate implementation of the corresponding gluing conditions, should again respect the localization-globalization process, conceived in this generalized categorical context.

Hence, there is every need to secure compatibility of information under the operations of restriction (from the global to the local level) and extension (from the local to the global level), where, the notions of local and global receive a meaning only with respect to a *suitable notion* of topology (categorical Grothendieck topology) defined on the category of differential $\mathbb{R}$-ringed spaces of states. Conclusively, at this stage, we may say that a differential $\mathbb{R}$-ringed space of states constitutes the *sheafification* of the model differential observable information *encoded* in its *functor of generalized points* (restricted to affine differential $\mathbb{R}$-

ringed spaces of states), with respect to a topology that explicates the localization-globalization process in a categorical context.

Further on, we are going to examine in detail the semantic role of the notion of a *categorical Grothendieck topology*. At the moment, we should take note that, in relation to the functorial viewpoint on differential $\mathbb{R}$-ringed spaces of states, the *Gelfand topology*, which has enebled the *sheafification* of observables (taking values in the affine differential $\mathbb{R}$-ringed spaces of states), gives rise to a *Grothendieck topology* on the category of differential $\mathbb{R}$-ringed spaces of states.

10.6 TOPOLOGY OF SIEVING: COMMUNICATION SITES

The notion of a categorical Grothendieck topology requires, first of all, the *abstraction* of the constitutive properties of *localization* in appropriate *categorical* terms, and then, the effectuation of these properties for the definition of *localization* systems. The crucial observation has to with the fact that the concept of sheaf, in terms of coverings, restrictions, and collation, can be defined and used *not just* in the *spatial* sense, namely on the usual topological spaces, but in a generalized spatial sense, on more *general topologies* (Grothendieck topologies). In the usual definition of a sheaf on a topological space we use the open neighbourhoods U of a point in a space X; such neighbourhoods are actually *injective* topological maps $U \infty X$. The neighbourhoods U in topological spaces can be replaced by morphisms $V \to X$ *not* necessarily injective, and this can be done in *any* category with *pullbacks*.

In effect, a covering by open sets can be replaced by a *new* notion of *covering* provided by a family of morphisms satisfying certain conditions. These conditions abstract the constitutive properties of a well-defined *localization* process in appropriate *categorical* terms. Our presentation applies to *any* small category $\mathcal{B}$, consisting of *base* reference categorical objects B, with structure-preserving morphisms between them, as arrows. Of course, in the classical topological case, $\mathcal{B}$ is tautosemous with $\mathcal{O}(X)$ and the reference contexts B are tautosemous with the open sets U of X, partially ordered by inclusion.

For an object B of $\mathcal{B}$, we consider *indexed families*:

$$\mathbf{S} = \{\psi_i : B_i \to B, i \in I\}$$

of maps to B, and we assume that for each object B of $\mathcal{B}$ we have a set $\mathbf{\Lambda}(B)$ of certain such families satisfying conditions to be specified later. These families play the role of coverings of B under those conditions. For the coverings provided, it is possible to construct the *analogue* of the *topological* definition of a sheaf, where as presheaves on $\mathcal{B}$ we consider the functors $\mathbf{P}:\mathcal{B}^{op} \to Sets$.

According to the topological definition of a *sheaf* on a space we demand that for each open cover $\{U_i, i \in I\}$ of some U, every *family* of elements $\{p_i \in \mathbf{P}(U_i)\}$ that satisfy the *compatibility* condition on the *intersections* $U_i \cap U_j, \forall i, j$, are *pasted together* as a *unique element* $p \in \mathbf{P}(U)$. Imitating the above procedure for any covering $\mathbf{S}$ of an object B, and *replacing* the *intersection* $U_i \cap U_j$ by the *pullback* $B_i \times_B B_j$ in the general case, according to the diagram;

$$\begin{array}{ccc} B_i \times_B B_j & \xrightarrow{g_{ij}} & B_j \\ {\scriptstyle h_{ij}}\downarrow & & \downarrow{\scriptstyle \psi_j} \\ B_i & \xrightarrow{\psi_i} & B \end{array}$$

we effectively obtain for a given *presheaf* $\mathbf{P}:\mathcal{B}^{op} \to Sets$ a *diagram* of sets as follows:

$$\begin{array}{ccc} \mathbf{P}(B_i \times_B B_j) & \xrightarrow{\mathbf{P}(g_{ij})} & \mathbf{P}(B_j) \\ {\scriptstyle \mathbf{P}(h_{ij})}\downarrow & & \downarrow{\scriptstyle \mathbf{P}(\psi_j)} \\ \mathbf{P}(\mathbf{B_i}) & \xrightarrow{\mathbf{P}(\psi_i)} & \mathbf{P}(B) \end{array}$$

In this case the compatibility condition for a sheaf takes the form: if $\{p_i \in \mathbf{P}_i, i \in I\}$ is a family of compatible elements, namely satisfy $p_i h_{ij} = p_j g_{ij}, \forall i, j$, then a *unique* element $p \in \mathbf{P}(B)$ is determined by the family such that $p \cdot \psi_i = p_i, \forall i \in I$, where the notational convention $p \cdot \psi_i = \mathbf{P}(\psi_i)(p)$ has been used . Equivalently this condition can be expressed in the categorical terminology by the requirement that in the diagram:

$$\prod_{i,j}\mathbf{P}(B_i\times_B B_j) \;\overset{\longleftarrow}{\longleftarrow}\; \prod_i \mathbf{P}(B_i) \;\xleftarrow{\;e\;}\; \mathbf{P}(B)$$

the arrow e, where:

$$e(p) = (p\cdot\psi_i, i\in I)$$

is an equalizer of the maps:

$$(p_i, i\in I)\rightarrow(p_i h_{ij}; i,j\in I\times I)$$

and

$$(p_i, i\in I)\rightarrow(p_i g_{ij}; i,j\in I\times I)$$

correspondingly.

The specific *conditions* that the elements of the set $\Lambda(B)$, or else the *coverings* of $\mathcal{B}$, have to satisfy naturally lead to the notion of a *Grothendieck pretopology* on the category $\mathcal{B}$ as follows:

A *Grothendieck pretopology* on $\mathcal{B}$ is an operation Λ which assigns to each object B in $\mathcal{B}$ a set $\Lambda(B)$. Each $\Lambda(B)$ contains indexed families of $\mathcal{B}$-morphisms with codomain B:

$$\mathbf{S} = \{\psi_i : B_i \rightarrow B, i\in I\}$$

such that, the following conditions are satisfied:

i For each B in $\mathcal{B}$, $\{id_B\}\in\Lambda(B)$;

ii If $C\rightarrow B$ in $\mathcal{B}$ and $\mathbf{S} = \{\psi_i : B_i \rightarrow B, i\in I\}\in\Lambda(B)$ then:

$$\{\psi_1 : C\times_B B_i \rightarrow B, i\in I\}\in\Lambda(C)$$

Note that ψ_1 is the pullback in $\mathcal{B}$ of ψ_i along $C\rightarrow B$;

iii If $\mathbf{S} = \{\psi_i : B_i \rightarrow B, i\in I\}\in\Lambda(B)$, and for each $i\in I$:

$$\{\psi_{ik}{}^{i} : C_{ik} \to B_i, k \in K_i\} \in \Lambda(B_i)$$

then

$$\{\psi_{ik}{}^{i} \circ \psi_i : C_{ik} \to B_i \to B, i \in I; k \in K_i\} \in \Lambda(B)$$

Note that C_{ik} is an example of a *double indexed* object rather than the intersection of C_i and C_k.

The notion of a Grothendieck pretopology on the category $\mathcal{B}$ is a *categorical generalization* of a system of set-theoretical *covers* on a topology T, where a cover for $U \in T$ is a set $\{U_i : U_i \in T, i \in I\}$ such that $\cup U_i = U$. The generalization is achieved by noting that the topology ordered by inclusion is a poset category and that any cover corresponds to a collection of inclusion arrows $U_i \to U$. Given this fact, any family of arrows of a pretopology contained in $\Lambda(B)$ is a cover as well.

Now, the notion of a Grothendieck topology on a small category $\mathcal{B}$, consisting of base reference categorical objects B, can be presented in terms of appropriate covering devices admitting a *functorial* interpretation. We emphasize that the notion of a Grothendieck topology requires, first of all, the abstraction of the constitutive properties of localization in appropriate categorical terms, and then, the effectuation of these properties for the definition of localization systems. Regarding these objectives, the sought abstraction is implemented by means of covering devices on the base category of reference contexts, called in categorical terminology *covering sieves*. The constitutive properties of *localization* abstracted categorically in terms of sieves, qualified as covering ones, satisfy the following basic requirements, as we will see subsequently:

i The covering sieves are *covariant* under *pullback* operations, that is, they are *stable* under *change* of a *base* reference context. Most importantly, the stability conditions are *functorial*;

ii The covering sieves are *transitive*.

From a physical perspective, we benefit from thinking of covering sieves as generalized *measures* of *localization* of states. The operation assigning to each reference context of the base category a collection of covering sieves satisfying the closure conditions stated previously, gives rise to the notion of a *Grothendieck topology* on the *base* category of contexts. The construction of a suitable Grothendieck topology on the base category of contexts is significant for the following reasons: Firstly, it lays out precisely and unambigously the conception of the *local* in a *categorical* measurement environment, such that this conception becomes *detached* from its restricted *spatial* connotation in terms of geometric point-spaces, rather finding expression exclusively in *relational information* terms. Secondly, it permits the *collation* of local information into global by utilization of the notion of a *sheaf* for a suitable Grothendieck topology.

Firstly, we shall explain the *general* notion of *sieves*, and afterwards, we shall focus more narrowly on the notion of *covering sieves*.

A B*-sieve* with respect to a reference *context* B in $\mathcal{B}$, is a *family* S of $\mathcal{B}$-morphisms with codomain B, such that if $C \to B$ belongs to S and $D \to C$ is any $\mathcal{B}$-morphism, then the composite $D \to C \to B$ belongs to S. We may think of a B-sieve as a *right* B*-ideal*. We notice that, in the case of $\mathcal{O}(X)$, since $\mathcal{O}(X)$-morphisms are *inclusions* of open loci, a right U-ideal equates to a *downwards closed* U*-subset*.

It is important to realize that a B-sieve is equivalent to a *subfunctor* $\mathbf{S} \infty \mathbf{y}[B]$ in $Sets^{\mathcal{B}^{op}}$, where $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$, denotes the *contravariant representable* functor of the reference locus B in $\mathcal{B}$.

More specifically, given a B-sieve S, we define:

$$\mathbf{S}(C) = \{g / g : C \to B, g \in S\} \subseteq \mathbf{y}[B](C)$$

This definition yields a functor $\mathbf{S}$ in $Sets^{\mathcal{B}^{op}}$, which is obviously a subfunctor of $\mathbf{y}[B]$. Conversely, given a subfunctor $\mathbf{S} \infty \mathbf{y}[B]$ in $Sets^{\mathcal{B}^{op}}$, the set:

$$S = \{g / g : C \to B, g \in \mathbf{S}(C)\}$$

for some reference loci C in $\mathcal{B}$, is a B-sieve. Thus, epigramatically, we state:

$$\langle \; B\text{-sieve: } S \; \rangle = \langle \; \text{Subfunctor of } \mathbf{y}[B]\text{: } \mathbf{S} \infty \mathbf{y}[B] \; \rangle$$

We notice that if S is a B-sieve and $h : C \to B$ is any arrow to the locus B, then:

$$h^*(S) = \{f / cod(f) = C, (h \circ f) \in S\}$$

is a C-sieve, called the *pullback* of S *along* h, where, $cod(f)$ denotes the codomain of f. Consequently, we may define a *presheaf* functor $\mathbf{\Omega}$ in $Sets^{\mathcal{B}^{op}}$, such that its *action* on loci B in $\mathcal{B}$, is given by:

$$\mathbf{\Omega}(B) = \{S / S : B - sieve\}$$

and on arrows $h : C \to B$, by $h^*(-) : \mathbf{\Omega}(B) \to \mathbf{\Omega}(C)$, given by:

$$h^*(S) = \{f / cod(f) = C, (h \circ f) \in S\}$$

We notice that for a context B in $\mathcal{B}$, the set of all arrows into B, is a B-sieve, called the *maximal sieve* on B, and denoted by, $t(B) := t_B$.

At a next stage of development, the key conceptual issue to be settled is the following: How is it possible to *restrict* $\mathbf{\Omega}(B)$, that is the *set* of all B-sieves for each reference context B in $\mathcal{B}$, such that *each* B-sieve of the restricted set can acquire the interpretation of a *covering* B-sieve, with respect to a generalized covering system?

Equivalently stated, we wish to impose the satisfaction of appropriate *conditions* on the set of B-sieves for each context B in $\mathcal{B}$, such that, the subset of B-sieves obtained, denoted by $\mathbf{\Omega}_\chi(B)$, implement the *constitutive* properties of *localization* in *functorial* terms. In this sense, the B-sieves of $\mathbf{\Omega}_\chi(B)$, for each locus B in $\mathcal{B}$, to be thought as covering B-sieves, can legitimately be used for the implementation of localization processes. The appropriate conditions depicting the covering B-sieves from the set of all B-sieves, for each reference context B in $\mathcal{B}$, are the following:

i We interpret an arrow $C \to B$, where C, B are contexts in $\mathcal{B}$, as a *figure* of B, and thus, we interpret B as an *extension* of C in $\mathcal{B}$. It is a natural requirement that the set of all figures of B should belong in $\mathbf{\Omega}_\chi(B)$ for each context B in $\mathcal{B}$.

ii The covering sieves should be *stable* under *pullback* operations, and most importantly, the stability conditions should be expressed *functorially*. This requirement means, in particular, that the *intersection* of covering sieves should also be a covering sieve, for each reference context B, in the base category $\mathcal{B}$.

iii Finally, it would be desirable to impose: (i) a *transitivity* requirement on the specification of the covering sieves, such that, intuitively stated, covering sieves of figures of a context in covering sieves of this context, should also be covering sieves of the context themselves, and (ii) a requirement of *common refinement* of covering sieves.

If we take into account the above requirements we can define a generalized covering system, called a Grothendieck topology, in the environment of $\mathcal{B}$ as follows:

A *Grothendieck topology* on $\mathcal{B}$ is an operation $\mathbf{J}$, which assigns to each reference context B in $\mathcal{B}$, a collection $\mathbf{J}(B)$ of B-sieves, called *covering* B*-sieves*, such that the following three conditions are satisfied:

i For every reference context B in $\mathcal{B}$, the maximal B-sieve $\{g : cod(g) = B\}$ belongs to $\mathbf{J}(B)$ (*maximality condition*).

ii If S belongs to $\mathbf{J}(B)$ and $h : C \to B$ is a figure of B, then $h^*(S) = \{f : C \to B, (h \circ f) \in S\}$ belongs to $\mathbf{J}(C)$ (*stability condition*).

iii If S belongs to $\mathbf{J}(B)$, and if for each figure $h : C_h \to B$ in S, there is a sieve R_h belonging to $\mathbf{J}(C_h)$, then the set of all composites $h \circ g$, with $h \in S$, and $g \in R_h$, belongs to $\mathbf{J}(B)$ (*transitivity condition*).

As a consequence of the definition above, we can easily check that any two B-covering sieves have a common refinement, that is: if S, R belong to $\mathbf{J}(B)$, then $S \cap R$ belongs to $\mathbf{J}(B)$.

It is important to notice that given a pretopology Λ we can define a topology $\mathbf{J}$ giving rise to the *same sheaves* on $\mathcal{B}$. More specifically, we say that for any $B \in \mathcal{B}$, we have R belongs to $\mathbf{J}(B)$ if and only if R contains a pretopology covering belonging to $\Lambda(B)$.

As a first application we may consider the partially ordered set of open subsets of a topological space X, viewed as the base category of *open* reference domains, $\mathcal{O}(X)$. Then we specify that S is a covering U-sieve if and only if U is contained in the *union* of open sets in S. The above specification fulfills the requirements of covering sieves posed above, and consequently, defines a topological covering system on $\mathcal{O}(X)$.

Obviously, a categorical covering system, i.e. a Grothendieck topology $\mathbf{J}$ exists as a *presheaf* functor $\mathbf{\Omega}_\chi$ in $Sets^{\mathcal{B}^{op}}$, such that: by acting on contexts B in $\mathcal{B}$, $\mathbf{J}$ gives the set of all covering B-sieves, denoted by $\mathbf{\Omega}_\chi(B)$, whereas by acting on figures $h : C \to B$, it gives a morphism $h^*(-) : \mathbf{\Omega}_\chi(B) \to \mathbf{\Omega}_\chi(C)$, expressed as:

$$h^*(S) = \{f / cod(f) = C, (h \circ f) \in S\}, \text{ for } S \in \mathbf{\Omega}_\chi(B).$$

A small category $\mathcal{B}$ together with a Grothendieck topology $\mathbf{J}$, is called a *site*, denoted by, $(\mathcal{B}, \mathbf{J})$.

A *sheaf on a site* $(\mathcal{B}, \mathbf{J})$ is a *contravariant functor* $\mathbf{P} : \mathcal{B}^{op} \to Sets$, satisfying an *equalizer* condition, expressed, in terms of covering B-sieves S, as in the following diagram in **Sets**:

$$\prod_{f \circ g \in S} \mathbf{P}(dom(g)) \leftleftarrows \prod_{f \in S} \mathbf{P}(dom(f)) \xleftarrow{\quad e \quad} \mathbf{P}(B)$$

If the above diagram is an equalizer for a particular covering sieve S, we obtain that $\mathbf{P}$ satisfies the *sheaf* condition *with respect* to the *covering sieve* S. The theoretical advantage of the above relies on the fact that it provides a description of sheaves entirely in terms of objects of the category of presheaves.

A *Grothendieck topos* over the small category $\mathcal{B}$ is a category which is equivalent to the *category of sheaves* $Sh(\mathcal{B}, \mathbf{J})$ on a site $(\mathcal{B}, \mathbf{J})$. The site can be conceived as a system of generators and relations for the topos. We note that a category of sheaves $Sh(\mathcal{B}, \mathbf{J})$ on a site $(\mathcal{B}, \mathbf{J})$ is a full subcategory of the functor category of presheaves $Sets^{\mathcal{B}^{op}}$.

The basic properties of a Grothendieck topos are the following:

i It admits *finite projective limits*; in particular, it has a terminal object, and it admits fibered products.

ii If $(K_i)_{i \in I}$ is a family of objects of the topos, then the *sum* $\coprod_{i \in I} K_i$ exists and is *disjoint.*

iii There exist *quotients* by equivalence relations and have the *same* good properties as in the category of sets.

10.7 QUANTUM-CLASSICAL COMMUNICATION: MODULATION VIA BOOLEAN FRAMES

The appropriate mathematical structure associated with the modelling of events is an *ordered* structure. In the Hilbert space formulation of quantum mechanics, events are considered as *closed subspaces* of a separable, complex Hilbert space. In this case, the quantum event structure is identified with the *lattice* of closed subspaces of the Hilbert space, ordered by inclusion, and carrying an *orthocomplementation* operation which is given by the orthogonal complement of the closed subspaces. In consequence, the quantum event structure is modelled in terms of a complete, atomic, *orthomodular* lattice.

The quantum event structure is isomorphic to the *partial Boolean* algebra of closed subspaces of the Hilbert space of the system or, alternatively the partial Boolean algebra of *projection* operators of the system. It models the *event structure* of a quantum mechanical system, just as the event structure of a classical system is modelled in terms of a *Boolean algebra* isomorphic to the Boolean algebra of Borel subsets of the phase space of the system or, equivalently the Boolean algebra of characteristic functions on the Borel subsets of the phase space.

The notion of an *event* is considered to be equivalent to a *proposition* regarding the behaviour of a physical system. The quantum logical formulation of Quantum theory is based on the *identification* of propositions with projection operators on a complex Hilbert space. Furthermore, the order relations and the lattice operations of the lattice of quantum propositions are associated with the logical implication relation and the logical operations of conjunction, disjunction, and negation of propositions. In effect, a *non-classical*, globally non-Boolean logical structure is induced which has its origins in Quantum theory.

On the contrary, the propositional logic of classical mechanics is Boolean logic. This means that the class of models over which validity and associated semantic notions are defined for the propositions of classical mechanics is the class of *Boolean logic* structures. We stress that Boolean logic refers to a Boolean algebra of propositions in which the Boolean lattice operators of join, meet, and complement, correspond to the logical operations of disjunction, conjunction and negation respectively. Moreover, the ordering in the lattice is interpreted as a logical relation of implication between the propositions of the algebra, and also, 1 and 0 are used to denote the greatest and lower elements of the lattice respectively.

It is a standard practice pertaining to the quantum-logical formalizations of quantum event structures that, due to the identification of quantum events with projection operators on a complex Hilbert space, the derived non-Boolean lattice structure is contrasted with the Boolean lattice structure referring to classical events. Although this is indeed the case *globally*, the pertinent problem is that every *single* observed event in the quantum domain requires taking explicitly into account the *complete Boolean algebra* of projection operators, which *spectrally resolves* the observable that this event refers to. Most significantly, such a complete Boolean algebra bears the status of a logical structural *invariant* characterizing a *whole algebra* of observables commuting with the one in question, and thus, potentially may give rise to the same event.

In other words, a complete Boolean algebra of projections is the logical structural *invariant* of a *commutative subalgebra* of observables, meaning that it instantiates the *simultaneous* spectral resolution of all these commuting observables. Given that observables exist which *do not commute* with any particular commutative subalgebra of observables, we encounter a multiplicity of possible Boolean algebras of orthogonal projections, which play the role of *an invariant* only in the *context* of all commuting observables that can be *simultaneously* resolved spectrally by this invariant.

In sum, the conceptual peculiarity characterizing a quantum event structure pertains to the fact that in the quantum domain there is *no* such thing as a *unique* and *universal* logical structural *invariant* with respect to which all possible observables can be spectrally resolved simultaneously. On the contrary, there exists a *multiplicity* of *spectral invariants* associated with *commutative* subalgebras of observables. Therefore, although a quantum event structure is globally non-Boolean, it can be spectrally qualified *only* in terms of Boolean event structures *attached* to it, in their *function* as logical structural invariants of co-measurable families of observables. Since these spectral invariants are *not* global, we consider them local, where the *locality* requirement refers precisely to the physical *context* of all commuting observables that can be simultaneously resolved spectrally by an invariant of this form.

We conclude that a complete Boolean algebra of projection operators in its function as a local spectral invariant of a commutative subalgebra of quantum observables plays the role of a *Boolean frame* with respect to which a quantum event can be qualified, and thus, lifted to the *empirical* level. Equivalently, each local Boolean frame serves as the local *pre-conditioning invariant* logical structure for the *evaluation* of events of all co-measurable observables in this frame. Due to the *non-availability* of a global uniquely defined Boolean frame, we must by default take into account *all possible* local Boolean frames *together* with their interrelations. The crucial problem is whether it is possible to identify a *universal* way to specify a quantum event structure through the literal *adjunction* of local Boolean spectral invariants to it, objectified in terms of *local Boolean* logical frames.

The existence of a universal solution essentially renders the global orthomodular lattice structure of quantum events physically and empirically vacuous without the *gnomonic adjunction* of local spectral invariants to it, effecting the *quantum-classical* communication through observability and measurement. The role of each locally adjoined Boolean frame is the instantiation of a *partial* or *local structural congruence* with a Boolean event structure pertaining to a context of measurement. The multiplicity of applicable local Boolean frames effects the *filtration*, *percolation*, and *separation* of several *resolution* sizes and types of quantum observable *grain* depending on the character of the corresponding spectral orthogonal projections.

The *objective* of a universal solution is the *derivation* of the *non-directly* accessible *quantum kind* of event structure by means of *all* possible partial or local structural *congruences* with the *directly* accessible *Boolean kind* of event structure, forced by means of *adjoining local spectral invariants* as probing frames to the former. In this setting, the major role is subsumed by all possible structural *relations* allowed *among* the probing *Boolean frames*, the spectra of which may be disjoint or nested or overlapping and interlocking together non-trivially. It is the realization that distinct Boolean frames may have non-trivial intersections, or more generally, *non-trivial pullback* compatibility relations that it is at the heart of the *so* called "quantum paradoxes".

The *universality* of the required solution poses the need to formulate the problem in *functorial* terms, to *circumvent* dependence on the artificial choice of particular Boolean frames. In turn, it requires a category-theoretic framework of interpretation, based on the aforementioned notion of partial or local structural congruence, which *locally* allows the *conjugation* of a *quantum* event structure by means of the *spectral invariants* adjoined as frames to it.

A *Boolean categorical* event structure is a *small* category, denoted by $\mathcal{B}$, which is called the *category of Boolean event algebras*. The objects of $\mathcal{B}$ are *complete* Boolean algebras of events, and the arrows are the corresponding Boolean algebraic *homomorphisms*.

A *quantum categorical* event structure is a *locally small co-complete* category, denoted by $\mathcal{L}$, which is called the *category of quantum event algebras*. The objects of $\mathcal{L}$ are the non- directly accessible global *quantum event algebras*, and the arrows are the corresponding quantum algebraic *homomorphisms*.

We consider a *Boolean shaping* functor of $\mathcal{L}$, $\mathbf{M}:\mathcal{B}\to\mathcal{L}$, which is to say a *forgetful* functor assigning to each Boolean event algebra the underlying quantum event algebra and to each Boolean homomorphism the underlying quantum algebraic homomorphism. Because of the fact that an *opposite*-directing functor from $\mathcal{L}$ to $\mathcal{B}$ is *not* feasible, since a quantum event algebra *cannot* be realized within *any* Boolean event algebra, we seek for an *extension* of $\mathcal{B}$ into a *larger* categorical environment, where *any* such realization becomes possible.

This *extension* should conform to the intended physical semantics adopted in *adjoining* a multiplicity of Boolean spectral invariants to a quantum event algebra, objectified as probing Boolean frames of the latter. For this reason, it is necessary to extend the *probes* from the categorical level of $\mathcal{B}$ to the categorical level of *diagrams* in $\mathcal{B}$, such that the initial probes can be *embedded* in the latter extended category.

This is accomplished by means of the *Yoneda embedding* $\mathbf{y}:\mathcal{B}\to Sets^{\mathcal{B}^{op}}$, which constitutes the *free completion* of $\mathcal{B}$ under the adoption of *colimits of diagrams* of Boolean structure probes, that is, of spectral invariants to be *adjoined* on a quantum event algebra.

An object $\mathbf{P}$ of $Sets^{\mathcal{B}^{op}}$ is thought of as a *right action* of the category $\mathcal{B}$ on a set of observables, which is *partitioned* into a variety of Boolean *spectral kinds* parameterized by the Boolean event algebras B in $\mathcal{B}$. Such an action $\mathbf{P}$ is equivalent to the specification of a diagram in $\mathcal{B}$, simply considered as a $\mathcal{B}$-variable set, called a *presheaf of sets* on $\mathcal{B}$. For each probe B of $\mathcal{B}$, $\mathbf{P}$(B) is a set, and for each arrow $f:C\to B$, $\mathbf{P}(f):\mathbf{P}(B)\to\mathbf{P}(C)$ is a set-theoretic function such that if $p\in\mathbf{P}(B)$, the value $\mathbf{P}(f)(p)$ for an arrow $f:C\to B$ in $\mathcal{B}$ is called the *restriction* of p along f and is denoted by $\mathbf{P}(f)(p)=p\cdot f$.

Each Boolean probe B in $\mathcal{B}$ gives rise to a *contravariant representable* Hom-functor $\mathbf{y}^B:=\mathbf{y}[B]:=Hom_{\mathcal{B}}(-,B)$. This functor defines a $\mathcal{B}$-variable set on $\mathcal{B}$, represented by B. The functor $\mathbf{y}$ is a *full* and *faithful* functor from $\mathcal{B}$ to the contravariant functors on $\mathcal{B}$, i.e.:

$$\mathbf{y}:\mathcal{B}\to Sets^{\mathcal{B}^{op}},$$

giving rise to the *Yoneda embedding* $\mathcal{B}\infty\ Sets^{\mathcal{B}^{op}}$.

The category of *presheaves* of sets on Boolean probes, denoted by $Sets^{\mathcal{B}^{op}}$, has objects all *functors* $\mathbf{P}:\mathcal{B}^{op}\to Sets$, and morphisms all *natural transformations* between such functors, where $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$, meaning all the arrows are inverted. In the setting of the functor category $Sets^{\mathcal{B}^{op}}$ it becomes possible to *realize* a quantum event algebra L in $\mathcal{L}$ in terms of a *distinguished* presheaf of Boolean event algebras, which models the *spectral capacity* of the latter to act as locally invariant logical probing frames of a quantum event algebra. These Boolean frames of an L in $\mathcal{L}$ are objectified as L-targeting morphisms in $\mathcal{L}$:

$$\psi_B:\mathbf{M}(B)\to L,$$

being *interrelated* by the operation of *restriction*. Explicitly, this means that for each Boolean *homomorphism* $f:C\to B$, if $\psi_B:\mathbf{M}(B)\to L$ is a

Boolean frame of L, the corresponding Boolean frame $\psi_C : \mathbf{M}(C) \to L$ is given by the restriction or pullback of ψ_B along f, denoted by $\psi_B \cdot f = \psi_C$. Thus, we obtain a contravariant presheaf functor $\mathbf{R}(L)(-) := \mathbf{R}_L(-) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$, called the *functor of Boolean frames* of L. Since the physical interpretation of the functor $\mathbf{R}(L)(-)$ refers to the functorial realization of a quantum event algebra L in $\mathcal{L}$ in terms of Boolean probes B in $\mathcal{B}$, we think of $\mathbf{R}_L(-)$ as the variable *Boolean spectral functor* of L through the Boolean frames adjoined to it as *local invariants*.

Due to the categorical *Yoneda Lemma*, an *injective* correspondence obtains between *elements* of the set $\mathbf{P}(B)$ and *natural transformations* in $Sets^{\mathcal{B}^{op}}$ from $\mathbf{y}[B]$ to $\mathbf{P}$ and this correspondence is natural in both $\mathbf{P}$ and B, for *every* presheaf of sets $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ and probe B in $\mathcal{B}$. The functor category of presheaves of sets on Boolean probes $Sets^{\mathcal{B}^{op}}$ is a complete and cocomplete category. Thus, the Yoneda embedding $\mathbf{y} : \mathcal{B} \to Sets^{\mathcal{B}^{op}}$ constitutes the free completion of $\mathcal{B}$ under colimits of diagrams of Boolean probes.

The significance of this boils down to the fact that, if we consider a Boolean shaping functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ there can be *precisely one unique*, up to isomorphism, *colimit-preserving functor* $\widehat{\mathbf{M}} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B} \arrow[d, tail, "\mathbf{y}"'] \arrow[dr, "\mathbf{M}"] & \\
\mathbf{S}ets^{\mathcal{B}^{op}} \arrow[r, dotted, shift left, "\mathbf{L}"] & \mathcal{L} \arrow[l, dotted, shift left, "\mathbf{R}"]
\end{tikzcd}$$

Consequently, *every* morphism from a Boolean probe B in $\mathcal{B}$ to a quantum event algebra L in $\mathcal{L}$ *factors uniquely* through the functor category $Sets^{\mathcal{B}^{op}}$ and the specification of the colimit-preserving functor $\widehat{\mathbf{M}} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$ is instrumental for understanding how the underlying structure of L in $\mathcal{L}$ *emerges* through communication, emerges in other words, through the *canonics* of adjoining Boolean probes as local spectral invariants to it.

More precisely, the functor $\widehat{\mathbf{M}}$ plays the role of a *left adjoint* $\mathbf{L}$, and thus serves as a *colimit-preserving* functor, from $Sets^{\mathcal{B}^{op}}$ to $\mathcal{L}$. Equivalently, the functor $\widehat{\mathbf{M}} := \mathbf{L}$ is the *left adjoint* of the categorical *adjunction* between the categories $Sets^{\mathcal{B}^{op}}$ and $\mathcal{L}$, where the *right adjoint* $\mathbf{R} : \mathcal{L} \to Sets^{\mathcal{B}^{op}}$, is interpreted as the *realization functor* of $\mathcal{L}$ in terms of variable Boolean probing frames.

More specifically, the variable Boolean probes-induced *realization functor* of $\mathcal{L}$ in $Sets^{\mathcal{B}^{op}}$ is defined as follows:

$$\mathbf{R} : \mathcal{L} \to Sets^{\mathcal{B}^{op}},$$

such that the contravariant presheaf functor $\mathbf{R}(L)(-) := \mathbf{R}_L(-) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$ in the image of $\mathbf{R}$ into in $Sets^{\mathcal{B}^{op}}$, for a fixed L in $\mathcal{L}$, is the presheaf functor of Boolean frames of L.

We conclude that the problem of specification of a quantum event algebra L in $\mathcal{L}$ by means of *diagrams of Boolean probes* B has a *universal solution*, which is provided by the *left adjoint* functor $\mathbf{L} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$ to the *realization* functor $\mathbf{R} : \mathcal{L} \to Sets^{\mathcal{B}^{op}}$. Equivalently, the existence of the left adjoint functor $\mathbf{L}$ paves the way for an explicit *inductive synthesis* of a quantum event algebra L in $\mathcal{L}$ by means of appropriate diagrams of Boolean probes in a functorial manner. Therefore, a quantum event algebra is essentially *generated*, and indirectly completely *specified*, by the canonics of *adjoining* locally invariant *Boolean spectral frames* thereto for the qualification and measurement of events, in short, through quantum-classical communication.

Technically speaking, a categorical adjunction pertains between the categories $Sets^{\mathcal{B}^{op}}$ and $\mathcal{L}$, called the *Boolean frames–quantum* adjunction. It is characterized by a pair of adjoint functors $\mathbf{L} - \mathbf{R}$ as follows:

$$\mathbf{L} : Sets^{\mathcal{B}^{op}} \rightleftarrows \mathcal{L} : \mathbf{R}$$

Thus, we obtain a *bijection*, which is *natural* in both $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ and L in $\mathcal{L}$:

$$Hom_{\mathbf{Sets}^{\mathcal{B}^{op}}}(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L)$$

abbreviated as follows:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L)$$

The "*category of elements*" of a presheaf plays an essential role in the canonics of this adjunction. We bear in mind that every presheaf functor $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ gives rise to a category, called the *category of elements* of $\mathbf{P}$ and denoted by $\int(\mathbf{P}, \mathcal{B})$.

The objects of this category are all pairs (B, p), and the morphisms $(B', p') \to (B, p)$ are those morphisms $u : B' \to B$ of the underlying category of probes $\mathcal{B}$, satisfying the condition that $p \cdot u = p'$, namely that the restriction, or the pullback of p along u is p'.

If we project onto the second coordinate of $\int(\mathbf{P}, \mathcal{B})$, we obtain a functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{B}) \to \mathcal{B}$. Therefore, every presheaf functor $\mathbf{P}$ induces a *split discrete* and *uniform* fibration, where $\mathcal{B}$ is the *base* category of the fibration.

The fibers are categories in which the only arrows are identity arrows. If B is a Boolean probe in $\mathcal{B}$, the inverse image under $\int_{\mathbf{P}}$ of B is simply the set $\mathbf{P}(\mathrm{B})$, although its elements are written as pairs so as to form a disjoint union.

$$\begin{tikzcd}
\int(\mathbf{P}, \mathcal{B}) \arrow[d, "\int_{\mathbf{P}}"'] & \\
\mathcal{B} \arrow[r, "\mathbf{P}"] & \mathbf{Sets}
\end{tikzcd}$$

A *natural transformation* τ between the presheaves $\mathbf{P}$ and $\mathbf{R}(L)$ on the category of Boolean probes $\mathcal{B}$, $\tau : \mathbf{P} \to \mathbf{R}(L)$ is equivalent to a *family* of compatible mappings between sets:

$$\tau_B : \mathbf{P}(B) \to Hom_{\mathcal{L}}(\mathbf{M}(B), L),$$

indexed by the probes B of $\mathcal{B}$. Each such mapping τ_B is identical with the following mapping:

$$\tau_B : (B,p) \to Hom_{\mathcal{L}}(\mathbf{M} \circ \int_{\mathbf{P}} (B,p), L).$$

Therefore, a natural transformation τ between the presheaves $\mathbf{P}$ and $\mathbf{R}(L)$ can be equivalently represented as a family of arrows of $\mathcal{L}$ targeting L, which is being indexed by the objects (B,p) of the *category of elements* of the presheaf $\mathbf{P}$, namely

$$\{\tau_B(p) : \mathbf{M}(B) \to L\}_{(B,p)}.$$

These arrows $\tau_B(p)$ considered *jointly* give rise to a *cocone* from the functor $\mathbf{M} \circ \int_{\mathbf{P}}$ to L. The categorical definition of a colimit determines that each such cocone emerges by the composition of the colimiting cocone with a *unique* arrow from the *colimit* $\mathbf{LP}$ to L. Equivalently, a bijection exists, which is natural in $\mathbf{P}$ and L:

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L).$$

Hence, the Boolean probes-induced *realization* functor of $\mathcal{L}$, realized for each L in $\mathcal{L}$ by the presheaf of Boolean probing frames $\mathbf{R}(L) = Hom_{\mathcal{L}}(\mathbf{M}(-), L)$ in $Sets^{\mathcal{B}^{op}}$, has a *left adjoint* functor $\mathbf{L} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$, which is defined for each presheaf of sets $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ as the *colimit* $\mathbf{L}(\mathbf{P})$.

The pair of *adjoint* functors $\mathbf{L} - \mathbf{R}$ formalizes category-theoretically the *functorial* process of *encoding* and *decoding* information between diagrams of Boolean probes B and quantum event algebras L through the action of Boolean probing frames $\psi_B : \mathbf{M}(B) \to L$.

The existence of an adjunction between two categories always gives rise to a family of universal morphisms, called *unit* and *counit* of the adjunction, one for each object in the first category and one for each object in the second. Furthermore, every adjunction gives rise to an *adjoint equivalence* of certain subcategories of the initial functorially correlated categories. It is precisely this category-theoretic fact which determines the necessary and sufficient conditions for the *isomorphic* representation of a quantum event algebra L in $\mathcal{L}$ by means of suitably qualified functors, forming *sheaves* of Boolean probing frames.

For any presheaf $\mathbf{P}$ in the functor category $Sets^{\mathcal{B}^{op}}$, the *unit* of the adjunction is defined as follows:

$$\delta_{\mathbf{P}} : \mathbf{P} \to \mathbf{RLP}.$$

On the other side, for each quantum event algebra L in $\mathcal{L}$ the *counit* is defined as follows:

$$\text{Ú}_L : \mathbf{LR}(L) \to L.$$

We conclude that the problem of establishing a *functorial representation* of a quantum event algebra in terms of Boolean logical probes has a *universal solution* in terms of quantum-classical natural communication, which is provided by the left adjoint functor $\mathbf{L} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$ to the Boolean realization functor $\mathbf{R} : \mathcal{L} \to Sets^{\mathcal{B}^{op}}$. In other words, the existence of the left adjoint functor $\mathbf{L}$ paves the way for an explicit articulation of quantum event algebras by means of suitably qualified diagrams of Boolean probing frames based on partial or local congruences between the Boolean and quantum kinds of event structure.

The *counit* natural transformation Ú_L defines the spectral enunciation of a quantum event algebra L in $\mathcal{L}$ through *metaphora* by means of the *colimiting–interconnection* of Boolean probing frames of L, whose domains are *partially congruent* Boolean event algebras to L.

In more detail, the left adjoint functor of the Boolean frames–quantum adjunction, $\mathbf{L} : Sets^{\mathcal{B}^{op}} \to \mathcal{L}$, is defined for each presheaf $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$ as the colimit $\mathbf{L}(\mathbf{P})$. The functorial enunciation of a quantum event algebra L in $\mathcal{L}$ by means of the counit natural transformation requires an explicit calculation of the colimit $\mathbf{LR}(L)$ of the presheaf functor of Boolean probing frames of a quantum event algebra L. The corresponding category of elements $\int(\mathbf{R}(L), \mathcal{B})$ has objects all pairs (B, ψ_B), where B is a Boolean event algebra and $\psi_B : \mathbf{M}(B) \to L$ is a Boolean frame of L defined over B. The morphisms of $\int(\mathbf{R}(L), \mathcal{B})$, denoted by $(B', \psi_{B'}) \to (B, \psi_B)$, are those Boolean event algebra homorphisms

$u : B' \to B$ of the base category $\mathcal{B}$ for which $\psi_B \cdot u = \psi_{B'}$, that is, the restriction or pullback of the Boolean frame ψ_B along u is $\psi_{B'}$.

The pertinent calculation of the colimit $\mathbf{LR}(L)$ is simplified by the observation that we have an *underlying colimit-preserving* faithful functor from the category $\mathcal{L}$ to the category $Sets$. Thus, the sought colimit can be calculated by means of *set-valued equivalence classes*, or partition blocks in $Sets$, under the constraint that the derived set of equivalence classes are able to carry the *structure* of a quantum event algebra:

$$\mathbf{L}(\mathbf{R}(L)) = \mathbf{L}_{\mathbf{M}}(\mathbf{R}(L)) = Colim\{\int(\mathbf{R}(L),\mathcal{B}) \to \mathcal{B} \to \mathcal{L} \to \mathbf{Sets}\}.$$

The *indexing* category corresponding to the functor $\mathbf{R}(L)$ is the category of its elements $\mathcal{I} \equiv \int(\mathbf{R}(L),\mathcal{B})$, whence the functor $[\mathbf{M} \circ \int_{\mathbf{R}(L)}]$ defines the *diagram* $\mathcal{I} \to \mathcal{L}$ over which the colimit should be calculated. Since a colimit-preserving functor from the category $\mathcal{L}$ to $Sets$ exists, the sought *colimit* is equivalent to the definition of the *tensor product* $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ of the set valued functors:

$$\mathbf{R}(L) : \mathcal{B}^{op} \to Sets, \qquad \mathbf{M} : \mathcal{B} \to Sets,$$

where the *contravariant* functor $\mathbf{R}(L)$ is considered a *right* $\mathcal{B}$*-module* and the *covariant* functor $\mathbf{M}$ a *left* $\mathcal{B}$*-module*, in analogia with the *algebraic* definition of the tensor product of a right $\mathcal{B}$-module with a left $\mathcal{B}$-module over a *ring* of coefficients $\mathcal{B}$. The above defines the *functorial* tensor product decomposition of the colimit in the category of elements of $\mathbf{R}(L)$ induced by the Boolean shaping functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ of $\mathcal{L}$.

Therefore, for a Boolean probing frame $\psi_B \in \mathbf{R}(L)(B)$, $v : B' \to B$ and $q' \in \mathbf{M}(B')$ the *elements* of the set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ are all of the form $\chi(\psi_B, q)$. This element can be written as:

$$\chi(\psi_B, q) = \psi_B \otimes q, \quad \psi_B \in \mathbf{R}(L)(B), q \in \mathbf{M}(B),$$

such that:

$$\psi_B \cdot v \otimes q' = \psi_B \otimes v(q'), \quad \psi_B \in \mathbf{R}(L)(B), q' \in \mathbf{M}(B'), v : B' \to B$$

We conclude that the set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ is actually the *quotient* of the set

$$\sum_{B} \mathbf{R}(L)(B) \times \mathbf{M}(B)$$

by the smallest *equivalence* relation generated by the above equations, whence the elements $\psi_B \otimes q$ of the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ are the *equivalence classes* of this relation. Since ψ_B denotes a probing Boolean frame of L and q denotes a projection operator $q \in \mathbf{M}(B)$, we conclude that the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ is a set of equivalence classes, or partition blocks of *pointed Boolean frames*, called *Boolean germs*.

Most important, this set can be naturally *endowed* with a *quantum* event algebraic structure by defining the *orthocomplementation* operator according to the assignment $(\psi_B \otimes q)^* := \psi_B \otimes q^*$, and the *unit* element according to $\mathbf{1} := \psi_B \otimes 1$. Notice that two equivalence classes in the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ can be *ordered* if and only if they have a *common refinement*. Consequently, the partial order structure is defined by the assignment,

$$(\psi_B \otimes q) \preceq (\psi_C \otimes r),$$

if and only if,

$$d_1 \preceq d_2,$$

where we have made the following identifications,

$$(\psi_B \otimes q) = (\psi_D \otimes d_1)$$
$$(\psi_C \otimes r) = (\psi_D \otimes d_2),$$

with d_1, d_2 $\in$ $\mathbf{M}(D)$, according to the *pullback* diagram,

$$
\begin{tikzcd}
\mathbf{M}(D) \arrow[r, "\beta"] \arrow[d, "\gamma"] & \mathbf{M}(B) \arrow[d, "\alpha"] \\
\mathbf{M}(C) \arrow[r, "\lambda"] & L
\end{tikzcd}
$$

such that $\beta(d_1) = q$, $\gamma(d_2) = r$, and $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$, $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$ denote the pullback of $\alpha : \mathbf{M}(B) \to L$ along $\lambda : \mathbf{M}(C) \to L$ in the category of quantum event algebras. Thus, the ordering relation between any two equivalence classes of pointed Boolean frames in the colimiting set $\mathbf{LR}(L) = \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ requires the existence of pullback compatibility conditions between the corresponding Boolean frames.

We conclude that the *spectral constitution* of a quantum event algebra L in $\mathcal{L}$ through the Boolean frames–quantum *adjunction* is based on the *action* of the *endofunctor* $\mathbf{G}$ on $\mathcal{L}$, defined by:

$$
\begin{gathered}
\mathbf{G} := \mathbf{LR} : \mathcal{L} \to \mathit{Sets}^{\mathcal{B}^{op}} \to \mathcal{L} \\
L \to \mathbf{R}(L) \to \mathbf{LR}(L) \to L,
\end{gathered}
$$

which acts as the global spectral constitution endofunctor of a quantum categorical event structure $\mathcal{L}$ via Boolean probing frames.

In particular, for each quantum event algebra L in $\mathcal{L}$ the counit universal morphism of the Boolean frames-quantum adjunction evaluated at L is expressed in terms of equivalence classes, or partition blocks of pointed Boolean frames, that is, in terms of Boolean germs:

$$
\epsilon_L : \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \to L.
$$

Thus, the counit ϵ_L fits into the following diagram:

$$
\begin{tikzcd}
\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \arrow[dr, "\epsilon_L"] & \\
\mathbf{M}(B) \arrow[u, "\psi_B \otimes (-)"] \arrow[r, "\psi_B"] & L
\end{tikzcd}
$$

Accordingly, for every Boolean frame $\psi_B : \mathbf{M}(B) \to L$ the *projection* operator $q \in \mathbf{M}(B)$ is mapped to an *event* in L only through its *factorization* via the adjoined *Boolean germ* $\psi_B \otimes q$, i.e. through the partition spectral block of pointed Boolean frames it belongs to, according to:

$$\acute{U}_L([\psi_B \otimes q]) = \psi_B(q), \qquad q \in \mathbf{M}(B).$$

This epitomizes the means of *communication* between the *Boolean* and the *quantum* structural kinds of event structure by means of germinal partial structural *congruence* through the canonics of the established adjunction. The crucial idea here is that the information encoded in the quantum structural kind can be accessed, decoded, qualified, and quantified, only indirectly in terms of *modulation* by Boolean germs, or partition spectral blocks of equivalent pointed Boolean frames.

In the same vein of ideas, it is revealing to examine the conditions that force the *counit* natural transformations of the *identity functor* in the category of quantum event algebras $\mathcal{L}$ into an *isomorphism*. The counit isomorphism expresses the property of *invariance* of a quantum event algebra under the two-step procedure of *encoding* in terms of appropriate families of Boolean event algebras through probing frames in $\mathbf{R}(L)$, and then *decoding* back by means of the action of the left adjoint on the former, denoted by $\mathbf{LR}(L)$.

Note that that if the *counit* evaluated at L is an *isomorphism*, then L can be considered as a *fixed* point of the corresponding global spectral constitution *endofunctor* of L through the action of Boolean probing frames. In general, the counit natural transformation $\acute{U}_L$ is a natural isomorphism, if and only if the *right adjoint* functor of the Boolean frames–quantum adjunction is *full* and *faithful*, or equivalently, if and only if the *cocone* from the functor $\mathbf{M} \circ \int_{\mathbf{R}(L)}$ to L is *universal* for each L in $\mathcal{L}$. In the latter case, the functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ is characterized as a *dense* Boolean shaping functor.

It is worth specifying in more detail the necessary and sufficient conditions which force the counit $\acute{U}_L$ to be an isomorphism. These conditions amount to the notion of *sheaf-theoretic localization* of L through the probing frames $\psi_B : \mathbf{M}(B) \to L$.

If the counit natural transformation $\acute{U}_L$ at L is *restricted* to an isomorphism:

$$\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \cong L$$

then the quantum structural information of L is completely *decoded* in terms of its *adjoined* Boolean *germs*, that is, through spectral equivalence classes of pointed Boolean frames. This will rest on the satisfaction of appropriate conditions by restricted families of Boolean frames in $\mathbf{R}(L)$, distinguished qualitatively by their function as local Boolean *covers* of L.

The requirements qualifying Boolean frames as local Boolean covers of L are the following: First, they should constitute a minimal *generating* class of Boolean frames instantiating a *sieve*, i.e. a subfunctor of the functor of Boolean frames $\mathbf{R}(L)$ of L. Second, they should jointly form an *epimorphic family covering* L entirely on their overlaps. Third, they should be *compatible* under refinement, or more generally *pullback* operations in L. Fourth, they should be *transitive* such that subcovers of covers of L can be qualified as covers themselves.

A *functor of Boolean coverings* for a quantum event algebra L in $\mathcal{L}$ is defined as a *subfunctor* T of the functor of Boolean frames $\mathbf{R}(L)$ of L, i.e. $\mathbf{T} \infty \ \mathbf{R}(L)$. For each Boolean algebra B in $\mathcal{B}$, a subfunctor $\mathbf{T} \infty \ \mathbf{R}(L)$ is equivalent to a *right ideal*, or equivalently a *spectral sieve* of quantum homomorphisms $\mathbf{T} \triangleright \mathbf{R}(L)$, defined by the requirement that, for each B in $\mathcal{B}$, the set of elements of $\mathbf{T}(B) \subseteq [\mathbf{R}(L)](B)$ is a set of Boolean frames $\psi_B : \mathbf{M}(B) \to L$ of $\mathbf{R}(L)(B)$, called *Boolean covers* of L, satisfying the following property:

If $[\psi_B : \mathbf{M}(B) \to L] \in \mathbf{T}(B)$, i.e. it is a Boolean cover of L, and $\mathbf{M}(v) : \mathbf{M}(B') \to \mathbf{M}(B)$ in $\mathcal{L}$ for $v : B' \to B$ in $\mathcal{B}$, then [$\psi_B \circ \mathbf{M}(v) : \mathbf{M}(B') \to \mathcal{L}] \in \mathbf{T}(B)$, i.e. it is also a Boolean cover of L}.

A *family* of Boolean covers $\psi_B : \mathbf{M}(B) \to L$, B in $\mathcal{B}$, is the *generator* of a *spectral sieve* of Boolean coverings T, if and only if, this sieve is the *smallest* among all containing that family. The spectral sieves of Boolean coverings for an L in $\mathcal{L}$ constitute a *partially ordered* set under inclusion of subobjects. The minimal sieve is the empty one, namely $\mathbf{T}(B) = \varnothing$ for all B in $\mathcal{B}$, whereas the maximal sieve is the set of all probing Boolean frames of L for all B in $\mathcal{B}$, considered as Boolean covers.

We recall that the ordering relation between any two equivalence classes of pointed Boolean frames in the colimiting set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ requires the property of *pullback* compatibility between the corresponding Boolean frames. Therefore, if we consider a functor of Boolean coverings $\mathbf{T}$ for a quantum event algebra L, we require that the generating family of Boolean covers they belong to is compatible under pullbacks.

The *pullback* of the Boolean covers $\psi_B : \mathbf{M}(B) \to L$ and $\psi_{B'} : \mathbf{M}(B') \to L$, where B and B' are Boolean event algebras in $\mathcal{B}$, with common codomain a *quantum* event algebra L, consists of the Boolean cover $\mathbf{M}(B) \times_L \mathbf{M}(B')$, together with the two *projections* $\psi_{BB'}$ and $\psi_{B'B}$, as shown in the diagram:

$$\begin{tikzcd}
\mathbf{M}(\acute{B}') \arrow[dr, "u"] \arrow[drr, "h"] \arrow[ddr, "g"'] & & \\
& \mathbf{M}(B) \times_L \mathbf{M}(\acute{B}) \arrow[r, "\psi_{B,\acute{B}}"] \arrow[d, "\psi_{\acute{B},B}"] & \mathbf{M}(B) \arrow[d, "\psi_B"] \\
& \mathbf{M}(\acute{B}) \arrow[r, "\psi_{\acute{B}}"] & L
\end{tikzcd}$$

If the Boolean probing frames ψ_B and $\psi_{B'}$ are *injective*, then their pullback is given by their *intersection*. Next, we define the pairwise *gluing isomorphism* of the Boolean probing frames ψ_B and $\psi_{B'}$, as follows:

$$\Omega_{B,B'} : \psi_{B'B}(\mathbf{M}(B) \times_L \mathbf{M}(B')) \to \psi_{BB'}(\mathbf{M}(B) \times_L \mathbf{M}(B'))$$
$$\Omega_{B,B'} = \psi_{BB'} \circ \psi_{B'B}{}^{-1}$$

From the previous definition, we derive the following *cocycle conditions*:

$$\Omega_{B,B} = id_B,$$
$$\Omega_{B,B'} \circ \Omega_{B',B''} = \Omega_{B,B''},$$
$$\Omega_{B,B'} = \Omega^{-1}{}_{B',B}$$

where, in the first condition id_B denotes the identity of $\mathbf{M}(B)$, in the second condition $\psi_B \times_L \psi_{B'} \times_L \psi_{B''} \neq 0$, and in the third condition $\psi_B \times_L \psi_{B'} \neq 0$.

Thus, the gluing isomorphism between any two Boolean frames of a spectral sieve $\mathbf{T}(L)$ assures that $\psi_{B'B}(\mathbf{M}(B) \times_L \mathbf{M}(B'))$ and $\psi_{BB'}(\mathbf{M}(B) \times_L \mathbf{M}(B'))$ probe L on their common refinement in a *compatible* way. This provides the sought-after *criterion* for the indirect *isomorphic* representation of a quantum event algebras in terms of a *spectral sieve* $\mathbf{T}(L)$ adjoined to it, under the proviso that the family of all Boolean covers $\psi_B : \mathbf{M}(B) \to L$, for variable B in $\mathcal{B}$, generating this spectral sieve jointly form an epimorphic family covering L completely:

$$T_L : \sum_{(B_j, \psi_j : \mathbf{M}(B_j) \to L)} \mathbf{M}(B_j) \twoheadrightarrow L,$$

where T_L is an *epimorphism* in $\mathcal{L}$ with codomain a quantum event algebra L.

A sieve *adjoined* to a quantum event algebra L is a Boolean *localizing* spectral sieve of L, or equivalently a functor of *Boolean localizations* of L, if and only if it is *closed* with respect to an *epimorphic* family of Boolean covers of L and the above *cocycle* conditions are satisfied. The conceptual significance of a Boolean localizing spectral sieve of L lies in the fact that the functor of Boolean probing frames $\mathbf{R}(L)$ becomes a *structure sheaf* of local Boolean frames when restricted to it. Then, for a dense epimorphic generating family of Boolean covers in a Boolean localization functor $\mathbf{T}$ of L, the counit of the Boolean frames-quantum adjunction is restricted to a quantum algebraic isomorphism, that is at once structure-preserving, injective and surjective.

In turn, the right adjoint functor of the adjunction restricted to a Boolean localization functor is full and faithful. This argument is formalized more precisely in *topos*-theoretic terminology by means of the *subcanonical* Grothendieck topology consisting of epimorphic families of covers on the base category of Boolean event algebras. Consequently, $\mathcal{L}$ becomes a *reflection* of the topos of variable local Boolean frames $Sets^{\mathcal{B}^{op}}$, and the structure of a quantum event algebra L in $\mathcal{L}$ is *preserved* by the action of a family of Boolean frames if and only if this family forms a *Boolean localization functor* of L. In this case, *any* compatible family of

Boolean frames ψ_j in the localized structure sheaf has a *unique amalgamation*, in the sense that there exists a unique colimiting Boolean frame $\psi : \mathbf{T}(L) \otimes_{\mathcal{B}} \mathbf{M} \to L$, such that the restriction of ψ along u_j gives ψ_j, that is $\psi \cdot u_j = \psi_j$.

$$
\begin{tikzcd}
\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M} \arrow[dr, "\epsilon_L"] & \\
\mathbf{M}(B) \arrow[u, "\psi_B\otimes(-)"] \arrow[r, "\psi_B"] & L
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} \arrow[d, tail, "\mathbf{y}"'] \arrow[dr, "\mathbf{M}"] & \\
\mathbf{S}ets^{\mathcal{B}^{op}} \arrow[r, dotted, shift left, "\mathbf{L}"] & \mathcal{L} \arrow[l, dotted, shift left, "\mathbf{R}"]
\end{tikzcd}
$$

FUNCTORIAL GNOMONICS: TEMPORAL PERCOLATION AND ALETHEIA

11

The functioning of a *localization* schema in the physical continuum is based on the operational specification of an appropriate *covering* categorical environment consisting of varying reference loci for the determination of observables. As we have already made clear previously, the functional role of localization systems serves to guarantee an efficient *pasting code* of the observable information between different localizing domains, effectuating the compatible transition from the *local* to the *global* regime.

Until this stage, we have not established any particular interpretation of an abstract localization schema in a continuum of observable events in terms of *spatial* or *temporal* relations. From a physical viewpoint, since we have assumed that some localizing categorical environment admits an operational specification, we ought to expand on the functioning of a localization schema in spatiotemporal terms, so that, a reference to individuated observable events in these terms can be made possible. In this sense, it is necessary to struggle for a well defined notion of a category equipped with covering families admitting a viable interpretation in terms of spatial and temporal relations, which will in consequence be suitable to provide the necessary and sufficient means for the manifestation of some localization schema in the physical continuum in terms of observable events individuated from it in a spatial or temporal way.

Significantly, we do not assume any *spatialization* of *temporal* concepts, as is usually the case. Accordingly, it becomes unavoidable to disentangle the defining requirements characterizing *spatial covering* systems, namely families consisting of spatial reference domains, from those characterizing *temporal covering* systems, that is, families consisting of temporal reference domains. This strategy will prove fruitful if we manage to associate, at a later stage, a localization schema in the physical continuum with notions of spatially and temporally distinguished *events*. The difference between spatial and temporal covering systems will be based on the distinctive meaning that the notion of *extension* acquires, when referring to temporal loci, as compared to spatial ones.

If we consider a general categorical environment $\mathcal{B}$, and an object B in $\mathcal{B}$ to be interpreted as a *spatial* reference locus, then the extensional aspects of B are captured by the *contravariant* Hom-functor of generalized point-elements of B in $\mathcal{B}$, denoted by $\mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$, which is a *representable presheaf* in $Sets^{\mathcal{B}^{op}}$. The functor $\mathbf{y}[B]$ gives a geometric form to the abstract extension of the spatial locus B in the environment of the category $\mathcal{B}$. In this sense, an arrow $C \to B$, such that C, B in $\mathcal{B}$, is interpreted as a *singular figure* of B. Stated differently, B is called a *spatial extension* of C in $\mathcal{B}$.

Note that, in this sense, B stands for the spatial extension of *many* different *spatial* loci, not necessarily constrained by relation to each other in any particular fashion, except that of the concrete conditions characterizing $\mathbf{y}[B]$ as a *presheaf functor*. Evidently, this is not the case if B is considered as a *temporal* reference domain. In this case, if B is considered as the *temporal extension* of C, then, any other locus D temporally extended by B, in the sense that an arrow $D \to B$ exists, is extended in this manner, by necessarily *factoring through* C. Equivalently stated, if a locus C is depicted in $\mathcal{B}$, and a locus B is considered to be a temporal extension of the specified locus C, then, in this temporal reference context, any other locus D for which an arrow exists with codomain B, must necessarily be a *proper part* of C, i.e. a *monic* arrow $D \rightarrowtail C$, or a *singular part* of C.

Hence, in the case of temporal extension, if we specify the locus C, as temporally extended to B, then any other locus also temporally extended to B, is so extended by factoring through C. Of course, the definition of temporal extension of a locus C by some other locus B in the category *does not* depend on which locus C is specified as the one being extended by B, but once a particular C is depicted as a reference domain, then the *factorization* condition of any other D -extended also by B- through C, guarantees the satisfaction of the quality of temporal extension.

The definition of the notion of temporal extension as *distinguished* by that of spatial extension saves the underlying intuitions making up the idea of *generalized history* of a locus. From this perspective, if a locus B temporally extends a locus C, in the sense of being its generalized history, it can also serve as the temporal extension, i.e. the *history* of only what can be considered as being proper or singular parts of C, that is, generalized point-elements of C.

Note that it is the *distinct* quality of spatial or temporal extension of a locus, by virtue of its *relation* to other loci in a categorical environment that points to a corresponding interpretation of its *character* as such, meaning as being spatial or temporal, and *not* any *intrinsic ad hoc* postulated character. In this sense, a locus in a categorical environment can be the referent of *both* spatial and temporal connotations *depending on* the way that it is related with other loci. Consequently, the construction of some covering schema in the depicted categorical environment, utilized for the substantiation of some corresponding localization schema in the physical continuum, manifested in terms of spatial or temporal covering relations, will depend *only* on the *relational* characteristics of the loci of each covering family, in turn making requisite, a corresponding non-exclusive interpretation in spatial or temporal terms.

11.2 FUNCTORIAL SPATIAL LOCALIZATION SCHEMATA

We begin our exposition of the notions referring to functorial spatiotemporal localization by introducing, first of all, the conception of a spatial covering system. The general notion of a category $\mathcal{B}$, equipped with a *spatial covering system*, interpreted as a structured family of reference domains used for the spatial localization of physical continuum events in the environment of $\mathcal{B}$, is based on the definition of appropriate covering devices of a spatial character, called *spatial covering sieves*. Let us recapitulate the notion of a sieve, then examine the requirements for an intended interpretation of a family of sieves as a *spatial* covering schema.

For a locus B in $\mathcal{B}$, a B-*sieve* is a family S of $\mathcal{B}$-morphisms with codomain B, such that if $C \to B$ belongs to S and $D \to C$ is any $\mathcal{B}$-morphism, then the composite $D \to C \to B$ belongs to S. We may think of a B-sieve as a right B-ideal. With reference to the functor of generalized points of B in $\mathcal{B}$, denoted by $\mathbf{y}[B] := Hom_{\mathcal{B}}(-,B)$, we have already proved previously that a B-sieve is *equivalent* to a subfunctor $\mathbf{S} \infty \mathbf{y}[B]$ in $Sets^{\mathcal{B}^{op}}$. Thus, epigrammatically, we state:

$$\langle \ B\text{-sieve: } S \ \rangle = \langle \ \text{Subfunctor of } \mathbf{y}[B] \text{: } \mathbf{S} \infty \mathbf{y}[B] \ \rangle$$

We recall that if S is a B-sieve and $h: C \to B$ is any arrow to the locus B, then:

$$h^*(S) = \{f \,/\, cod(f) = C, (h \cdot f) \in S\}$$

is a C-sieve, called the *pullback* of S *along* f. Consequently, we may define a presheaf functor $\mathbf{\Omega}$ in $Sets^{\mathcal{B}^{op}}$, such that its action on loci B in $\mathcal{B}$, is given by:

$$\mathbf{\Omega}(B) = \{S \,/\, S : B - sieve\}$$

and on arrows $h: C \to B$, by $h^*(-): \mathbf{\Omega}(B) \to \mathbf{\Omega}(C)$, given by:

$$h^*(S) = \{f / cod(f) = C, (h \cdot f) \in S\}$$

We stress again that for a locus B in $\mathcal{B}$, the set of all arrows into B is a B-sieve, called the *maximal sieve* on B, and denoted by $t(B) := t_B$.

The natural question that arises in our context of enquiry is the following: How is it possible to restrict $\mathbf{\Omega}(B)$, that is the set of B-sieves for each locus B in $\mathcal{B}$, such that each B-sieve of the restricted set can assume the interpretation of a *spatial covering system* of B. In other words, we look for those appropriate conditions on the set of B-sieves, for each locus B in $\mathcal{B}$, so that the subset of B-sieves obtained, denoted by $\mathbf{\Omega}_\chi(B)$, respect the quality of spatial extension. In this way, the B-sieves of $\mathbf{\Omega}_\chi(B)$, for each locus B in $\mathcal{B}$, to be thought as spatial covering B-sieves, can legitimately be used for the definition of a spatial localization scheme in the physical continuum. The clue for an answer comes from the following observations:

1 We have seen in the discussion of the quality of spatial extension that it constitutes a relational property between reference loci B in $\mathcal{B}$. In this sense, an arrow $C \to B$, such that C, B in $\mathcal{B}$, is interpreted as a singular figure of B, and thus B, is interpreted as a spatial extension of C in $\mathcal{B}$. It is a natural requirement that the set of all figures of B should belong in $\mathbf{\Omega}_\chi(B)$ for each locus B in $\mathcal{B}$;

2 It is important to keep in mind that each spatial covering sieve on a locus B in $\mathcal{B}$, is going to serve as a model of a spatial localization system in the physical continuum, such that localized events are endowed with an interpretation in terms of spatial relations in the environment of $\mathcal{B}$. If we recall the relevant discussion about localization systems and their compatibility requirements, we realize that spatial covering sieves should be stable under pullback operations, and most importantly, the stability conditions should be expressed functorially;

3 Finally, it would be desirable to impose:

i a *transitivity* requirement on the specification of the spatial covering sieves, such that, spatial covering sieves of figures of a locus in spatial covering sieves of this locus should be spatial covering sieves of the locus themselves, and

ii a requirement of *common refinement* of spatial covering sieves.

If we take into account the above requirements we define a spatial covering scheme in the environment of $\mathcal{B}$ as follows:

A *spatial covering schema* on $\mathcal{B}$ is an operation J, which assigns to each locus B in $\mathcal{B}$, a collection $J(B)$ of B-sieves, called *spatial covering* B-sieves, such that, the following three conditions are satisfied:

1 For every locus B in $\mathcal{B}$ the maximal sieve $\{g : cod(g) = B\}$ belongs to $J(B)$ (maximality condition);

2 If S belongs to $J(B)$ and $h : C \to B$ is a figure of B, then $h^*(S) = \{f : C \to B, (h \cdot f) \in S\}$ belongs to $J(C)$ (stability condition);

3 If S belongs to $J(B)$, and if for each figure $h : C_h \to B$ in S there is a sieve R_h belonging to $J(C_h)$, then the set of all composites $h \circ g$, with $h \in S$, and $g \in R_h$, belongs to $J(B)$ (transitivity condition).

As a consequence of the conditions above, we verify that any two spatial covering sieves have a common refinement: if S, R belong to $J(B)$, then $S \cap R$ belongs to $J(B)$.

The operation J satisfying the aforementioned conditions, can be equivalently characterized in terms of a *Grothendieck topology* on the category $\mathcal{B}$, where the covering sieves implicate the requirements of spatial extension. A Grothendieck topology J may be thought of in the *shape* of a *presheaf* functor Ω_χ in $Sets^{\mathcal{B}^{op}}$, such that, by acting on loci B in $\mathcal{B}$, J gives the set of all spatial covering B-sieves, denoted by $\Omega_\chi(B)$, whereas by acting on figures $h : C \to B$, it gives a morphism of sets

$$h^*(-) : \Omega_\chi(B) \to \Omega_\chi(C),$$

expressed as:

$$h^*(S) = \{f / cod(f) = C, (h \cdot f) \in S\},$$

for $S \in \Omega_\chi(B)$. Clearly, Ω_χ is a *subobject* of Ω, that is, $\Omega_\chi \infty \Omega$.

11.3 FUNCTORIAL TEMPORAL LOCALIZATION SCHEMATA

Analogously with the conception of spatial covering schemata on the category $\mathcal{B}$, we can introduce the notion of temporal covering schemata on $\mathcal{B}$ consisting of temporal covering sieves. These covering systems should be construed in such a way that the relational quality of *temporal extension* between their reference domains is explicated properly. Accordingly, temporal covering sieves can be used for the *temporal localization* of physical continuum events in the environment of $\mathcal{B}$.

We have seen previously that the effectuation of spatial covering sieves required the imposition of certain restrictive conditions on the set of all B-sieves, for each locus B in $\mathcal{B}$, in order to qualify as denotations of spatial extension. In this sense, it is clear that an *analogous* interpretation of covering relations in terms of temporal extension between reference loci would require the satisfaction of all the relevant conditions. At a first stage, we may notice that the quality of temporal extension (in the sense of a generalized history of a locus) constitutes a *constrained* form of spatial extension, as discussed in detail previously. Thus, temporal covering B-sieves for each locus B in $\mathcal{B}$, should satisfy the conditions obeyed by spatial covering B-sieves, and additionally, a constraint signifying the *temporal* character of the relevant included extensive relations. Developing this line of reasoning, it is necessary to express the quality of temporal extension between reference domains in terms of sieves.

We consider a *Grothendieck topology* J on $\mathcal{B}$, such that, the maximality, stability and transitivity conditions are satisfied among covering sieves of reference loci B in $\mathcal{B}$. Hence, if S is a B-sieve that belongs to $J(B)$, we say that S is a *covering* B-sieve. It is also convenient to provide the following definition:

A B-sieve S *covers an arrow* $h: C \rightarrow B$ in $\mathcal{B}$, if and only if $h^*(S)$ is a *covering* C-sieve:

$$\langle S \triangleright [h: C \rightarrow B] \rangle \Leftrightarrow \langle h^*(S) \triangleright C \rangle$$

We notice that, as a consequence of the stability condition, if the arrow $h:C \to B$ belongs to S itself, then $id_C : C \to C$ belongs to $h^*(S)$, and thus, $h^*(S) = t_C$, i.e. $h^*(S)$ is the maximal covering C-sieve.

We may formulate this observation, given a covering B-sieve S and any arrow $h:C \to B$, as follows:

$$\langle h \in S \rangle \Leftrightarrow h^*(S) = t_C$$

This is a very convenient setting to explicate the notion of temporal extension among reference loci B in $\mathcal{B}$ in terms of covering sieves, if we further define;

A B-sieve is J*-closed*, if and only if, for all $\mathcal{B}$-arrows $h:C \to B$:

$$\langle\ S \text{ covers the } \mathcal{B}\text{-arrow } h:C \to B\ \rangle \ \Rightarrow\ \langle h \in S \rangle$$

In a suggestive notation, given J we say:

$$\langle S = [S] \rangle_J \Leftrightarrow \langle \langle S \rhd h \rangle \Rightarrow \langle h \in S \rangle \rangle_J$$

Given a Grothendieck topology J on $\mathcal{B}$, J*-closed sieves* constitute a *presheaf* functor Ω_κ in $Sets^{\mathcal{B}^{op}}$, such that, by acting on loci B in $\mathcal{B}$, Ω_κ gives the set of all J-closed B-sieves, denoted by $\Omega_\kappa(B)$, whereas by acting on arrows $h:C \to B$, it gives a morphism of sets $h^*(-):\Omega_\kappa(B) \to \Omega_\kappa(C)$, expressed as: $h^*(S) = \{f / cod(f) = C, (h \cdot f) \in S\}$, for $S \in \Omega_\kappa(B)$.

Indeed, we can immediately verify the following: For any B-sieve R and any $\mathcal{B}$-arrow $h:C \to B$;

$$\langle\ R:\ J\text{-closed } B\text{-sieve}\ \rangle \ \Rightarrow\ \langle\ h^*(R):\ J\text{-closed } C\text{-sieve}\ \rangle$$

In order to see that this is actually the case, we assume that $h^*(R)$ covers a $\mathcal{B}$-arrow $g:C \to B$. This means, by definition, that R covers the composition hg; taking into account that R is J-closed B-sieve, $hg \in R$, or equivalently, $g \in h^*(R)$. Hence, we obtain that $h^*(R)$ is J-closed C-sieve.

Clearly, Ω_κ is a *subobject* of Ω, that is $\Omega_\kappa \infty \Omega$.

The quality of *temporal extension* in terms of covering sieves is captured precisely by the defining requirement of J-*closed sieves*, if we re-express it as follows:

The locus B is a *temporal extension* of a locus C in $\mathcal{B}$, if a $\mathcal{B}$-arrow $h: C \to B$ exists, which is covered by a J-closed B-sieve.

Thus, *all* $\mathcal{B}$-arrows denoting the quality of temporal extension must necessarily be members of corresponding J-closed covering sieves. Consequently, a temporal covering scheme on $\mathcal{B}$, has to be properly expressed in the *restrictive* terms of J-closed covering sieves. These sieves obviously satisfy the maximality and transitivity conditions required. Moreover, since the property of being closed respects the stability condition under pullback operations, J-closed covering sieves also remain stable under pullback.

Now, by considering all the relevant requirements we can define a temporal covering schema in the environment of $\mathcal{B}$, as follows:

A *temporal covering schema* on $\mathcal{B}$ is an operation T, which assigns to each arrow $g: C \to B$ in $\mathcal{B}$ (interpreted as a *temporal extent* C - *irreducible duration* of B), a collection $T(g)$ of B-sieves, to be called *local time-forcing* B-sieves of temporal *resolution* unit $dom(g)$, such that, the following four conditions are satisfied:

i If S is B-sieve and $g \in S$, then S is a covering B-sieve (*maximality condition*);

ii If S covers an arrow $g: C \to B$, it also covers the composition $g \circ f$, for any arrow $D \to C$ (*stability condition*);

iii If S covers an arrow $g: C \to B$, and R is a B-sieve which covers all arrows of S, then R covers g (*transitivity condition*);

iv If S covers an arrow $g: C \to B$, then g belongs to S (*closure condition*).

We may again easily check that any two temporal covering sieves have a common refinement, that is: if R and S both cover $h: C \to B$, then $R \bigcap S$ covers h.

A temporal covering schema on $\mathcal{B}$, formulated in the above arrow-form, obviously satisfies the equivalent conditions [1]-[3] specifying in this form a spatial covering schema, and also additionally the closure constraint [4]. Due to this constraint, characteristic of temporal extension, we conclude that a J-closed B-sieve of temporal *resolution* unit B is necessarily the *maximal* B-sieve. This is clear, if we consider the identity arrow $id_B : B \to B$ and apply the closure constraint. The concept of temporal resolution unit will be analyzed in detail further on.

A *maximal* temporal covering scheme T on $\mathcal{B}$, exists as a presheaf functor Ω_τ in $Sets^{\mathcal{B}^{op}}$, such that: By acting on loci B in $\mathcal{B}$, T gives $\Omega_\tau(B)$, constituted only from the maximal B-sieve for each B, whereas by acting on arrows $h : C \to B$, it gives a morphism of sets $h^*(-) : \Omega_\tau(B) \to \Omega_\tau(C)$, expressed as:

$$h^*(t_B) = \{f \,/\, cod(f) = C, (h \cdot f) \in t_B\} = t_C$$

where $t_B \in \Omega_\tau(B)$. Clearly, Ω_τ is a subobject of Ω, that is $\Omega_\tau \infty \Omega$.

It is useful to notice that from *any* given B-sieve S we can construct a corresponding J-*closed* B-sieve, denoted by $[S]$, simply as follows:

$$[S] = \{f \,/\, cod(f) = B, S \rhd f\}$$

where f is any $\mathcal{B}$-arrow with codomain B, and the notation $S \rhd f$ denotes that S covers f. The above prescription means that the J-closed sieve $[S]$, corresponding to a given sieve S, is construed by *adding* in S *all* arrows that it *covers*. Furthermore, $[S]$ is the *smallest* closed sieve that contains S, named accordingly the *closure* of S.

11.4 PARADIGMATIC CATEGORICAL SPATIOTEMPORAL RELATIONS

After having explicated the defining requirements of spatial and temporal covering schemata in the environment of $\mathcal{B}$, bearing in mind that they are going to provide precise functorial concepts of spatial and temporal localization in a continuum of observable events, it is appropriate to concentrate our attention to some consequences of their functioning.

We suppose that $\mathcal{B}$ has a terminal object denoted by 1, that is for any locus B in $\mathcal{B}$ there exists a unique arrow $!_B : B \to 1$. Then we can view the locus B as a domain of irreducible durations (of temporal extent 1), if we define B as the temporal extension of 1, as follows:

The locus B is a domain of 1-irreducible durations in $\mathcal{B}$, if there exist $\mathcal{B}$-arrows $m:1 \to B$, such that, each one of them is covered by a J-closed B-sieve.

We notice that, according to the above definition, a locus B signifying a domain of *durations* by means of $\mathcal{B}$-arrows $1 \to B$ *cannot* be conceived *separately* from all J-closed B-sieves covering its point-durations. Put differently, an arrow $1 \to B$ obtains the semantics of an *irreducible duration* by being a *member* of a J*-closed* B-sieve that *covers* it, and subsequently, the locus B is interpreted as a domain of point-durations (*history of point-durations*). From a converse perspective, the definition of a point-duration of the reference domain B, forces an interpretation of the terminal object 1 as an *instantaneous* locus at point-duration m, denoted as 1_m, by means of the unique arrow $!_B : B \to 1$. Now, by definition, if we consider a J-closed B-sieve, covering m, its pullback along m is the maximal sieve on the instantaneous locus at m, denoted by t_{1_m}. This fact has the following consequences:

For every J-closed B-sieve covering m, where B stands for a domain of point-durations, 1_m consists of a *snapshot* of each and every locus in $\mathcal{B}$, a role which consolidates its interpretation as an instantaneous locus at point-duration m pretty clearly. Subsequently, the domain of point-durations B stands for the *temporal* extension of the instantaneous space at anyone of its specified durations $m:1 \to B$. It is important to notice that, all the above arguments are independent of any specific locus B used to illustrate them, since only *relational* properties, expressed in terms J*-closed* sieves, actually matter.

Thus, any reference locus in $\mathcal{B}$, equipped with arrows from the terminal object 1 of $\mathcal{B}$, being covered by J-closed sieves, acquires the status of a *temporal* domain of *point-durations*, and in each case, 1 becomes the *instantaneous* locus for some depicted duration. Subsequently, the instantaneous locus at any point-duration *cannot* signify the temporal extension of any other locus except of its own identity, identified with that point-duration, hence, it signifies only the pure quality of *spatial* extension.

This observation permits the characterization of 1_m as an *instantaneous space* at m. Concretely, the instantaneous space 1_m constitutes the *spatial* extension of all other loci C in $\mathcal{B}$, by means of the unique arrows $!_C : C \to 1_m$, contained in a J-closed sieve covering m. Precisely speaking, these unique arrows have been interpreted above as figurative snapshots at m corresponding to each and every locus in $\mathcal{B}$, displayed at 1_m.

11.5 SIMULTANEITY OF FIGURES AND LOCAL TIME DOMAINS OF DURATIONS

We have seen above that a domain of point-durations is interpreted as the temporal extension of the terminal object in $\mathcal{B}$, characterized as the instantaneous space at any depicted duration. The instantaneous space displays the quality of spatial extension in the purest sense, since it is the referent of diminished temporal extension, substantiated in the form of an irreducible point-duration.

In the intuitive sense, a domain of point-durations is the *generalized history* of the terminal locus, as instantiated at each duration in the form of an instantaneous space for that duration. Notice again that, by definition, an irreducible point-duration $m:1 \to B$ of a temporal domain B is covered by a *time-forcing* B-sieve of temporal *resolution* unit 1, such that, its pullback along m is the *maximal* sieve on the instantaneous space at m, denoted by t_{1_m}. Hence, in the perspective of a J-closed sieve covering $m:1 \to B$, the terminal 1_m is conceived as a *hole* of the sieve, such that every locus C is extended to B by *factoring* through the hole 1_m. More precisely, 1_m is the *maximal* hole of the J-*closed* B-sieve covering m, and since there exist unique arrows $!_C : C \to 1_m$, for every C in $\mathcal{B}$, all these C being extended to the temporal domain B, achieve this translation by passing through the maximal hole 1_m of this J-closed B-sieve. This observation permits the definition of the concept of *simultaneity* of figures with respect to a domain of irreducible point-durations B as follows:

Two figures $C \to B$ and $D \to B$ of a domain of point-durations B are *simultaneous* at a moment m, if and only if, they both *factor* through the *maximal* hole 1_m (temporal resolution unit) of *any* J-closed B-sieve covering m.

It is instructive again to clarify the truly *relational* spatiotemporal sense that a locus B acquires by means of the intended interpretation. In the case discussed above, the locus B is both; the denotator of temporal extension of the terminal 1 by its specification as a domain of irreducible point-durations, and also, the denotator of spatial extension of the figures $C \to B$ and $D \to B$ at one of their respective moments.

Furthermore, the same case concerning the concept of simultaneity, points to the conclusion that the decisive factor that determines *simultaneity*, with respect to a temporal domain of durations, is *factorization* through the *maximal hole* at a depicted duration of any J-closed sieve covering that duration. Nevertheless, it is important to clarify that temporal extension is *not* restricted exclusively to domains of point-durations; the latter should be considered only as *paradigmatic* cases. In this sense, it is possible to expand our argumentation and talk about generalized durations.

The locus B is a domain of *generalized* L*-durations* (temporal extent L-irreducible durations) in $\mathcal{B}$, if $\mathcal{B}$-arrows $m_L : L \to B$ exist, such that, each one of them is covered by a J-closed B-sieve. In that case, some $\mathcal{B}$-arrow $m_L : L \to B$ is interpreted as a temporal extent L-irreducible duration of B by means of being a member of a J-closed B-sieve covering it. Subsequently, the locus B is interpreted as a domain (history) of durations of temporal extent L.

Again, in the perspective of a J-closed B-sieve covering m_L, the locus L is considered a *hole*, such that every locus K that can be temporally extended to B, extends via factorization through the hole L. In this generalized sense, the hole L specifies the temporal *resolution* unit of a covering J-closed B-sieve, represented by the corresponding L-durations, that in turn, can be considered as denotations of simultaneity relations with respect to the applied temporal resolution unit of the domain B.

Note that *all* the J-closed B-sieves covering *all* the generalized durations of B contain *complete* information about all questions concerning *temporal* extension with respect to B, under *varying* temporal resolution units. In conclusion, the operational role of time in the present framework, is completely incorporated in the *functioning* of J-closed sieves. Thus, given a covering scheme J, for any J-closed B-sieve we define:

$$T_B(S) = \{h / cod(h) = B, S \triangleright h\}$$

or equivalently, since S stands for a J-closed B-sieve;

$$T_B(S) = \{h / cod(h) = B, h^*(S) = t_{dom(h)}\}$$

The set $T_B(S)$ is interpreted as the set of *generalized durations* of B covered by the J-closed sieve S, under varying temporal resolution units specified by the *holes* $dom(h)$. We can immediately verify that actually $T_B(S) = S$ for every J-closed B-sieve. Thus the sets S and $T_B(S)$ respectively, stand for the *active* and *passive* interpretation of the same entity, being the operation of local time on loci B in $\mathcal{B}$, transforming them into *local time domains* of generalized irreducible durations in the environment of $\mathcal{B}$. Due to this identification, J*-closed* sieves are interpreted as *local time-forcing* sieves for each locus B in $\mathcal{B}$.

Moreover, the temporal *resolution* unit of a J-closed sieve S is determined by the locus dom(h), specified as temporally extended to B by means of h, if the latter is covered by S. The locus $dom(h)$ can be thought of as a *hole* in the B-sieve S, that specifies the kind of generalized durations of a local time domain being covered by S, with respect to the relation of temporal extension between loci. In this sense, we may define a *local time operator* $\hat{T}_S$, associated with a J*-closed* B-sieve, for each locus B in $\mathcal{B}$ as follows:

$$\hat{T}_S \mid B\rangle = h \mid B\rangle$$

where $h \in T_B(S)$. The local time operator $\hat{T}_S$ acting on a locus B, denoted in the -so called- Dirac notation as the eigenstate $\mid B\rangle$, takes for eigenvalues the generalized durations being covered by S. Consequently, the locus B is interpreted as a local time domain endowed with generalized durations under *varying* temporal resolution units.

11.6 SIEVING SPATIAL FIGURES AT DURATIONS OF LOCAL TIME DOMAINS

A natural question that arises in this context of enquiry is the following: Given the means of functorial spatiotemporal localization, formulated in terms of *spatial* and *temporal* covering schemata, how is it possible to *spectrally classify spatial figures* of a reference locus B in $\mathcal{B}$ at *generalized durations* of that locus, considered as a *local time* domain?

In order to tackle this fundamental problem, we are going to use the notion of *subobject classifier* in a topos. First of all, it is useful to clarify the notion of *subobjects* in *any* categorical environment, since it is going to be the main conceptual tool in our argumentation.

A *subobject* of an object X in any category $\mathcal{X}$, is an *equivalence class* of *monic* arrows targeting X, denoted by $\mu : M \rightarrowtail X$. The set of all subobjects of X in the category $\mathcal{X}$, denoted by $\Theta_{\mathcal{X}}(X)$, is a *partially ordered* set under inclusion of subobjects.

The functor $\Theta_{\mathcal{X}}$ can be construed as a *presheaf* functor in the *topos* $Sets^{\mathcal{X}^{op}}$ by the operation of *pulling back* as follows; Given an arrow $g : Y \to X$ in $\mathcal{X}$, the pullback of any monic arrow $\mu : M \rightarrowtail X$ along the arrow g is a *new* monic arrow $\mu' : M' \rightarrowtail Y$, that is a subobject of Y, and obviously the assignment $\mu \mapsto \mu'$, defines a function $\Theta_{\mathcal{X}}(g) : \Theta_{\mathcal{X}}(X) \to \Theta_{\mathcal{X}}(Y)$.

An immediate question that arises here is related to the possibility of representing the subobject functor $\Theta_{\mathcal{X}}$ in the topos $Sets^{\mathcal{X}^{op}}$ by an *object* Ω in $\mathcal{X}$, considered as a category with pullbacks, such that for each X in $\mathcal{X}$, there exists a *natural isomorphism*:

$$\iota_X : \Theta_{\mathcal{X}}(X) \cong Hom_{\mathcal{X}}(X, \Omega)$$

If the subobject functor becomes *representable* with *representing* object Ω in $\mathcal{X}$, then we say that, the category $\mathcal{X}$ is equipped with a *subobject classifier*. By this term we mean a *universal* monic arrow:

$$\mathrm{T} := True : 1 \rightarrowtail \Omega$$

such that, to every monic arrow, $\mu : M \rightarrowtail X$ in $\mathcal{X}$, there is a *unique characteristic* arrow ϕ_μ, which, with the given monic arrow μ, forms a *pullback* diagram:

$$\begin{tikzcd}
M \arrow[r, "!"] \arrow[d, "\mu"] & 1 \arrow[d, "T"] \\
X \arrow[r, "\phi_\mu"] & \Omega
\end{tikzcd}$$

This is equivalent to saying that every subobject of X in $\mathcal{X}$, is *uniquely* a *pullback* of the *universal* monic $\top$. Conversely, satisfaction of this property amounts to saying that the *subobject* functor $\boldsymbol{\Theta}_{\mathcal{X}}$ is *representable* by the object Ω, that is, isomorphic to $Hom_{\mathcal{X}}(-,\Omega)$. Note that the bijection ι_X sends each subobject $\mu : M \infty X$ of X to its unique characteristic arrow $\phi_\mu : X \to \Omega$ and conversely.

After these necessary introductory remarks, we turn to our main objective concerning the problem of *classification* of *figures* of reference loci B in $\mathcal{B}$, referring to B-sieves of corresponding spatial schemata, which provide the functorial means for spatial localization in a continuum of events. The starting point of our enquiry is determined by the realization that a spatial covering schema on $\mathcal{B}$ exists as a presheaf $\boldsymbol{\Omega}_{\chi}$ in $Sets^{\mathcal{B}^{op}}$.

Clearly, by the defining requirement of spatial covering sieves the following subobject relation holds: $\boldsymbol{\Omega}_{\chi} \infty \boldsymbol{\Omega}$, where, $\boldsymbol{\Omega}(B)$, denotes the set of all B-sieves for each locus B in $\mathcal{B}$. The connective link with our initial remarks appears if we recall that:

$$\langle \ B\text{-sieve: } S \ \rangle = \langle \ \text{Subfunctor of } \mathbf{y}[B]\text{: } \mathbf{S} \infty \mathbf{y}[B] \ \rangle$$

It is immediately evident that, because of the above correspondence, the presheaf functor $\boldsymbol{\Omega}$ may be used, at a first stage, for *classification* purposes in $Sets^{\mathcal{B}^{op}}$, since in particular, it could classify subobjects, that is, subfunctors of $\mathbf{y}[B]$ for each locus B in $\mathcal{B}$, according to the pullback square in $Sets^{\mathcal{B}^{op}}$:

$$\begin{tikzcd}
\mathbf{S} \arrow[r, "!"] \arrow[d, "S"] & \mathbf{1} \arrow[d, "T"] \\
\mathbf{y}[B] \arrow[r, "\phi_S"] & \boldsymbol{\Omega}
\end{tikzcd}$$

where:

$$\mathbf{\Theta}_{Sets^{\mathcal{B}^{op}}}(\mathbf{y}[B]) := \mathbf{\Theta}_{\hat{\mathcal{B}}}(\mathbf{y}[B]) = Hom_{\hat{\mathcal{B}}}(\mathbf{y}[B], \mathbf{\Omega}) = \mathbf{\Omega}(B)$$

by using the *Yoneda lemma*. The morphism $True : \mathbf{1} \to \mathbf{\Omega}$ is a natural transformation in $Sets^{\mathcal{B}^{op}}$, given by components $True_B$ for all reference loci B in $\mathcal{B}$. The functor $\mathbf{1}$ is given by the assignment $B \mapsto \{\varnothing\}$ with the obvious restriction morphisms. It is clear that $\mathbf{1}$ is the *terminal* object in $Sets^{\mathcal{B}^{op}}$, and the components of the morphism $True$ are defined by $True_B : \{\varnothing\} \to \mathbf{\Omega}(B)$; where $True_B(\{\varnothing\}) = t_B$, that is, the maximal B-sieve. Thus, $True_B$ is the map that picks out the *maximal* B-sieve.

For any subfunctor of $\mathbf{y}[B]$, $S : \mathbf{S} \infty \mathbf{y}[B]$, the classifying arrow ϕ_S is a natural transformation $\phi_S : \mathbf{y}[B] \to \mathbf{\Omega}$, given by components $(\phi_S)_C : \mathbf{y}[B](C) \to \mathbf{\Omega}(C)$, such that for any figure $\chi : C \to B$ of the locus B, belonging to the set $\mathbf{y}[B](C)$, we have:

$$(\phi_S)_C(\chi) = \{h / (\chi \cdot h) \in \mathbf{S}(dom(h))\}$$

where h is *any* $\mathcal{B}$-arrow with codomain C. Then, obviously, $(\phi_S)_C(\chi)$ is a C-sieve. Note that a $\mathcal{B}$-arrow with codomain C, for instance $h : D \to C$, determines a set theoretical morphism:

$$\mathbf{y}[B](h) : Hom_{\mathcal{B}}(C, B) \to Hom_{\mathcal{B}}(D, B)$$
$$\mathbf{y}[B](h)(\chi) = \chi \cdot h$$

that may *or* may not take the figure $\chi \in \mathbf{y}[B](C)$, by means of $\chi \cdot h$ into $\mathbf{S}(dom(h)) = \mathbf{S}(D) \subseteq \mathbf{y}[B](D)$. In this sense, the C-sieve $(\phi_S)_C(\chi) \in \mathbf{\Omega}(C)$, contains all, and only those, $\mathcal{B}$-arrows h that *actually* take the figure χ into the subobject $\mathbf{S}(dom(h))$. It is clear that $(\phi_S)_C(\chi) = t_C$, namely it is the *maximal* C-sieve in the set $\mathbf{\Omega}(C)$, if and only if the figure χ belonging to the set $\mathbf{y}[B](C)$, belongs to $\mathbf{S}(C)$ as well.

Furthermore, if we *replace* the representable functor $\mathbf{y}[B]$ by *any* presheaf functor $\mathbf{P}$, and the subfunctors $S : \mathbf{S} \infty \mathbf{y}[B]$ by corresponding subfunctors $Q : \mathbf{Q} \infty \mathbf{P}$, entirely analogous arguments lead to the conclusion that $\mathbf{\Omega}$ is the *subobject classifier* in the category of presheaves $\mathit{Sets}^{\mathcal{B}^{op}}$, by which we mean that the diagram:

$$
\begin{tikzcd}
\mathbf{Q} \arrow[r, "!"] \arrow[d, "Q"] & \mathbf{1} \arrow[d, "T"] \\
\mathbf{P} \arrow[r, "\phi_Q"] & \mathbf{\Omega}
\end{tikzcd}
$$

is a *pullback* diagram in $\mathit{Sets}^{\mathcal{B}^{op}}$, where $\mathbf{\Omega}(B) = \Theta_{\hat{\mathcal{B}}}(\mathbf{y}[B])$, namely the set of all B-sieves for each locus B in $\mathcal{B}$, or equivalently the set of all subfunctors of $\mathbf{y}[B]$. The set $\mathbf{\Omega}(B)$ for each locus B in $\mathcal{B}$, is a partially ordered set under the relation of inclusion of B-sieves, whereas the maximal element of the poset $\mathbf{\Omega}(B)$ is the maximal B-sieve.

The crucial observation for our purposes is related to the fact that a spatial covering schema on the categorical environment $\mathcal{B}$ bears thr status of a presheaf subfunctor of $\mathbf{\Omega}$, due to the subobject inclusion $\mathbf{\Omega}_\chi \infty \mathbf{\Omega}$, established previously, and consequently it can be characterized in terms of some *classifying arrow* into $\mathbf{\Omega}$.

Before we specify the description of spatial covering schemata in $\mathcal{B}$ in terms of characteristic arrows into $\mathbf{\Omega}$, it is convenient to introduce some terminology related to the semantics of the identity arrow, $id_{\mathbf{\Omega}_\chi} : \mathbf{\Omega}_\chi \to \mathbf{\Omega}_\chi$. Let us consider, a spatial covering B-sieve Σ, $\Sigma \in \mathbf{\Omega}_\chi(B)$. Then, we define the *identical* assignment;

$$\mathbf{\Omega}_\chi(B) \not\ni \Sigma \mapsto [id_{\mathbf{\Omega}_\chi}]_B(\Sigma) := \Sigma \in \mathbf{\Omega}_\chi(B)$$

According to the above, Σ acquires a dual interpretation, which can be expressed equivalently as both:

i In the active sense Σ is interpreted as a spatial covering B-sieve; that is as a device that acts on the locus B by covering it in spatial terms.

ii In the passive sense Σ, that is $[id_{\mathbf{\Omega}_\chi}]_B(\Sigma) := \Sigma$, is interpreted as a generalized variable spatial point of the locus B, where the latter has obtained a spatial reference by means of Σ, that covers it spatially.

Thus, by means of the identity arrow $id_{\mathbf{\Omega}_\chi} : \mathbf{\Omega}_\chi \to \mathbf{\Omega}_\chi$, for each locus B in $\mathcal{B}$, what gets spatially covered is identified with what spatially covers.

Let us call $\chi : \mathbf{\Omega} \to \mathbf{\Omega}$, the *classifying arrow* characterizing the subobject inclusion of a spatial covering scheme $J : \mathbf{\Omega}_\chi \infty \mathbf{\Omega}$ according to the pullback diagram in $Sets^{\mathcal{B}^{op}}$

$$\begin{tikzcd}
\mathbf{\Omega}_\chi \arrow[r, "!"] \arrow[d, "J"] & \mathbf{1} \arrow[d, "True"] \\
\mathbf{\Omega} \arrow[r, "\chi"] & \mathbf{\Omega}
\end{tikzcd}$$

Obviously, the characteristic arrow $\phi_J := \chi$, such that $\chi : \mathbf{\Omega} \to \mathbf{\Omega}$, determines the spatial covering scheme $J : \mathbf{\Omega}_\chi \infty \mathbf{\Omega}$ that classifies, and conversely, it is uniquely determined by that relation. In order to understand the semantics of χ, we consider a spatial covering scheme on the categorical environment $\mathcal{B}$, such that $J(B)$ denotes the set of spatial covering B-sieves for each locus B in $\mathcal{B}$. Next, we define the natural transformation $\chi : \mathbf{\Omega} \to \mathbf{\Omega}$ as follows:

$$\chi_B(S) = \{h \,/\, cod(h) = B, S \triangleright h\}$$

or equivalently:

$$\chi_B(S) = \{h \,/\, cod(h) = B, h^*(S) \in J(dom(h))\}$$

that is, $\chi_B(S)$ denotes the set of all $h : C \to B$, such that S covers h. Thus, for any $h : C \to B$ being covered by S, h belongs to $\chi_B(S)$;

$$\langle S \triangleright h \rangle \Rightarrow h \in \chi_B(S)$$

Clearly, this prescription specifies $\chi_B(S)$ as a B-sieve, that is;

$$\mathbf{\Omega}(B) \ni S \mapsto \chi_B(S) \in \mathbf{\Omega}(B)$$

We notice that the classifying arrow at the locus B serves to specify *exactly* what each B-sieve *covers* with respect to a *spatial* covering scheme J. If we focus our attention on the definition of $\chi_B(S) \in \mathbf{\Omega}(B)$, we notice the following:

For any $\mathcal{B}$-arrow $g: C \to B$, we obtain the logical conjugation relation:

$$\chi_C(g^*(S)) = g^*(\chi_B(S))$$

for any B-sieve, therefore χ is actually a natural transformation $\chi: \mathbf{\Omega} \to \mathbf{\Omega}$ as required. Furthermore, if in the definition of $\chi_B(S)$, we employ the maximal B-sieve $t_B \in \mathbf{\Omega}(B)$, that by its specification is a covering sieve of all arrows with codomain B, we obtain:

$$\chi_B(t_B) = t_B$$

This relation holds for every locus B in $\mathcal{B}$, and thus in functional terms we further obtain:

$$\chi \circ True = True$$

Moreover, the classifying arrow χ_B clearly preserves order, in the sense that for B-sieves R, S:

$$R \subset S \Rightarrow \chi_B(R) \subset \chi_B(S)$$

Thus, for any B-sieves R and S we obtain:

$$\chi_B(R \bigcap S) \subseteq \chi_B(R) \bigcap \chi_B(S)$$

and conversely:

$$\chi_B(R) \bigcap \chi_B(S) \subseteq \chi_B(R \bigcap S)$$

Hence, for each B in $\mathcal{B}$ we obtain the equality:

$$\chi_B(R \bigcap S) = \chi_B(R) \bigcap \chi_B(S)$$

meaning that the operation of *spatial classification commutes* with the operation of *intersection* of sieves, expressed in suggestive functional terms as follows:

$$\chi \circ \wedge = \wedge \circ (\chi \times \chi)$$

Finally, since the classifying arrow χ_B preserves order, and also $R \subset \chi_B(R)$, if we operate on this inclusion by acting with χ_B, we obtain:

$$\chi_B(R) \subseteq \chi_B(\chi_B(R))$$

Conversely, if $h \in \chi_B(\chi_B(R))$, then by definition $\chi_B(R) \triangleright R$, that is $\chi_B(R)$ covers R. Moreover, for each $g \in \chi_B(R)$, by definition, $R \triangleright g$. If we bear in mind the transitivity property of a spatial covering scheme, then $R \triangleright h$, or else, $h \in \chi_B(R)$. Thus, if $h \in \chi_B(\chi_B(R))$, then $h \in \chi_B(R)$ or equivalently:

$$\chi_B(\chi_B(R)) \subseteq \chi_B(R)$$

Hence, for each B in $\mathcal{B}$ we obtain the equality:

$$\chi_B(\chi_B(R)) = \chi_B(R)$$

that is, the operation of classification is *idempotent*, expressed in functional terms simply as follows:

$$\chi \circ \chi = \chi$$

Furthermore, if we consider a spatial covering B-sieve R, then, any $h: C \to B$ covered by R, belongs to $\chi_B(R)$, that is, $h \in \chi_B(R)$. Now, let us assume that, $\chi_B(R)$ covers an arrow f. By definition of $\chi_B(R)$, R covers all arrows in $\chi_B(R)$. Thus, by the transitivity condition of covering sieves, we obtain that R covers the arrow f, and hence, $f \in \chi_B(R)$. We conclude, in this sense that:

$$\langle \chi_B(R) \triangleright f \rangle \Rightarrow \langle f \in \chi_B(R) \rangle$$

If we recall that for any $f : D \rightarrow B$:

$$\langle f \in \chi_B(R) \rangle \Leftrightarrow f^*(\chi_B(R)) = t_D$$

we conclude that:

$$\langle \chi_B(R) \triangleright f \rangle \Rightarrow f^*(\chi_B(R)) = t_D$$

The above condition means that, $\chi_B(R)$ is a J-*closed sieve*; Most importantly, $\chi_B(R)$ is the *closure* of R in the covering scheme J. This conclusion can be equivalently stated as follows:

$$\chi_B(R) = [R]$$

Returning to the interpretation given above, according to which the classifying arrow at the locus B serves to specify exactly what each B-sieve R covers, by means of the B-sieve $\chi_B(R)$, we conclude that:

Spatial generalized points of the locus B, where the latter has obtained a spatial reference with respect to a covering sieve R of a *spatial covering* scheme J operating on $\mathcal{B}$, are being *classified* with respect to the *generalized irreducible durations* of B, covered by $[R]$.

The B-sieve $\chi_B(R)$, identified as a *local time forcing* B-sieve, signifies the set of *generalized durations* of the locus B, *classifying spatial generalized points* of B depicted through R. Remarkably from this perspective, local time-forcing sieves have a dual role. On the one hand, they are the *constituents* of *temporal covering* schemata, and on the other, they are used as *devices* for *spatial classification*.

This interpretation reveals the two-fold operational role of local time forcing in the categorical environment of $\mathcal{B}$ as both, the *generator* of a *temporal covering* schema endowing loci B with a *relational* temporal reference in terms of generalized durations, and also, as the *generator* of a *classification* schema characterizing spatial relations in terms of the durations they cover.

Let us now concentrate our attention on the presheaf of time-forcing sieves Ω_κ, being the presheaf of J-closed sieves for a covering schema J. There is clearly a subobject inclusion $T:\Omega_{\tau} \infty \Omega$. The *classifying* arrow characterizing T, is denoted by $\phi_T := \kappa$ according to the pullback diagram in $Sets^{\mathcal{B}^{op}}$:

$$
\begin{tikzcd}
\Omega_{\kappa} \arrow[r, "!"] \arrow[d, "T"] & \mathbf{1} \arrow[d, "True"] \\
\Omega \arrow[r, "\kappa"] & \Omega
\end{tikzcd}
$$

Evidently, the characteristic arrow $\phi_T := \kappa$, such that, $\kappa : \Omega \to \Omega$ determines the subobject $T:\Omega_{k} \infty \Omega$ that classifies, and conversely, is uniquely determined by that relation. We define the natural transformation $\kappa : \Omega \to \Omega$ as follows:

$$\kappa_B(S) = \{h / cod(h) = B, h^*(S) = t_{dom(h)}\}$$

It is clear that $\kappa_B(S) = S$ for every J-closed B-sieve. The set $\kappa_B(S)$ is interpreted as the set of generalized durations of B covered by a J-closed sieve S, under varying temporal resolution units specified by the sieve holes $dom(h)$. We notice again that the classifying arrow at the locus B serves to specify exactly what each B-sieve covers with respect to T. In an analogous fashion we obtain the conditions:

$$\kappa \circ True = True$$
$$\kappa \circ \wedge = \wedge \circ (\kappa \times \kappa)$$
$$\kappa \circ \kappa = \kappa$$

It is instructive to define the map $B \mapsto \hat{\kappa}_R(B)$ for each B-sieve R, given a covering scheme J, as follows:

$$\hat{\kappa}_R(B) = \kappa_B(R) \qquad iff \qquad R = [R]_J$$

which, assigns to each locus B in $\mathcal{B}$, its set of generalized durations, if and only if the B-sieve R is J-covering them and is also J-closed.

This assignment endows the reference domain B precisely with the semantics of temporal relations, that is generalized durations, and also, permits the interpretation of the locus B as a local time domain.

Similarly, let us call $\tau:\mathbf{\Omega}\to\mathbf{\Omega}$, the classifying arrow characterizing the subobject inclusion of a maximal temporal covering schema $T:\mathbf{\Omega}_\tau \infty \mathbf{\Omega}$ according to the pullback diagram in $Sets^{\mathcal{B}^{op}}$:

$$\begin{tikzcd}
\mathbf{\Omega}_\tau \arrow[r, "!"] \arrow[d, "T"] & \mathbf{1} \arrow[d, "True"] \\
\mathbf{\Omega} \arrow[r, "\tau"] & \mathbf{\Omega}
\end{tikzcd}$$

We define the natural transformation $\tau:\mathbf{\Omega}\to\mathbf{\Omega}$ as follows:

$$\tau_B(S)=t_B$$

This prescription specifies $\tau_B(S)$ as the maximal B-sieve, and obviously, the analogous conditions for the characteristic arrow τ are trivially satisfied. The set $\tau_B(S)$ is interpreted as the maximal set of generalized durations of B, identified with the set of generalized elements of B being covered by a J-closed sieve covering the identity of B. Clearly, such a J-closed sieve possesses maximal temporal resolution capability.

Under these circumstances, we define a complete local time-domain as follows: If the generalized durations of a local time-domain are covered by a J-closed sieve that *covers* the *identity* of B, then we call it a *complete local time domain*. It is also evident from the explicit description of the classifying arrow referring to time-forcing sieves, that if such a sieve covers the identity of a locus B in $\mathcal{B}$, then it is necessarily the maximal temporal covering B-sieve, and the locus B acquires the interpretation of a complete local time-domain. In this sense, a maximal temporal covering scheme on $\mathcal{B}$ is equivalent to the specification of complete local time-domains in the environment of $\mathcal{B}$, described functorially by means of $\mathbf{\Omega}_\tau$.

From the above detailed description of the cases considered, we conclude that we may form a unified framework of classification, according to which, an arbitrary subobject Ω_λ of $\mathbf{\Omega}$ amenable to a spatial or maximal temporal covering qualification, denoted by the inclusion $\mathbf{\Omega}_\lambda \infty \mathbf{\Omega}$ can be characterized by means of a classifying arrow $\lambda : \mathbf{\Omega} \to \mathbf{\Omega}$, defined by a pullback diagram in $Sets^{\mathcal{B}^{op}}$ as follows:

$$\begin{array}{ccc} \mathbf{\Omega}_\lambda & \xrightarrow{!} & \mathbf{1} \\ \Big\downarrow{\scriptstyle \Lambda} & & \Big\downarrow{\scriptstyle True} \\ \mathbf{\Omega} & \xrightarrow{\lambda} & \mathbf{\Omega} \end{array}$$

where, the natural transformation $\lambda : \mathbf{\Omega} \to \mathbf{\Omega}$ is defined for each locus B in $\mathcal{B}$ by the set;

$$\lambda(S) = \{h / cod(h) = B, S \triangleright h\}$$

If S is a *spatial covering* B-sieve we obtain its *closure* $[S]$ in the image of the *classifying* arrow λ, whereas, if S is a *time-forcing* B-sieve, which, also covers the identity of B, we obtain the maximal *temporal* resolution id_B-covering B-sieve t_B, identifying the locus B as a *complete local time-domain*. The classifying arrow λ is order preserving, idempotent and commutes with the operation of finite intersections of covering sieves.

It is important to emphasize that, from a *logical* point of view, the *subobject classifier* $\mathbf{\Omega}$ is interpreted as a domain of *truth values*, partially ordered by inclusion, where the maximal truth value, for each locus B in $\mathcal{B}$, is represented by the maximal B-sieve. Hence, $\mathbf{\Omega}$ extends and enriches the classical static, absolute, and rigid set-theoretic two-valued object of truth values.

Since these truth values are *sieves*, which operate in the context of a communication topos, the proper and precise interpretation of the *representing* object $\mathbf{\Omega}$ is in the logical terms of the ancient Greek term "*aletheia*", which bears the meaning of *unveiling* temporally through the *holes* of a *covering* sieve. We conceive of unveiling as a process of temporal *percolation*, which can be localized with respect to complete local time-domains. For this reason, "*aletheia*" is amenable to *localization* enunciated in terms of both *spatial* and *temporal sieving* relations, and thus acquires the characteristics of a logical *sheaf*.

From the preceding, we may draw the following conclusions:

i The classifying arrow $\lambda : \Omega \to \Omega$ can be interpreted as an *operator* of *localization* of "*aletheia*" with respect to a complete local time domain. Equivalently, the operator λ, together with the above prescribed properties, induces a *topology* on the *communication topos* $\widehat{\mathcal{B}}$.

ii Since the *maximal truth value* for each locus B in $\mathcal{B}$ is the *maximal* B*-sieve* t_B, and also, in case t_B operated as a time-forcing B-sieve would have been id_B-covering, the localization induced by the action of λ on t_B, encapsulates the association of a *complete local-time domain* with a local reference frame, where a *complete local description* of reality can be effectuated appropriately in the environment of $\mathcal{B}$, signified simultaneously in terms of the maximal truth value t_B.

In this sense, the maximal B-sieve t_B plays *three* interwoven roles; (a) it is a *maximal covering* B-sieve; (b) if considered as time-forcing, it forces the interpretation of the locus B as a *complete local time-domain* by means of its associated local time operator $\widehat{T}_{t_B}$; and (c) it *unveils* through its *holes* a *complete local description* of reality, being *forced* by $\widehat{T}_{t_B}$, with respect to the complete local time-domain B in $\mathcal{B}$, the latter being subsequently called a local reference frame.

Note that such a local reference frame has meaning *only in time* in its function as a complete local time-domain. It expresses the intuition that in a localization schema of the physical continuum events can be individuated *simultaneously* from the continuum only in time, that is, over *complete local-time domains* in the localizing environment, such that a complete local description of reality is legitimate in their own descriptive terms. It is worth stressing that the notion of simultaneity with respect to a complete local time-domain, $id_B : B \to B$, refers to B as a *totality*, since the maximal temporal resolution unit or hole of the time-forcing B-sieve covering id_B, is clearly extended to extent B.

In the following, we shall realize that the *effectuation* of a complete *local* description in *every* local reference frame can be implemented appropriately in terms of *spatial covering* sieves, that is, through variable generalized points of a locus B (interpreted *spatially* by means of each covering B-sieve), *classified* by the generalized durations of B (interpreted accordingly as a complete *local time-domain*). In order to substantiate this argument we need some further notions that we explicate in detail below.

First of all, we note that we have defined the classifying arrow $\lambda:\mathbf{\Omega}\to\mathbf{\Omega}$ as a localization operator on the functor $\mathbf{\Omega}$ of "*aletheia*". Of course, *besides* λ, we could obviously consider the *identity* arrow $id_{\Omega}:\mathbf{\Omega}\to\mathbf{\Omega}$. Thus, it is reasonable to ask for their *equalizer*, denoted by $\mathbf{\Omega}^{\Lambda}$, according to the diagram:

$$\begin{tikzcd} \mathbf{\Omega}^{\Lambda} \arrow[r, "\epsilon"] & \mathbf{\Omega} \arrow[r, shift left, "id_{\mathbf{\Omega}}"] \arrow[r, shift right, "\lambda"'] & \mathbf{\Omega} \end{tikzcd}$$

where, $\mathbf{\Omega}^{\Lambda}$ is defined, for each locus B in $\mathcal{B}$, as follows:

$$\mathbf{\Omega}^{\Lambda}(B)=\{S: S\in\mathbf{\Omega}(B)\wedge[id_{\Omega}]^{B}(S)=\lambda_{B}(S)\}$$

The condition imposed on the defining requirement of $\mathbf{\Omega}^{\Lambda}$, is satisfied *only for those* B-sieves that are *local time-forcing*; that is for the B-sieves S that are J*-closed*, $S=[S]_{J}$, with respect to a *spatial covering* scheme J on $\mathcal{B}$. It is also clear that the maximal B-sieve t_{B} belongs in $\mathbf{\Omega}^{\Lambda}(B)$. We furthermore notice that if a sieve R belonging to $\mathbf{\Omega}^{\Lambda}(B)$ is also id_{B}-covering, then it is necessarily the *maximal* B-sieve t_{B}, which forces and consolidates the interpretation of B as a *complete local time-domain.*

From these observations, we conclude that the *equalizer* $\mathbf{\Omega}^{\Lambda}\to\mathbf{\Omega}$ is actually the same as the *subobject* $T:\mathbf{\Omega}_{\kappa}\infty\,\mathbf{\Omega}$ consisting of *local time-forcing sieves* $R=[R]$, or equivalently, local time operators $\hat{T}_{[R]}$.

Furthermore, since the operator λ is *idempotent*, from the universal property of the equalizer $\mathbf{\Omega}^{\Lambda}\to\mathbf{\Omega}$ we derive the existence of a *unique* arrow, $\iota:\mathbf{\Omega}\to\mathbf{\Omega}^{\Lambda}$, such that $\mathbf{\Omega}^{\Lambda}$ is the *image* of the operator λ;

$$\mathbf{\Omega} \xrightarrow{\iota} \mathbf{\Omega}^{\Lambda} \xrightarrow{\text{Ú}} \mathbf{\Omega}$$

The above defines then the *epimorphic-monomorphic* factorization of the operator $\lambda : \mathbf{\Omega} \to \mathbf{\Omega}$.

Additionally, we may consider the following pullback diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "!"] \arrow[d, "True^{\Lambda}"] & \mathbf{1} \arrow[d, "True"] \\
\mathbf{\Omega^{\Lambda}} \arrow[r, "\epsilon"] & \mathbf{\Omega}
\end{tikzcd}$$

Since we have that $\lambda \circ True = True$, the arrow $True$ *factors through* $\mathbf{\Omega}^{\Lambda}$ as

$$True^{\Lambda} : \mathbf{1} \to \mathbf{\Omega}^{\Lambda}$$

where the *equalizer* arrow $\text{Ú} : \mathbf{\Omega}^{\Lambda} \to \mathbf{\Omega}$ actually characterizes the *subobject* $\mathbf{1} \infty \mathbf{\Omega}^{\Lambda}$, according to the above pullback diagram.

Moreover, if $\Xi : \mathbf{\Omega}_{\Xi} \infty \mathbf{\Omega}$ denotes an arbitrary subobject of $\mathbf{\Omega}$, characterized by means of $\xi : \mathbf{\Omega}_{\Xi} \to \mathbf{\Omega}$ then its *closure* with respect to a covering schema Λ, denoted by $[\mathbf{\Omega}_{\Xi}]_{\Lambda}$ is characterized by $\lambda \circ \xi$, and also clearly, $\mathbf{\Omega}_{\Xi}$ is Λ *-closed*, if and only if $\lambda \circ \xi = \xi$, that is, equivalently, if and only if ξ *factors through* $\mathbf{\Omega}^{\Lambda} \infty \mathbf{\Omega}$.

The above description can be formulated suitably, in order to apply to *subobjects* of an *arbitrary* functor $\mathbf{P}$ in the *topos* $Sets^{\mathcal{B}^{op}}$ if we define that a *subobject* $Q : \mathbf{Q} \infty \mathbf{P}$ characterized by means of $\phi_Q : \mathbf{P} \to \mathbf{\Omega}$ is λ*-closed*, if and only if its λ*-closure*, specified by $\lambda \circ \phi_Q$ satisfies the condition $\lambda \circ \phi_Q = \phi_Q$.

Then we are able to claim that the functor $\mathbf{\Omega}^{\Lambda}$ *classifies* the λ-closed subobjects, in the sense that, for each functor $\mathbf{P}$ in $Sets^{\mathcal{B}^{op}}$, there exists a *natural bijection*:

$$Hom_{\widehat{\mathcal{B}}}(\mathbf{P}, \mathbf{\Omega}^{\Lambda}) \cong [\mathbf{\Theta}_{\widehat{\mathcal{B}}}(\mathbf{P})]_{\lambda}$$

where $[\Theta_{\widehat{\mathcal{B}}}(\mathbf{P})]_\lambda$ denotes the set of λ-*closed* subobjects of $\mathbf{P}$, according to the *pullback* diagram:

$$\begin{tikzcd}
{[\mathbf{Q}]} \arrow[r, "!"] \arrow[d, "{[Q]}"] & \mathbf{1} \arrow[d, "True^\Lambda"] \\
\mathbf{P} \arrow[r, "\phi_{[Q]}"] & \mathbf{\Omega}^\Lambda
\end{tikzcd}$$

where $\mathbf{\Omega}^\Lambda(B) \equiv [\Theta_{\widehat{\mathcal{B}}}(\mathbf{y}[B])]_\lambda$, denotes the set of *all* λ-*closed* B-sieves for each locus B in $\mathcal{B}$, or equivalently, the set of all λ-closed subfunctors of $\mathbf{y}[B]$. In other words, this is precisely the set of *local time-forcing* B-sieves, or local time-forcing operators with respect to the *localization* operator λ associated with the *covering* schema Λ.

It is important to notice that the notions of *closure* with respect to the *localization* operator λ and the *covering* schema Λ respectively, are actually *equivalent*. The set $\mathbf{\Omega}^\Lambda(B)$, for each locus B in $\mathcal{B}$, is a *partially ordered* set under the relation of inclusion of λ-*closed* B-sieves. Moreover, the elements of $\mathbf{\Omega}^\Lambda(B)$ can be interpreted as *truth values* with respect to λ, where the maximal truth value is t_B, which is λ-closed.

Furthermore, $\mathbf{\Omega}^\Lambda(B)$, for each locus B in $\mathcal{B}$, can be endowed with the logical operations of conjunction, disjunction and implication, and thus, acquire the structure of an *Arend Heyting algebra* with respect to these operations. In this sense, we say that the functor $\mathbf{\Omega}^\Lambda$ is a Heyting algebra object in the topos $\mathcal{S}ets^{\mathcal{B}^{op}}$, which *classifies*, in particular, λ-*closed* subobjects of *any presheaf* functor in this category.

Here it is essential to consider the *pullback* diagram that describes the classification of λ-closed subobjects of the representable functor $\mathbf{y}[B]$ in $\mathcal{S}ets^{\mathcal{B}^{op}}$, being precisely those subobjects that play the role of *local time-forcing* B-sieves in the environment of $\mathcal{B}$. The characteristic arrow $\phi_{[S]}$, such that $\phi_{[S]}: \mathbf{y}[B] \to \mathbf{\Omega}^\Lambda$, determines the local-time forcing B-sieve $[S]: \mathbf{[S]} \infty\ \mathbf{y}[B]$ that classifies, and conversely, it is uniquely determined by that, as follows:

$$\begin{tikzcd}
{[\mathbf{S}]} \arrow[r, "!"] \arrow[d, "{[S]}"] & \mathbf{1} \arrow[d, "True^{\Lambda}"] \\
\mathbf{y}[B] \arrow[r, "\phi_{[S]}"] & \mathbf{\Omega}^{\Lambda}
\end{tikzcd}$$

We introduce the following terminology: An arbitrary B-sieve $R : \mathbf{R} \infty \mathbf{y}[B]$ is λ*-dense* in $\mathbf{y}[B]$, if $[\mathbf{R}] = \mathbf{y}[B]$, or equivalently, if $[R] = t_B$. This is the case if and only if R is an id_B-covering B-sieve. Thus, an arbitrary B-sieve is λ*-dense* in $\mathbf{y}[B]$, if and only if it belongs to the family of *spatial covering* B-sieves of a spatial covering schema Λ on $\mathcal{B}$.

Consequently, since *any spatial covering* B-sieve S in the schema Λ is λ*-dense*, in the sense that the induced local-time forcing B-sieve $[S]$ is an id_B-covering B-sieve, and thus, the *maximal local time-forcing* B-sieve t_B, we have the possibility of a *complete local description* of reality, identified as the *maximal truth value* in $\mathbf{\Omega}^{\Lambda}$. This expresses the completed temporal *percolation* of "*aletheia*" with respect to the *maximal* local time-forcing B-sieve t_B. The latter is effectuated *in terms* of *any* depicted *spatial covering* B-sieve S over the simultaneously substantiated - by means of $[S] = t_B$ - *complete local time-domain* B, identified previously with a local reference frame. Conclusively we say that:

A complete local description of reality is legitimate for any covering B-sieve of a spatial covering schema on $\mathcal{B}$, with respect to the correspondingly induced complete local time-frame B.

To slightly rephrase the above for reasons of clarity, we assert the following:

A complete local description of reality is forced by the action of the maximal local-time operator $\widehat{T}_{t_B}$ on a locus B, generated from any spatial covering B-sieve of a schema Λ by the process of λ-closure, that is, by the process of temporal percolation, and expressed in the descriptive terms of the spatially covering objects over the correspondingly induced complete local time-frame B.

At this stage, it is necessary to make clear the precise manner in which the *maximal truth value* t_B, for each locus B in $\mathcal{B}$, *reflects* a *complete local* description of reality, in case B stands for a complete local time-frame, being induced by a spatial covering B-sieve of a scheme Λ, or by its associated localization operator λ. The clarification of this issue necessitates the introduction of the concept of λ-sheaf.

In general, if $R:\mathbf{R} \infty \mathbf{y}[B]$ is a B-sieve, then a presheaf $\mathbf{P}$ is defined to be a λ*-sheaf* if and only if the induced map

$$R^* : Hom_{\hat{\mathcal{B}}}(\mathbf{y}[B],\mathbf{P}) \xrightarrow{\cong} Hom_{\hat{\mathcal{B}}}(\mathbf{R},\mathbf{P})$$

is an *isomorphism* for every λ*-dense subfunctor* of $\mathbf{y}[B]$. We mention that if we imposed on the map R^* as above, the relevant requirement according to which R^* was just a *monomorphism*, then the presheaf $\mathbf{P}$ would be λ*-separated*.

Taking into account our previous remarks on the notion of subobjects being λ-dense, a presheaf $\mathbf{P}$ is a λ-sheaf, if and only if the above map R^* is an isomorphism for every covering B-sieve R of a spatial covering schema Λ on $\mathcal{B}$. The definition of a λ-sheaf essentially means that an arrow from a λ*-dense subfunctor* of $\mathbf{y}[B]$, namely a *spatial covering* B-sieve, to a functor Δ qualified as a λ-sheaf, can be extended *uniquely* to an arrow on all of $\mathbf{y}[B]$ targeting Δ, such that according to the diagram below $r = y \circ R$:

$$\begin{tikzcd}
\mathbf{R} \arrow[r, "r"] \arrow[d, "R"] & \mathbf{\Delta} \arrow[d, "id_{\mathbf{\Delta}}"] \\
\mathbf{y}[B] \arrow[r, "y"] & \mathbf{\Delta}
\end{tikzcd}$$

In this setting, we impose the condition that for a *spatial covering* schema Λ on $\mathcal{B}$ all *representable* presheaves on $\mathcal{B}$ are λ*-sheaves*. Thus, spatial covering schemata correspond to the so called *subcanonical* and *canonical* Grothendieck topologies.

Let $\hat{\mathcal{B}}_\lambda := \mathbf{Sh}[\mathcal{B}]$ be the full *subcategory* of $\hat{\mathcal{B}}$, the objects of which are the λ-sheaves, and let $I : \mathbf{Sh}[\mathcal{B}] \infty \hat{\mathcal{B}}$ be the *inclusion* functor. A convenient way to think about the category $\mathbf{Sh}[\mathcal{B}]$ is provided by the insight that it is the subcategory of $\hat{\mathcal{B}}$, which is *closed* under *isomorphisms* induced by the action of an *operator-functor* L, commuting with pull-back operations, on the objects and arrows of $\hat{\mathcal{B}}$, in the following sense:

If an object in $\hat{\mathcal{B}}$ is *isomorphic* to one in $\mathbf{Sh}[\mathcal{B}]$, then it is itself in $\mathbf{Sh}[\mathcal{B}]$. We express this idea, given a *localization* operator λ, by constructing an operator-functor $\mathbf{L} : \hat{\mathcal{B}} \to \mathbf{Sh}[\mathcal{B}]$ which is *left adjoint* to the inclusion functor $I : \mathbf{Sh}[\mathcal{B}] \infty \hat{\mathcal{B}}$ and also *preserves pull-backs*. The induced adjunction, i.e. the *encoding/decoding* functorial relations between the category of presheaves $\hat{\mathcal{B}}$ and the category of sheaves $\mathbf{Sh}[\mathcal{B}]$ establishes concretely the functorial schema of *metaphora* characterizing the *localization* of "*aletheia*" through temporal percolation.

By this specification we mean precisely that L *reflects* each functor in $\hat{\mathcal{B}}$ into the subcategory $\mathbf{Sh}[\mathcal{B}]$, such that any Δ in $\hat{\mathcal{B}}$ is a λ*-sheaf*, if and only if the map $\Delta \to \mathbf{L}\Delta$ is an *isomorphism*.

Furthermore we define the set:

$$\Psi = \{\boldsymbol{\psi} / \langle \boldsymbol{\psi} : \mathbf{Y} \to \mathbf{X} \rangle \in \hat{\mathcal{B}}_1 \wedge \langle \mathbf{L}\boldsymbol{\psi} : \mathbf{LX} \xrightarrow{\cong} \mathbf{LY} \rangle\}$$

where, $\hat{\mathcal{B}}_1$ denotes the set of arrows of $\hat{\mathcal{B}}$. We notice that an arrow $\psi \in \hat{\mathcal{B}}_1$ belongs to the set Ψ, if and only if it is taken by the *action* of L, that is $\mathbf{L}\boldsymbol{\psi}$, to an *isomorphism*. Thus, for every Δ in $\hat{\mathcal{B}}$ the map $\Delta \to \mathbf{L}\Delta$ belongs to the set Ψ. In this light we can also verify that:

i Any presheaf Δ in $\hat{\mathcal{B}}$ is a λ-sheaf if and only if for each $\langle \psi : \mathbf{Y} \to \mathbf{X} \rangle \in \Psi$, the induced map ψ^* is an isomorphism:

$$\psi^* : Hom_{\hat{\mathcal{B}}}(\mathbf{X}, \Delta) \xrightarrow{\cong} Hom_{\hat{\mathcal{B}}}(\mathbf{Y}, \Delta)$$

ii Any map $\langle \psi : \mathbf{Y} \to \mathbf{X} \rangle \in \widehat{\mathcal{B}}_1$ belongs to Ψ if and only if for each λ-sheaf Δ in $\mathbf{Sh}[\mathcal{B}]$, the induced map ψ^* is an isomorphism:

$$\psi^* : Hom_{\widehat{B}}(\mathbf{X}, \Delta) \xrightarrow{\cong} Hom_{\widehat{B}}(\mathbf{Y}, \Delta)$$

Taking into account the original definition of a λ-sheaf, in conjunction with propositions [i] and [ii] above, we conclude that the set Ψ is *completely determined* by its *restriction* to λ-*dense* subfunctors of $\mathbf{y}[B]$. In other words, Ψ is completely determined by its restriction to covering B-sieves, whereas the operator-functor $\mathbf{L}$ expresses the process of λ-*closure*, or equivalently, the process of *closure* with respect to a *spatial covering* schema Λ. From this perspective, the role of the localization operator λ is precisely incorporated in the complete determination of the set Ψ by its restriction to the set of covering B-sieves for each locus B in $\mathcal{B}$. Furthermore, for any monic arrow $\psi : \mathbf{Y} \infty \mathbf{X}$ in $\mathbf{\Psi}$, we say that, the subobject $\mathbf{Y}$ is λ-dense in $\mathbf{X}$.

From the property of Λ-closure, referring to Λ-closed B-sieves, as an operation preserved by pulling-back, we conclude that the operator-functor $\mathbf{L}$ should *commute* with pull-backs. Thus, if we recall that Λ-closed B-sieves, for each locus B in $\mathcal{B}$, stand for local time-forcing B-sieves, or equivalently local time-forcing operators, we conclude that:

The λ-*sheaf reflection* functor $\mathbf{L} : \widehat{\mathcal{B}} \to \mathbf{Sh}[\mathcal{B}]$, which is *left adjoint* to the *inclusion* functor $I : \mathbf{Sh}[\mathcal{B}] \infty \widehat{\mathcal{B}}$, and also *commutes* with pull-back operations, expresses precisely the functioning of *Time* as a process of λ-*closure*, which is to say a process of *temporal percolation* with respect to a *spatial covering* schema Λ on $\mathcal{B}$. Again, this is enacted by the generation of *local-time forcing* B-sieves, for each locus B in $\mathcal{B}$, qualified as *complete local time-frames* if and only if they are also id_B-covering.

In this sense, the process of λ-*closure* with respect to a *spatial covering* scheme Λ, for every *subobject* $\Delta : \mathbf{\Delta} \infty \mathbf{E}$, and in particular, for every B-sieve $R : \mathbf{R} \infty \mathbf{y}[B]$, is simply expressed by the following *pullback* diagram:

$$\begin{array}{ccc} [\mathbf{\Delta}] & \longrightarrow & \mathbf{L\Delta} \\ \Big\downarrow{\scriptstyle [\Delta]} & & \Big\downarrow{\scriptstyle \mathbf{L}\Delta} \\ \mathbf{E} & \longrightarrow & \mathbf{LE} \end{array}$$

We notice that since L *commutes* with pull-back operations, and thus preserves subobject inclusions, we obtain for each subfunctor $\Delta : \mathbf{\Delta} \infty \mathbf{E}$, a new subfunctor $[\Delta] : [\mathbf{\Delta}] \infty \mathbf{E}$, identified as the λ*-closure* of Δ, such that $[\Delta]$ contains Δ. In case that E is itself a λ-sheaf, then a subfunctor of E is also a λ-sheaf if and only if it is λ*-closed.* As a consequence, we reach the important conclusion that *local time-forcing* B-sieves, for each locus B in $\mathcal{B}$, are λ*-sheaves* themselves.

Let us now consider the *subobject classifier* of λ *-closed* subobjects, that is, the functor expressing the *sheaf-theoretic localization* of "*aletheia*" $\mathbf{\Omega}^{\Lambda}$. We recall that for each functor E in $\mathit{Sets}^{\mathcal{B}^{op}}$, the following natural bijection pertains:

$$Hom_{\widehat{\mathcal{B}}}(\mathbf{E}, \mathbf{\Omega}^{\Lambda}) \cong [\mathbf{\Theta}_{\widehat{\mathcal{B}}}(\mathbf{E})]_{\lambda}$$

where, $[\mathbf{\Theta}_{\widehat{\mathcal{B}}}(\mathbf{E})]_{\lambda}$ denotes the set of λ-closed subfunctors of E, according to the pullback diagram:

$$\begin{array}{ccc} [\mathbf{\Delta}] & \xrightarrow{\ !\ } & \mathbf{1} \\ \Big\downarrow{\scriptstyle [\Delta]} & & \Big\downarrow{\scriptstyle True^{\Lambda}} \\ \mathbf{E} & \xrightarrow{\phi_{[\Delta]}} & \mathbf{\Omega}^{\Lambda} \end{array}$$

It is conceptually clear from the discussion above that the *functor* $\mathbf{\Omega}^{\Lambda}$ is actually a λ*-sheaf*, which remarkably plays the role of the *subobject classifier* in the *topos* of λ*-sheaves*, such that the *natural bijection* above takes the following form, if *restricted* to the subcategory $\mathbf{Sh}[\mathcal{B}]$:

$$Hom_{\mathbf{Sh}[\mathcal{B}]}(\mathbf{E}, \mathbf{\Omega}^{\Lambda}) \cong \mathbf{\Theta}_{\mathbf{Sh}[\mathcal{B}]}(\mathbf{E})$$

where, $\Theta_{\mathbf{Sh}[\mathcal{B}]}(\mathrm{E})$ denotes the set of λ-subsheaves of the λ-sheaf E.

11.9 TEMPORAL GAUGES: *ALETHEIA* IN THE REFLECTION OF SHEAVES

We shall provide a simple argument, which actually proves that $\mathbf{\Omega}^{\Lambda}$ is a λ-*sheaf*, and also that it is the *subobject classifier* in $\mathbf{Sh}[\mathcal{B}]$. From the natural bijection referring to λ-closed subobjects of a functor E in $\widehat{\mathcal{B}}$ it is evident that if E is a λ-sheaf, then the natural transformations $\phi_{\Gamma}:\mathbf{E}\to\mathbf{\Omega}^{\Lambda}$ correspond to the subfunctors $\mathbf{\Gamma}$ which are also λ-sheaves, if and only if $\mathbf{\Omega}^{\Lambda}$ is a λ-sheaf itself.

For this purpose, we consider an arrow $\langle\xi:\Xi\to\mathbf{\Pi}\rangle\in\Psi$, i.e. a natural transformation $\langle\xi:\Xi\to\mathbf{\Pi}\rangle\in\widehat{\mathcal{B}}_1$, qualified as an isomorphism $\mathbf{L}\xi:\mathbf{L}\Xi\xrightarrow{\cong}\mathbf{L}\mathbf{\Pi}$, when L acts upon it. Then, the presheaf $\mathbf{\Omega}^{\Lambda}$ in $\widehat{\mathcal{B}}$ is a λ-sheaf, if we show that for $\langle\xi:\Xi\to\mathbf{\Pi}\rangle\in\Psi$ the induced map ξ^* is an isomorphism:

$$\xi^*:Hom_{\widehat{\mathcal{B}}}(\mathbf{\Pi},\mathbf{\Omega}^{\Lambda})\xrightarrow{\cong}Hom_{\widehat{\mathcal{B}}}(\Xi,\mathbf{\Omega}^{\Lambda})$$

Equivalently, if we use the natural bijection characterizing $\mathbf{\Omega}^{\Lambda}$ as the subobject classifier of λ-closed subobjects, and also further consider λ-closed subobjects $[\mathbf{\Sigma}]$ and $[\mathbf{\Upsilon}]$ of Ξ and $\mathbf{\Pi}$ respectively, i.e. $[\mathbf{\Sigma}]\propto\Xi$; $[\mathbf{\Upsilon}]\propto\mathbf{\Pi}$, it will be enough to show that there exists a surjective and injective correspondence between them, concluding thereby that $\mathbf{\Omega}^{\Lambda}$ is actually a λ-sheaf. We consider the composition of pullback diagrams:

$$\begin{tikzcd}
{[\mathbf{\Sigma}]} \arrow[r] \arrow[d, "{[\sigma]}"] & {[\mathbf{\Upsilon}]} \arrow[r] \arrow[d, "{[\upsilon]}"] & \mathbf{L}[\mathbf{\Sigma}]\cong\mathbf{L}[\mathbf{\Upsilon}] \arrow[d, "{\mathbf{L}[\sigma]}"] \\
\Xi \arrow[r] & \mathbf{\Pi} \arrow[r] & \mathbf{L}\Xi\cong\mathbf{L}\mathbf{\Pi}
\end{tikzcd}$$

If $[\upsilon]:[\mathbf{\Upsilon}]\propto\mathbf{\Pi}$ is a λ-closed subfunctor of $\mathbf{\Pi}$, then upon restriction to Ξ along the arrow $\langle\xi:\Xi\to\mathbf{\Pi}\rangle\in\Psi$, it gives a λ-closed subfunctor of Ξ, that is, $[\sigma]:[\mathbf{\Sigma}]\propto\Xi$. Conversely, if $[\sigma]:[\mathbf{\Sigma}]\propto\Xi$ is a λ-closed subfunctor of Ξ, then, by pulling-back $\mathbf{L}[\sigma]$ along the arrow $\langle \mathbf{\Pi} \to \mathbf{L\Pi} \cong \mathbf{L\Xi} \rangle \in \Psi$, we obtain a λ-closed subfunctor of $\mathbf{\Pi}$, i.e. $[\upsilon]:[\mathbf{\Upsilon}] \infty\ \mathbf{\Pi}$. This completes the proof of the argument.

After having established that the *functor* $\mathbf{\Omega}^{\Lambda}$ is a λ*-sheaf*, and also operates as the *subobject classifier* in the category of λ-sheaves $\mathbf{Sh}[\mathcal{B}]$, it is instructive to consider the following *pullback* diagram, as reflected in $\mathbf{Sh}[\mathcal{B}]$:

$$\begin{array}{ccc} [\mathbf{S}] & \xrightarrow{\ !\ } & \mathbf{1} \\ {\scriptstyle [S]}\downarrow & & \downarrow{\scriptstyle True^{\Lambda}} \\ \mathbf{y}[B] & \xrightarrow{\ \phi_{[S]}\ } & \mathbf{\Omega}^{\Lambda} \end{array}$$

where, given a *spatial covering* schema Λ on $\mathcal{B}$, $[S]$ is a *local time-forcing* B-sieve, classified in the category of λ-sheaves by the *characteristic arrow* $\phi_{[S]}:\mathbf{y}[B] \to \mathbf{\Omega}^{\Lambda}$. Note that the classifier object $\mathbf{\Omega}^{\Lambda}$ comprehends *only* the *closed subobjects* $[S]$ of $\mathbf{y}[B]$, identified as local time-forcing B-sieves, actually being λ*-sheaves* themselves, and consequently characterizes them in terms of *truth values*.

Thus, if S is *any* B-sieve, $\mathbf{\Omega}^{\Lambda}$ perceives and classifies *only* its λ-closure $[S]$, given a covering scheme Λ, as this is precisely *reflected* in $\mathbf{Sh}[\mathcal{B}]$ by the action of the *left adjoint* operator $\mathbf{L}$, providing a faithful manifestation of the temporal percolation process, that is, of the process of λ-closure. For reasons of clarity, we recall that from any given B-sieve S we can construct a corresponding λ-closed B-sieve, denoted by $[S]$, simply as follows:

$$[S] = \{f \,/\, cod(f) = B, S \rhd^{\Lambda} f\}$$

where f is any $\mathcal{B}$-arrow with codomain B, and the notation $S \rhd^{\Lambda} f$ denotes that S covers f according to a covering scheme Λ. Note that $[S]$ is the smallest closed sieve that contains S, called accordingly the λ-closure of S.

Let us examine, precisely what is expressed by the *maximal truth value* in the universe of λ-sheaves, where "*aletheia*" is localized. We notice that, given a B-sieve S, if it has the property of being λ-dense, then $[S]=t_B$; in other words its λ-closure is assigned the maximal truth value in $\mathbf{\Omega}^{\Lambda}(B)$. Of course, this is the case if S is an id_B-covering B-sieve, or simply, a spatial covering B-sieve, specified according to a spatial covering schema Λ. Stated equivalently, in the reflection of the subcategory of λ-sheaves, the B-sieve S is perceived as being the maximal B-sieve t_B through its λ-closure, and most importantly, this is the case if and only if S is a spatial covering B-sieve.

Thus, the maximal truth value t_B in $\mathbf{\Omega}^{\Lambda}(B)$, i.e. the maximal λ-closed B-sieve t_B, interpreted as a truth value, for each locus B in $\mathcal{B}$, expresses the fact that it is $\langle \lambda - True \rangle$, hence $\langle [true] \rangle^{\lambda}{}_B \equiv t_B$ in the λ-sheaves reflection, that S is all of $\mathbf{y}[B]$, if and only if S is a spatial covering B-sieve. Consequently, the truth-values object $\mathbf{\Omega}^{\Lambda}(B)$ in $\mathbf{Sh}[\mathcal{B}]$, for each locus B in $\mathcal{B}$, perceives every spatial covering B-sieve, as the maximal λ-closed B-sieve t_B.

We bear in mind now that the maximal λ-closed B-sieve, as the maximal id_B-covering local time-forcing B-sieve, forces the interpretation of the locus B as a complete local time-frame, where a complete local description of reality is feasible by means of its identification with the maximal truth value in $\mathbf{\Omega}^{\Lambda}(B)$. Thus, we form the following conclusion:

In the "*aletheia*" localization-environment of the category of *sheaves* $\mathbf{Sh}[\mathcal{B}]$, we can substantiate a *complete local description* of reality, formulated in terms of *every spatial covering* B-sieve Σ of a spatial covering schema Λ, for each locus B in $\mathcal{B}$, qualified by the temporal λ-closure process of local-time forcing as a *complete local time-frame*. This is simply expressed by the equation:

$$\lambda \circ \phi_{\Sigma} = \langle [true] \rangle^{\lambda}$$

We also recall that the maximal λ-closed, id_B-covering B-sieve t_B, can equivalently be thought of, in operator form, as the maximal local time-operator $\widehat{T}_{t_B}$, which, by *acting* on the locus B *forces* the interpretation of B as a complete local time-frame. Moreover, every other λ-closed, h-covering B-sieve $[\Gamma]$, that is, time-forcing B-sieve covering durations $h: dom(h) \rightarrow B$, acts as a local-time operator $\widehat{T}_{[\Gamma]}$ on the locus B, identifying h in $[\Gamma]$, with irreducible durations of the so-forced time-domain B, according to the "eigenvalues" characteristic equation:

$$\widehat{T}_{[\Gamma]} \mid B\rangle = h \mid B\rangle$$

where, $h \in [\Gamma]$ are the generalized irreducible durations of the local time domain B, covered by Γ, or equivalently, forced by $\widehat{T}_{[\Gamma]}$. Moreover, each locus $dom(h)$ is thought of as a hole in the B-sieve $[\Gamma]$, which specifies the kind of generalized durations of a local time domain covered by Γ, under $dom(h)$-varying temporal resolution units, with respect to the relation of temporal extension between loci. Obviously, $\widehat{T}_{t_B}$, corresponding to the maximal B-sieve t_B, possesses the maximal set of "eigenvalues" h, since it is covering the id_B, being themselves elements of t_B, and thus, durations of the forced local time-frame B.

In this setting, it must be stressed that the notion of a *hole*, or a temporal resolution unit, of a covering sieve is the denotator of a relation of *simultaneity*. This is an important notion that will permit us to understand precisely the meaning of all the truth values contained in $\Omega^{\Lambda}(B)$. First of all, note that the notion of simultaneity with respect to a complete local time-domain, $id_B : B \rightarrow B$, refers to B temporally as a simultaneous *totality*, since the maximal temporal resolution unit or hole of the time-forcing B-sieve covering id_B, is obviously extended to level B.

In this sense, concerning the relation of simultaneity, a complete local time-domain B is completely characterized by its *maximal* temporal resolution unit, which is B itself. Hence, in the corresponding local time-frame B, a spatial covering B-sieve, that is an id_B-covering B sieve, incorporating a complete description of reality in terms of the *maximal truth value* in $\Omega^{\Lambda}(B)$, is perceived as a *simultaneously*-existing object in its λ-*closure* t_B. This is precisely what we mean by characterizing it as a λ-*dense* object in its λ-closure. Thus, the maximal truth value in $\Omega^{\Lambda}(B)$ encapsulates precisely the fact that an id-B

covering B sieve is a simultaneity in its λ- closure - forced complete local time-frame B by means of being λ-dense in its λ-closure.

As a result, we establish a threefold association in the reflections' topos of λ-sheaves for each locus B in $\mathcal{B}$: the maximal truth value in $\mathbf{\Omega}^{\Lambda}(B)$ is associated with a id_B-covering B-sieve Σ as a simultaneous object in its λ-closure; Σ is associated in its λ-closure as a simultaneity with the maximal hole B of Σ; the maximal hole B of the id_B-covering B-sieve Σ is associated with the establishment of a complete local time-frame B in the λ-closure of Σ.

Of course, there exist B-sieves Φ that are *not necessarily* id_B-covering. For example, we may consider a B-sieve that spatially covers the arrow $h: C \to B$, meaning that $h^*(\Phi) \in \Lambda(C)$ for a spatial covering schema Λ on $\mathcal{B}$. In the reflection of λ-sheaves it shows up as the corresponding λ-closed B-sieve $[\Phi]$ covering h, which consequently is interpreted as a generalized $\langle$ $dom(h)$-duration $\rangle$, denoted by $\langle$ $h_{dom(h)}$ $\rangle$ at temporal resolution unit $dom(h) = C$. By saying that B is a temporal extension of C, we fix the maximal temporal resolution unit of Φ at level C, and associate the hole C with the denotation of simultaneity at level C. Parenthetically, note that a generalized duration at temporal resolution unit B is a complete local time-domain B, and the term $\langle$ B-duration $\rangle$ expresses precisely the signified simultaneity at maximal hole B, which is denoted by $\langle [true] \rangle^{\lambda}{}_{B}$.

From the perspective of the complete local time-frame B, the B-sieve Φ, comprehended through its λ-closure $[\Phi]$, is assigned a *truth value* $\langle [true] \rangle^{\lambda}{}_{B\downarrow C}$, which means that it expresses a simultaneity at maximum temporal resolution unit equal to the *hole* C. Hence, from the viewpoint of a complete local time-frame B the B-sieve Φ *spatially* covering $h: C \to B$, where C stands for the maximal hole of Φ, provides a *partial* description of reality up to simultaneity level, specified by the maximal hole $dom(h) = C$. Of course, from the perspective of the complete local time-frame C, the restriction of the B-sieve Φ to C, since it is id_C-covering by means of $h^*(\Phi) \in \Lambda(C)$, for $h: C \to B$, it provides a complete description of reality as perceived through its λ-

closure, denoted by the maximal truth value $\langle [true] \rangle^{\lambda}{}_{C}$, signifying simultaneity at maximal hole C. Thus, we form the conclusion:

The object of "*aletheia*", identified as the *truth-values* object in the universe of sheaves incorporates the precise means of *localization* of "aletheia", regarding *local* descriptions of reality with respect to *local time-frames*, on the basis of *simultaneity* relations, whose maximal extent is signified by the maximal *holes* of *spatial covering* sieves, perceived through their *closure*.

Finally, in order to emphasize the meaning of *simultaneity* associated with the maximal hole $\langle dom(h) \rangle_{max}$ of a local time-forcing B-sieve $[\Gamma]$ covering durations $h : dom(h) \to B$, we define for every local time operator $\widehat{T}_{[\Gamma]}$ on the locus B, identifying h in $[\Gamma]$, with a duration of the so-forced time-domain B, an associated local time-operator $\widehat{T_{s[\Gamma]}}$ of the maximal extent of simultaneity relations incorporated in the action of $\widehat{T}_{[\Gamma]}$ as follows:

$$\widehat{T_{s[\Gamma]}} \mid B \rangle = \langle dom(h) \rangle_{max} \mid B \rangle$$

where, $\langle dom(h) \rangle_{max}$ denotes the maximal temporal resolution unit of the local time-operator $\widehat{T}_{[\Gamma]}$.

The development of the ideas regarding the *gnomonic* and *functorial* conceptualization of "*aletheia*", as a process of temporal percolation, which can be localized sheaf-theoretically with respect to a variety of local time-frames, together with their logical relational descriptive rules, inspires the following claims:

We claim that neglecting the relational functioning of local time-frames in nature, together with their respective simultaneity relations and truth values assignments, is a source of paradoxes that subsequently generate defective epicyclic interpretations of various forms. It is enough to point out that reducing the temporal closure process to simultaneity relations at point-durations $1 \to B$, thus accepting only the existence of point time-frames in nature, or equivalently only instantaneous spaces through which motion can be qualified, is the source of serious conceptual and technical problems in attempts to reconcile classical theories, including general relativity, with quantum theories. From this standpoint, we emphasize that spatial covering, as well as time-forcing schemata, have been developed both for the purpose of explicating, and qualifying in logical terms, the process of localization of physical continuum events in relational spatiotemporal terms within a suitably specified categorical environment.

The further claim we wish to state concerns the ontology of localized events. We assert that it is precisely the local time-frame of simultaneity, or equivalently, the maximal hole of a local time-forcing covering sieve on a locus, forcing its former temporal interpretation, which determines the ontology of individuated events from the continuum by means of their localization over the locus. In this sense, continuum events are individuated as simultaneity-determined entities in the descriptive terms of local time-frames, where the maximal extent of simultaneity relations involved in the action of a time-forcing sieve $[\Gamma]$ on a locus B is determined by the maximal hole-"eigenvalue" of the corresponding local time-operator $\widehat{T}_{s[\Gamma]}$ acting on B. The ontology of individuated events, in this sense, is specified exactly by the nature of the maximal hole in the associated covering sieve.

Thus, localized events are not restricted in any way to point-events. The logical rules used for the understanding of the signified simultaneity-relations should comply with the sheaf-theoretic localization of truth-values assignments, expressing their spectral classification in local time-frames of corresponding temporal resolution units, and should never be reduced uncritically and exclusively to the descriptive terms of point time-frames. Otherwise, a variety of paradoxes of mixed ontological and logical inconsistencies arise, precisely as the net-effect of the reduction of the temporal percolation process to time-frames of merely point-durations.

11.10 NATURAL SPECTRAL SPATIOTEMPORAL OBSERVATION IN A TOPOS

Up to present, we have determined all the necessary spatial and temporal concepts needed for the individuation of observable events in the physical continuum through the localizing environment of a category of sheaves. Individuated events from the physical continuum become comprehensible through spatiotemporal observation that respects the norms of closed-sieve temporal percolation taking place over loci that have been qualified as local time-frames. In this manner, if we try to enunciate the means of functorial spatiotemporal localization in precise *spectral* terms, we can legitimately identify the above *loci* with the *spectra* of commutative observable algebras sheaf-theoretically.

From this perspective, spatiotemporal *observation* is the process that detects, and subsequently, organizes the set of generalized points of a spatial h-covering B-sieve R on a locus B, corresponding to *events* localized over that locus, by means of an A_B*-co-sieve* $\circlearrowright$ consisting of *local* ς*-linear epimorphic* representations $A_B \to A_C$ of commutative, associative and unital algebras A_C, for $h: C \to B$, defined over an *algebraic number field* ς, whose elements are identified as *observables* taking co-final values in ς, as follows: in every A_B-co-sieve $\circlearrowright$ a local epimorphisms of ς-algebras $A_B \to A_C$, corresponds to an A_C-state of A_B, interpreted as the A_C-state of an observable in A_B at duration $h: C \to B$ of the corresponding local time-frame B, forced by the temporal closure $[R]$ of R.

We call each commutative, associative and unital algebra without zero divisors, defined over an algebraic field ς, contained in an A_B-co-sieve $\circlearrowright$, a *commutative* ς*-arithmetic* effectuating *observation* of events at duration $h: C \to B$ of a local time-frame B, or equivalently, at temporal resolution unit $dom(h)$ of the local time operator $\widehat{T}_{[R]}$ acting on the locus B. Without loss of generality, we may assume that the field ς is identical with the field R of the real numbers, or its algebraic closure C.

Note that if the spatial B-sieve is id_B-covering, and consequently, the corresponding time-forcing B-sieve is the *maximal* λ*-closed* B-sieve, making B a complete local time-frame, then the A_B-co-sieve $\circlearrowright$ contains the identity R-linear representation $id_{A_B}: A_B \to A_B$. Thus, a complete local time-frame B is the *simultaneity* locus, or id_B-duration of A_B-evaluated observables in A_B. Furthermore, the R-states of observables in A_B, that is, the R-linear representations of A_B into the R-algebra R, namely $A_B \to \mathrm{R}$, correspond to *states* at point-durations $1 \to B$ of a local time-frame B, identified with spectrally observable point-figures or *spatial* 1-points of the corresponding *instantaneous* spaces 1_m at 1*-durations* $m: 1 \to B$. We call the set of all observable point-figures at all 1-durations of a local time-frame B, the R*-spectrum* of the commutative R-arithmetic A_B, meaning the set of individuated events at temporal resolution unit 1, which are precisely detectable by means of *evaluations* of observables into the R-algebra R.

In a completely analogous fashion, we define the A_C*-spectrum* of the commutative R-arithmetic A_B, where A_C is a commutative arithmetic in $\circlearrowright$, as the set of individuated events at *temporal resolution* unit C, that are detectable as C-*figures* at *durations* $h:C \to B$, by means of *evaluations* of observables belonging to A_B into the R-algebra A_C.

In general algebraic terms, all A_C-states of the commutative R-arithmetic A_B, where $A_B \to A_C$ are local R-linear epimorphic representations of commutative R-arithmetics in the A_B-co-sieve $\mathcal{O}$, are in *bijective correspondence* with *prime ideals* of A_B, as follows:

$$\langle H : A_B \to A_C \rangle \xrightarrow{\cong} Ker(H)$$

The set of all prime ideals of the commutative arithmetic A_B, specified as above, constitutes the *prime spectrum* of A_B. Thus, the prime spectrum of A_B consists of all individuated events under varying temporal resolution units of the local time-frame B, being detectable as figures at the corresponding durations of B. We recall that the *maximal* spectrum of any commutative ς-arithmetic A_B is the set of *maximal ideals* of A_B, where an ideal α is maximal if and only if A_B / α is an algebraic field.

It is clear that spatiotemporal observation, understood as a process by means of which individuated events become *spectrally* detectable through evaluations of observables in local commutative arithmetics, constitutes a dual or opposite categorical perspective to the one corresponding to the localization of events in relational spatial and temporal terms. We may say that it constitutes the *algebraic encoding* of the information encapsulated in the spatial and temporal covering schemata, which in turn can be characterized accordingly as *geometrical.*

In this sense, taking into account the *duality* between sieves and co-sieves, spatiotemporal observation is categorically equivalent to *co-localization* of the *covariant* functor $\mathcal{O}$, which is by specification a *closed dense subobject* of $Hom_{\mathcal{A}}(A_B,-) := \bar{\mathbf{y}}[A_B]$, namely a $\bar{\lambda}$-co-sheaf locally isomorphic to the covariant representable functor $\bar{\mathbf{y}}[A_B]$ in the category $\bar{\mathbf{Sh}}[\mathcal{A}]$. Most importantly, from the *sheaf-theoretic* perspective the elements of the A_B-co-sieve $\mathcal{O}$, that is the observables, are identified with *local sections* of this $\bar{\lambda}$-co-sheaf $\mathcal{O}$. We may refer to all relevant functors as sheaves, under the condition that the distinction of the algebraic from the geometrical perspective becomes clear from the covariance or contravariance respectively of these functors.

In conclusion, natural spatiotemporal observation is essentially the algebraic manifestation of the temporal percolation process with respect to a spatial covering schema, or equivalently, the algebraic transcription of the action of local time-operators on loci B in a categorical environment $\mathcal{B}$ by means of information encoding referring to event-figures at durations $h: C \to B$ of local time-frames B, in terms of corresponding local commutative arithmetic evaluations of observables belonging to A_B into A_C, which take place at respective durations h of B.

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