Current Natural Sciences

Yuming QIN

1D Radiative Fluid and Liquid Crystal Equations

SCIENCE PRESS

edp sciences

Printed in France

EDP Sciences – ISBN(print): 978-2-7598-2903-3 – ISBN(ebook): 978-2-7598-2904-0
DOI: 10.1051/978-2-7598-2903-3

The printed edition is not for sale in Chinese mainland. Customers in Chinese mainland please order the print book from Science Press. ISBN of the China edition: Science Press 978-7-03-072137-2

In memory of my father, Zhenrong QIN and my mother, Xilan XIA

To my wife, Yu YIN, my son, Jia QIN

To my elder sister, Yujuan QIN, younger brother, Yuxing QIN and younger sister, Yuzhou QIN

Contents

Foreword

In this book, we shall present some recent results on the global well-posedness of strong solutions to 1D radiative fluid equations and liquid crystal equations. Most of the contents of this book are based on the research carried out by the authors and their collaborators in recent years, which have been previously published only in original papers; but some contents of the book have never been published until now.

There are four chapters in this book.

Chapter 1 will recall some basic properties of Sobolev spaces, some differential integral inequalities in analysis, some of which will be used in the subsequent chapters.

In chapter 2, we shall study one-dimensional compressible infrarelativistic radiation equations and further prove the global existence and the large-time behavior of solutions to this system. Novelties of this chapter are: (1) Using a suitable expression of specific volume and the delicate priori estimates, we establish the positively lower bound and upper bound of the specific volume. (2) Using the embedding theorems and the delicate interpolation inequalities, we have overcome some mathematical difficulties caused by the higher order of partial derivatives to prove the global well-posedness of solutions in higher regular spaces. It is a remarkable fact that the difficulties we encounter in chapter 2 are how to deal with the radiative term, which makes the analysis in this book different from those in Qin [104], where the author studied some models without the radiative term.

Chapters 3 and 4 will study one-dimensional compressible liquid crystal fluid equations. In chapter 3, we shall establish the existence of global solutions in H^i $(i = 1, 2, 4)$ in Lagrangian coordinates. In chapter 4, we shall first establish the large-time behavior of solutions to one-dimensional compressible liquid crystal fluid equations. The novelty in this chapter is that using a suitable expression of the specific volume, we shall establish uniform bound of the specific volume by the embedding theorems and a sequence of delicate interpolation techniques and then prove the long-time behavior of solutions to the system using the Shen–Zheng inequality.

DOI: 10.1051/978-2-7598-2903-3.c901

For the contents of chapter 1, we refer the reader to [1, 2, 5–8, 37, 38, 40, 41, 45, 55, 75, 76, 95, 96, 105–108, 135, 137, 138, 141, 148, 149, 155]. For the theory of radiation hydrodynamical equations, we refer the reader to the monographs [12, 94, 99, 100] and [10, 17–21, 44, 59, 60, 74, 79, 80, 89, 111, 112, 116, 117, 127, 128]. For the theory of equations of liquid crystal, we refer the reader to [9, 11, 16, 23–25, 56, 57, 77, 78, 81–83, 85–88, 110, 118, 124, 143, 144]. Since the compressible Navier–Stokes equations are closely related to the radiation hydrodynamical equations and the liquid crystal equations under consideration of this book, we also refer the reader to related references of the compressible Navier–Stokes equations [3, 4, 13–15, 22, 26–36, 39, 42, 43, 46–50, 52–54, 58, 61–73, 84, 91–93, 97, 98, 102–105, 113, 114, 119–123, 125, 126, 129, 130–134, 136, 139, 140, 142, 146, 147, 150–154, 156], some techniques of which can be used to deal with our problems in this book.

We sincerely wish that the reader will learn the essential ideas, basic theories and methods in deriving the global existence, asymptotic behavior and regularity of solutions for the systems considered in this book. We also wish that the reader can undertake the further research of these systems after having read this book. This book was financially supported in part by the NNSF of China with contract number 12171082, the Fundamental Research Funds for the Central Universities with contract number 2232022G-13 and by the Graduate Course (Textbook) Construction Project of Donghua University.

We also take this opportunity to thank all the people who were once concerned about me.

Last but not least, Yuming QIN hopes to express his deepest thanks to his parents (Zhenrong QIN and Xilan XIA), sisters (Yujuan QIN and Yuzhou QIN), brother (Yuxing QIN), wife (Yu YIN) and son (Jia QIN) for their great help, constant concern and advice in his career.

Professor Yuming QIN
Department of Mathematics
College of Science
Institute for Nonlinear Sciences
Research Institute of Science and Technology
Donghua University
Shanghai 201620, P. R. China
yuming_qin@hotmail.com, yuming@dhu.edu.cn

Chapter 1

Preliminary

This chapter will introduce some basic results, most of which will be used in the following chapters. First we shall recall some basic inequalities whose detailed proofs can be found in the related literature, see, *e.g.*, Adams [1, 2], Friedman [37, 38], Gagliardo [40, 41], Nirenberg [95, 96], Yosida [148], etc.

1.1 Some Basic Inequalities

1.1.1 The Sobolev Inequalities

We shall first introduce some basic concepts of Sobolev spaces.

Definition 1.1.1. *Assume $\Omega \subseteq \mathbb{R}^n$ is a bounded or an unbounded domain with a smooth boundary Γ. For $1 \le p \le +\infty$ and m a non-negative integer, $W^{m,p}(\Omega)$ is defined to be the space of functions u in $L^p(\Omega)$ whose distribution derivatives of order up to m are also in $L^p(\Omega)$. That is,*

$$W^{m,p}(\Omega) = L^p(\Omega) \cap \{u : D^\alpha u \in L^p(\Omega), \ \ |\alpha| \le m\}.$$

The space $W^{m,p}(\Omega)$, called a Sobolev space, is equipped with a norm

$$\|u\|_{m,p,\Omega} = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha u|^p dx \right)^{1/p}, \ \text{ if } \ 1 \le p < +\infty, \ \text{ or} \tag{1.1.1}$$

$$\|u\|_{m,p,\Omega} = \max_{|\alpha| \le m} \operatorname*{esssup}_{x \in \Omega} |D^\alpha u(x)|, \ \text{ if } \ p = +\infty \tag{1.1.2}$$

DOI: 10.1051/978-2-7598-2903-3.c001

which is clearly equivalent to

$$\sum_{|\alpha|\le m} \|D^\alpha u\|_{p,\Omega}. \tag{1.1.3}$$

If $\Omega = \mathbb{R}^n$, we only denote

$$\|u\|_{m,p} = \|u\|_{m,p,\mathbb{R}^n}, \quad \|u\|_{0,p} = \|u\|_p.$$

$W^{m,p}(\Omega)$ is a Banach space. The space $W_0^{m,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ relative to the norm (1.1.3). Clearly,

$$W^{0,p}(\Omega) = L^p(\Omega)$$

with norm $\|\cdot\|_{0,p,\Omega} \equiv \|\cdot\|_{p,\Omega}$. For $p = 2$, $W^{m,2}(\Omega) \doteq H^m(\Omega)$, is a Hilbert space with respect to the scalar product

$$(u, v)_m = \sum_{|\alpha|\le m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}$$

with $(f, g)_{L^2(\Omega)} = \int_\Omega f\bar{g}dx$, here $\bar{g}$ is the conjugate function of g.

It is well-known that the Sobolev inequalities are important tools in the study of nonlinear evolutionary equations. First, we shall introduce these inequalities for functions in the space $W_0^{1,p}(\Omega)$.

Theorem 1.1.1 (The Sobolev Inequality). *Assume that* $\Omega\subseteq\mathbb{R}^n, n > 1$, *is an open domain. There exists a constant* $C = C(n, p) > 0$ *such that*

(1) if $n > p \geq 1$, *and* $u \in W_0^{1,p}(\Omega)$, *then* $u \in L^{p^*}(\Omega)$ *and*

$$\|u\|_{p^*,\Omega} \le \frac{p(n-1)}{2(n-p)\sqrt{n}} \|Du\|_{p,\Omega} \tag{1.1.4}$$

where $p^* = np/(n-p)$;

(2) if $p > n$ *and* Ω *is bounded, and* $u \in W_0^{1,p}(\Omega)$, *then* $u \in C(\overline{\Omega})$ *and*

$$\sup_\Omega |u| \le C|\Omega|^{\left(\frac{1}{n}-\frac{1}{p}\right)} \|Du\|_{p,\Omega}. \tag{1.1.5}$$

While, if $\Omega = \mathbb{R}^n$, *then*

$$\sup_{\mathbb{R}^n} |u| \le C\omega_n^{-\frac{1}{p}} \|u\|_{1,p,\mathbb{R}^n} \tag{1.1.6}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ *is the measure of the n-dimensional unit ball,* Γ *is the Euler gamma function and* $C = \max\left\{1, \left(\frac{p-1}{p-n}\right)^{(p-1)/p}\right\}$.

Remark 1.1.1. *The Sobolev inequality* (1.1.4) *does not hold for* $p = n$, $p^* = +\infty$.

(1.1.4) was first proved by Sobolev [138] in 1938. Sobolev [138] stated that the L^{p^*} norm of u can be estimated by $\|u\|_{1,p,\Omega}$ or $\|Du\|_{p,\Omega}$, the Sobolev norm of u. However, we can bound a higher L^p norm of u by exploiting higher order derivatives of u as shown in the next theorem which generalizes theorem 1.1.1 from $m = 1$, $p > n$ to $m \geq 1$ an integer.

Theorem 1.1.2. *Assume* $\Omega \subseteq \mathbb{R}^n$ *is an open domain. There exists a constant* $C = C(n, m, p) > 0$ *such that*

(1) if $mp < n$, $p \geq 1$, *and* $u \in W_0^{m,p}(\Omega)$, *then* $u \in L^{p^*}(\Omega)$ *and*

$$\|u\|_{p^*,\Omega} \leq C\|u\|_{m,p,\Omega} \tag{1.1.7}$$

where $p^* = \frac{np}{n-mp}$;
(2) if $mp > n$, *and* $u \in W_0^{m,p}(\Omega)$, *then* $u \in C(\overline{\Omega})$ *and*

$$\sup_{\Omega} |u| \leq C|K|^{\frac{1}{p}} \left[\sum_{|\alpha|=0}^{m-1} (diamK)^{|\alpha|} \frac{1}{\alpha!} \|D^\alpha u\|_{p,K} \right.$$
$$\left. + (diamK)^m \frac{1}{(m-1)!} (m - n/p)^{-1} \|D^m u\|_{p,K} \right] \tag{1.1.8}$$

where $K = suppu$, $C = C(m, p, n)$ *and* $diamK$ *is the diameter of* K.

Remark 1.1.2. *An important case considered in theorems 1.1.1 and 1.1.2 is* $\Omega = \mathbb{R}^n$. *In this situation,* $W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n)$ *and therefore the results of theorems 1.1.1 and 1.1.2 apply to* $W^{m,p}(\mathbb{R}^n)$.

For $p > n$, the results of theorems 1.1.1 and 1.1.2 imply the fact that u is bounded. Indeed, u is Höder continuous, which we shall state as follows.

Theorem 1.1.3. *If* $u \in W_0^{1,p}(\Omega), p > n$, *then* $u \in C^{0,\alpha}(\overline{\Omega})$ *where* $\alpha = 1 - n/p$.

Generally, the embedding theorems are closely related to the smoothness of the domain considered, which means that when we study the embedding theorems, we need some smoothness conditions for the domain. These conditions include that the domain Ω possesses the cone property, and it is a uniformly regular open set in $\mathbb{R}^n$, etc. For example, when $\Omega \in C^1$ or $\partial\Omega \in$ Lip, Ω has the cone property. Mathematically, we need to define the special meaning of the word "embedding" or "compact embedding".

Definition 1.1.2. *Assume* A *and* B *are two subsets of some function space. Set* A *is said to be embedded into* B *if and only if*

(1) $A \subseteq B$;
(2) the identity mapping I: $A \to B$ *is continuous, i.e., there exists a constant* $C > 0$ *such that for any* $x \in A$, *there holds that*

$$\|Ix\|_B \leq C\|x\|_A.$$

If A is embedded into B, then we simply denote by $A \hookrightarrow B$.
A is said to be compactly embedded into B if and only if

(1) *A is embedded into B;*
(2) *the identity mapping I:* $A \mapsto B$ *is a compact operator.*

If A is compactly embedded into B, then we simply denote by $A \hookrightarrow\hookrightarrow B$.

Now we draw some consequences from theorem 1.1.1. In fact, exploiting theorem 1.1.1, we have the following result which is an embedding theorem.

Corollary 1.1.1. *If* $u \in W_0^{1,p}(\Omega)$, *then* $u \in L^q(\Omega)$ *with* $p \le q \le \frac{np}{n-p}$ *if* $1 \le p < n$, *and* $p \le q < +\infty$ *if* $p = n$. *Moreover, if* $p > n$, *u coincides a.e. in* Ω *with a (uniquely determined) function of* $C(\overline{\Omega})$. *Finally, there holds that*

$$\|u\|_{q,\Omega} \le C\|u\|_{1,p,\Omega} \quad if \quad 1 \le p < n, p \le q \le \frac{np}{n-p}, \tag{1.1.9}$$

$$\|u\|_{q,\Omega} \le C\|u\|_{1,p,\Omega} \quad if \quad p = n, p \le q < +\infty, \tag{1.1.10}$$

$$\|u\|_{C} \le C\|u\|_{1,p,\Omega} \quad if \quad p > n, \tag{1.1.11}$$

where $C = C(n, p, q) > 0$ *is a constant.*

We can generalize corollary 1.1.1 to functions from $W_0^{m,p}(\Omega)$ which can be stated as the following embedding theorem.

Theorem 1.1.4. *Let* $u \in W_0^{m,p}(\Omega), p \ge 1, m \ge 0$. *Then*

(1) *if* $mp < n$, *then we have, for all* $q \in \left[p, \frac{np}{n-mp}\right]$,

$$W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \tag{1.1.12}$$

and there is a constant $C_1 > 0$ *depending only on m, p, q and n such that for all* $q \in \left[p, \frac{np}{n-mp}\right]$,

$$\|u\|_{q,\Omega} \le C_1\|u\|_{m,p,\Omega}; \tag{1.1.13}$$

(2) *if* $mp = n$, *then we have, for all* $q \in [p, +\infty)$,

$$W_0^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \tag{1.1.14}$$

and there is a constant $C_2 > 0$ *depending only on m, p, q and n such that for all* $q \in [p, +\infty)$,

$$\|u\|_{q,\Omega} \le C_2\|u\|_{m,p,\Omega}; \tag{1.1.15}$$

(3) if $mp > n$*, each* $u \in W_0^{m,p}(\Omega)$ *is equal a.e. in* Ω *to a unique function in* $C^k(\overline{\Omega})$*, for all* $k \in [0, m - n/p)$ *and there is a constant* $C_3 > 0$ *depending only on* m*,* p*,* q *and* n *such that*

$$\|u\|_{C^k} \leq C_3 \|u\|_{m,p,\Omega}. \tag{1.1.16}$$

Remark 1.1.3. *In case (2) of theorem 1.1.4, the following exception case holds for* $m = n$*,* $p = 1$*,* $q = +\infty$*:*

$$W^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega). \tag{1.1.17}$$

Now we give the following compact embedding theorem.

Theorem 1.1.5 (Embedding and Compact Embedding Theorem). *Assume that* Ω *is a bounded domain of class* C^m*. Then we have*

(i) If $mp < n$*, then* $W^{m,p}(\Omega)$ *is continuously embedded in* $L^{q^*}(\Omega)$ *with* $\frac{1}{q^*} = \frac{1}{p} - \frac{m}{n}$*:*

$$W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega). \tag{1.1.18}$$

In addition, the embedding is compact for any q*,* $1 \leq q < q^*$*.*
(ii) If $mp = n$*, then* $W^{m,p}(\Omega)$ *is continuously embedded in* $L^q(\Omega)$*,* $\forall q, 1 \leq q < +\infty$*:*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega). \tag{1.1.19}$$

In addition, the embedding is compact, $\forall q, 1 \leq q < +\infty$*. If* $p = 1$*,* $m = n$*, then the above still holds for* $q = +\infty$*.*
(iii) If $k + 1 > m - \frac{n}{p} > k, k \in \mathbb{N}$*, then writing* $m - \frac{n}{p} = k + \alpha, \alpha \in (0,1)$*,* $W^{m,p}(\Omega)$ *is continuously embedded in* $C^{k,\alpha}(\overline{\Omega})$*:*

$$W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\overline{\Omega}), \tag{1.1.20}$$

where $C^{k,\alpha}(\overline{\Omega})$ *is the space of functions in* $C^k(\overline{\Omega})$ *whose derivatives of order* k *are Hölder continuous with exponent* α*. Moreover, if* $n = m - k - 1$*, and* $\alpha = 1$*,* $p = 1$*, then (1.1.20) holds for* $\alpha = 1$*, and the embedding is compact from* $W^{m,p}(\Omega)$ *to* $C^{k,\beta}(\overline{\Omega})$*, for all* $0 \leq \beta < \alpha$*.*

1.1.2 The Interpolation Inequalities

In this subsection, we shall present the Gagliardo–Nirenberg interpolation inequalities (see, *e.g.*, Friedman [38] and Nirenberg [96]) which play a very important role in the theory of nonlinear evolutionary equations.

For $p > 0$, $|u|_{p,\Omega} = \|u\|_{L^p(\Omega)}$. For $p < 0$, set $-\frac{n}{p} = h + \alpha$ with $h = \left[-\frac{n}{p}\right]$ and $\alpha \in [0,1)$. Define

$$|u|_{p,\Omega} = \sup_{\Omega} |D^h u| \equiv \sum_{|\beta|=h} \sup_{\Omega} |D^\beta u|, \ \text{if} \ \alpha = 0,$$
$$|u|_{p,\Omega} = [D^h u]_{\alpha,\Omega} \equiv \sum_{|\beta|=h} \sup_{\Omega} [D^\beta u]_\alpha$$
$$\equiv \sum_{|\beta|=h} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}, \ \text{if} \ \alpha > 0.$$

If $\Omega = \mathbb{R}^n$, we simply write $|u|_p$ instead of $|u|_{p,\Omega}$.

Theorem 1.1.6 (The Gagliardo–Nirenberg Interpolation Inequalities). *Let j, m be any integers satisfying $0 \le j < m$, and let $1 \le q, r \le +\infty$, and $p \in \mathbb{R}$, $\frac{j}{m} \le \alpha \le 1$ such that*

$$\frac{1}{p} - \frac{j}{n} = \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + \frac{1-\alpha}{q}.$$

Then

(i) For any $u \in W^{m,r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, there is a positive constant $C = C(m, n, j, q, r, \alpha)$ such that

$$|D^j u|_p \le C |D^m u|_r^\alpha |u|_q^{1-\alpha} \tag{1.1.21}$$

with the following exception: if $1 < r < +\infty$ and $m - j - n/p$ is a non-negative integer, then (1.1.21) holds only for α satisfying $j/m \le \alpha < 1$.
(ii) For any $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$ where Ω is a bounded domain with smooth boundary, there are two positive constants C_1, C_2 such that

$$|D^j u|_{p,\Omega} \le C_1 |D^m u|_{r,\Omega}^\alpha |u|_{q,\Omega}^{1-\alpha} + C_2 |u|_{q,\Omega} \tag{1.1.22}$$

with the same exception as in (i).

In particular, for any $u \in W_0^{m,p}(\Omega) \cap L^q(\Omega)$, the constant C_2 in (1.1.22) can be taken as zero.

1.1.3 The Poincaré Inequality

In this subsection, we shall recall the Poincaré inequality in different forms.

Theorem 1.1.7. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in H_0^1(\Omega)$. Then there is a positive constant $C = C(\Omega, n)$ such that for all $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \le C \|\nabla u\|_{L^2(\Omega)}. \tag{1.1.23}$$

Theorem 1.1.8. *Let* $\Omega\subseteq\mathbb{R}^n$ *be a bounded domain of* C^1*. Then there is a positive constant* $C = C(\Omega,\ n)$ *such that for any* $u \in H^1(\Omega)$,

$$\|u-\bar{u}\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \tag{1.1.24}$$

where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx$ *is the integral average of* u *over* Ω*, and* $|\Omega|$ *is the volume of* Ω*.*

Theorem 1.1.9. *Under assumptions of theorem 1.1.8, for any* $u \in H^1(\Omega)$*, then*

$$\|u\|_{L^2(\Omega)} \le C\bigg(\|\nabla u\|_{L^2(\Omega)} + \bigg|\int_\Omega u\,dx\bigg|\bigg). \tag{1.1.25}$$

1.1.4 The Classical Bellman–Gronwall Inequality

In this subsection, we shall give the following classical Bellman–Gronwall inequality which plays an important role in the study of global well-posedness of solutions to evolutionary differential equations. For more details, we can refer to Bellman [5–8] and Gronwall [45].

Theorem 1.1.10 (The Classical Bellman–Gronwall Inequality). *If* $y(t)$ *and* $g(t)$ *are non-negative, continuous functions on* $0 \leq t \leq \tau$*, which satisfy the inequality*

$$y(t) \le \eta + \int_0^t g(s)y(s)\,ds, \quad 0 \le t \le \tau, \tag{1.1.26}$$

where η *is a non-negative constant, then for all* $0 \leq t \leq \tau$*,*

$$y(t) \le \eta \exp\bigg(\int_0^t g(s)\,ds\bigg). \tag{1.1.27}$$

Gronwall [45] first proved the special case of (1.1.26) with $g(t) =$ constant ≥ 0. Later on, Bellman [6] (see also Kuang [75]) extended this result to the form of theorem 1.1.10, which is a crucial tool in the analysis of differential equations. Until now, more and more improvements and generalizations of the classical Bellman–Gronwall inequality have been made. Specially, Bellman proved another inequality which can be stated as follows (see, *e.g.*, Kuang [75]).

Remark 1.1.4. *Let* $u(t)$*,* $b(t)$ *be continuous on* $(\alpha,\ \beta)$*, and* $b(t)$ *be non-negative. If for all* $t \geq t_0$*,* t_0*,* $t \in (\alpha,\ \beta)$*,*

$$u(t) \le u(t_0) + \int_{t_0}^t b(s)u(s)\,ds,$$

then for any $t \geq t_0$*,*

$$u(t_0)\exp\bigg(-\int_{t_0}^t b(s)u(s)\,ds\bigg) \le u(t) \le u(t_0)\exp\bigg(\int_{t_0}^t b(s)u(s)\,ds\bigg).$$

The above theorem gives bounds on solution of (1.1.26) in terms of the solution of a related linear integral equation

$$v(t) = \eta + \int_0^t g(s)v(s)\,ds \tag{1.1.28}$$

and is one of the basic tools in the theory of differential equations. Based on the basis of various motivations, we know that it has been extended and used considerably in various contexts. For instance, in the Picard–Cauchy type of iteration for establishing existence and uniqueness of solutions, this inequality and its various variants play a significant role. Inequalities of this type (1.1.26) are also encountered frequently in the perturbation and stability theory of differential equations.

1.1.5 The Generalized Bellman–Gronwall Inequalities

In this subsection, we shall review the following generalized Bellman–Gronwall inequalities which can be found in Qin [104, 106, 123, 128].

Theorem 1.1.11 (The Generalized Bellman–Gronwall Inequality). *Assume that $f(t)$, $g(t)$ and $y(t)$ are non-negative integrable functions in $[\tau, T]$ $(\tau < T)$ verifying the following integral inequality for all $t \in [\tau, T]$,*

$$y(t) \le g(t) + \int_\tau^t f(s)y(s)\,ds.$$

Then it holds, for all $t \in [\tau, T]$,

$$y(t) \le g(t) + \int_\tau^t \exp\left(\int_s^t f(\theta)\,d\theta\right) f(s)g(s)\,ds. \tag{1.1.29}$$

In addition, if $g(t)$ is a nondecreasing function in $[\tau, T]$, then, for all $t \in [\tau, T]$,

$$y(t) \le g(t)\left[1 + \int_\tau^t \exp\left(\int_s^t f(\theta)\,d\theta\right) f(s)\,ds\right] \tag{1.1.30}$$

$$\le g(t)\left[1 + \int_\tau^t f(s)\,ds \exp\left(\int_\tau^t f(\theta)\,d\theta\right)\right]. \tag{1.1.31}$$

If further $T = +\infty$ and $\int_\tau^{+\infty} f(s)\,ds < +\infty$, then

$$y(t) \le Cg(t) \tag{1.1.32}$$

where $C = 1 + \int_\tau^{+\infty} f(s)\,ds \exp(\int_\tau^{+\infty} f(\theta)\,d\theta)$ is a positive constant.

The next result is a corollary of theorem 1.1.11, it can be found in Racke [135].

Corollary 1.1.2. *Let $a > 0$, ϕ, $h \in C([0, a])$, $h \ge 0$ and $g : [0, a] \to \mathbb{R}$ is increasing. If for any $t \in [0, a]$,*

$$\phi(t) \le g(t) + \int_0^t h(s)\phi(s)\,ds, \tag{1.1.33}$$

then for all $t \in [0,\ a]$,

$$\phi(t) \le g(t)\exp\left(\int_0^t h(s)\,ds\right). \tag{1.1.34}$$

1.1.6 The Uniform Bellman–Gronwall Inequality

In this subsection, we shall introduce some uniform Gronwall inequalities which provide uniform bounds or decay rates. This type of integral inequalities plays a very crucial role in the study of the global existence and the large-time behavior of solutions to evolutionary equations.

We start with the following theorem which is cited in Temam [141].

Theorem 1.1.12 (The Uniform Bellman–Gronwall Inequality). *Assume that* $g(t)$, $h(t)$ *and* $y(t)$ *are three positive locally integrable functions on* $(t_0,\ +\infty)$ *such that* $y'(t)$ *is locally integrable on* $(t_0,\ +\infty)$ *and there holds that for all* $t \ge t_0$,

$$\frac{dy}{dt} \le gy + h,$$

$$\int_t^{t+r} g(s)\,ds \le a_1, \quad \int_t^{t+r} h(s)\,ds \le a_2, \quad \int_t^{t+r} y(s)\,ds \le a_3,$$

where r, $a_i (i = 1,\ 2,\ 3)$ *are positive constants. Then, for all* $t \ge t_0$,

$$y(t+r) \le \left(\frac{a_3}{r} + a_2\right)e^{a_1}.$$

Next, we shall introduce some uniform generalizations which may provide some large-time behavior of functions. This class of inequalities is a very powerful tool in establishing the large-time behavior of solution when we use the energy methods to study problems of partial differential equations.

We now give the familiar results in the classical calculus for the single real variable analysis.

Lemma 1.1.1. *(1) Assume* $y(t) \in L^1(0,\ +\infty)$ *with* $y(t) \ge 0$ *for a.e.* $t \ge 0$, $y'(t) \in L^1(0,\ +\infty)$. *Then*

$$\lim_{t\to+\infty} y(t) = 0.$$

(2) Assume $y(t) \in L^1(0,\ +\infty)$ *with* $y(t) \ge 0$ *for a.e.* $t \ge 0$, *and* $\lim_{t\to+\infty} y(t)$ *exists. Then*

$$\lim_{t\to+\infty} y(t) = 0.$$

(3) Assume $y(t)$ *is uniformly continuous on* $[0, +\infty)$, $y(t) \in L^1(0, +\infty)$. *Then*

$$\lim_{t \to +\infty} y(t) = 0.$$

(4) Assume $y(t)$ *is a monotone function on* $[0, +\infty)$ *and* $y(t) \in L^1(0, +\infty)$. *Then*

$$\lim_{t \to +\infty} y(t) = 0$$

and

$$y(t) = o(1/t) \quad as \quad t \to +\infty.$$

Obviously, the above lemma provides the asymptotic behavior of $y(t)$ for the large time.

The next theorem related to the uniform Gronwall inequality was first established by Shen and Zheng [137] in 1993 (see, *e.g.*, Zheng [155]) which is very useful and powerful in dealing with the global well-posedness and asymptotic behavior of solutions to some evolutionary partial differential equations. We shall apply it frequently in the subsequent context of this book (see, chapters 2–4).

Lemma 1.1.2 (The Shen–Zheng Inequality). *Assume* T *is an arbitrarily given constant with* $0 < T \leq +\infty$, *and* y *and* h *are non-negative continuous functions defined on* $[0, T]$ *and satisfy the following conditions*

$$\frac{dy}{dt} \leq A_1 y^2(t) + A_2 + h(t), \quad \textit{for all } t \geq 0, \tag{1.1.35}$$

$$\int_0^T y(s)\,ds \leq A_3, \quad \int_0^T h(s)\,ds \leq A_4, \quad \textit{for all } T > 0, \tag{1.1.36}$$

where A_1, A_2, A_3, A_4 *are given non-negative constants. Then for any* $r > 0$, *with* $0 < r < T$, *for all* $t \geq 0$,

$$y(t+r) \leq \left(\frac{A_3}{r} + A_2 r + A_4\right) \cdot e^{A_1 A_3}. \tag{1.1.37}$$

Furthermore, if $T = +\infty$, *then*

$$\lim_{t \to +\infty} y(t) = 0. \tag{1.1.38}$$

Proof. We can find the proof in [137]. However, for reader's convenience, we shall give the detailed proof. The proof is similar to that of the Uniform Gronwall Lemma (see lemma 1.1 in [141], p. 89, or theorem 1.1.12). Assume $0 \leq t \leq s \leq t + r$ with any given $r > 0$. We multiply (1.1.35) by $\exp(-\int_t^s A_1 y(\tau)\,d\tau)$ and obtain the relation

$$\frac{d}{ds}\left[y(s)\exp\left(-\int_t^s A_1 y(\tau)\,d\tau\right)\right] \le (A_2+h(s))\exp\left(-\int_t^s A_1 y(\tau)\,d\tau\right) \le A_2+h(s). \tag{1.1.39}$$

Then integrating it over $[s,\ t+r]$ yields

$$\begin{aligned} y(t+r) &\le y(s)\exp\left(\int_s^{t+r} A_1 y(\tau)\,d\tau\right) + (A_2 r + A_4)\exp\left(\int_t^{t+r} A_1 y(\tau)\,d\tau\right) \\ &\le (y(s)+A_2 r + A_4)e^{A_1 A_3}. \end{aligned} \tag{1.1.40}$$

Integrating this inequality, with respect to s between t and $t+r$, gives us (1.1.37). From (1.1.35) and (1.1.37), it follows

$$\begin{aligned} \frac{dy}{dt} &\le A_1\left[\left(\frac{A_3}{r}+A_2 r + A_4\right)e^{A_1A_3}\right]^2 + A_2 + h(t) \\ &= A_r + h(t), \quad \text{for all} \quad t \ge r, \end{aligned} \tag{1.1.41}$$

where

$$A_r := A_1\left[\left(\frac{A_3}{r}+A_2 r + A_4\right)e^{A_1A_3}\right]^2 + A_2.$$

To prove (1.1.38), we use the contradiction argument. Assume it were not true. Then there would exist a monotone increasing sequence $\{t_n\}$ and a constant $a>0$ such that for all $n \in \mathbb{N}$,

$$t_n \ge r + \frac{a}{4A_r}, \quad t_{n+1} \ge t_n + \frac{a}{4A_r}, \tag{1.1.42}$$

$$\lim_{n\to+\infty} t_n = +\infty, \tag{1.1.43}$$

$$y(t_n) \ge \frac{a}{2} > 0. \tag{1.1.44}$$

On the other hand, from (1.1.41) we have

$$y(t_n) - y(t) \le A_r(t_n - t) + \int_t^{t_n} h(\tau)\,d\tau, \ \text{ as } \ t_n - \frac{a}{4A_r} \le t < t_n. \tag{1.1.45}$$

Combining (1.1.44) and (1.1.45) yields

$$\frac{a}{2} - y(t) \le y(t_n) - y(t) \le \frac{a}{4}\int_{t_n-\frac{a}{4A_r}}^{t_n} h(\tau)\,d\tau, \ \text{ as } \ t_n - \frac{a}{4A_r} \le t < t_n. \tag{1.1.46}$$

Therefore,

$$y(t)+\int_{t_n-\frac{a}{4A_r}}^{t_n} h(\tau)\,d\tau \geq \frac{a}{4}, \quad \text{as} \quad t_n-\frac{a}{4A_r} \leq t < t_n. \tag{1.1.47}$$

Let

$$n_T = \max\left\{ n \mid n \in \mathbb{N},\ r+\frac{a}{4A_r} \leq t_n < T \right\}. \tag{1.1.48}$$

Thus,

$$\lim_{T\to+\infty} n_T = +\infty. \tag{1.1.49}$$

It turns out from (1.1.47) that for all $T>0$,

$$\begin{aligned} A_3+\frac{aA_4}{4A_r} &\geq \int_0^T y(\tau)\,d\tau+\frac{a}{4A_r}\int_0^T y(\tau)\,d\tau \\ &\geq \sum_{1\leq n\leq n_T}\left(\int_{t_n-\frac{a}{4A_r}}^{t_n} h(\tau)\,d\tau+\frac{a}{4A_r}\int_{t_n-\frac{a}{4A_r}}^{t_n} h(\tau)\,d\tau\right) \geq \frac{a^2}{16A_r}n_T \end{aligned} \tag{1.1.50}$$

which contradicts (1.1.36). Thus this completes the proof. □

In the sequel, we shall collect other useful inequalities which play important roles in classical calculus. These inequalities include the Young inequality, the Höder inequality, and the Minkowski inequality.

1.1.7 The Young Inequalities

Theorem 1.1.13. *Suppose f is a positive, real-valued, continuous and strictly increasing function on $[0, c]$ with $c>0$. If $f(0)=0$, $a\in[0, c]$ and $b\in[0, f(c)]$, then*

$$\int_0^a f(x)\,dx+\int_0^b f^{-1}(x)\,dx \geq ab \tag{1.1.51}$$

with f^{-1} is the inverse function of f. Equality holds in (1.1.51) if and only if $b=f(a)$.

This is a classical result called "the Young inequality" whose proof can be found in Young [149].

If we take $f(x)=x^{p-1}$ with $p>1$ in the above theorem, then we can conclude the following corollary.

Corollary 1.1.3. *There holds that*

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q} \tag{1.1.52}$$

where $a,\ b\geq 0$, $p>1$ and $1/p+1/q=1$.

If $0 < p < 1$, *then*

$$ab \geq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.1.53}$$

The equalities in (1.1.52) *and* (1.1.53) *hold if and only if* $b = a^{p-1}$.

In corollary 1.1.3, if we consider a and b as εa and $\varepsilon^{-1} b$, respectively, we can get the next corollary.

Corollary 1.1.4. *For any* $\varepsilon > 0$, *there holds that*

$$ab \leq \frac{\varepsilon^p a^p}{p} + \frac{b^q}{q\varepsilon^q}$$

where $a,\ b \geq 0$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

In fact, the Young inequality has the following several variants.

Corollary 1.1.5. *(1) Let* $a,\ b > 0,\ \frac{1}{p} + \frac{1}{q} = 1,\ 1 < p < +\infty$. *Then*

(i) $a^{1/p} b^{1/q} \leq a/p + b/q$;
(ii) $a^{1/p} b^{1/q} \leq a/(p\varepsilon^{1/q}) + b\varepsilon^{1/p}/q$, *for all* $\varepsilon > 0$;
(iii) $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b,\ 0 < \alpha < 1$.

(2) Let $a_k \geq 0, p_k > 0, \sum_{k=1}^m p_k = 1$. *Then* $\prod_{k=1}^m a_k^{p_k} \leq \sum_{k=1}^m p_k a_k$.

1.1.8 The Hölder Inequalities

This subsection will introduce some Hölder inequalities. The following is the discrete Hölder inequality which was proved by Hölder in 1889 (see *e.g.*, Hölder [55]). However, as pointed out by Lech [76] that in fact it should be called the Roger inequality or Roger–Hölder inequality since Roger established the inequality (1.1.54) in 1888 earlier than Hölder did in 1889. However, we still call it here the Hölder inequality.

Theorem 1.1.14. *If* $a_k \geq 0$, $b_k \geq 0$ *for* $k = 1, 2, \ldots, n$, *and* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$, *then*

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}}. \tag{1.1.54}$$

If $0 < p < 1$, *then*

$$\sum_{k=1}^n a_k b_k \geq \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q\right)^{\frac{1}{q}}. \tag{1.1.55}$$

Here the equalities in (1.1.54) *and* (1.1.55) *hold if and only if* $\alpha a_k^p = \beta b_k^q$ *for* $k = 1, 2, \ldots, n$ *where* α *and* β *are real non-negative constants with* $\alpha^2 + \beta^2 > 0$.

Remark 1.1.5. *If $p = 1$ or $p = +\infty$, we have the trivial case.*

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k\right) \sup_{1\leq k\leq n} b_k, \quad \text{if} \quad p = 1;$$
$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} b_k\right) \sup_{1\leq k\leq n} a_k, \quad \text{if} \quad p = +\infty.$$

Remark 1.1.6. *When $p = q = 2$, (1.1.54) and (1.1.55) are called to be the Cauchy inequality, the Schwarz inequality, the Cauchy–Schwarz inequality or the Bunyakovskii inequality.*

By virtue of the discrete Hölder inequality (theorem 1.1.14), we can easily obtain the integral form of the Hölder inequality, namely.

Theorem 1.1.15. *If $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ is a measurable set, then*

$$fg \in L^1(\Omega)$$

and

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} \tag{1.1.56}$$

with $1 \leq p \leq +\infty, \frac{1}{p} + \frac{1}{q} = 1$ and

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}};$$
$$\|f\|_{L^\infty(\Omega)} = \operatorname*{esssup}_{x\in\Omega} |f(x)|.$$

If $0 < p < 1$, then

$$\|fg\|_{L^1(\Omega)} \geq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{1.1.57}$$

The equalities in (1.1.56) and (1.1.57) hold if and only if there exist $\beta \in \mathbb{R}$ and real numbers C_1, C_2 which are not all zeros such that $C_1|f(x)|^p = C_2|g(x)|^q$ and $arg(f(x)g(x)) = \beta$ a.e. on Ω hold.

Remark 1.1.7. *We have the corresponding weighted Hölder inequality of the integral form. Let $1 < p < +\infty, f \in L^p(\Omega), g \in L^q(\Omega), \frac{1}{p} + \frac{1}{q} = 1$, $\omega(x) > 0$ on Ω. Then*

$$\int_\Omega |fg|\omega(x)\, dx \leq \left(\int_\Omega |f(x)|^p \omega(x)\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \omega(x)\, dx\right)^{\frac{1}{q}}.$$

1.1.9 The Minkowski Inequalities

Note that, in 1896, Minkowski established the following famous inequality.

Theorem 1.1.16. *Let $a = \{a_1, \ldots, a_n\}$ or $a = \{a_1, \ldots, a_n, \ldots\}$ be a real sequence or complex sequence. Define*

$$\|a\|_p = \left(\sum_k |a_k|^p\right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty;$$

$$\|a\|_\infty = \sup_k |a_k| \text{ if } p = +\infty.$$

Then for $1 \leq p \leq +\infty$,

$$\|a+b\|_p \leq \|a\|_p + \|b\|_p. \tag{1.1.58}$$

If $0 \neq p < 1$, then

$$\|a+b\|_p \geq \|a\|_p + \|b\|_p \tag{1.1.59}$$

where when $p < 0$, we require that a_k, b_k, $a_k + b_k \neq 0$ $(k = 1, 2, \ldots)$. Moreover, when $p \neq 0, 1$, the equality in (1.1.58) holds if the sequences a and b are proportional. When $p = 1$, the equalities in (1.1.58) and (1.1.59) hold if and only if $\arg a_k = \arg b_k$, for all k.

Remark 1.1.8. *If we replace p by $1/p$ in (1.1.58), we can obtain the following assertion:*

(1) if $1 \leq p < +\infty$, then there holds

$$\left(\sum_k |a_k + b_k|^{\frac{1}{p}}\right)^p \geq \left(\sum_k |a_k|^{\frac{1}{p}}\right)^p + \left(\sum_k |b_k|^{\frac{1}{p}}\right)^p;$$

(2) if $0 < p < 1$, then there holds

$$\left(\sum_k |a_k + b_k|^{\frac{1}{p}}\right)^p \leq \left(\sum_k |a_k|^{\frac{1}{p}}\right)^p + \left(\sum_k |b_k|^{\frac{1}{p}}\right)^p.$$

In the applications, the following integral form of the Minkowski inequality is used frequently.

Theorem 1.1.17. *Assume that Ω is a smooth open set in $\mathbb{R}^n$ and $f, g \in L^p(\Omega)$ with $1 \leq p \leq +\infty$. Then*

$$f + g \in L^p(\Omega)$$

and

$$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \tag{1.1.60}$$

If $0 < p < 1$, *then*

$$\|f+g\|_{L^p(\Omega)} \geq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}. \tag{1.1.61}$$

If $p > 1$, *the equality in* (1.1.60) *holds if and only if there exists a constant* $C_1 \neq 0$ *such that* $C_1 f(x) = g(x)$ *a.e. in* $\mathbf{\Omega}$.

If $p = 1$, *then the equality in* (1.1.60) *holds if and only if* $\arg f(x) = \arg g(x)$ *a.e. in* $\mathbf{\Omega}$ *or there exists a non-negative measurable function* h *such that* $fh = g$ *a.e. in the set* $A = \{x \in \mathbf{\Omega} \mid f(x)g(x) \neq 0\}$.

Chapter 2

Asymptotic Behavior of Solutions for the One-Dimensional Infrarelativistic Model of a Compressible Viscous Gas with Radiation

2.1 Main Results

This chapter will be devoted to the study of the large-time behavior of global solutions to the one-dimensional infrarelativistic model of a compressible viscous gas with radiation. The content of this chapter is adopted from Qin *et al.* [109], which has improved the results of Qin *et al.* [111]. We note that the existence of global solutions to such a model has been proved by Ducomet and Nečasová [18] and Qin *et al.* [111]. It is well-known that the radiative model in the one-dimensional case can be reduced into the following equations (see, Ducomet and Nečasová [19–21])

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + (\rho v^2)_y + p_y = \mu v_{yy} - (S_F)_R, \\ \left[\rho\left(e + \frac{1}{2}v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \\ \frac{1}{c}I_t + \omega I_y = S. \end{cases} \tag{2.1.1}$$

Now we assume that the fluid motion is small enough with respect to the velocity of light c so that we can drop all the $\frac{1}{c}$ factors in the previous formulation and then get an "infrarelativistic" model of a compressible Navier–Stokes system for a one-dimensional flow coupled to the radiative transfer equation given in the following system

$$\begin{cases} \rho_\tau + (\rho v)_y = 0, \\ (\rho v)_\tau + (\rho v^2)_y + p_y = \mu v_{yy}, \\ \left[\rho\left(e + \frac{1}{2}v^2\right)\right]_\tau + \left[\rho v\left(e + \frac{1}{2}v^2\right) + pv - \kappa\theta_y - \mu v v_y\right]_y = -(S_E)_R, \\ \omega I_y = S. \end{cases} \tag{2.1.2}$$

DOI: 10.1051/978-2-7598-2903-3.c002

Under the Lagrangian coordinates, *i.e.*,

$$x = \int_0^y \rho(\xi, \tau)\, d\xi, \qquad t = \tau,$$

system (2.1.2) reduces to the following system

$$\begin{cases} \eta_t = v_x, & (2.1.3) \\ v_t = \sigma_x, & (2.1.4) \\ \left(e + \dfrac{1}{2}v^2\right)_t = (\sigma v - Q)_x - \eta(S_E)_R, & (2.1.5) \\ \omega I_x = \eta S, & (2.1.6) \end{cases}$$

where $x \in [0,\, 1]$, η is the specific volume (*i.e.*, $\eta = \frac{1}{\rho}$), v denotes the velocity, θ is the temperature, I represents the radiative intensity depending on the Lagrangian mass coordinates $(x,\, t)$ and also on two extra variables: the radiation frequency $\nu \in \mathbb{R}_+ = (0,\, +\infty)$ and the angular variable $\omega \in S^1 := [-1,\, 1]$, $\sigma := -p + \mu \frac{v_x}{\eta}$ is the stress and $Q := -\kappa \frac{\theta_x}{\eta}$ is the heat flux with the heat conductivity κ and μ is the viscosity coefficient. Source term S in the last equation is expressed as

$$\begin{aligned} S(x,\, t;\, \nu,\, \omega) &= \sigma_a(\nu,\, \omega;\, \eta,\, \theta)[B(\nu;\, \theta) - I(x,\, t;\, \nu,\, \omega)] \\ &\quad + \sigma_s(\nu;\, \eta,\, \theta)[\tilde{I}(x,\, t;\, \nu) - I(x,\, t;\, \nu,\, \omega)], \end{aligned} \tag{2.1.7}$$

where $\tilde{I}(x,\, t,\, \nu) := \frac{1}{2}\int_{-1}^{1} I(x,\, t;\, \nu,\, \omega)\, d\omega$ and B is a function of temperature and frequency describing the equilibrium state.

We define the radiative energy as

$$E_R = \int_{-1}^{1}\int_0^{+\infty} I(x,\, t;\, \nu,\, \omega)\, d\nu d\omega, \tag{2.1.8}$$

the radiative flux

$$F_R = \int_{-1}^{1}\int_0^{+\infty} \omega I(x,\, t;\, \nu,\, \omega)\, d\nu d\omega, \tag{2.1.9}$$

and the radiative energy sourceradiative energy source

$$(S_E)_R = \int_{-1}^{1}\int_0^{+\infty} S(x,\, t;\, \nu,\, \omega)\, d\nu d\omega. \tag{2.1.10}$$

We now consider a typical initial boundary value problem for (2.1.3)–(2.1.6) in the reference domain $\boldsymbol{\Omega} \times [0,\, +\infty) = (0,\, 1) \times [0,\, +\infty)$ under the Dirichlet–Neumann boundary conditions for the fluid unknowns

$$v(0,\, t) = v(1,\, t) = 0, \quad Q(0,\, t) = Q(1,\, t) = 0, \quad \forall\, t \geq 0, \tag{2.1.11}$$

and transparent boundary conditions for the radiative intensity

$$\begin{cases} I(0,\ t; v,\ \omega) = 0 & \text{for } \omega \in (0,\ 1), \quad \forall\ t \geq 0, \\ I(1,\ t; v,\ \omega) = 0 & \text{for } \omega \in (-1,\ 0), \quad \forall\ t \geq 0, \end{cases} \tag{2.1.12}$$

and initial conditions

$$\eta(x,\ 0) = \eta_0(x), \quad v(x,\ 0) = v_0(x), \quad \theta(x,\ 0) = \theta_0(x) \quad \text{on } \mathbf{\Omega}, \tag{2.1.13}$$

and

$$I(x,\ 0;\ v,\ \omega) = I_0(x;\ v,\ \omega) \quad \text{on } \mathbf{\Omega} \times \mathbb{R}_+ \times S^1. \tag{2.1.14}$$

Pressure and energy of the matter are related by the thermodynamical relation

$$e_\eta(\eta,\ \theta) = -p(\eta,\ \theta) + \theta p_\theta(\eta,\ \theta). \tag{2.1.15}$$

For the system (2.1.3) and (2.1.14), Ducomet and Nečasová [18] proved the global existence of solutions in $\mathcal{H}_i$ $(i = 1,\ 2)$. However, estimates obtained there depend on any given time T, so they could not establish the large-time behavior of global solutions in $\mathcal{H}_i$ $(i = 1,\ 2)$ based on their estimates. Moreover, in Ducomet and Nečasová [18], all estimates hold only for $q \geq 2r + 1$. Later on, Qin *et al.* [111] had improved the results in [18]. Recently, Qin *et al.* [109] have improved the results in [111] by establishing the uniform-in-time estimates of $(\eta(t),\ v(t),\ \theta(t),\ I(t))$ in $\mathcal{H}_i$ $(i = 1,\ 2,\ 4)$, which hold for q and r satisfying (2.1.16). Furthermore, the system considered here is different from that in Qin [104], so our uniform-in-time estimates are also different from those in Qin [104].

We now assume that e, p, σ and κ are twice continuously differential on $0 < \eta < +\infty$ and $0 \leq \theta < +\infty$, and there are exponents q and r satisfying one of the following relations

$$\begin{cases} 0 < r \leq \dfrac{1}{2}, & \dfrac{1}{2} < q, \\ \dfrac{1}{2} < r \leq \dfrac{5}{2}, & \dfrac{2r+1}{4} < q, \\ \dfrac{5}{2} < r \leq \dfrac{17}{5}, & \dfrac{5r+1}{9} < q, \\ \dfrac{17}{5} \leq r, & \dfrac{10r+4}{19} < q \end{cases} \tag{2.1.16}$$

and we suppose the following growth conditions:

$$\begin{cases} e(\eta,\ 0) \geq 0, \quad c_1(1+\theta^r) \leq e_\theta(\eta,\ \theta) \leq C_1(1+\theta^r), \\ -c_2\eta^{-2}(1+\theta^{1+r}) \leq p_\eta(\eta, \theta) \leq -\ C_2\eta^{-2}(1+\theta^{1+r}), \\ |p_\theta(\eta,\ \theta)| \leq C_3\eta^{-1}(1+\theta^r), \\ c_4(1+\theta^{1+r}) \leq \eta p(\eta,\ \theta) \leq C_4(1+\theta^{1+r}), \quad p_\eta(\eta,\ \theta) < 0, \\ 0 \leq p(\eta,\ \theta) \leq C_5(1+\theta^{1+r}), \\ c_6(1+\theta^q) \leq \kappa(\eta, \theta) \leq C_6(1+\theta^q), \\ |\kappa_\eta(\eta,\ \theta)| + |\kappa_{\eta\eta}(\eta,\ \theta)| \leq C_7(1+\theta^q), \end{cases} \tag{2.1.17}$$

and that the absorption–emission coefficient $\sigma_a(\nu,\ \omega;\ \eta,\ \theta)$ and the scattering coefficient $\sigma_s(\nu;\ \eta,\ \theta)$ satisfy the following conditions:

$$\begin{cases} \eta\sigma_a(\nu,\ \omega;\ \eta,\ \theta)B^m(\nu,\ \theta) \le C_8|\omega|\theta^{\alpha+1}f(\nu,\omega) \quad \text{for } m=1,\ 2, \\ 0<\sigma_a(\nu,\ \omega;\ \eta,\ \theta) \le C_9|\omega|^2 g(\nu,\ \omega), \\ [\sigma_a+|(\sigma_a)_\eta|+|(\sigma_a)_\theta|](\nu,\ \omega;\ \eta,\ \theta)[1+B(\nu,\ \theta)+|B_\theta(\nu,\ \theta)|+|B_{\theta\theta}(\nu,\ \theta)|] \le C_{10}|\omega|h(\nu,\ \omega), \\ 0<\sigma_s(\nu;\ \eta,\ \theta) \le C_{11}|\omega|^2 k(\nu), \\ [|(\sigma_a)_{\eta\eta}|+|(\sigma_a)_{\eta\theta}|+|(\sigma_a)_{\theta\theta}|](\nu,\ \omega;\ \eta,\ \theta)(1+B(\nu,\ \theta)+|B_\theta(\nu,\ \theta)|) \le C_{12}|\omega|l(\nu,\ \omega), \\ [|(\sigma_s)_\eta|+|(\sigma_s)_\theta|+|(\sigma_s)_{\eta\eta}|+|(\sigma_s)_{\eta\theta}|+|(\sigma_s)_{\theta\theta}|](\nu;\ \eta,\ \theta) \le C_{13}|\omega|\mathcal{M}(\nu,\ \omega), \end{cases} \tag{2.1.18}$$

where $0 \le \alpha \le r$, the numbers $c_i,\ C_j,\ (i=1,\ldots,7,\ j=1,\ldots,13)$ are positive constants and the non-negative functions $f,\ g,\ h,\ k,\ l,\ \mathcal{M}$ are such that

$$f,\ g,\ h,\ k,\ l,\ \mathcal{M} \in L^1(\mathbb{R}_+\times S^1)\cap L^\infty(\mathbb{R}_+\times S^1).$$

We assume that the viscosity coefficient μ is a positive constant. In the following, we denote

$$\mathcal{I}(x,t) := \int_0^{+\infty}\int_{S^1} I(x,t;\nu,\omega)\,d\omega d\nu$$

for the integrated radiative intensity. In particular,

$$\mathcal{I}(x,0) \equiv \mathcal{I}_0 = \int_0^{+\infty}\int_{S^1} I(x,\ 0;\ \nu,\ \omega)\,d\omega d\nu.$$

We define

$$\begin{aligned} \mathcal{H}_1 = \{(\eta,\ v,\ \theta,\ I) &\in H^1(0,\ 1)\times H_0^1(0,\ 1)\times H^1(0,\ 1) \\ &\times L^1(\mathbb{R}_+\times S^1, H^2(0,\ 1)):\ \eta(x)>0,\ \theta(x)>0,\ x\in[0,\ 1], \\ &v|_{x=0,1}=0,\ I|_{x=0}=0 \text{ for } \omega\in(0,1),\ I|_{x=1}=0 \text{ for } \omega\in(-1,0)\}, \end{aligned}$$

$$\begin{aligned} \mathcal{H}_i = \{(\eta,\ v,\ \theta,\ I) &\in H^i(0,\ 1)\times H_0^i(0,\ 1)\times H^i(0,\ 1) \\ &\times L^1(\mathbb{R}_+\times S^1, H^{i+1}(0,\ 1)):\ \eta(x)>0,\ \theta(x)>0,\ x\in[0,\ 1],\ v|_{x=0,1}=0, \\ &\theta_x|_{x=0,1}=0,\ I|_{x=0}=0 \text{ for } \omega\in(0,\ 1),\ I|_{x=1}=0 \text{ for } \omega\in(-1,\ 0)\},\quad i=2,\ 4. \end{aligned}$$

The main aim of this chapter was to establish the global existence and the large-time behavior of solutions in $\mathcal{H}_i$ $(i=1,\ 2,\ 4)$ to the system (2.1.3) and (2.1.14).

The notation in this chapter will be as follows: $L^q, 1\le q\le +\infty,\ W^{m,q},\ m\in\mathbb{N}$, $H^1=W^{1,2},\ H_0^1=W_0^{1,2}$ denote the usual (Sobolev) spaces on $[0,\ 1]$. In addition, $\|\cdot\|_B$ denotes the norm in space B; we also put $\|\cdot\|=\|\cdot\|_{L^2[0,\ 1]}$. Subscripts t and x denote the (partial) derivatives with respect to t and x, respectively. We use C_i $(i=1,\ 2,\ 4)$ to denote the generic positive constants depending on the $\|(\eta_0,\ v_0,\ \theta_0,\ \mathcal{I}_0)\|_{\mathcal{H}_i}$, $\min_{x\in[0,\ 1]}\eta_0(x)$, $\min_{x\in[0,\ 1]}\theta_0(x)$, but not depending on t.

Our main results read as follows (see also Qin *et al.* [109]), which has improved the result in [111]. The next result concerns the global existence and asymptotic behavior of solutions in $\mathcal{H}_1$.

Theorem 2.1.1. *Suppose that* $(\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_1$ *and the compatibility conditions hold. Under assumptions* (2.1.15)–(2.1.18)*, there exists a unique global solution* $(\eta(t), v(t), \theta(t), I(t)) \in L^\infty([0, +\infty), \mathcal{H}_1)$ *to the problem* (2.1.3)–(2.1.14) *such that for all* $(x, t) \in [0, 1] \times [0, +\infty)$,

$$0 < C_1^{-1} \leq \eta(x, t) \leq C_1, \tag{2.1.19}$$

and for all $t > 0$,

$$\begin{aligned} &\|\eta(t) - \overline{\eta}\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{H^1}^2 + \|I(t)\|_{L^1(\mathbb{R}_+\times S^1, H^2(0,1))}^2 \\ &\quad + \int_0^t (\|\eta - \overline{\eta}\|_{H^1}^2 + \|v\|_{H^2}^2 + \left\|\theta - \overline{\theta}\right\|_{H^2}^2 + \|\theta_t\|^2)(s)\, ds \\ &\quad + \int_0^t \int_0^1 \int_0^{+\infty} \int_{S^1} I_t^2 \, d\omega dv dx ds \leq C_1. \end{aligned} \tag{2.1.20}$$

Moreover, we have, as $t \to +\infty$,

$$\|\eta(t) - \overline{\eta}\|_{H^1} \to 0, \quad \|v(t)\|_{H^1} \to 0, \quad \left\|\theta(t) - \overline{\theta}\right\|_{H^1} \to 0, \quad \|I(t)\|_{L^1(\mathbb{R}_+\times S^1, H^2(0,1))} \to 0, \tag{2.1.21}$$

where $\overline{\eta} = \int_0^1 \eta(x,t)\,dx = \int_0^1 \eta_0\,dx$, $\overline{\theta} > 0$ *is determined by* $e(\overline{\eta}, \overline{\theta}) = \int_0^1 (\frac{1}{2} v_0^2 + e(\eta_0, \theta_0) + F_R(0))\,dx$.

In the next theorem, we shall establish the global existence and asymptotic behavior of solutions in $\mathcal{H}_2$.

Theorem 2.1.2. *Suppose that* $(\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_2$ *and the compatibility conditions hold. Under assumptions* (2.1.15)–(2.1.18)*, there exists a unique global solution* $(\eta(t), v(t), \theta(t), I(t)) \in L^\infty([0, +\infty), \mathcal{H}_2)$ *to the problem* (2.1.3)–(2.1.14) *satisfying for any* $t > 0$,

$$\begin{aligned} &\|\eta(t) - \overline{\eta}\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \left\|\theta(t) - \overline{\theta}\right\|_{H^2}^2 + \|I(t)\|_{L^1(\mathbb{R}_+\times S^1, H^3(0,1))}^2 + \|v_t(t)\|^2 \\ &\quad + \|\theta_t(t)\|^2 + \int_0^t (\|v_{xt}\|^2 + \|\theta_{xt}\|^2 + \left\|\theta - \overline{\theta}\right\|_{H^3}^2 + \|v\|_{H^3}^2 \\ &\quad + \|\eta - \overline{\eta}\|_{H^2}^2)(s)\, ds \leq C_2. \end{aligned} \tag{2.1.22}$$

Moreover, we have, as $t \to +\infty$,

$$\|\eta(t) - \overline{\eta}\|_{H^2} \to 0, \quad \|v(t)\|_{H^2} \to 0, \quad \left\|\theta(t) - \overline{\theta}\right\|_{H^2} \to 0, \quad \|I(t)\|_{L^1(\mathbb{R}_+\times S^1, H^3(0,1))} \to 0. \tag{2.1.23}$$

Theorem 2.1.3. *Suppose that* $(\eta_0,\ v_0,\ \theta_0,\ I_0)\in\mathcal{H}_4$ *and the compatibility conditions hold. Under assumptions* (2.1.15)–(2.1.18)*, there exists a unique global solution* $(\eta(t),\ v(t),\ \theta(t),\ I(t))\in L^\infty([0,\ +\infty),\ \mathcal{H}_4)$ *to the problem* (2.1.3) *and* (2.1.14) *verifying that for any* $t>0$,

$$\begin{aligned}&\|\eta(t)-\overline{\eta}\|^2_{H^4}+\|\eta_t(t)\|^2_{H^3}+\|\eta_{tt}(t)\|^2_{H^1}+\|v(t)\|^2_{H^4}+\|v_{tt}(t)\|^2\\&\quad+\left\|\theta(t)-\overline{\theta}\right\|^2_{H^4}+\|\theta_t(t)\|^2_{H^2}+\|\theta_{tt}(t)\|^2+\|I\|^2_{L^1(\mathbb{R}_+\times S^1,H^5(0,1))}\\&\quad+\int_0^t(\|\eta-\overline{\eta}\|^2_{H^4}+\|v\|^2_{H^5}+\|v_t\|^2_{H^3}+\|v_{tt}\|^2_{H^1}+\left\|\theta-\overline{\theta}\right\|^2_{H^5}\\&\quad+\|\theta_t\|^3_{H^3}+\|\theta_{tt}\|^2_{H^1})(s)\,ds\le C_4,\end{aligned}\tag{2.1.24}$$

$$\int_0^t(\|\eta_t\|^2_{H^4}+\|\eta_{tt}\|^2_{H^2}+\|\eta_{ttt}\|^2)(s)\,ds\le C_4.\tag{2.1.25}$$

Moreover, we have as $t\to+\infty$,

$$\|\eta(t)-\overline{\eta}\|_{H^4}\to 0,\quad \|v(t)\|_{H^4}\to 0,\quad \left\|\theta(t)-\overline{\theta}\right\|_{H^4}\to 0,\quad \|I(t)\|_{L^1(\mathbb{R}_+\times S^1,H^5(0,1))}\to 0,\tag{2.1.26}$$

where $\overline{\eta}=\int_0^1\eta(x,\ t)\,dx=\int_0^1\eta_0\,dx$, $\overline{\theta}>0$ *is determined by* $e(\overline{\eta},\ \overline{\theta})=\int_0^1(\frac{1}{2}v_0^2+e(\eta_0,\ \theta_0)+F_R(0))\,dx$.

Corollary 2.1.1. *The global solution* $(\eta(t),\ v(t),\ \theta(t),\ \mathcal{I}(t))$ *obtained in theorem 2.1.3 is, in fact, a classical solution such that as* $t\to\ +\infty$,

$$\left\|(\eta(t)-\overline{\eta},v(t),\theta(t)-\overline{\theta})\right\|_{(C^{3+\frac{1}{2}}(0,1))^3}\to 0,\ \|I(t)\|_{L^1(\mathbb{R}_+\times S^1,C^{4+\frac{1}{2}}(0,1))}\to 0.$$

Remark 2.1.1. *Theorems 2.1.1–2.1.3 also hold for the boundary conditions* (2.1.12) *and*

$$v(0,\ t)=v(1,\ t)=0,\quad \theta(0,\ t)=\theta(1,\ t)=T_0=const.>0,$$

where $\overline{\theta}$ *can be replaced by* T_0.

2.2 Global Existence and Uniform-in-Time Estimates in $\mathcal{H}_1$

We note that the global existence of solutions in $\mathcal{H}_1$ has been established in [18]. This section will study the global existence and asymptotic behavior of global solutions in $\mathcal{H}_1$. To this end, we shall first establish some uniform-in-time estimates in $\mathcal{H}_1$.

Lemma 2.2.1. *Under assumptions in theorem 2.1.1, there holds that*

$$\theta(x,\ t) > 0, \qquad \forall\ (x,\ t) \in [0,\ 1] \times [0,\ +\infty), \tag{2.2.1}$$

$$\int_0^1 \eta(x,\ t)\,dx = \int_0^1 \eta_0(x)\,dx \equiv \overline{\eta}_0, \quad \forall\ t > 0, \tag{2.2.2}$$

$$\int_0^1 (\theta + \theta^{1+r})(x,\ t)\,dx \le C_1, \quad \forall\ t > 0, \tag{2.2.3}$$

$$\int_0^1 [(\theta - \log\theta - 1) + \theta^{1+r} + v^2](x,\ t)\,dx + \int_0^t \int_0^1 \left(\frac{(1+\theta^q)\theta_x^2}{\eta\theta^2} + \frac{\mu v_x^2}{\eta\theta}\right)(x,\ s)\,dxds \le C_1. \tag{2.2.4}$$

Proof. Inequality (2.2.1) is a consequence of the generalized maximum principle [3] and one can find the proof in [19–21].

Integrating (2.1.3) over $Q_t = (0,\ 1) \times (0,\ t)$, and using the boundary conditions, we can easily deduce (2.2.2).

From (2.1.6), (2.1.9) and (2.1.10), we can infer

$$(F_R)_x = \eta(S_E)_R. \tag{2.2.5}$$

Inserting (2.2.5) into (2.1.5), we arrive at

$$\left(e + \frac{1}{2}v^2\right)_t = (\sigma v - Q - F_R)_x. \tag{2.2.6}$$

Integrating (2.2.6) over Q_t and using boundary conditions (2.1.11) and (2.1.12), we have

$$\int_0^1 \left(e + \frac{1}{2}v^2\right)(x,t)\,dx + \int_0^t F_R|_{x=0}^{x=1}\,ds = \int_0^1 \left(e_0 + \frac{1}{2}v_0^2\right)(x)\,dx. \tag{2.2.7}$$

Using (2.1.12), the contribution of the radiation term reads (see, *e.g.*, [18])

$$\int_0^t F_R|_{x=0}^{x=1}\,ds = \int_0^t \left[\int_0^{+\infty}\int_0^1 \omega I(1,\ t;\ v,\ \omega)\,d\omega dv - \int_0^{+\infty}\int_{-1}^0 \omega I(0,\ t;\ v,\ \omega)\,d\omega dv\right] ds \ge 0,$$

which, together with (2.2.7), implies

$$\int_0^1 \left(e + \frac{1}{2}v^2\right)(x,\ t)\,dx \le C_1. \tag{2.2.8}$$

Combining (2.2.8) with (2.1.17) yields (2.2.3).

Noting that radiative term $\eta(S_E)_R$ appears in (2.1.5), our estimate (2.2.8) is different from the one in [17] where there is no radiative term.

We define the free energy $\psi := e - \theta S$ with $\psi_\theta = -S$ and $\psi_\eta = -p$ with the specific entropy S. Let us consider the auxiliary function

$$E(\eta, \theta) := \psi(\eta, \theta) - \psi(1, 1) - (\eta - 1)\psi_\eta(1, 1) - (\theta - 1)\psi_\theta(\eta, \theta). \tag{2.2.9}$$

We have the following estimate (see, *e.g.*, Ducomet and Nečasová [18] for details)

$$\int_0^1 \left(E + \frac{1}{2}v^2\right)dx + \int_0^t\int_0^1\left(\frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}\right)dxds + \int_0^t\int_0^1\frac{\eta}{\theta}\int_0^{+\infty}\int_{S^1}\sigma_a I d\omega d\nu dxds$$
$$+ \int_0^t\left[\int_0^{+\infty}\int_0^1 \omega I(1, t; \nu, \omega)d\omega d\nu - \int_0^{+\infty}\int_{-1}^0 \omega I(0, t; \nu, \omega)d\omega d\nu\right]ds \le C_1. \tag{2.2.10}$$

Using the Taylor theorem and the definition of $E(\eta, \theta)$, we can conclude

$$\begin{aligned} &E(\eta, \theta) - \psi(\eta, \theta) + \psi(\eta, 1) + (\theta - 1)\psi_\theta(\eta, \theta)\\ &\quad = \psi(\eta, 1) - \psi(1, 1) - \psi_\eta(\eta, \theta)\\ &\quad = (\eta - 1)^2\int_0^1 (1-\xi)\psi_{\eta\eta}(1 + \xi(\eta - 1), 1)d\xi \ge 0. \end{aligned}$$

Thus,

$$\begin{aligned} E(\eta, \theta) &\ge \psi(\eta, \theta) - \psi(\eta, 1) - (\theta - 1)\psi_\theta(\eta, \theta)\\ &= -(1-\theta)^2\int_0^1 (1-\tau)\psi_{\theta\theta}(\eta, \theta + \tau(1-\theta))d\tau\\ &\ge C_1^{-1}(1-\theta)^2\int_0^1 \frac{(1-\tau)\{1 + [\theta + \tau(1-\theta)]^r\}}{\theta + \tau(1-\theta)}d\tau\\ &= \begin{cases} C_1^{-1}(\theta - \log\theta - 1) + \frac{C_1^{-1}(1-\theta^r)}{r} + \frac{C_1^{-1}(1-\theta^{r+1})}{r+1}, & \text{for } r > 0,\\ 2C_1^{-1}(\theta - \log\theta - 1), & \text{for } r = 0, \end{cases}\\ &\ge C_1^{-1}(\theta - \log\theta - 1) + C_1^{-1}\theta^{r+1} - C_1^{-1}. \end{aligned} \tag{2.2.11}$$

Combining (2.2.11) and (2.2.10), and using (2.1.17) yields (2.2.4). The proof is now complete. □

The following two lemmas concerning the uniform-in-time estimate of specific volume η play a very crucial role in this chapter. The uniform-in-time estimate is different from the one in Ducomet and Nečasová [18], where estimates are dependent on any given time $T > 0$.

Lemma 2.2.2. *For any $t \ge 0$, there exists one point $x_1 = x_1(t) \in [0, 1]$ such that the solution $\eta(x, t)$ to the problem* (2.1.3)–(2.1.6), (2.1.11)–(2.1.14) *possesses the following expression:*

$$\eta(x, t) = D(x, t)Z(t)\left\{1 + \frac{1}{\mu}\int_0^t \eta(x, s)p(x, s)D^{-1}(x, s)Z^{-1}(s)\,ds\right\} \tag{2.2.12}$$

where

$$\begin{aligned} D(x, t) = \eta_0(x)\exp\left\{\frac{1}{\mu}\left(\int_{x_1(t)}^x v(y, t)\,dy - \int_0^x v_0(y)\,dy\right.\right. \\ \left.\left. + \frac{1}{\overline{\eta}_0}\int_0^1 \eta_0(x)\int_0^x v_0(y)\,dydx\right)\right\}, \end{aligned} \tag{2.2.13}$$

$$Z(t) = \exp\left\{-\frac{1}{\mu\overline{\eta}_0}\int_0^t\int_0^1 (v^2 + \eta p)(y, s)\,dyds\right\}. \tag{2.2.14}$$

Proof. The proof is the same as that of lemma 2.1.3 in Qin [104]. But for the book's self-contained, we copy its proof here. Let

$$h(x, t) = \int_0^x v_0(y)\,dy + \int_0^t \sigma(x, \tau)\,d\tau.$$

The from (2.1.13), $h(x, t)$ satisfies

$$h_x = v, \quad h_t = \sigma \tag{2.2.15}$$

and from (2.1.11) it solves the equation

$$h_t = -p + \frac{\mu h_{xx}}{\eta} \tag{2.2.16}$$

with

$$x = 0, 1 : h_x = v = 0. \tag{2.2.17}$$

Hence we derive from (2.2.16) that

$$(\eta h)_t = hv_x - \eta p + \mu h_{xx}. \tag{2.2.18}$$

Integrating (2.2.18) over $[0, 1] \times [0, t]$ and using (2.2.17), we arrive at

$$\int_0^1 \eta h\,dx = \int_0^1 \eta_0 h_0\,dx - \int_0^t\int_0^1 (\eta p + v^2)\,dxd\tau \equiv \phi(t). \tag{2.2.19}$$

Then for any $t \geq 0$, there exists one point $x_1 = x_1(t) \in [0, 1]$ such that

$$\phi(t) = \int_0^1 \eta h\,dx = \int_0^1 \eta\,dx \cdot h(x_1(t), t) = \overline{\eta}_0 \cdot h(x_1(t), t),$$

i.e.,

$$\int_0^t p(x_1(t), \tau)\,d\tau = \int_0^{x_1(t)} v_0(y)\,dy + \mu\log\frac{\eta(x_1(t), t)}{\eta_0(x_1(t))} - \frac{\phi(t)}{\overline{\eta}_0} \tag{2.2.20}$$

with

$$\phi(t) = -\int_0^t \int_0^1 (v^2 + \eta p)(x, s)\, dxds + \int_0^1 \eta_0(x) \int_0^x v_0(y)\, dydx. \tag{2.2.21}$$

Moreover, (2.1.4) can be rewritten as

$$v_t - \mu(\log \eta)_{xt} = -p_x = -p_x^* \tag{2.2.22}$$

with $p^* = p - \int_0^1 p(x, t)\, dx$. Integrating (2.2.22) over $[x_1(t), x] \times [0, t]$ for fixed $t > 0$, we get

$$\begin{aligned}\eta(x, t) = \frac{\eta_0(x)\eta(x_1(t), t)}{\eta_0(x_1(t))} \exp\Bigg\{ \frac{1}{\overline{\eta}_0} \Bigg[\int_{x_1(t)}^x (v(y, t) - v_0(y))\, dy \\ + \int_0^t (p(x, \tau) - p(x_1(t), \tau))\, d\tau \Bigg] \Bigg\}.\end{aligned} \tag{2.2.23}$$

Inserting (2.2.20) into (2.2.23) and noting (2.2.13), (2.2.14) and (2.2.21), we conclude

$$\eta^{-1}(x, t) \exp\left\{ \frac{1}{\eta_0} \int_0^t p(x, s)\, ds \right\} = D^{-1}(x, t) Z^{-1}(t) \tag{2.2.24}$$

which implies that

$$\exp\left\{ \frac{1}{\eta_0} \int_0^t p(x, s)\, ds \right\} = 1 + \frac{1}{\eta_0} \int_0^t D^{-1}(x, s) Z^{-1}(s) \eta(x, s) p(x, s)\, ds. \tag{2.2.25}$$

Thus (2.2.12) follows from (2.2.24) and (2.2.25). □

Lemma 2.2.3. *There holds that*

$$0 < C_1^{-1} \le \eta(x,\ t) \le C_1, \quad \forall\ (x,\ t) \in [0,\ 1] \times [0,\ +\infty), \tag{2.2.26}$$

$$\int_0^t \|v(s)\|_{L^\infty}^2\, ds \le C_1, \quad \forall\ t > 0. \tag{2.2.27}$$

Proof. Let

$$M_\eta(t) = \max_{x \in [0,1]} \eta(x, t).$$

Using the Young inequality, the Hölder inequality and lemma 2.2.1, we get

$$\begin{aligned}&\left| \int_{x_1(t)}^x v(y,\ t)\, dy - \int_0^x v_0(y)\, dy + \frac{1}{\overline{\eta}_0} \int_0^1 \eta_0(x) \int_0^x v_0(y)\, dydx \right| \\ &\le \left(\int_0^1 v^2\, dy \right)^{\frac{1}{2}} + \left(\int_0^1 v_0^2\, dy \right)^{\frac{1}{2}} + \frac{1}{\overline{\eta}_0} \int_0^1 \eta_0(x) \left(\int_0^1 v_0^2\, dy \right)^{\frac{1}{2}} dx \\ &\le C_1 \|v\|^2 + C_1 \le C_1.\end{aligned} \tag{2.2.28}$$

Equations (2.2.28) and (2.2.13) yield the existence of some positive constant $C_1 > 0$ such that

$$0 < C_1^{-1} \le D(x,t) \le C_1, \quad \forall\ (x,t) \in [0,\ 1] \times [0,\ +\infty).$$

From (2.1.14) and (2.2.1), we deduce

$$\int_0^1 (v^2 + \eta p)(x,\ t)\,dx \ge \int_0^1 \eta p(x,\ t)\,dx \ge \int_0^1 (c_4 + \theta^{1+r})\,dx \ge C_1^{-1}, \quad \forall\ t > 0. \tag{2.2.29}$$

Using lemma 2.2.1, we get

$$\int_0^1 (v^2 + \eta p)(x,\ t)\,dx \le \|v\|^2 + C_4 \int_0^1 (1 + \theta^{1+r})\,dx \le C_1, \quad \forall\ t > 0. \tag{2.2.30}$$

Thus, from (2.2.29) and (2.2.30) it follows that for all $0 \le s \le t$,

$$C_1^{-1}(t-s) \le \int_s^t \int_0^1 (v^2 + \eta p)(x,\ s)\,dxds \le C_1(t-s), \tag{2.2.31}$$

which, together with (2.2.14), gives that for any $0 \le s \le t$,

$$e^{-C_1(t-s)} \le Z(t)Z^{-1}(s) = \exp\left\{ -\frac{1}{\mu\overline{\eta}_0} \int_s^t \int_0^1 (v^2 + \eta p)(y,\ s)\,dyds \right\} \le e^{-C_1^{-1}(t-s)}. \tag{2.2.32}$$

It thus derives from (2.2.4) and the convexity of the function $-\log y$ that

$$\int_0^1 \theta\,dx - \log \int_0^1 \theta\,dx - 1 \le \int_0^1 (\theta - \log\theta - 1)\,dx \le C_1$$

which results in the existence of $a(t) \in [0,\ 1]$ and two positive roots of the equation $y - \log y - 1 = C_1$ such that

$$0 < r_1 \le \int_0^1 \theta(x,t)\,dx = \theta(a(t),t) \le r_2.$$

This gives, for any $t > 0$, such that

$$\begin{aligned} |\theta^{m_1}(x,t) - \theta^{m_1}(a(t),t)| &= \left| \int_{a(t)}^x (\theta^{m_1}(x,t))_x\,dy \right| \le C_1 \int_0^1 \theta^{m_1 - 1}|\theta_x|\,dx \\ &\le C_1 \left(\int_0^1 \frac{1+\theta^q}{\eta\theta^2}\theta_x^2\,dx \right)^{\frac{1}{2}} \left(\int_0^1 \eta \frac{\theta^{2m_1}}{1+\theta^q}\,dx \right)^{\frac{1}{2}} \\ &\le C_1 V^{\frac{1}{2}}(t) M_\eta^{\frac{1}{2}}(t) \end{aligned} \tag{2.2.33}$$

where $V(t) = \int_0^1 \frac{1+\theta^q}{\eta\theta^2}\theta_x^2\,dx$ and $0 \le m_1 \le m = (q+r+1)/2$.

Then for any $(x, t) \in [0, 1] \times [0, +\infty)$, we get

$$C_1^{-1} - C_1^{-1} V(t) M_\eta(t) \le \theta^{2m_1}(x, t) \le C_1 + C_1 V(t) M_\eta(t). \tag{2.2.34}$$

Thus we conclude from lemma 2.2.1 and (2.2.32)–(2.2.34)

$$\begin{aligned}\eta(x, t) &= D(x, t)\left[Z(t) + \frac{1}{\mu}\int_0^t \eta(x, s)p(x, s)D^{-1}(x, s)Z(t)Z^{-1}(s)\,ds\right]\\ &\le C_1\left[e^{-C_1 t} + \int_0^t (1 + V(s)M_\eta(s))e^{-C_1(t-s)}\,ds\right]\\ &\le C_1 + C_1\int_0^t M_\eta(s)V(s)\,ds,\end{aligned}$$

i.e.,

$$M_\eta(t) \le C_1 + C_1\int_0^t M_\eta(s)V(s)\,ds,$$

which, with (2.2.4) and using the Gronwall inequality, yields

$$M_\eta(t) \le C_1. \tag{2.2.35}$$

Using (2.2.12) and (2.2.32), we find that there exists a large time t_0 such that as $t \ge t_0$, $x \in [0, 1]$,

$$\begin{aligned}\eta(x, t) &= D(x, t)Z(t)\left\{1 + \frac{1}{\mu}\int_0^t \eta(x, s)p(x, s)D^{-1}(x, s)Z^{-1}(s)\,ds\right\}\\ &\ge C_1^{-1}\left[e^{-C_1 t} + \int_0^t e^{-C_1(t-s)}\,ds\right]\\ &\ge C_1^{-1}\int_0^t e^{-C_1(t-s)}\,ds \ge (2C_1)^{-1}.\end{aligned} \tag{2.2.36}$$

Note now that $D(x, t) \ge C_1^{-1}$, $Z(t) \ge \exp(-C_1 t)$ and infer that for any $(x, t) \in [0, 1] \times [0, t_0]$,

$$\eta(x,\ t) \ge D(x,\ t)Z(t) \ge C_1^{-1}\exp(-C_1 t) \ge C_1^{-1}\exp(-C_1 t_0),$$

which, together with (2.2.36), gives that for any $(x, t) \in [0, 1] \times [0, +\infty)$

$$\eta(x,\ t) \ge C_1^{-1}. \tag{2.2.37}$$

Combining (2.2.35) and (2.2.37) gives immediately (2.2.26).

Using the Hölder inequality, (2.2.26) and lemma 2.2.1, we obtain for any $t \ge 0$

$$\int_0^t \|v(s)\|_{L^\infty}^2\,ds \le \int_0^t\left(\int_0^1 |v_x|\,dx\right)^2 ds \le \int_0^t\left(\int_0^1 \frac{v_x^2}{\theta}dx\right)\left(\int_0^1 \theta\,dx\right)ds \le C_1.$$

The proof is now complete. □

Corollary 2.2.1. *If assumptions in theorem 2.1.1 hold, then there holds*

$$C_1 - C_1 V(t) \le \theta^{2m}(x,\ t) \le C_1 + C_1 V(t), (x,\ t) \in [0,\ 1] \times [0,\ \infty), \tag{2.2.38}$$

where

$$V(t) = \int_0^1 \frac{(1+\theta^q)\theta_x^2}{\theta^2} dx, \quad \int_0^\infty V(s)\, ds \le C_1.$$

Corollary 2.2.2. *If assumptions in theorem 2.1.1 hold, then there holds that*

$$\int_0^t \int_0^1 (1+\theta)^{2m} v^2\, dxds \le C_1, \quad \forall t > 0. \tag{2.2.39}$$

Proof. Using the Poincaré inequality, lemma 2.2.1 and (2.2.38), we obtain

$$\begin{aligned}\int_0^t \int_0^1 (1+\theta)^{2m} v^2\, dxds &\le C_1 \int_0^t \int_0^1 v^2\, dxds + C_1 \int_0^t \int_0^1 V(s) v^2\, dxds \\ &\le C_1 + C_1 \int_0^t V(s)\, ds \le C_1.\end{aligned}$$

The proof is now complete. □

Set $\Lambda = \sup\limits_{0 \le s \le t} \|\theta(s)\|_{L^\infty}$.

Lemma 2.2.4. *If assumptions in theorem 2.1.1 hold, then the following estimates hold for any* $t > 0$,

$$\|\eta_x(t)\|^2 + \int_0^t \int_0^1 (1+\theta^{1+r})\eta_x^2\, dxds \le C_1(1+\Lambda)^\beta, \tag{2.2.40}$$

$$\int_0^t \|v_x(s)\|^2\, ds \le C_1(1+\Lambda)^{\beta/2}, \tag{2.2.41}$$

with $\beta = \max(r+1-q, 0)$.

Proof. Obviously, equation (2.1.3) can be written as

$$\left(v - \mu\frac{\eta_x}{\eta}\right)_t + p_\eta \eta_x = -p_\theta \theta_x. \tag{2.2.42}$$

Multiplying (2.2.42) by $\left(v - \mu\frac{\eta_x}{\eta}\right)$ and then integrating the result over $[0,\ 1] \times (0,\ t)$, we have

$$\frac{1}{2}\left\|v-\mu\frac{\eta_x}{\eta}\right\|^2+\int_0^t\int_0^1\frac{-\mu p_\eta\eta_x^2}{\eta}dxds$$
$$=\frac{1}{2}\left\|v_0-\mu\frac{\eta_{0x}}{\eta_0}\right\|^2-\int_0^t\int_0^1\left[p_\eta\eta_x v+p_\theta\theta_x\left(v-\mu\frac{\eta_x}{\eta}\right)\right]dxds.$$

Now using the Young inequality, (2.1.17) and lemmas 2.2.1–2.2.3, we can infer for any $\epsilon>0$,

$$\begin{aligned}
&\frac{1}{2}\left\|v-\mu\frac{\eta_x}{\eta}\right\|^2+\int_0^t\int_0^1(1+\theta^{1+r})\eta_x^2dxds\\
&\leq C_1+C_1\int_0^t\int_0^1\left[(1+\theta^{1+r})|\eta_x v|+(1+\theta^r)\left|\theta_x\left(v-\mu\frac{\eta_x}{\eta}\right)\right|\right]dxds\\
&\leq C_1+C_1\int_0^t\int_0^1(1+\theta^{1+r})(\varepsilon\eta_x^2+C_1v^2)dxds\\
&\quad+C_1\left(\int_0^t V(s)ds\right)^{\frac{1}{2}}\left(\int_0^t\int_0^1\frac{\theta^2(1+\theta^r)^2}{(1+\theta^q)}v^2dxds\right)^{\frac{1}{2}}\\
&\quad+\epsilon\int_0^t\int_0^1(1+\theta^{1+r})\eta_x^2dxds+C_1\int_0^t\int_0^1\frac{(1+\theta^r)^2\theta_x^2}{1+\theta^{1+r}}dxds\\
&\leq C_1(1+\Lambda)^{\max(\beta,\frac{\delta}{2})}+\epsilon\int_0^t\int_0^1(1+\theta^{1+r})\eta_x^2dxds\\
&\leq C_1(1+\Lambda)^\beta+\epsilon\int_0^t V(s)\|\eta_x\|^2ds+\epsilon\int_0^t\int_0^1\theta^{r+1}\eta_x^2dxds\\
&\quad+C_1\int_0^t\int_0^1(1+\theta^r)|\theta_x\eta_x|dxds
\end{aligned}\tag{2.2.43}$$

with $\delta=\max(r+1-2q,0)\leq\beta$.

Now we estimate the last term in (2.2.43). Using the Young inequality and lemmas 2.2.1–2.2.3, we can conclude for any $\epsilon>0$,

$$\begin{aligned}
\int_0^t\int_0^1(1+\theta^r)|\theta_x\eta_x|&\leq\frac{\epsilon}{2}\int_0^t\int_0^1(1+\theta^{1+r})\eta_x^2dxds+C_1(\epsilon)\int_0^t\int_0^1\frac{(1+\theta^r)^2}{1+\theta^{1+r}}\theta_x^2dxds\\
&\leq\frac{\epsilon}{2}\int_0^t\int_0^1(1+\theta^{1+r})\eta_x^2dxds+C_1+C_1\Lambda^\beta,
\end{aligned}$$

which, together with (2.2.43), yields

$$\left\|v-\mu\frac{\eta_x}{\eta}\right\|^2+\int_0^t\int_0^1(1+\theta^{1+r})\eta_x^2dxds\leq\epsilon\int_0^t\int_0^1(1+\theta^{1+r})\eta_x^2dxds+C_1(1+\Lambda)^\beta.$$

Thus for small $\epsilon>0$ in the above inequality, and applying the generalized Bellman–Gronwall inequality, we conclude (2.2.40).

Multiplying (2.1.4) by v, integrating the result over Q_t, and using the Young inequality, lemmas 2.2.1–2.2.3 and (2.2.38), we derive

$$\begin{aligned}
&\frac{1}{2}\int_0^1 v^2\,dx + \int_0^t\int_0^1 \mu\frac{v_x^2}{\eta}\,dxds\\
&\quad = \frac{1}{2}\int_0^1 v_0^2\,dx - \int_0^t\int_0^1 (p_\eta\eta_x + p_\theta\theta_x)v dxds\\
&\quad \le C_1 + C_1\int_0^t\int_0^1 [(1+\theta^{1+r})|\eta_x v| + (1+\theta^r)|\theta_x v|]\,dxds\\
&\quad \le C_1 + C_1\left(\int_0^t\int_0^1 (1+\theta^{1+r})\eta_x^2\,dxds\right)^{1/2}\left(\int_0^t\int_0^1 (1+\theta^{1+r})v^2\,dxds\right)^{1/2}\\
&\qquad + C_1\left(\int_0^t\int_0^1 \frac{(1+\theta^q)\theta_x^2}{\theta^2}dxds\right)^{1/2}\left(\int_0^t\int_0^1 \frac{\theta^2(1+\theta^r)^2v^2}{1+\theta^q}dxds\right)^{1/2}\\
&\quad \le C_1\Lambda^{\beta/2} + C_1\Lambda^{\delta/2} \le C_1 + C_1\Lambda^{\beta/2}
\end{aligned}$$

which yields (2.2.41). Thus this proves the proof. □

Lemma 2.2.5. *There holds that for any* $t > 0$,

$$\|v_x(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2\,ds \le C_1(1+\Lambda)^{\beta_1}, \tag{2.2.44}$$

$$\int_0^t \|v_x(s)\|_{L^\infty}^2\,ds \le C_1(1+\Lambda)^{\beta_2}, \tag{2.2.45}$$

$$\|v_x(t)\|^2 + \int_0^t \|v_t(s)\|^2\,ds \le C_1(1+\Lambda)^{\beta_3} \tag{2.2.46}$$

with

$$\begin{aligned}
&\beta_1 = \max\left\{r+1+\beta, \frac{5\beta}{2}, \max(2r+2-q,0)\right\}, \quad \beta_2 = \frac{\beta}{4} + \frac{\beta_1}{2},\\
&\beta_3 = \max\left\{r+1+\beta, \max(2r+2-q,0), \frac{3}{4}\beta_1 + \frac{3}{8}\beta\right\}.
\end{aligned}$$

Proof. Multiplying (2.1.4) by v_{xx}, and then integrating the resultants over [0, 1], we get

$$\frac{1}{2}\frac{d}{dt}\|v_x\|^2 = \int_0^1 p_x v_{xx}\,dx - \int_0^1 \left[\left(\frac{\mu}{\eta}\right)'\eta_x v_x + \frac{\mu}{\eta}v_{xx}\right]v_{xx}\,dx, \tag{2.2.47}$$

i.e., for any $\epsilon > 0$

$$\begin{aligned}
&\|v_x(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \\
&\quad \le C_1 + C_1 \int_0^t \int_0^1 [|\eta_x v_x v_{xx}| + (1+\theta^{1+r})|\eta_x v_{xx}| + (1+\theta^r)|\theta_x v_{xx}|]\, dxds \\
&\quad \le C_1 + C_1 \int_0^t \int_0^1 v_x^2 \eta_x^2 dxds + C_1 \int_0^t \int_0^1 (1+\theta^{1+r})^2 \eta_x^2 dxds \\
&\qquad + C_1 \int_0^t \int_0^1 (1+\theta)^{2r} \theta_x^2 dxds \\
&\quad \le C_1 + C_1 (1+\Lambda)^{r+1} \int_0^t \int_0^1 (1+\theta^{1+r}) \eta_x^2 dxds + C_1 \int_0^t \|v_x\| \|v_{xx}\| \|\eta_x\|^2 ds \\
&\qquad + C_1 \int_0^t \int_0^1 \frac{(1+\theta^q)\theta_x^2}{\theta^2} \frac{\theta^2 (1+\theta)^{2r}}{(1+\theta^q)} dxds \\
&\quad \le C_1 + C_1 (1+\Lambda)^{r+1+\beta} + C_1 (1+\Lambda)^{\beta} \left(\int_0^t \|v_x(s)\|^2 ds \right)^{1/2} \left(\int_0^t \|v_{xx}(s)\|^2 ds \right)^{1/2} \\
&\qquad + C_1 \sup_{0 \le s \le t} \left\| \frac{\theta^2 (1+\theta)^{2r}}{1+\theta^q} \right\|_{L^\infty} \\
&\quad \le C_1 + C_1 (1+\Lambda)^{r+1+\beta} + C_1 (1+\Lambda)^{\frac{5}{2}\beta} + C_1 (1+\Lambda)^{\max(2r+2-q,0)} \\
&\qquad + \epsilon \int_0^t \|v_{xx}(s)\|^2 ds \\
&\quad \le C_1 (1+\Lambda)^{\beta_1} + \epsilon \int_0^t \|v_{xx}(s)\|^2 ds,
\end{aligned}$$

which gives for small $\epsilon > 0$,

$$\|v_x(t)\|^2 + \int_0^t \|v_{xx}(s)\|^2 ds \le C_1 (1+\Lambda)^{\beta_1}. \tag{2.2.48}$$

Thus

$$\begin{aligned}
\int_0^t \|v_x(s)\|_{L^\infty}^2 ds &\le C_1 \left(\int_0^t \|v_x(s)\|^2 ds \right)^{1/2} \left(\int_0^t \|v_{xx}(s)\|^2 ds \right)^{1/2} \\
&\le C_1 (1+\Lambda)^{\beta_2}.
\end{aligned} \tag{2.2.49}$$

Multiplying (2.1.4) by v_t, integrating the resultants over [0, 1], and using lemmas 2.2.1–2.2.4, we get

$$\|v_t\|^2 = -\int_0^1 p_x v_t dx - \int_0^1 \left[\left(\frac{\mu}{\eta} \right)' \eta_x v_x + \frac{\mu}{\eta} v_{xx} \right] v_t dx \tag{2.2.50}$$

Integrating (2.2.50) in t gives

$$\begin{aligned}
&\|v_x(t)\|^2 + \int_0^t \|v_t(s)\|^2 ds \\
&\le C_1 + C_1 \int_0^t \|p_x\|^2 ds + \int_0^t \|v_x\|_{L^3}^3 dxds \\
&\le C_1 + C_1 \int_0^t \int_0^1 (1+\theta^{1+r})^2 \eta_x^2 dxds + C_1 \int_0^t \int_0^1 \frac{\theta^2(1+\theta)^{2r}}{1+\theta^q} \frac{\theta_x^2(1+\theta^q)}{\theta^2} dxds \\
&\quad + C_1 \sup_{0\le s\le t} \|v_x(s)\| \left(\int_0^t \|v_x\|^2 ds\right)^{3/4} \left(\int_0^t \|v_{xx}(s)\|^2 ds\right)^{1/4} \\
&\le C_1(1+\Lambda)^{r+1+\beta} + C_1(1+\Lambda)^{\max(2r+2-q,0)} + C_1(1+\Lambda)^{\frac{3}{4}\beta_1+\frac{3}{8}\beta} \\
&\le C_1(1+\Lambda)^{\beta_3}.
\end{aligned}$$

□

Corollary 2.2.3. *If assumptions in theorem 2.1.1 hold, then the following estimates hold for any* $t > 0$,

$$\int_0^t \int_0^1 (1+\theta)^{2m} v_x^2 dxds \le C_1(1+\Lambda)^{\beta_1}, \tag{2.2.51}$$

$$\int_0^t \int_0^1 (1+\theta)^{2m+1} v_x^2 dxds \le C_1(1+\Lambda)^{\beta_1+1}, \tag{2.2.52}$$

$$\int_0^t \int_0^1 (1+\theta)^{q+1} |v_x|^3 dxds \le C_1(1+\Lambda)^{\beta_4}, \tag{2.2.53}$$

$$\int_0^t \int_0^1 (1+\theta)^{q-r} v_x^4 dxds \le C_1(1+\Lambda)^{\beta_6}, \tag{2.2.54}$$

where

$$\begin{aligned}
&\delta_1 = \frac{1}{4}\max(q-3r+1,\, 0), \quad \delta_2 = \max\left(\frac{q-3r-1}{2},\, 0\right), \\
&\beta_4 = \delta_1 + \frac{3\beta_1}{2}, \quad \beta_5 = \max(q-r,\, 0) + \frac{3\beta_1}{2} + \frac{\beta}{4}, \\
&\beta_6 = \min(\delta_2 + 2\beta_1,\, \beta_5).
\end{aligned}$$

Proof. Using corollary 2.2.1 and lemmas 2.2.1–2.2.5, we can derive

$$\begin{aligned}\int_0^t\int_0^1 (1+\theta)^{2m} v_x^2\,dxds &\le \int_0^t \|v_x\|^2\,ds + \int_0^t V(s)\|v_x\|^2\,ds\\ &\le C_1(1+\Lambda)^{\frac{\beta}{2}} + C_1(1+\Lambda)^{\beta_1}\\ &\le C_1(1+\Lambda)^{\beta_1}.\end{aligned}$$

Note that

$$\begin{aligned}\int_0^t\int_0^1 (1+\theta)^{2m+1} v_x^2\,dxds &\le (1+\Lambda)\int_0^t\int_0^1 (1+\theta)^{2m} v_x^2\,dxds\\ &\le C_1(1+\Lambda)^{\beta_1+1},\end{aligned}$$

and

$$\begin{aligned}\int_0^t\int_0^1 (1+\theta)^{q+1}|v_x|^3\,dxds &\le C_1(1+\Lambda)^{\delta_1}\int_0^t\int_0^1 (1+\theta)^{\frac{3}{4}(q+r+1)}|v_x|^3\,dxds\\ &\le C_1(1+\Lambda)^{\delta_1}\left[\int_0^t \|v_x\|_{L^3}^3\,ds + \int_0^t V(s)^{\frac{3}{4}}\|v_x\|_{L^3}^3\,ds\right]\\ &\le C_1(1+\Lambda)^{\delta_1}\sup_{s\in[0,\,t]}\|v_x(s)\|\left(\int_0^t \|v_x\|^2\,ds\right)^{\frac{3}{4}}\left(\int_0^t \|v_{xx}\|^2\,ds\right)^{\frac{1}{4}}\\ &\quad + C_1(1+\Lambda)^{\delta_1}\max_{s\in[0,\,t]}\|v_x(s)\|^{\frac{5}{2}}\left(\int_0^t \|v_{xx}\|^2\,ds\right)^{\frac{1}{4}}\\ &\le C_1(1+\Lambda)^{\beta_4}\end{aligned}$$

where we have used

$$\begin{aligned}\int_0^t \|v_x\|_{L^3}^3\,ds &\le C_1\int_0^t \|v_x\|^{\frac{5}{2}}\|v_{xx}\|^{\frac{1}{2}}\,ds \le C_1\left(\int_0^t \|v_x\|^{\frac{10}{3}}\,ds\right)^{\frac{3}{4}}\left(\int_0^t \|v_{xx}\|^2\,ds\right)^{\frac{1}{4}}\\ &\le C_1\sup_{s\in[0,\,t]}\|v_x(s)\|\left(\int_0^t \|v_x\|^2\,ds\right)^{\frac{3}{4}}\left(\int_0^t \|v_{xx}\|^2\,ds\right)^{\frac{1}{4}}\\ &\le C_1(1+\Lambda)^{\frac{3\beta_1}{4}+\frac{3\beta}{8}},\end{aligned}$$

$$\begin{aligned}\int_0^t\int_0^1 V(s)^{\frac{3}{4}}|v_x|^3\,dxds &\le \int_0^t V(s)^{\frac{3}{4}}\|v_x\|_{L^3}^3\,ds \le C_1\int_0^t V(s)^{\frac{3}{4}}\|v_x\|^{\frac{5}{2}}\|v_{xx}\|^{\frac{1}{2}}\,ds\\ &\le C_1\left(\int_0^t V(s)\,ds\right)^{\frac{3}{4}}\left(\int_0^t \|v_x\|^{10}\|v_{xx}\|^2\,ds\right)^{\frac{1}{4}}\\ &\le C_1\sup_{s\in[0,\,t]}\|v_x(s)\|^{\frac{5}{2}}\left(\int_0^t \|v_{xx}\|^2\,ds\right)^{\frac{1}{4}}\\ &\le C_1(1+\Lambda)^{\frac{5}{4}\beta_1+\frac{1}{4}\beta_1} = C_1(1+\Lambda)^{\frac{3}{2}\beta_1}\end{aligned}$$

and

$$\begin{aligned}
\int_0^t\int_0^1 (1+\theta)^{q-r} v_x^4 dxds &\le C_1(1+\Lambda)^{\delta_2}\int_0^t\int_0^1 (1+\theta^m)v_x^4 dxds\\
&\le C_1(1+\Lambda)^{\delta_2}\int_0^t\int_0^1 (1+V(s)^{\frac{1}{2}})v_x^4 dxds\\
&\le C_1(1+\Lambda)^{\delta_2}\Bigg[\int_0^t \|v_x\|^3\|v_{xx}\|ds+\left(\int_0^t V(s)ds\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_0^t \|v_x\|^6\|v_{xx}\|^2 ds\right)^{\frac{1}{2}}\Bigg]\\
&\le C_1(1+\Lambda)^{\delta_2}\Bigg[\sup_{s\in[0,\,t]}\|v_x(s)\|^2\left(\int_0^t\|v_x\|^2 ds\right)^{\frac{1}{2}}\left(\int_0^t\|v_{xx}\|^2 ds\right)^{\frac{1}{2}}\\
&\quad+\sup_{s\in[0,\,t]}\|v_x(s)\|^3\left(\int_0^t\|v_{xx}\|^2 ds\right)^{\frac{1}{2}}\Bigg]\\
&\le C_1(1+\Lambda)^{\delta_2}\left[(1+\Lambda)^{\frac{1}{4}\beta+\frac{3}{2}\beta_1}+(1+\Lambda)^{2\beta_1}\right]\\
&\le C_1(1+\Lambda)^{\delta_2+2\beta_1}.
\end{aligned}$$

However, we also know that

$$\begin{aligned}
\int_0^t\int_0^1 (1+\theta)^{q-r} v_x^4 dxds &\le C_1(1+\Lambda)^{\max(q-r,0)}\int_0^t \|v_x\|_{L^4}^4 ds\\
&\le C_1(1+\Lambda)^{\max(q-r,0)}\int_0^t \|v_x\|^3\|v_{xx}\|ds\\
&\le C_1(1+\Lambda)^{\max(q-r,0)}\left(\int_0^t\|v_x\|^6 ds\right)^{\frac{1}{2}}\left(\int_0^t\|v_{xx}\|^2 ds\right)^{\frac{1}{2}}\\
&\le C_1(1+\Lambda)^{\max(q-r,0)}\sup_{s\in[0,\,t]}\|v_x(s)\|^2\left(\int_0^t\|v_x(s)\|^2 ds\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_0^t\|v_{xx}(s)\|^2 ds\right)^{\frac{1}{2}}\\
&\le C_1(1+\Lambda)^{\max(q-r,0)+\frac{3}{2}\beta_1+\frac{1}{4}\beta} = C_1(1+\Lambda)^{\beta_5}.
\end{aligned}$$

Therefore,

$$\int_0^t\int_0^1 (1+\theta)^{q-r} v_x^4 dxds \le C_1(1+\Lambda)^{\beta_6}. \tag{2.2.55}$$

□

In the next lemma, we shall derive new uniform-in-time estimates on radiative term $I(x,\ t;\ v,\ \omega)$ given in the following lemma, which are more complicated, delicate than and quite different from those in [111], where estimates are not uniform-in-time.

Lemma 2.2.6. *There holds that for any* $t > 0$,

$$\int_0^t \int_0^1 \int_0^\infty \int_{S^1} v\sigma_a I^2 d\omega dv dx ds + \int_0^t \int_0^1 \int_0^\infty \int_{S^1} v\sigma_s (\tilde{I} - I)^2 d\omega dv dx ds \\ + \int_0^\infty \int_{S^1} v\sigma_s I^2(x,t;v,\omega) d\omega dv \leq C_1 \int_0^1 \theta^{1+\alpha} dx \leq C_1, \tag{2.2.56}$$

$$\int_0^\infty \int_{S^1} I(x,t;v,\omega) d\omega dv \leq C_1. \tag{2.2.57}$$

Proof. Multiplying (2.1.6) by I, integrating the result over $(0, 1) \times S^1 \times (0, \infty)$ and using boundary conditions (2.1.11) and (2.1.12), we get for any $\epsilon > 0$,

$$\frac{1}{2}\int_0^\infty \int_{S^1} \omega I^2(1,t;v,\omega) d\omega dv - \frac{1}{2}\int_0^\infty \int_{S^1} \omega I^2(0,t;v,\omega) d\omega dv \\ + \int_0^1 \int_0^\infty \int_{S^1} \eta\sigma_a I^2 d\omega dv dx + \int_0^1 \int_0^\infty \int_{S^1} \eta\sigma_s (\tilde{I} - I)^2 d\omega dv dx \\ \leq C_1(\epsilon) \int_0^1 \int_0^\infty \int_{S^1} \eta\sigma_a B^2 d\omega dv dx + \epsilon \int_0^1 \int_0^\infty \int_{S^1} \eta\sigma_a I^2 d\omega dv dx \\ \leq C_1(\epsilon) \int_0^1 \theta^{1+\alpha} dx \int_0^\infty \int_{S^1} f(v,\omega) d\omega dv + \epsilon \int_0^1 \int_0^\infty \int_{S^1} \eta\sigma_a I^2 d\omega dv dx \\ \leq C_1(\epsilon) \int_0^1 \theta^{1+\alpha} dx + \epsilon \int_0^1 \int_0^\infty \int_{S^1} \eta\sigma_a I^2 d\omega dv dx \\ \leq C_1 + \epsilon \int_0^1 \int_0^\infty \int_{S^1} \eta\sigma_a I^2 d\omega dv dx.$$

Similarly, we also deduce that

$$\int_0^1 \int_0^\infty \int_{S^1} \eta(\sigma_a + \sigma_s) I^2 d\omega dv dx \leq \int_0^1 \theta^{1+\alpha} dx \leq C_1. \tag{2.2.58}$$

In order to derive (2.2.5), we consider now the following integro-differential equation

$$\begin{cases} \omega \dfrac{\partial}{\partial x} I(x,t;v,\omega) = \eta\sigma_a(v,\omega;\eta,\theta)[B(v,\theta) - I(x;v,\omega)] + \eta\sigma_s(v;\eta,\theta) \\ \qquad\qquad \times [\tilde{I}(x;v) - I(x;v,\omega)] \text{ on } \mathbf{\Omega} \times [0,\ t] \times R_+ \times S^1, \\ I(0,t;v,\omega) = 0 \quad \text{for all } \omega \in (0,1), \\ I(1,t;v,\omega) = 0 \quad \text{for all } \omega \in (-1,0), \\ I(x,0;v,\omega) = I_0(x;v,\omega) \text{ on } \mathbf{\Omega} \times \mathbb{R}_+ \times S^1. \end{cases} \tag{2.2.59}$$

Solving explicitly the ordinary differential equation and using boundary condition (2.1.1), we arrive at (see [19] for details)

$$I(x,t;v,\omega)=\begin{cases}\displaystyle\int_0^x e^{\int_x^y \frac{\eta(\sigma_a+\sigma_s)}{\omega}dz}\frac{\eta}{\omega}(\sigma_a B+\sigma_s\tilde{I})dy \quad \text{for} \quad \text{all } \omega\in(0,\,1),\\ \displaystyle -\int_x^1 e^{\int_x^y \frac{\eta(\sigma_a+\sigma_s)}{\omega}dz}\frac{\eta}{\omega}(\sigma_a B+\sigma_s\tilde{I})dy \quad \text{for} \quad \text{all } \omega\in(-1,\,0).\end{cases} \tag{2.2.60}$$

Using the Young inequality, (2.1.18), (2.2.3), and (2.2.56), we have for all $\omega\in(0,\,1)$,

$$\begin{aligned}
&\int_0^\infty\int_{S^1} I\,dv\,d\omega\\
&=\int_0^\infty\int_{S^1}\Big(\int_0^x e^{\int_x^y \frac{\eta(\sigma_a+\sigma_s)}{\omega}dz}\frac{\eta}{\omega}(\sigma_a B+\sigma_s\tilde{I})dy\Big)dv\,d\omega\\
&\le\Big|\int_0^\infty\int_{S^1}\int_0^1\frac{\eta}{\omega}\big(\sigma_a B+\sigma_s\tilde{I}\big)\,dy\,dv\,d\omega\Big|\\
&=\Big|\int_0^1\eta\int_0^\infty\int_{S^1}\frac{1}{\omega}\sigma_a B\,d\omega\,dv\,dx+\int_0^1\eta\int_0^\infty\int_{S^1}\frac{1}{\omega}\sigma_s(\tilde{I}-I+I)\,d\omega\,dv\,dx\Big|\\
&\le C_1\int_0^1\theta^{1+\alpha}dx+C_1\Big|\int_0^1\eta\int_0^\infty\int_{S^1}\Big[\frac{1}{\omega^2}\sigma_s+\sigma_s(\tilde{I}-I)^2+\sigma_s I^2\Big]d\omega\,dv\,dx\Big|\\
&\le C_1\int_0^1\theta^{1+\alpha}dx\le C_1.
\end{aligned} \tag{2.2.61}$$

In the same manner, we have the same result for all $\omega\in(-1,\,0)$. This completes the proof. □

Obviously, we can obtain the following result by lemmas 2.2.1 and 2.2.6.

Lemma 2.2.7. *If assumptions in theorem 2.1.1 hold, then there holds that for any* $t>0$,

$$\left\|\theta+\theta^{1+r}+v^2\right\|^2+\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2\,dx\,ds\le C_1(1+\Lambda)^{\beta_8}, \tag{2.2.62}$$

where

$$\begin{aligned}
&\beta_7=\max(2r+1-2q,0),\\
&\beta_8=\max\Big\{\beta_7,\frac{1}{2}(\beta+\max(2r+2-q,0)),\,q+1+\beta,\frac{3\beta_1}{4},\frac{\beta_2+\beta+r+1}{2},\\
&\qquad\qquad \max(r-q,0)+\frac{\beta_2}{2},\max(q-r,0)+\frac{\beta_2}{2}\Big\}.
\end{aligned}$$

Proof. Multiplying (2.1.5) by $e+\frac{1}{2}v^2$, integrating the resultant over $[0, 1]\times(0, t)$ and using lemmas 2.2.1–2.2.6 and (2.1.15), we get

$$\begin{aligned}
&\left\|\theta(t)+\theta^{1+r}(t)+v^2\right\|^2+\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2dxds\\
&\le C_1+C_1\int_0^t\int_0^1(1+\theta)^{2r+1}|v\theta_x|dxds+C_1\int_0^t\int_0^1(1+\theta)^{2r+2}|v\eta_x|dxds\\
&\quad+C_1\int_0^t\int_0^1(1+\theta)^{q+r+1}|\theta_x\eta_x|dxds+C_1\int_0^t\int_0^1(1+\theta)^{r+1}v^2|v_x|dxds\\
&\quad+C_1\int_0^t\int_0^1(1+\theta)^{r+1}|vv_x\eta_x|dxds+C_1\int_0^t\int_0^1(1+\theta)^{r}|vv_x\theta_x|dxds\\
&\quad+C_1\int_0^t\int_0^1(1+\theta)^{q}|vv_x\theta_x|dxds+C_1\int_0^t\int_0^1\left|\eta(S_E)_R\left(e+\frac{1}{2}v^2\right)\right|dxds\\
&:=C_1+\sum_{i=1}^{8}D_i.
\end{aligned}\tag{2.2.63}$$

Similarly to those in [104], we have estimates of D_1, D_2, and D_3, for any $\epsilon>0$,

$$\begin{aligned}
&D_1\le\epsilon\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2dxds+C_1(\epsilon)(1+\Lambda)^{\beta_7},\\
&D_2\le C_1(1+\Lambda)^{\frac{1}{2}(\beta+\max(2r+2-q,0))},\\
&D_3\le\epsilon\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2dxds+C_1(\epsilon)(1+\Lambda)^{\beta+1+q}.
\end{aligned}$$

Here we only give estimates of D_i, $(i=4, 5, 6, 7, 8)$, for any $\epsilon>0$,

$$\begin{aligned}
D_4&\le C_1\Big(\int_0^t\int_0^1(1+\theta)^{2m}v_x^2dxds\Big)^{\frac{1}{2}}\Big(\int_0^t\int_0^1(1+\theta)^{r+1-q}v^4dxds\Big)^{\frac{1}{2}}\\
&\le C_1(1+\Lambda)^{\frac{\beta_1}{2}}\sup_{s\in[0,\,t]}\|v(s)\|_{L^\infty}\Big(\int_0^t\int_0^1(1+\theta)^{2m}v^2dxds\Big)^{\frac{1}{2}}\\
&\le C_1(1+\Lambda)^{\frac{\beta_1}{2}}\sup_{s\in[0,\,t]}\|v_x(s)\|^{\frac{1}{2}}\Big(\int_0^t\int_0^1(1+\theta)^{2m}v^2dxds\Big)^{\frac{1}{2}}\\
&\le C_1(1+\Lambda)^{\frac{3\beta_1}{4}},\\
D_5&\le C_1\Big(\int_0^t\int_0^1(1+\theta)^{r+1}\eta_x^2dxds\Big)^{\frac{1}{2}}\Big(\int_0^t\int_0^1(1+\theta)^{r+1}v^2v_x^2dxds\Big)^{\frac{1}{2}}\\
&\le C_1(1+\Lambda)^{\frac{\beta}{2}+\frac{r+1}{2}}\Big(\int_0^t\|v_x\|_{L^\infty}^2\int_0^1v^2dxds\Big)^{\frac{1}{2}}\\
&\le C_1(1+\Lambda)^{\frac{\beta+\beta_2}{2}+\frac{r+1}{2}},
\end{aligned}$$

$$\begin{aligned}
D_6 &\le \epsilon \int_0^t \int_0^1 (1+\theta)^{r+q} \theta_x^2 dxds + C_1(\epsilon) \int_0^t \int_0^1 (1+\theta)^{r-q} v^2 v_x^2 dxds \\
&\le \epsilon \int_0^t \int_0^1 (1+\theta)^{r+q} \theta_x^2 dxds + C_1(\epsilon)(1+\Lambda)^{\max(r-q,0)+\frac{\beta_2}{2}}, \\
D_7 &\le \epsilon \int_0^t \int_0^1 (1+\theta)^{r+q} \theta_x^2 dxds + C_1(\epsilon) \int_0^t \int_0^1 (1+\theta)^{q-r} v^2 v_x^2 dxds \\
&\le \epsilon \int_0^t \int_0^1 (1+\theta)^{r+q} \theta_x^2 dxds + C_1(\epsilon)(1+\Lambda)^{\max(q-r,0)+\frac{\beta_2}{2}}.
\end{aligned}$$

At last, we use corollary 2.2.1, definitions of S and $\tilde{I}$, (2.1.17) and (2.2.61) to get

$$\begin{aligned}
D_8 &\le \Big| \int_0^t \int_0^1 \eta (S_E)_R \Big(e + \frac{1}{2} v^2 \Big) dxds \Big| \\
&= \Big| \int_0^t \int_0^1 \eta \Big(e + \frac{1}{2} v^2 \Big) \Big\{ \int_{-1}^1 \int_0^\infty \sigma_a (B - I) dv d\omega + \int_{-1}^1 \int_0^\infty \sigma_s (\tilde{I} - I) dv d\omega \Big\} dxds \Big| \\
&\le \Big| \int_0^t \int_0^1 \eta \Big(e + \frac{1}{2} v^2 \Big) \int_{-1}^1 \int_0^\infty \sigma_a (B - I) dv d\omega dxds \Big| \\
&\le \int_0^t \int_0^1 \eta \Big| \Big(e + \frac{1}{2} v^2 \Big) \int_{-1}^1 \int_0^\infty \sigma_a (B - I) dv d\omega \Big| dxds \\
&\le \int_0^t \int_0^1 \Big| \Big(e + \frac{1}{2} v^2 \Big) \Big(\int_{-1}^1 \int_0^\infty \theta^{1+\alpha} f dv d\omega + \int_{-1}^1 \int_0^\infty \eta \sigma_a I dv d\omega \Big) \Big| dxds \\
&\le C_1 \int_0^t V(s) ds + C_1 \int_0^t \|v\|_{L^\infty}^2 \Big(\int_0^1 \theta^{1+\alpha} dx + C_1 \Big) ds \\
&\le C_1 \int_0^t \Big(V(s) + \|v\|_{L^\infty}^2 \Big) ds \le C_1 + \int_0^t \|v\| \|v_x\| ds \\
&\le C_1 + \Big(\int_0^t \|v_x\|^2 ds \Big)^{\frac{1}{2}} \le C_1 (1+\Lambda)^{\frac{\beta}{4}}.
\end{aligned}$$

Inserting estimates of D_1, D_2, D_3, D_4, and D_5 into (2.2.63), using the Young inequality and taking $\varepsilon > 0$ small enough, we conclude

$$\big\| \theta(t) + \theta^{1+r}(t) + v^2 \big\|^2 + \int_0^t \int_0^1 (1+\theta)^{q+r} \theta_x^2 dxds \le C_1 \Lambda^{\beta_8}.$$

The proof is hence complete. □

In the next lemma, we shall prove the new uniform-in-time upper bound on temperature $\theta(x, t)$. The difficulty of the proof is how to derive the uniform-in-time estimate on the last term $\int_0^t \int_0^1 \eta (S_E)_R K_t dxds$ in (2.2.67) below.

Lemma 2.2.8. *If assumptions in theorem 2.1.1 hold, the there holds that for any* $t > 0$,

$$\int_0^1 (1+\theta^{2q})\theta_x^2(x,t)\,dx + \int_0^t \int_0^1 (1+\theta^{q+r})\theta_t^2\,dxds \le C_1(1+\Lambda)^{\beta_{13}}, \tag{2.2.64}$$

$$\Lambda \le C_1, \tag{2.2.65}$$

where

$$\begin{aligned}
\beta_9 &= \{2\max(q-r,0)+2\beta+\beta_8, \max(q-r,0)+\beta+(\beta_8+\beta_1+1)/2,\\
&\quad \max(q-r,0)+\beta+(\beta_8+\beta_6)/2, \max(q-r,0)\},\\
\beta_{10} &= \max\{\max(q-r,0)+q+2+\beta, 2\max(q-r,0)+r+2+2\beta,\\
&\quad \max(q-r,0)+\beta+(\beta_1+r+3)/2, \max(q-r,0)+\beta+(\beta_6+r+2)/2,\\
&\quad (\max(q-r),0)\},\\
\beta_{11} &= \min(\beta_9,\beta_{10}),\\
\beta_{12} &= \left\{\frac{\max(3q+2-r,0)+\beta_1+\beta_8}{2}, \frac{\max(3q+2-r,0)+\beta_2+\beta+\beta_8}{2}\right\},\\
\beta_{13} &= \max\left\{\frac{3q+4+\beta_1}{2}, \frac{3q+4+\beta_2+\beta}{2}\right\},\\
\beta_{14} &= \min\{\beta_{12},\beta_{13}\},\\
\beta_{15} &= \max\{\beta_1+1, \beta_4, \beta_6, \beta_{11}, \beta_{14}, \max(2q+2,\beta_1)\}.
\end{aligned}$$

Proof. Let

$$K(\eta,\theta) = \int_0^\theta \frac{\kappa(\eta,u)}{\eta}du,$$

$$X(t) = \int_0^t \int_0^1 (1+\theta^{q+r})\theta_t^2\,dxds, \quad Y(t) = \int_0^1 (1+\theta^{2q})\theta_x^2\,dx.$$

Then it is easy to verify that

$$K_t = K_\eta v_x + \frac{\kappa}{\eta}\theta_t, \quad K_{xt} = \left(\frac{\kappa\theta_t}{\eta}\right)_t + K_{\eta\eta}v_x\eta_x + \left(\frac{\kappa}{\eta}\right)_\eta \eta_x\theta_t + K_\eta v_{xx}.$$

We know from (2.1.18) that

$$|K_\eta| + |K_{\eta\eta}| \le C_1(1+\theta^{q+1}).$$

The equation (2.1.5) can be rewritten as

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}v_x^2 = \left(\frac{\kappa\theta_x}{\eta}\right)_x - \eta(S_E)_R. \tag{2.2.66}$$

Multiplying (2.2.66) by K_t and integrating the result over $[0, 1] \times (0, t)$, we arrive at

$$\int_0^t\int_0^1 \Big(e_\theta\theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}v_x^2\Big)K_t\,dxds + \int_0^t\int_0^1 \Big[\Big(\frac{\kappa\theta_x}{\eta}\Big)\Big\{\Big(\frac{\kappa\theta_t}{\eta}\Big)_t + K_{\eta\eta}v_x\eta_x + K_\eta v_{xx} + \Big(\frac{\kappa}{\eta}\Big)_\eta\eta_x\theta_t\Big\} + \eta(S_E)_R K_t\Big]dxds = 0. \tag{2.2.67}$$

Now we estimate each term in (2.2.67). Similarly to those in [104], we have

$$\int_0^t\int_0^1 e_\theta\theta_t K_t\,dxds \geq C_1X(t) - C_1(1+\Lambda)^{\beta_1+1}, \tag{2.2.68}$$

$$\Big|\int_0^t\int_0^1 \Big(\theta p_\theta v_x - \frac{\mu_0 v_x^2}{\eta}\Big)K_t\,dxds\Big| \leq \frac{C_1}{8}X(t) + C_1(1+\Lambda)^{\beta_1+1} + C_1(1+\Lambda)^{\beta_4} + C_1(1+\Lambda)^{\beta_6}, \tag{2.2.69}$$

$$\int_0^t\int_0^1 \frac{\kappa\theta_x}{\eta}\Big(\frac{\kappa\theta_x}{\eta}\Big)_t dxds \geq C_1Y(t) - C_1, \tag{2.2.70}$$

$$\begin{aligned}
&\Big|\int_0^t\int_0^1 \frac{\kappa\theta_x}{\eta}\Big(K_\eta v_{xx} + K_{\eta\eta}v_x\eta_x\Big)dxds\Big| \\
&\leq C_1\int_0^t\int_0^1 (1+\theta)^{2q+1}|\theta_x v_{xx} + \theta_x v_x\eta_x|\,dxds \\
&\leq C_1(1+\Lambda)^{\frac{\max(3q+2-r,0)}{2}}\Big(\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2\,dxds\Big)^{\frac12}\Big(\int_0^t\int_0^1 v_{xx}^2\,dxds\Big)^{\frac12} \\
&\quad + C_1(1+\Lambda)^{\frac{\max(3q+2-r,0)}{2}}\Big(\int_0^t\int_0^1 (1+\theta)^{q+r}\theta_x^2\,dxds\Big)^{\frac12}\Big(\int_0^t \|v_x\|_{L^\infty}^2\|\eta_x\|^2\,dxds\Big)^{\frac12} \\
&\leq C_1(1+\Lambda)^{\frac{\max(3q+2-r,0)+\beta_1+\beta_8}{2}} + C_1(1+\Lambda)^{\frac{\max(3q+2-r,0)+\beta_2+\beta+\beta_8}{2}} \\
&\leq C_1(1+\Lambda)^{\beta_{12}}.
\end{aligned}$$

On the other hand, it is easy to verify

$$
\begin{aligned}
&\Big|\int_0^t\int_0^1 \frac{\kappa\theta_x}{\eta}(K_\eta v_{xx}+K_{\eta\eta}v_x\eta_x)\,dxds\Big|\\
&\le C_1\int_0^t\int_0^1(1+\theta)^{2q+1}|\theta_x v_{xx}+\theta_x v_x\eta_x|\,dxds\\
&\le C_1(1+\Lambda)^{\frac{3q+4}{2}}\Big(\int_0^t\int_0^1\frac{(1+\theta)^q\theta_x^2}{\theta^2}dxds\Big)^{\frac12}\Big(\int_0^t\int_0^1 v_{xx}^2\,dxds\Big)^{\frac12}\\
&\quad + C_1(1+\Lambda)^{\frac{3q+4}{2}}\Big(\int_0^t\int_0^1\frac{(1+\theta)^q\theta_x^2}{\theta^2}dxds\Big)^{\frac12}\Big(\int_0^t\|v_x\|_{L^\infty}^2\|\eta_x\|^2\,dxds\Big)^{\frac12}\\
&\le C_1(1+\Lambda)^{\frac{3q+4+\beta_1}{2}}+C_1(1+\Lambda)^{\frac{3q+4+\beta_2+\beta}{2}}\\
&\le C_1(1+\Lambda)^{\beta_{13}}.
\end{aligned}
$$

Therefore

$$
\Big|\int_0^t\int_0^1\frac{\kappa\theta_x}{\eta}(K_\eta v_{xx}+K_{\eta\eta}v_x u_x)\,dxds\Big|\le C_1(1+\Lambda)^{\beta_{14}}. \tag{2.2.71}
$$

Now we estimate the last two terms in (2.2.67). First, by lemmas 2.2.1–2.2.7, we have

$$
\begin{aligned}
\Big|\int_0^t\int_0^1\frac{\kappa\theta_x}{\eta}\Big(\frac{\kappa}{\eta}\Big)_\eta\eta_x\theta_t\,dxds\Big| &\le C_1\int_0^t\int_0^1(1+\theta)^q\Big|\frac{\kappa\theta_x}{\eta}\eta_x\theta_t\Big|\,dxds\\
&\le \frac{C_1}{8}X(t)+C_1\int_0^t\int_0^1\Big(\frac{\kappa\theta_x}{\eta}\Big)^2(1+\theta)^{q-r}\eta_x^2\,dxds\\
&\le \frac{C_1}{8}X(t)+C_1(1+\Lambda)^{\max(q-r,0)+\beta}\int_0^t\Big\|\frac{\kappa\theta_x}{\eta}\Big\|_{L^\infty}^2 ds\\
&\le \frac{C_1}{8}X(t)+C_1(1+\Lambda)^{\max(q-r,0)+\beta}\int_0^t\Big[\Big\|\frac{\kappa\theta_x}{\eta}\Big\|^2\\
&\quad+\int_0^1\Big|\frac{\kappa\theta_x}{\eta}\Big(\frac{\kappa\theta_x}{\eta}\Big)_x\Big|dx\Big](s)\,ds\\
&\le \frac{C_1}{8}X(t)+C_1(1+\Lambda)^{\max(q-r,0)+\beta}\\
&\quad\times\Big\{(1+\Lambda)^{\max(q-r,0)}\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2\,dxds\\
&\quad+\Big(\int_0^t\int_0^1(1+\theta)^{q+r}\theta_x^2\,dxds\Big)^{1/2}\\
&\quad\times\Big(\int_0^t\int_0^1(1+\theta)^{q-r}\Big|\Big(\frac{\kappa\theta_x}{\eta}\Big)_x\Big|^2dxds\Big)^{1/2}\Big\},
\end{aligned}
$$

which, along with (2.2.66), lemmas 2.2.4, 2.2.5 and corollary 2.2.2, leads to

$$
\begin{aligned}
&\Big|\int_0^t\int_0^1 \frac{\kappa\theta_x}{\eta}\Big(\frac{\kappa}{\eta}\Big)_\eta \eta_x\theta_t\,dxds\Big| \\
&\quad\le \frac{C_1}{8}X(t) + C_1(1+\Lambda)^{2\max(q-r,0)+\beta+\beta_8} + C_1(1+\Lambda)^{\max(q-r,0)+\beta+\beta_8/2} \\
&\qquad\times\Big\{X(t)+\int_0^t\int_0^1\Big[(1+\theta)^{q+r+2}v_x^2+(1+\theta)^{q-r}v_x^4+(1+\theta)^{q-r}\eta^2(S_E)_R^2\Big]dxds\Big\}^{1/2} \\
&\quad\le \frac{C_1}{4}X(t) + C_1(1+\Lambda)^{2\max(q-r,0)+2\beta+\beta_8} + C_1(1+\Lambda)^{\max(q-r,0)+\beta+(\beta_8+\beta_1+1)/2} \\
&\qquad+ C_1(1+\Lambda)^{\max(q-r,0)+\beta+(\beta_8+\beta_6)/2} + C_1(1+\Lambda)^{\max(q-r,0)+\beta+\beta_7} \\
&\quad\le \frac{C_1}{4}X(t) + C_1(1+\Lambda)^{\beta_9}.
\end{aligned}
\tag{2.2.72}
$$

However, we also know that

$$
\begin{aligned}
&\Big|\int_0^t\int_0^1 \frac{\kappa\theta_x}{\eta}\Big(\frac{\kappa}{\eta}\Big)_\eta \eta_x\theta_t\,dxds\Big| \\
&\quad\le \frac{C_1}{8}X(t) + C_1\int_0^t\int_0^1\Big(\frac{\kappa\theta_x}{\eta}\Big)^2(1+\theta)^{q-r}\eta_x^2\,dxds \\
&\quad\le \frac{C_1}{8}X(t) + C_1(1+\Lambda)^{\max(q-r,0)+\beta} \\
&\qquad\times\Big\{\int_0^t\int_0^1(1+\theta)^{2q}\theta_x^2\,dxds + \int_0^t\int_0^1(1+\theta)^q|\theta_x|\Big|\Big(\frac{\kappa\theta_x}{\eta}\Big)_x\Big|dxds\Big\} \\
&\quad\le \frac{C_1}{8}X(t) + C_1(1+\Lambda)^{\max(q-r,0)+\beta}\Big\{\Lambda^{q+2}\int_0^t V(s)\,ds \\
&\qquad+\Big(\int_0^t V(s)\,ds\Big)^{1/2}\Big(\int_0^t\int_0^1\theta^2(1+\theta)^q\Big|\Big(\frac{\kappa\theta_x}{\eta}\Big)_x\Big|^2dxds\Big)^{1/2}\Big\} \\
&\quad\le \frac{C_1}{8}X(t) + C_1(1+\Lambda)^{\max(q-r,0)+q+2+\beta} \\
&\qquad+ C_1(1+\Lambda)^{\max(q-r,0)+\beta+(r+2)/2}\Big\{\int_0^t\int_0^1(1+\theta)^{q-r}\Big|\Big(\frac{\kappa\theta_x}{\eta}\Big)_x\Big|^2dxds\Big\}^{1/2} \\
&\quad\le \frac{C_1}{4}X(t) + C_1(1+\Lambda)^{\max(q-r,0)+q+2+\beta} + C_1(1+\Lambda)^{2\max(q-r,0)+2\beta+r+2} \\
&\qquad+ C_1(1+\Lambda)^{\max(q-r,0)+\beta+(r+3+\beta_1)/2} + C_1(1+\Lambda)^{\max(q-r,0)+\beta+(r+2+\beta_6)/2} \\
&\qquad+ C_1(1+\Lambda)^{\max(q-r,0)} \\
&\quad\le \frac{C_1}{4}X(t) + C_1(1+\Lambda)^{\beta_{10}}.
\end{aligned}
$$

Next, we estimate the last term,

$$\begin{aligned}\Big|\int_0^t\int_0^1 \eta(S_E)_R K_t dxds\Big| \le &\int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_a B dv d\omega\Big)|K_t|dxds\\ &+\int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_a I dv d\omega\Big)|K_t|dxds\\ &+\int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_s|\tilde{I}-I| dv d\omega\Big)|K_t|dxds\\ =:&\, P+Q+R. \end{aligned}\tag{2.2.73}$$

Using (2.1.18), lemma 2.2.6, the Young and the Hölder inequalities, we conclude

$$\begin{aligned}|P| &\le \int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_a B dv d\omega\Big)\Big|K_\eta v_x+\frac{\kappa}{\eta}\theta_t\Big|dxds\\ &\le C_1\int_0^t\int_0^1(1+\theta^{q+1+\alpha})|v_x|dxds+C_1\int_0^t\int_0^1(1+\theta^{q+\alpha})|\theta_t|dxds\\ &\le \frac{1}{8}X(t)+C_1\Lambda^{q+2\alpha-r},\end{aligned}$$

$$\begin{aligned}Q&\le \int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_a I dv d\omega\Big)\Big|K_\eta v_x+\frac{\kappa}{\eta}\theta_t\Big|dxds\\ &\le C_1\int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_a d\omega dv\Big)^{\frac12}\Big(\int_0^\infty\int_{S^1}\eta\sigma_a I^2 d\omega dv\Big)^{\frac12}(1+\theta^{q+1})|v_x|dxds\\ &\quad +C_1\int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_a d\omega dv\Big)^{\frac12}\Big(\int_0^\infty\int_{S^1}\eta\sigma_a I^2 d\omega dv\Big)^{\frac12}(1+\theta^{q+1})|\theta_t|dxds\\ &\le C_1(1+\Lambda)^{2q+2}\int_0^t\int_0^1|v_x|^2dxds+\epsilon X(t)+C_1(\epsilon)\int_0^t\int_0^1\Big(\int_0^\infty\int_{S^1}\eta\sigma_a I^2 d\omega dv\Big)\\ &\le \epsilon X(t)+C_1(1+\Lambda)^{2q+2}.\end{aligned}$$

Using the same technique, we also get

$$R\le \frac{1}{8}X(t)+C_1\Lambda^{\max(q-r,0)}.$$

Inserting estimates on P, Q and R into (2.2.73), using the Young inequality and taking $\epsilon>0$ small enough, we derive

$$X(t)+Y(t)\le C_1(1+\Lambda)^{\beta_{15}}.\tag{2.2.74}$$

By lemmas 2.2.1–2.2.7 and the Hölder inequality, there exists a point $a(t)\in[0,1]$ such that for any $t>0$, $\int_0^1\theta(x,\,t)dx=\theta(a(t),\,t)$, we can hence deduce

$$
\begin{aligned}
|\theta^{q+\frac{r+3}{2}}(x,t)-\theta^{q+\frac{r+3}{2}}(a(t),t)| &\le C_1\int_0^1 \theta^{q+\frac{r+3}{2}}|\theta_x|\,dx\\
&\le C_1\Big(\int_0^1 (1+\theta^{2q})\theta_x^2\,dx\Big)^{\frac12}\Big(\int_0^1 \theta^{r+1}\,dx\Big)^{\frac12}\\
&\le C_1\,Y^{\frac12}(t).
\end{aligned}
$$

Then

$$
\Lambda \le C_1\,Y^{\frac{1}{2q+r+3}} + C_1 \le C_1(1+\Lambda)^{\frac{\beta_{15}}{2q+r+3}}. \tag{2.2.75}
$$

After a lengthy calculation, it is easy to verify that assumption (2.1.16) implies that $\beta_{15} < 2q + r + 3$. Therefore, by the Young inequality, it follows from (2.2.75) that

$$
\Lambda \le C_1.
$$

This thus completes the proof. □

The following lemmas are concerned with the new arguments on uniform-in-time estimates of radiative term $I(x,\,t;\,\nu,\,\omega)$.

Lemma 2.2.9. *There holds*

$$
\Big\|\int_0^\infty\int_{S^1} I^2\,d\omega d\nu\Big\|_{L^\infty(Q_t)} \le C_1,\quad \forall\ t>0, \tag{2.2.76}
$$

$$
\Big\|\int_0^\infty\int_{S^1} |I_x|\,d\omega d\nu\Big\|_{L^\infty(Q_t)} \le C_1,\quad \forall\ t>0, \tag{2.2.77}
$$

$$
\int_0^t\int_0^1\int_0^\infty\int_{S^1} I_t^2\,d\omega d\nu dx ds \le C_1,\quad \forall\ t>0. \tag{2.2.78}
$$

Proof. By (2.2.57) and (2.2.66), we can easily get (2.2.76). By the definition of $\tilde I$, we can derive from (2.2.76) that

$$
\Big\|\int_0^\infty\int_{S^1} \tilde I\,d\omega d\nu\Big\|_{L^\infty(Q_t)} \le C_1. \tag{2.2.79}
$$

From (2.2.60), using the Young and the Hölder inequalities, (2.1.18) and lemmas 2.2.7, 2.2.8, we derive

$$
\begin{aligned}
\int_0^\infty\int_{S^1} I^2\,d\nu d\omega &\le \int_0^\infty\int_{S^1}\Big(\int_0^1 \frac{\eta}{\omega}(\sigma_a B+\sigma_s\tilde I)\,dx\Big)^2 d\nu d\omega\\
&\le C_1\int_0^\infty\int_{S^1}\Big(\frac{\eta^2}{\omega^2}\sigma_a\,dx\cdot\int_0^1 \sigma_a B^2\,dx\Big)d\omega d\nu\\
&\quad + C_1\int_0^\infty\int_{S^1}\Big(\int_0^1\frac{\eta^2}{\omega^2}\sigma_s\,dx\cdot\int_0^1\sigma_s\tilde I^2\,dx\Big)d\omega d\nu\\
&\le C_1 + C_1\int_0^\infty\int_{S^1}\int_0^1 (\sigma_s(\tilde I-I)^2+\sigma_s I^2)\,dxd\omega d\nu \le C_1,
\end{aligned} \tag{2.2.80}
$$

which, by the definition of $\tilde{I}$, implies

$$\int_0^\infty \int_{S^1} \tilde{I}^2 d\omega dv \le C_1. \tag{2.2.81}$$

From (2.1.6), we get

$$I_x = -\frac{\eta}{\omega}(\sigma_a + \sigma_s)I + \frac{\eta}{\omega}(\sigma_a B + \sigma_s \tilde{I}).$$

Integrating the above equality and using (2.1.18), we obtain

$$\begin{aligned}\int_0^\infty \int_{S^1} |I_x| d\omega dv &\le C_1 \int_0^\infty \int_{S^1} \frac{1}{|\omega|}(\sigma_a + \sigma_s)|I| d\omega dv \\ &\quad + C_1 \int_0^\infty \int_{S^1} \frac{1}{|\omega|}(\sigma_a B + \sigma_s \tilde{I}) d\omega dv \\ &\le C_1 + C_1 \int_0^\infty \int_{S^1} |I| d\omega dv + C_1 \int_0^\infty \int_{S^1} |\tilde{I}| d\omega dv \end{aligned} \tag{2.2.82}$$

which, using (2.2.76) and (2.2.81), gives (2.2.77).

By (2.2.60), we have for any $\omega \in (0, 1)$,

$$\begin{aligned} I_t &= \int_0^x e^{\int_x^y \frac{\eta}{\omega}(\sigma_a + \sigma_s) dz} \Big(\int_x^y \frac{\eta}{\omega}(\sigma_a + \sigma_s) dz \Big)_t \frac{\eta}{\omega}(\sigma_a B + \sigma_s \tilde{I}) \\ &\quad + \int_0^x e^{\int_x^y \frac{\eta}{\omega}(\sigma_a + \sigma_s) dz} \Big(\frac{\eta}{\omega}(\sigma_a B + \sigma_s \tilde{I}) \Big)_t dy =: A_2 + B_2. \end{aligned} \tag{2.2.83}$$

Using the Young inequality, (2.2.81), (2.1.18), lemmas 2.2.4 and 2.2.8, we deduce

$$\begin{aligned} &\int_0^t \int_0^\infty \int_{S^1} A_2^2 d\omega dv ds \\ &\quad \le C_1 \int_0^t \int_0^\infty \int_{S^1} \Big[\int_0^x \Big(\int_x^y \frac{v_x}{\omega}(\sigma_a + \sigma_s) + \frac{\eta}{\omega}((\sigma_a + \sigma_s)_\eta v_x \\ &\qquad + (\sigma_a + \sigma_s)_\theta \theta_t) dz \Big) \frac{\eta}{\omega}(\sigma_a B + \sigma_s \tilde{I}) dy \Big]^2 d\omega dv ds \\ &\quad \le C_1 \int_0^t \int_0^\infty \int_{S^1} \Big(\int_0^1 \frac{v_x^2}{\omega^2}(\sigma_a + \sigma_s + (\sigma_a)_\eta + (\sigma_s)_\eta)^2 \\ &\qquad + \frac{\theta_t^2}{\omega^2}((\sigma_a)_\theta + (\sigma_s)_\theta)^2 dx \cdot \int_0^1 \frac{\eta^2}{\omega^2}\sigma_a^2 B^2 dx \Big) d\omega dv ds \\ &\qquad + C_1 \int_0^t \int_0^\infty \int_{S^1} \Big(\int_0^1 \frac{v_x^2}{\omega^2}(\sigma_a + \sigma_s + (\sigma_a)_\eta + (\sigma_s)_\eta)^2 \\ &\qquad + \frac{\theta_t^2}{\omega^2}((\sigma_a)_\theta + (\sigma_s)_\theta)^2 dx \cdot \int_0^1 \frac{\eta^2}{\omega^2}\sigma_s^2 \tilde{I}^2 dx \Big) d\omega dv ds \\ &\quad \le C_1 \int_0^t \int_0^1 (v_x^2 + \theta_t^2) dx ds + C_1 \int_0^t \int_0^\infty \int_{S^1} \int_0^1 \tilde{I}^2 dx d\omega dv ds \\ &\quad \le C_1. \end{aligned} \tag{2.2.84}$$

Analogously,

$$\int_0^t\int_0^\infty\int_{S^1} B_2^2\,d\omega dv ds \le C_1\int_0^t\int_0^\infty\int_{S^1}\Big[\int_0^x\Big(\frac{v_x}{\omega}(\sigma_a B+\sigma_s\tilde{I})+\frac{\eta}{\omega}((\sigma_a)_\eta v_x B$$
$$+(\sigma_a)_\theta\theta_t B+\sigma B_\theta\theta_t+(\sigma_s)_\eta\tilde{I}v_x+(\sigma_s)_\theta\theta_t\tilde{I}+\sigma_s\tilde{I}_t)\Big)dx\Big]^2 d\omega dv ds$$
$$\le C_1\int_0^t\int_0^1(v_x^2+\theta_t^2)dxds+C_1\int_0^x\int_0^t\int_0^\infty\int_{S^1}I_t^2\,d\omega dv ds dy$$
$$\le C_1+C_1\int_0^x\int_0^t\int_0^\infty\int_{S^1}I_t^2\,d\omega dv ds dy,$$

which, together with (2.2.84), implies

$$\int_0^t\int_0^\infty\int_{S^1}I_t^2\,d\omega dv ds\le C_1+C_1\int_0^x\int_0^t\int_0^\infty\int_{S^1}I_t^2\,d\omega dv ds dy.$$

Fixing $t>0$ and using the Gronwall inequality, we get

$$\int_0^t\int_0^\infty\int_{S^1}I_t^2\,d\omega dv ds\le C_1e^{C_1x}\le C_1e^{C_1}\le C_1,\quad \forall\, x\in[0,\,1].$$

This proves the proof. □

Lemma 2.2.10. *If assumptions in theorem 2.1.1 hold, then there holds that*

$$\|\mathcal{I}_{xx}(t)\|\le C_2,\quad \forall\, t>0. \tag{2.2.85}$$

Proof. By virtue of the direct computation, we have

$$\begin{aligned}\|\mathcal{I}_{xx}(t)\|^2&=\int_0^1\Big(\int_0^\infty\int_{S^1}I_{xx}\,d\omega dv\Big)^2dx\\&=\int_0^1\Big(\int_0^\infty\int_{S^1}\frac{1}{\omega}(\eta_xS+\eta S_x)\,d\omega dv\Big)^2dx\\&\le C_1\int_0^1\Big[\Big(\int_0^\infty\int_{S^1}\frac{1}{\omega}\eta_xS\,d\omega dv\Big)^2+\Big(\int_0^\infty\int_{S^1}\frac{1}{\omega}\eta S_x\,d\omega dv\Big)^2\Big]dx\\&=:G+H.\end{aligned}\tag{2.2.86}$$

Using (2.1.18), (2.2.40) and lemma 2.2.9, we see that

$$\begin{aligned}G&=\int_0^1\Big(\int_0^\infty\int_{S^1}\frac{1}{\omega}\eta_xS\,d\omega dv\Big)^2dx\\&=\int_0^1\eta_x^2\Big(\int_0^\infty\int_{S^1}\frac{1}{\omega}(\sigma_a(B-I)+\sigma_s(\tilde{I}-I))\,d\omega dv\Big)^2dx\\&\le C_1\int_0^1\eta_x^2dx\le C_1.\end{aligned}\tag{2.2.87}$$

Similarly,

$$
\begin{aligned}
H &= \int_0^1 \left(\int_0^\infty \int_{S^1} \frac{\eta}{\omega} \{[(\sigma_a)_\eta \eta_x + (\sigma_a)_\theta \theta_x](B - I) + \sigma_a (B_\theta \theta_x - I_x) \right. \\
&\quad \left. + [(\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x](\tilde{I} - I) + \sigma_s (\tilde{I} - I)_x\} d\omega dv \right)^2 dx \\
&\le C_1 \int_0^1 (\eta_x^2 + \theta_x^2) dx + C_1 \\
&\le C_1. \qquad (2.2.88)
\end{aligned}
$$

Plugging (2.2.87) and (2.2.88) into (2.2.86), we can get (2.2.85). The proof is complete. □

2.3 Asymptotic Behavior of Solutions in $\mathcal{H}_1$

This will establish the large-time behavior of global solutions in $\mathcal{H}_1$, which completes the proof of theorem 2.1.1.

Lemma 2.3.1. *If assumptions in theorem 2.1.1 hold, then we have*

$$\lim_{t \to +\infty} \|\eta(t) - \overline{\eta}\|_{H^1} = 0, \qquad (2.3.1)$$

$$\lim_{t \to +\infty} \|v(t)\|_{H^1} = 0, \qquad (2.3.2)$$

where $\overline{\eta} = \int_0^1 \eta(y,t) dy = \int_0^1 \eta_0(y) dy$.

Proof. By lemmas 2.2.1–2.2.8, we can derive from (2.2.47)

$$\frac{d}{dt} \|v_x\|^2 + C_1^{-1} \|v_{xx}\|^2 \le C_1. \qquad (2.3.3)$$

Applying lemma 1.1.2 to (2.3.3), and using the Poincaré inequality, we obtain

$$\|v(t)\|_{H^1}^2 \le C_1 \|v_x\|^2 \to 0 \quad as \quad t \to \infty.$$

Similarly, by lemmas 2.2.1–2.2.8, it follows from (2.2.42) that

$$\frac{d}{dt} \left\| v - \mu \frac{\eta_x}{\eta} \right\|^2 + C_1^{-1} \int_0^1 (1 + \theta^{1+r}) \eta_x^2 dx \le C_1. \qquad (2.3.4)$$

Applying lemma 1.1.2 to (2.3.4) again, and using lemmas 2.2.1–2.2.8 and (2.3.1), we conclude that as $t \to \infty$,

$$\|\eta_x\|^2 \le C_1 \left(\left\| v - \mu \frac{\eta_x}{\eta} \right\|^2 + \|v\|^2 \right) \to 0$$

which gives (2.3.2). □

Lemma 2.3.2. *If assumptions in theorem 2.1.1 hold, then we have*

$$\lim_{t\to+\infty}\left\|\theta(t)-\overline{\theta}\right\|_{H^1}=0, \tag{2.3.5}$$

where $\overline{\theta}>0$ *is determined by* $e(\overline{\eta},\overline{\theta})=\int_0^1\left(\frac{1}{2}v_0^2+e(\eta_0,\theta_0)+F_R(0)\right)dx.$

Proof. Equation (2.1.5) can be rewritten as

$$e_\theta\theta_t+(-p+\theta p_\theta)v_x-\sigma v_x+q_x+\eta(S_E)_R=0. \tag{2.3.6}$$

Multiplying (2.3.6) by $e_\theta^{-1}\theta_{xx}$, integrating the result over (0, 1) and using the Young inequality, the interpolation inequality and lemmas 2.2.1–2.2.8, we can conclude for any $\varepsilon>0$,

$$\begin{aligned}
&\frac{d}{dt}\|\theta_x(t)\|^2+2\int_0^1\frac{\kappa\theta_{xx}^2}{e_\theta\eta}\\
&=2\int_0^1\Big[\frac{\theta p_\theta v_x}{e_\theta}-\frac{\mu v_x^2}{e_\theta\eta}-\frac{(\frac{\kappa}{\eta})_x\theta_x}{e_\theta}+\frac{\eta(S_E)_R}{e_\theta}\Big]\theta_{xx}\,dx\\
&\le\frac{\varepsilon}{2}\|\theta_{xx}(t)\|^2+C_1(\|v_x\|^2+\|v_x\|_{L^4}^4+\|\theta_x\|_{L^4}^4+\|\eta_x\theta_x\|^2+\left\|(S_E)_R\right\|^2)\\
&\le\frac{\varepsilon}{2}\|\theta_{xx}(t)\|^2+C_1(\|v_x\|^2+\|v_x\|^3\|v_{xx}\|+\|v_x\|^4+\|\theta_x\|^3\|\theta_{xx}\|+\|\theta_x\|^4\\
&\qquad+\|\theta_x\|_{L^\infty}+\left\|(S_E)_R\right\|^2)\\
&\le\varepsilon\|\theta_{xx}(t)\|^2+C_1(\|v_x\|^2+\|v_{xx}\|^2+\|\theta_x\|^2+\left\|(S_E)_R\right\|^2).
\end{aligned} \tag{2.3.7}$$

Now we need to estimate the new radiative term $\eta(S_E)_R$ in (2.3.7), which did not appear in the system considered in Qin [104] ($(S_E)_R=0$).

From (2.1.18), the Young and the Hölder inequalities and lemmas 2.2.7, 2.2.8, we derive

$$\begin{aligned}
\left\|(S_E)_R\right\|^2&=\int_0^1\Big(\int_0^{+\infty}\int_{S^1}(\sigma_a(B-I)+\sigma_s(\tilde{I}-I))\,d\omega\,dv\Big)^2dx\\
&\le C_1\int_0^1\Big[\Big(\int_0^{+\infty}\int_{S^1}\sigma_a(B-I)\,d\omega\,dv\Big)^2\\
&\qquad+\Big(\int_0^{+\infty}\int_{S^1}\sigma_s(\tilde{I}-I)\,d\omega\,dv\Big)^2\Big]dx\\
&\le C_1\int_0^1\Big[\Big(\int_0^{+\infty}\int_{S^1}\sigma_a\,d\omega\,dv\Big)\Big(\int_0^{+\infty}\int_{S^1}\sigma_a(B^2+I^2)\,d\omega\,dv\Big)\\
&\qquad+C_1\Big(\int_0^{+\infty}\int_{S^1}\sigma_s\,d\omega\,dv\Big)\Big(\int_0^{+\infty}\int_{S^1}\sigma_s(\tilde{I}-I)^2\,d\omega\,dv\Big)\Big]dx\\
&\le C_1\int_0^1\theta^{\alpha+1}\,dx+C_1\le C_1,
\end{aligned} \tag{2.3.8}$$

which, together with (2.3.7), yields

$$\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}(s)\|^2 ds \le C_1.$$

Plugging (2.3.8) into (2.3.7), using lemmas 2.2.1–2.2.8 and taking $\varepsilon > 0$ small enough, we have

$$\frac{d}{dt}\|\theta_x(t)\|^2 + C_1 \int_0^1 (1+\theta^{q-r})\theta_{xx}^2 dx \le C_1(\|v_{xx}\|^2 + 1), \tag{2.3.9}$$

which, together with lemma 1.1.2 and (2.2.44), (2.2.65), yields

$$\lim_{t\to+\infty} \|\theta_x(t)\|^2 = 0. \tag{2.3.10}$$

By the Poincaré inequality, we deduce

$$\left\|\theta(t) - \overline{\theta}\right\|_{H^1} \le C_1\|\theta_x(t)\|,$$

which, combined with (2.3.10), gives (2.3.5). The proof is thus complete. □

The following lemma is the new argument on the large-time behavior of radiative term $I(x,\, t;\, v,\, \omega)$.

Lemma 2.3.3. *If assumptions in theorem 2.1.1 hold, then we have*

$$\lim_{t\to+\infty} \|\mathcal{I}(t)\|_{H^2} = 0. \tag{2.3.11}$$

Proof. By (2.2.76) and (2.1.6), the direct computation yields

$$\begin{aligned}
\frac{d}{dt}\|\mathcal{I}_x(t)\|^2 &= \frac{d}{dt}\int_0^1 \Big(\int_0^{+\infty}\int_{S^1} I_x d\omega dv\Big)^2 dx \\
&= 2\int_0^1 \Big(\int_0^{+\infty}\int_{S^1} I_x d\omega dv\Big)\Big(\int_0^{+\infty}\int_{S^1} I_{xt} d\omega dv\Big) dx \\
&\le C_1 \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} |I_{xt}| d\omega dv\Big) dx \\
&\le C_1 \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} \frac{1}{\omega}|v_x S + \eta S_t| d\omega dv\Big) dx =: A_3 + B_3.
\end{aligned} \tag{2.3.12}$$

We denote

$$\begin{aligned}
A_3 &= \int_0^1\int_0^{+\infty}\int_{S^1} \left|\frac{1}{\omega} v_x S\right| d\omega dv dx \\
&= \int_0^1\int_0^{+\infty}\int_{S^1} \left|\frac{1}{\omega} v_x(\sigma_a(B-I) + \sigma_s(\tilde{I} - I))\right| d\omega dv dx =: \widehat{C} + \widehat{D}.
\end{aligned}$$

Using (2.1.18), lemmas 2.2.7–2.2.9 and the Young inequality, we can derive

$$\begin{aligned}
\widehat{C} &\le \int_0^1 \int_0^{+\infty} \int_{S^1} \Big|\frac{1}{\omega} v_x \sigma_a (B - I)\Big| d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|}(|\omega|\theta^{\alpha+1} f(v,\omega) + |\omega| g(v,\omega) I) d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| dx \le C_1 \int_0^1 v_x^2 dx + C_1, \\
\widehat{D} &\le \int_0^1 \int_0^{+\infty} \int_{S^1} \Big|\frac{1}{\omega} v_x \sigma_s (\tilde{I} - I)\Big| d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} |\omega| k(v,\omega) |\tilde{I} - I| d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} I d\omega dv dx \le C_1 \int_0^1 |v_x| dx \le C_1 \int_0^1 v_x^2 dx + C_1.
\end{aligned}$$

We denote

$$\begin{aligned}
B_3 &= \int_0^1 \int_0^{+\infty} \int_{S^1} \Big|\frac{1}{\omega} \eta S_t\Big| d\omega dv dx \\
&= \int_0^1 \int_0^{+\infty} \int_{S^1} \Big|\frac{\eta}{\omega}\Big\{\Big[(\sigma_a)_\eta v_x + (\sigma_a)_\theta \theta_t\Big](B - I) + \sigma_a (B_\theta \theta_t - I_t) \\
&\quad + \Big[(\sigma_s)_\eta v_x + (\sigma_s)_\theta \theta_t\Big](\tilde{I} - I) + \sigma_s (\tilde{I} - I)_t\Big\}\Big| d\omega dv dx \\
&=: E + F,
\end{aligned}$$

where

$$\begin{aligned}
E &= \int_0^1 \int_0^{+\infty} \int_{S^1} \Big|\frac{\eta}{\omega}\Big\{\Big[(\sigma_a)_\eta v_x + (\sigma_a)_\theta \theta_t\Big](B - I) + \sigma_a (B_\theta \theta_t - I_t)\Big\}\Big| d\omega dv dx =: \sum_{i=1}^{6} P_i, \\
F &=: \int_0^1 \int_0^{+\infty} \int_{S^1} \Big|\frac{\eta}{\omega}\Big\{\Big[(\sigma_s)_\eta v_x + (\sigma_s)_\theta \theta_t\Big](\tilde{I} - I) + \sigma_s (\tilde{I} - I)_t\Big\}\Big| d\omega dv dx.
\end{aligned}$$

Using (2.1.18), lemmas 2.2.7–2.2.9 and the Young inequality, we can conclude

$$\begin{aligned}
|P_1| &\le C_1 \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} |(\sigma_a)_\eta B| d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} |\omega| h(v,\omega) d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| dx \le C_1 \int_0^1 v_x^2 dx + C_1.
\end{aligned}$$

Analogously, by lemmas 2.2.7–2.2.9,

$$
\begin{aligned}
|P_2| &\le C_1 \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} |(\sigma_a)_\eta I| d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| dx \le C_1 \int_0^1 v_x^2 dx + C_1, \\
|P_3| &\le C_1 \int_0^1 \theta_t \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} |(\sigma_a)_\theta| B d\omega dv dx \\
&\le C_1 \int_0^1 |\theta_t| dx \le C_1 \int_0^1 \theta_t^2 dx + C_1, \\
|P_4| &\le C_1 \int_0^1 |\theta_t| \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} |(\sigma_a)_\theta I| d\omega dv dx \\
&\le C_1 \int_0^1 |\theta_t| dx \le C_1 \int_0^1 \theta_t^2 dx + C_1, \\
|P_5| &\le C_1 \int_0^1 |\theta_t| \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} |\sigma_a B_\theta| d\omega dv dx \\
&\le C_1 \int_0^1 |\theta_t| dx \le C_1 \int_0^1 \theta_t^2 dx + C_1, \\
|P_6| &\le C_1 \int_0^1 \int_0^{+\infty} \int_{S^1} \frac{1}{|\omega|} \sigma_a |I_t| d\omega dv dx \\
&\le C_1 + \int_0^1 \int_0^{+\infty} \int_{S^1} I_t^2 d\omega dv dx.
\end{aligned}
$$

Now we estimate F as follows, by lemmas 2.2.7–2.2.9,

$$
\begin{aligned}
F \le& \int_0^1 \int_0^{+\infty} \int_{S^1} \frac{\eta}{|\omega|} |(\sigma_s)_\eta v_x| |\tilde{I} - I| d\omega dv dx + \int_0^1 \int_0^{+\infty} \int_{S^1} \frac{\eta}{|\omega|} |(\sigma_s)_\theta \theta_t| |\tilde{I} - I| d\omega dv dx \\
&+ \int_0^1 \int_0^{+\infty} \int_{S^1} \frac{\eta}{|\omega|} \sigma_s |(\tilde{I} - I)_t| d\omega dv dx =: \sum_{i=1}^3 M_i.
\end{aligned}
$$

By (2.1.18) and lemmas 2.2.7–2.2.9, we deduce

$$
\begin{aligned}
M_1 &\le \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} \frac{\eta}{|\omega|} |(\sigma_s)_\eta| |\tilde{I} - I| d\omega dv dx \\
&\le C_1 \int_0^1 |v_x| \int_0^{+\infty} \int_{S^1} |I| d\omega dv dx \le C_1 \int_0^1 v_x^2 dx + C_1, \\
M_2 &\le C_1 \int_0^1 |\theta_t| \int_0^{+\infty} \int_{S^1} |I| d\omega dv dx \le C_1 \int_0^1 \theta_t^2 dx + C_1, \\
M_3 &\le C_1 + C_1 \int_0^1 \int_0^{+\infty} \int_{S^1} I_t^2 d\omega dv dx.
\end{aligned}
$$

Inserting all previous estimates into (2.3.12) implies

$$\frac{d}{dt}\|\mathcal{I}_x(t)\|^2 \le C_1(\|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|I_t\|^2_{L^2(S^1\times\mathbb{R}_+\times\Omega)}) + C_1, \tag{2.3.13}$$

which, together with lemma 1.1.2, (2.2.41), (2.2.64), (2.2.65) and (2.2.78), yields

$$\lim_{t\to+\infty}\|\mathcal{I}_x(t)\|^2 = 0. \tag{2.3.14}$$

From (2.1.6), we derive

$$\begin{aligned}\|\mathcal{I}_{xx}(t)\|^2 &= \left\|\int_0^{+\infty}\int_{S^1} I_{xx}\,d\omega dv\right\|^2 = \left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}(\eta_x S + \eta S_x)\,d\omega dv\right\|^2\\ &\le C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_x S d\omega dv\right\|^2 + C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta S_x d\omega dv\right\|^2\\ &=: N_1 + N_2.\end{aligned} \tag{2.3.15}$$

Using (2.1.18) and lemmas 2.2.4–2.2.9, we deduce

$$\begin{aligned}N_1 &\le C_1\int_0^1 \eta_x^2\Big(\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}[\sigma_a(B-I) + \sigma_s(\tilde{I}-I)]\,d\omega dv\Big)^2 dx\\ &\le C_1\|\eta_x(t)\|^2,\end{aligned} \tag{2.3.16}$$

$$\begin{aligned}N_2 \le\ & C_1\int_0^1\Big(\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\Big(((\sigma_1)_\eta\eta_x + (\sigma_a)_\theta\theta_x)(B-I) + \sigma_a(B_\theta\theta_x - I)\\ &+ ((\sigma_s)_\eta\eta_x + (\sigma_s)_\theta\theta_x)(\tilde{I}-I) + \sigma_s(\tilde{I}-I)_x\Big)\,d\omega dv\Big)^2 dx\\ \le\ & C_1(\|\eta_x(t)\|^2 + \|\theta_x(t)\|^2 + \|\mathcal{I}_x(t)\|^2).\end{aligned} \tag{2.3.17}$$

Inserting (2.3.16) and (2.3.17) into (2.3.15) and using (2.3.1), (2.3.10) and (2.3.14), we get

$$\lim_{t\to+\infty}\|\mathcal{I}_{xx}(t)\| = 0. \tag{2.3.18}$$

Thus (2.3.11) follows from (2.3.14), (2.3.18). □

Proof of Theorem 2.1.1. Combining lemmas 2.2.1–2.2.8 and 2.3.1–2.3.3, we can complete the proof of theorem 2.1.1. □

2.4 Global Existence and Uniform-in-Time Estimates in $\mathcal{H}_2$

This section will prove the global existence of solutions and uniform-in-time estimates in $\mathcal{H}_2$. The next lemma concerns the uniform-in-time global (in time) positive lower bound (independent of t) of the absolute temperature θ.

Lemma 2.4.1. *If assumptions in theorem 2.1.1 hold, then the generalized global solution* $(\eta(t),\ v(t),\ \theta(t),\ \mathcal{I}(t))$ *to the problem* (2.1.3)–(2.1.6) *and* (2.1.11)–(2.1.14) *satisfies for all* $(x,\ t) \in [0,\ 1] \times [0,\ +\infty)$,

$$0 < C_1^{-1} \le \theta(x,\ t). \tag{2.4.1}$$

Proof. We prove (2.4.1) by contradiction. If (2.4.1) is not true, that is,

$$\inf_{(x,t)\in[0,\,1]\times[0,\,+\infty)} \theta(x,\ t) = 0,$$

then there exists a sequence $(x_n,\ t_n) \in [0,\ 1] \times [0,\ +\infty)$ such that as $n \to +\infty$,

$$\theta(x_n, t_n) \to 0. \tag{2.4.2}$$

If sequence $\{t_n\}$ has a subsequence, denoted also by t_n, converging to $+\infty$, then by the asymptotic behavior results in theorem 2.1.1, we know that as $n \to +\infty$,

$$\theta(x_n, t_n) \to \overline{\theta} > 0$$

which contradicts (2.4.2). If sequence $\{t_n\}$ is bounded, *i.e.*, there exists constant $M > 0$, independent of n, such that for any $n = 1,\ 2,\ 3,\ldots,\ 0 < t_n \le M$. Thus there exists point $((x^*, t^*),\ t^*) \in [0,\ 1] \times [0,\ M]$ such that $(x_n, t_n) \to (x^*, t^*)$ as $n \to +\infty$. On the other hand, by (2.4.2) and the continuity of solutions in lemmas 2.2.1 and 2.2.2, we conclude that $\theta(x_n, t_n) \to \theta(x^*, t^*) = 0$ as $n \to +\infty$, which contradicts (2.2.1). Thus the proof is complete. □

Lemma 2.4.2. *If assumptions in theorem 2.1.2 hold, then the following estimates hold for all* $t > 0$,

$$\|\theta_t(t)\|^2 + \|v_t(t)\|^2 + \int_0^t (\|v_{xt}\|^2 + \|\theta_{xt}\|^2)(s)\,ds \le C_2, \tag{2.4.3}$$

$$\|v_{xx}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \int_0^t (\|v_{xxx}\|^2 + \|\theta_{xxx}\|^2)(s)\,ds \le C_2, \tag{2.4.4}$$

$$\|\eta_{xx}(t)\|^2 + \int_0^t \|\eta_{xx}(s)\|^2\,ds \le C_2. \tag{2.4.5}$$

Proof. Differentiating (2.1.4) with respect to t, multiplying the result by v_t and integrating over $(0,\ 1)$, we infer that

$$\begin{aligned}\frac{d}{dt}\|v_t(t)\|^2 + C_1^{-1}\|v_{xt}(t)\|^2 &\le \frac{1}{2C_1}\|v_{xt}(t)\|^2 + C_1(\|v_x(t)\|^2 + \|v_x(t)\|_{L^4}^4 + \|\theta_t(t)\|^2)\\ &\le \frac{1}{2C_1}\|v_{xt}(t)\|^2 + C_1(\|v_{xx}(t)\|^2 + \|\theta_t(t)\|^2)\end{aligned}$$

which, together with lemmas 2.2.1–2.2.8, yields

$$\|v_t(t)\|^2 + \int_0^t \|v_{xt}\|^2(\tau)\,d\tau \le C_2 + C_1\int_0^t (\|v_{xx}\|^2 + \|\theta_t\|^2)(\tau)\,d\tau \le C_2. \qquad (2.4.6)$$

Differentiating (2.2.42) with respect to x, using (2.1.3) ($\eta_{txx} = v_{xxx}$), we see that

$$\mu\frac{\partial}{\partial t}\Big(\frac{\eta_{xx}}{\eta}\Big) - p_\eta\eta_{xx} = v_{tx} + E(x,\ t) \qquad (2.4.7)$$

with

$$E(x,t) = (p_{\eta\eta}\eta_x^2 + 2p_{\theta\eta}\theta_x\eta_x + p_{\theta\theta}\theta_x^2) + p_\theta\theta_{xx} - 2\mu_0 v_x\eta_x^2/\eta^3 + 2\mu_0\eta_x v_{xx}/\eta^2.$$

Multiplying (2.4.7) by η_{xx}/η, and by the Young inequality, lemmas 2.2.1–2.2.3 and (2.1.18), we can deduce that

$$\begin{aligned} &\frac{d}{dt}\Big\|\frac{\eta_{xx}}{\eta}(t)\Big\|^2 + C_1^{-1}\Big\|\frac{\eta_{xx}}{\eta}(t)\Big\|^2 \\ &\quad\le \frac{1}{4C_1}\Big\|\frac{\eta_{xx}}{\eta}\Big\|^2 + C_1(\|\theta_x(t)\|_{L^4}^4 + \|\eta_x(t)\|_{L^4}^4 + \|v_{xt}(t)\|^2 \\ &\qquad + \|\theta_{xx}(t)\|^2 + \big\|v_x\eta_x^2(t)\big\|^2) \\ &\quad\le \frac{1}{2C_1}\Big\|\frac{\eta_{xx}}{\eta}(t)\Big\|^2 + C_2(\|\theta_{xx}(t)\|^2 + \|\eta_x(t)\|^2 + \|v_{xt}(t)\|^2) \end{aligned} \qquad (2.4.8)$$

which, combined with lemmas 2.2.1–2.2.8, gives

$$\|\eta_{xx}(t)\|^2 + \int_0^t \|\eta_{xx}(\tau)\|^2\,d\tau \le C_2, \quad \forall t > 0. \qquad (2.4.9)$$

By the embedding theorem, (2.1.4), (2.1.5) and (2.4.6), we conclude for all $t > 0$,

$$\|v_x\|_{L^\infty} \le C_1\|v_{xx}\| \le C_1(\|v_x\| + \|\theta_x\| + \|\eta_x\| + \|v_x\|) \le C_1, \qquad (2.4.10)$$

$$\int_0^t \|v_{xxx}\|^2\,ds \le C_1\int_0^t (\|v_{tx}\|^2 + \|\theta_{xx}\|^2 + \|\eta_{xx}\|^2 + \|v_{xx}\|^2)\,ds \le C_2. \qquad (2.4.11)$$

Using equation (2.1.5), lemmas 2.2.1–2.2.8, (2.3.8), the Gagliardo–Nirenberg interpolation inequality and the Young inequality, we have

$$\|\theta_{xx}(t)\| \le C_1(\|\theta_t(t)\| + \big\|\eta(S_E)_R\big\|) \le C_1(\|\theta_t(t)\| + 1). \qquad (2.4.12)$$

Differentiating (2.1.5) with respect to t, multiplying the result by θ_t and integrating over (0, 1), we infer that for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{e_\theta}\theta_t(t)\|^2 + C_1^{-1}\|\theta_{xt}(t)\|^2 \\
&\quad \le \varepsilon\|\theta_{xt}(t)\|^2 + C_1\Big\{\|\theta_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_t(t)\|_{L^3}^3 + \|\theta_t(t)\|^2 \\
&\qquad + \|v_{xt}(t)\|^2 + (\|\theta_t(t)\| + \|\theta_t(t)\|^{\frac{1}{2}}\|\theta_{tx}(t)\|^{\frac{1}{2}})\|\theta_{xt}(t)\| \\
&\qquad + C_1\big\|[\eta(S_E)_R]_t\big\|^2\Big\}.
\end{aligned} \tag{2.4.13}$$

Integrating (2.4.12) with respect to t and using lemmas 2.2.1–2.2.8 and the Young inequality, we derive for any $\varepsilon > 0$,

$$\begin{aligned}
&\|\theta_t(t)\|^2 + \int_0^t \|\theta_{xt}(s)\|^2 ds \\
&\quad \le C_2 + \varepsilon\int_0^t \|\theta_{xt}(s)\|^2 ds + C_1\int_0^t (\|\theta_t\|^{\frac{5}{2}}\|\theta_{tx}\|^{\frac{1}{2}} + \|\theta_t\|^3)(s)\,ds \\
&\qquad + C_1\int_0^t \big\|[\eta(S_E)_R]_t\big\|^2(s)\,ds \\
&\quad \le C_2 + \varepsilon\int_0^t \|\theta_{xt}(s)\|^2 ds + C_1\sup_{0\le s\le t}\|\theta_t(s)\|^{\frac{4}{3}} + C_1\int_0^t \big\|[\eta(S_E)_R]_t\big\|^2(s)\,ds \\
&\quad \le C_2 + \varepsilon\int_0^t \|\theta_{xt}(s)\|^2 ds + \frac{1}{2}\sup_{0\le s\le t}\|\theta_t(s)\|^2 + C_1\int_0^t \big\|[\eta(S_E)_R]_t\big\|^2(s)\,ds.
\end{aligned} \tag{2.4.14}$$

Noting that the new radiative term $\int_0^t \big\|[\eta(S_E)_R]_t\big\|^2(s)\,ds$, we need to obtain the uniform-in-time estimate.

From (2.2.41), (2.2.46), (2.2.65) and (2.2.78), we can derive

$$\begin{aligned}
&\int_0^t \big\|[\eta(S_E)_R]_t\big\|^2(s)\,ds \\
&\quad = \int_0^t\int_0^1 [v_x(S_E)_R + \eta[(S_E)_R]_t]^2\,dxds \\
&\quad \le C_1\int_0^t\int_0^1 v_x^2\Big(\int_0^{+\infty}\int_{S^1} \sigma_a(B-I) + \sigma_s(\tilde{I}-I)\,d\omega dv\Big)^2 dxds \\
&\qquad + C_1\int_0^t\int_0^1 \Big\{\int_0^{+\infty}\int_{S^1} [(\sigma_a)_\eta v_x + (\sigma_a)_\theta\theta_t](B-I) + \sigma_a(B_\theta\theta_t - I_t) \\
&\qquad + [(\sigma_s)_\eta v_x + (\sigma_s)_\theta\theta_t](\tilde{I}-I) + \sigma_s(\tilde{I}-I)_t\,d\omega dv\Big\}^2 dxds \\
&\quad \le C_1\int_0^t (\|v_x\|^2 + \|\theta_t\|^2)(s)\,ds + C_1\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1} I_t^2\,d\omega dv dx ds \\
&\quad \le C_1.
\end{aligned} \tag{2.4.15}$$

Inserting (2.2.15) into (2.2.14), then taking supremum in t on the left-hand side of (2.2.14), picking $\varepsilon > 0$ small enough, we can get for all $t > 0$,

$$\|\theta_t(t)\|^2 + \int_0^t \|\theta_{xt}(s)\|^2 ds \le C_2, \tag{2.4.16}$$

which, together with (2.2.12), implies

$$\|\theta_{xx}(t)\| \le C_2. \tag{2.4.17}$$

Differentiating (2.1.5) with respect to x, using the Young inequality, the Gagliardo–Nirenberg interpolation and the Poincaré inequality and lemmas 2.2.1–2.2.8, (2.2.16), (2.2.17) and (2.4.5), we deduce

$$\begin{aligned}\int_0^t \|\theta_{xxx}(s)\|^2 ds &\le C_2 \int_0^t (\|\eta_x\|^2 + \|\eta_{xx}\|^2 + \|v_x\|^2 + \|v_{xx}\|^2 + \|\theta_t\|^2 \\ &\quad + \|\theta_{xt}\|^2)(s)\,ds + C_2 \int_0^t \left\|[\eta(S_E)_R]_x\right\|^2(s)\,ds \\ &\le C_2 + C_2 \int_0^t \left\|[\eta(S_E)_R]_x\right\|^2(s)\,ds. \end{aligned} \tag{2.4.18}$$

The same estimate as (2.4.16) yields

$$\begin{aligned}\int_0^t \left\|[\eta(S_E)_R]_x\right\|^2(s)\,ds &= \int_0^t \int_0^1 (\eta_x(S_E)_R + \eta[(S_E)_R]_x)^2\,dxds \\ &\le C_1 \int_0^t (\|\eta_x\|^2 + \|\theta_x\|^2)(s)\,ds \\ &\quad + C_1 \left\| \int_0^{+\infty} \int_{S^1} |I_x|\,d\omega\,dv \right\|^2_{L^\infty(Q_t)} \\ &\le C_1, \end{aligned} \tag{2.4.19}$$

which, together with (2.4.18), implies that for all $t > 0$,

$$\int_0^t \|\theta_{xxx}(s)\|^2\,dxds \le C_2. \tag{2.4.20}$$

Thus (2.4.3), (2.4.4) follow from (2.4.6), (2.4.16), (2.4.17) and (2.4.20). The proof follows immediately. □

The following two lemmas are the new arguments on the uniform-in-time estimates of radiative term $I(x, t; v, \omega)$.

Lemma 2.4.3. *There holds that for all* $t > 0$,

$$\left\| \int_0^{+\infty} \int_{S^1} |I_t|\,d\omega\,dv \right\|_{L^\infty(Q_t)} \le C_2. \tag{2.4.21}$$

Proof. Using (2.1.18), the Young inequality, lemmas 2.2.5 and 2.2.9, we derive from (2.2.83)

$$\begin{aligned}
&\int_0^{+\infty}\int_{S^1}|A_2|d\omega dv\\
&\quad\le C_1\int_0^{+\infty}\int_{S^1}\Big|\int_0^x\Big(\int_x^y\frac{v_x}{\omega}(\sigma_a+\sigma_s+(\sigma_a+\sigma_s)_\eta)\\
&\qquad+\frac{\theta_t}{\omega}(\sigma_a+\sigma_s)_\theta dz\Big)\frac{\eta}{\omega}(\sigma_aB+\sigma_s\tilde{I})\,dy\Big|d\omega dv\\
&\quad\le C_1\int_0^{+\infty}\int_{S^1}\Big|\Big[\int_0^x\Big(\int_x^y\frac{v_x}{\omega}(\sigma_a+\sigma_s+(\sigma_a+\sigma_s)_\eta)\\
&\qquad+\frac{\theta_t}{\omega}(\sigma_a+\sigma_s)_\theta dz\Big)^2dy+\int_0^x\frac{\eta^2}{\omega^2}\sigma_a^2B^2\,dy+\int_0^x\sigma_s^2\tilde{I}^2\,dy\Big]\Big|d\omega dv\\
&\quad\le C_1\int_0^1(v_x^2+\theta_t^2)\,dx\le C_2.
\end{aligned}\tag{2.4.22}$$

Analogously,

$$\begin{aligned}
\int_0^{+\infty}\int_{S^1}|B_2|d\omega dv&\le C_1\int_0^{+\infty}\int_{S^1}\Big|\Big[\int_0^x\Big(\frac{v_x}{\omega}(\sigma_aB+\sigma_s\tilde{I})+\frac{\eta}{\omega}\Big((\sigma_a)_\eta v_xB\\
&\qquad+(\sigma_a)_\theta\theta_tB+\sigma B_\theta\theta_t+(\sigma_s)_\eta\tilde{I}v_x+(\sigma_s)_\theta\theta_t\tilde{I}+\sigma_s\tilde{I}_t\Big)\Big)dy\Big]\Big|d\omega dv\\
&\le C_1\int_0^1(|v_x|+|\theta_t|)\,dx+C_1\int_0^x\int_0^{+\infty}\int_{S^1}\Big|\frac{\eta}{\omega}\sigma_s\tilde{I}_t\Big|d\omega dv dy\\
&\le C_2+C_1\int_0^x\int_0^{+\infty}\int_{S^1}|I_t|d\omega dv dy,
\end{aligned}$$

which, together with (2.4.22) and (2.4.18) and using the Gronwall inequality, implies

$$\int_0^{+\infty}\int_{S^1}|I_t|d\omega dv\le C_2.$$

Thus the proof is hence complete. □

Lemma 2.4.4. *If assumptions in theorem 2.1.2 hold, then there holds that*

$$\|\mathcal{I}_{xxx}(t)\|\le C_2.\tag{2.4.23}$$

Proof. The elementary computation yields

$$\begin{aligned}
\|\mathcal{I}_{xxx}(t)\|^2&=\int_0^1\Big(\int_0^{+\infty}\int_{S^1}I_{xxx}d\omega dv\Big)^2dx\\
&=\int_0^1\Big(\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}(\eta_{xx}S+2\eta_xS_x+\eta S_{xx})\,d\omega dv\Big)^2dx\\
&\le C_1\int_0^1\Big[\Big(\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_{xx}Sd\omega dv\Big)^2+\Big(\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_xS_xd\omega dv\Big)^2\\
&\qquad+\Big(\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta S_{xx}d\omega dv\Big)^2\Big]dx=:\sum_{i=1}^3J_i.
\end{aligned}\tag{2.4.24}$$

Employing the Gagliardo–Nirenberg interpolation inequality and using (2.1.18), lemma 2.2.9 and (2.4.5), we conclude

$$
\begin{aligned}
J_1 &\le C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_{xx} S d\omega dv \Big)^2 dx \\
&\le C_1 \int_0^1 \eta_{xx}^2 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} [\sigma_a (B - I) + \sigma_s (\tilde{I} - I)] d\omega dv \Big)^2 dx \\
&\le C_1 \int_0^1 \eta_{xx}^2 dx \le C_2,
\end{aligned} \tag{2.4.25}
$$

$$
\begin{aligned}
J_2 &\le C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_x \Big\{ [(\sigma_a)_\eta \eta_x + (\sigma_a)_\theta \theta_x](B - I) \\
&\quad + \sigma_a (B_\theta \theta_x - I_x) + [(\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x](\tilde{I} - I) + \sigma_s (\tilde{I} - I)_x \Big\} d\omega dv \Big)^2 dx \\
&\le C_1 \int_0^1 (\eta_x^4 + \eta_x^2 \theta_x^2 + \eta_x^2) dx \\
&\le C_1 \max_{\Omega} \eta_x^2 \Big(\int_0^1 (\eta_x^2 + \theta_x^2) dx \Big) + C_1 \\
&\le C_1 (\|\eta_x\| \|\eta_{xx}\| + \|\eta_x\|^2) + C_1 \\
&\le C_2.
\end{aligned} \tag{2.4.26}
$$

Analogously, we infer from (2.1.18), (2.4.4), (2.4.5) and (2.2.85) that

$$
\begin{aligned}
J_3 &\le C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} \frac{\eta}{\omega} \Big\{ [(\sigma_a)_\eta \eta_x + (\sigma_a)_\theta \theta_x](B - I) \\
&\quad + \sigma_a (B_\theta \theta_x - I_x) + [(\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x](\tilde{I} - I) + \sigma_s (\tilde{I} - I)_x \Big\}_x d\omega dv \Big)^2 dx \\
&\le C_1 \int_0^1 (\eta_x^4 + \theta_x^4 + \eta_x^2 \theta_x^2 + \eta_{xx}^2 + \theta_{xx}^2 + \mathcal{I}_{xx}^2) dx \\
&\le C_1 (\|\eta_x\| \|\eta_{xx}\| + \|\eta_x\|^2 + \|\theta_x\| \|\theta_{xx}\| + \|\theta_x\|^2) \int_0^1 (\eta_x^2 + \theta_x^2) dx + C_2 \\
&\le C_2,
\end{aligned}
$$

which, along with (2.4.25) and (2.4.26), gives (2.4.23). The proof is thus complete. □

By lemmas 2.4.1–2.4.4, we conclude that the global existence of solutions $(\eta(t), v(t), \theta(t), \mathcal{I}(t))$ exists in $\mathcal{H}_2$ such that for all $t > 0$,

$$
\|(\eta(t), v(t), \theta(t), \mathcal{I}(t))\|_{\mathcal{H}_2} \le C_2. \tag{2.4.27}
$$

2.5 Asymptotic Behavior of Solutions in $\mathcal{H}_2$

This section will derive the asymptotic behavior of global solutions $(\eta, v, \theta, \mathcal{I})$ in $\mathcal{H}_2$ based on uniform-in-time estimates in section 2.4.

Lemma 2.5.1. *If assumptions in theorem 2.1.2 hold, then we have*

$$\lim_{t\to+\infty}\|\eta(t)-\overline{\eta}\|_{H^2}=0, \tag{2.5.1}$$

$$\lim_{t\to+\infty}\|v(t)\|_{H^2}=0, \tag{2.5.2}$$

where $\overline{\eta}=\int_0^1\eta(y,t)\,dy=\int_0^1\eta_0(y)\,dy.$

Proof. Applying lemma 1.1.2 to (2.4.8) and using lemmas 2.2.1–2.2.8 and 2.4.1, 2.4.2, we get, as $t\to\infty$,

$$\|\eta_{xx}\|\to 0. \tag{2.5.3}$$

Thus (2.5.1) follows from (2.5.3) and (2.3.1) in lemma 2.3.1. Applying lemma 1.1.2 to (2.4.6), using lemmas 2.2.1–2.2.8, we conclude, as $t\to\infty$,

$$\|v_t(t)\|\to 0$$

which, with (2.4.10), gives, as $t\to\infty$,

$$\|v_{xx}(t)\|\to 0. \tag{2.5.4}$$

Then (2.5.2) follows from (2.5.3) and (2.3.2) in lemma 2.3.1. The proof is now complete. □

Since radiative term $(S_E)_R$ is present in our model, whose uniform-in-time estimates are more complicated than those in Qin [104], we have to estimate a new radiative term $\|(S_E)_R\|^2$ in the next lemma.

Lemma 2.5.2. *If assumptions in theorem 2.1.2 hold, then we have*

$$\lim_{t\to+\infty}\|\theta(t)-\overline{\theta}\|_{H^2}=0, \tag{2.5.5}$$

where $\overline{\theta}>0$ *is determined by* $e(\overline{\eta},\overline{\theta})=\int_0^1\left(\frac{1}{2}v_0^2+e(\eta_0,\theta_0)+F_R(0)\right)dx.$

Proof. By (2.4.13), we can get

$$\begin{aligned}\frac{d}{dt}\|\sqrt{e_\theta}\theta_t(t)\|^2+C_1^{-1}\|\theta_{xt}(t)\|^2\le C_2(\|\theta_x(t)\|^2+\|v_x(t)\|^2+\|\theta_t(t)\|^2+\|v_{xt}(t)\|^2\\+\|[\eta(S_E)_R]_t\|^2),\end{aligned}$$

which, combined with (2.2.41), (2.2.62), (2.2.64), (2.2.65), (2.4.3), (2.4.15), and lemma 1.1.2, we can conclude

$$\lim_{t\to+\infty}\|\theta_t(t)\|^2=0. \tag{2.5.6}$$

Using (2.1.18) and lemma 2.2.9, we can deduce

$$\begin{aligned}\frac{d}{dt}\left\|(S_E)_R\right\|^2 &= 2\int_0^1 \Big(\int_0^{+\infty}\int_{S^1} S\,d\omega\,d\nu\Big)\Big(\int_0^{+\infty}\int_{S^1} S_t\,d\omega\,d\nu\Big)dx\\ &\le C_2(1+\|v_x(t)\|^2+\|\theta_t(t)\|^2),\end{aligned}$$

which, together with (2.2.41), (2.2.64) and lemma 1.1.2, implies

$$\lim_{t\to+\infty}\left\|(S_E)_R\right\|^2 = 0. \tag{2.5.7}$$

By equation (2.1.5), we see that

$$\|\theta_{xx}(t)\| \le C_2(\|v_x(t)\|+\|\theta_t(t)\|+\|\theta_x(t)\|+\left\|(S_E)_R\right\|),$$

which, along with (2.3.2), (2.3.10), (2.5.6) and (2.5.7), gives

$$\lim_{t\to+\infty}\|\theta_{xx}(t)\| = 0. \tag{2.5.8}$$

Thus (2.5.5) follows from (2.5.7) and (2.5.8). The proof is then complete. □

A new asymptotic behavior of solutions of radiative term $I(x, t; \nu, \omega)$ in H^3 will be given in the following lemma.

Lemma 2.5.3. *If assumptions in theorem 2.1.2 hold, then we have*

$$\lim_{t\to+\infty}\|\mathcal{I}(t)\|_{H^3} = 0. \tag{2.5.9}$$

Proof. By (2.1.6), we denote

$$\begin{aligned}\|\mathcal{I}_{xxx}(t)\|^2 &\le C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_{xx}S\,d\omega\,d\nu\right\|^2 + C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_x S_x\,d\omega\,d\nu\right\|^2\\ &\quad+\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta S_{xx}\,d\omega\,d\nu\right\|^2 =: R_1+R_2+R_3.\end{aligned} \tag{2.5.10}$$

Similarly, by (2.1.18), lemma 2.2.9 and the more delicated computation, we see that

$$R_1 \le C_1\|\eta_{xx}(t)\|^2, \tag{2.5.11}$$

$$R_2 \le C_1(\|\eta_x(t)\|^2+\|\theta_x(t)\|^2), \tag{2.5.12}$$

$$R_3 \le C_1(\|\eta_{xx}(t)\|^2+\|\theta_{xx}(t)\|^2+\|\mathcal{I}_{xx}(t)\|^2). \tag{2.5.13}$$

Plugging (2.5.11)–(2.5.13) into (2.5.10) and using (2.3.1), (2.3.5), (2.3.11), (2.5.1), (2.5.5), (2.5.8) and (2.3.18), we obtain

$$\lim_{t\to+\infty}\|\mathcal{I}_{xxx}(t)\|^2=0,$$

which, combined with (2.3.11), yields (2.5.9). □

Proof of Theorem 2.1.2. Combining lemmas 2.4.1–2.4.4 and 2.5.1–2.5.3, we can complete the proof of theorem 2.1.2. □

2.6 Global Existence and Uniform-in-Time Estimates in $\mathcal{H}_4$

In this section, we shall first establish the global existence solutions in $\mathcal{H}_4$.

Lemma 2.6.1. *If assumptions in theorem 1.1.3 hold, then there holds that for any* $(\eta_0, v_0, \theta_0, \mathcal{I}_0)\in\mathcal{H}_4$*, the following estimates hold for any* $t>0$,

$$\|v_{xt}(x,0)\|+\|\theta_{xt}(x,0)\|\le C_3, \tag{2.6.1}$$

$$\|v_{tt}(x,0)\|+\|\theta_{tt}(x,0)\|+\|v_{xxt}(x,0)\|+\|\theta_{xxt}(x,0)\|\le C_4, \tag{2.6.2}$$

$$\|v_{tt}(t)\|^2+\int_0^t\|v_{xtt}(s)\|^2ds\le C_4+C_4\int_0^t\|\theta_{xxt}(s)\|^2ds, \tag{2.6.3}$$

$$\begin{aligned}\|\theta_{tt}(t)\|^2+\int_0^t\|\theta_{xtt}(s)\|^2ds\le C_4\varepsilon^{-3}+C_2\varepsilon^{-1}\int_0^t\|\theta_{xxt}(s)\|^2ds\\ +C_1\varepsilon\int_0^t(\|v_{xtt}\|^2+\|v_{xxt}\|^2)(s)\,ds.\end{aligned} \tag{2.6.4}$$

Proof. Using theorems 2.1.1 and 2.1.2, we derive from (2.1.4)

$$\begin{aligned}\|v_t(t)\|&\le C_1(\|\eta_x(t)\|+\|\theta_x(t)\|+\|v_x(t)\|_{L^\infty}\|\eta_x(t)\|+\|v_{xx}(t)\|)\\ &\le C_2(\|v_x(t)\|_{H^1}+\|\eta_x(t)\|+\|\theta_x(t)\|).\end{aligned} \tag{2.6.5}$$

Differentiating (2.1.4) with respect to x and using theorems 2.1.1 and 2.1.2, we deduce

$$\|v_{xt}(t)\|\le C_2(\|v_x(t)\|_{H^2}+\|\eta_x(t)\|_{H^1}+\|\theta_x(t)\|_{H^1}) \tag{2.6.6}$$

or

$$\|v_{xxx}(t)\|\le C_2(\|v(t)\|_{H^2}+\|\eta_x(t)\|_{H^1}+\|\theta_x(t)\|_{H^1}+\|v_{xt}(t)\|). \tag{2.6.7}$$

Similarly, differentiating (2.1.4) with respect to x twice, using theorems 2.1.1 and 2.1.2 and the embedding theorem, we get

$$\begin{aligned}\|v_{xxt}(t)\| &\le C_2(\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{L^\infty}\|\eta_{xxx}(t)\| \\ &\quad + \|\eta_x(t)\|_{L^\infty}\|v_{xxx}(t)\| + \|v_{xx}(t)\|_{L^\infty}\|\eta_{xx}(t)\|) \\ &\le C_2(\|\eta_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^3}) \end{aligned} \tag{2.6.8}$$

or

$$\|v_{xxxx}(t)\| \le C_2(\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_{xxt}\|). \tag{2.6.9}$$

Using theorems 2.1.1, 2.1.2, (2.1.18) and the Hölder inequality, we have

$$\begin{aligned}\int_0^1 [\eta(S_E)_R]_x^2 dx &= \int_0^1 \left\{\eta(S_E)_R + \eta[(S_E)_R]_x\right\}^2 dx \\ &\le C_1 \int_0^1 \eta_x^2 (S_E)_R^2 dx + C_1 \int_0^1 [(S_E)_R]_x^2 dx \\ &\le C_1 \int_0^1 \|\eta_x\|^2 \Big(\int_0^{+\infty}\int_{S^1} \sigma_a(B-I) + \sigma_s(\tilde{I}-I)\, d\omega dv\Big)^2 dx \\ &\quad + C_1 \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} \Big\{[(\sigma_a)_\eta \eta_x + (\sigma_a)_\theta \theta_x](B-I) + \sigma_a(B_\theta\theta_x - I_x) \\ &\quad + [(\sigma_s)_\eta \eta_x + (\sigma_s)_\theta \theta_x](\tilde{I}-I) + \sigma_s(\tilde{I}-I)_x\Big\} d\omega dv\Big)^2 dx \\ &\le C_1 \int_0^1 \|\eta_x\|^2 \Big(\int_0^{+\infty}\int_{S^1} \sigma_a B^2 d\omega dv + \int_0^{+\infty}\int_{S^1} I^2 d\omega dv\Big) dx \\ &\quad + C_1 \int_0^1 \eta_x^2 \Big[\Big(\int_0^{+\infty}\int_{S^1} (\sigma_a)_\eta B d\omega dv\Big)^2 + \Big(\int_0^{+\infty}\int_{S^1} (\sigma_a)_\eta I d\omega dv\Big)^2 \\ &\quad + \Big(\int_0^{+\infty}\int_{S^1} (\sigma_a)_\eta (\tilde{I}-I) d\omega dv\Big)^2\Big] dx + C_1 \int_0^1 \theta_x^2 \Big[\Big(\int_0^{+\infty}\int_{S^1} (\sigma_a)_\theta I d\omega dv\Big)^2 \\ &\quad + \Big(\int_0^{+\infty}\int_{S^1} (\sigma_a)_\theta B d\omega dv\Big)^2 + \Big(\int_0^{+\infty}\int_{S^1} \sigma_a B_\theta d\omega dv\Big)^2 \\ &\quad + \Big(\int_0^{+\infty}\int_{S^1} (\sigma_s)_\theta (\tilde{I}-I) d\omega dv\Big)^2\Big] dx + C_1 \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} \sigma_a I_x d\omega dv\Big)^2 dx \\ &\quad + C_1 \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} \sigma_s (\tilde{I}-I)_x d\omega dv\Big)^2 dx \\ &\le C_1(\|\eta_x(t)\|^2 + \|\theta_x(t)\|^2 + \|\mathcal{I}_x(t)\|^2),\end{aligned}$$

i.e.,

$$\big\|[\eta(S_E)_R]_x\big\| \le C_1(\|\eta_x(t)\| + \|\theta_x(t)\| + \|\mathcal{I}_x(t)\|). \tag{2.6.10}$$

Similarly, we can derive

$$\big\|[\eta(S_E)_R]_{xx}\big\| \le C_2(\|\eta_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|\mathcal{I}_x(t)\|_{H^1}) \tag{2.6.11}$$

and

$$\big\|[\eta(S_E)_R]_t\big\| \le C_2(\|v_x(t)\| + \|\theta_t(t)\| + \|\mathcal{I}_t(t)\|) \tag{2.6.12}$$

or

$$\big\|[\eta(S_E)_R]_t\big\| \le C_2(\|v_x(t)\| + \|\theta_t(t)\| + \|I_t(t)\|_{L^2(\Omega\times(-1,1)\times\mathbb{R}_+)}). \tag{2.6.13}$$

From (2.1.5) and theorems 2.1.1 and 2.1.2 it follows that

$$\begin{aligned}\|\theta_t(t)\| &\le C_1(\|v_x(t)\| + \|v_x(t)\|_{L^\infty}\|v_x(t)\| \\ &\quad + (\|\eta_x(t)\| + \|\theta_x(t)\|)\|\theta_x(t)\|_{L^\infty} + \|\theta_{xx}(t)\| + \big\|(S_E)_R\big\|) \\ &\le C_1(\|\theta_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \big\|(S_E)_R\big\|). \end{aligned} \tag{2.6.14}$$

Differentiating (2.1.5) with respect to x, and using theorems 2.1.1 and 2.1.2, we arrive at

$$\begin{aligned}\|\theta_{xt}(t)\| &\le C_2(\|\theta_t(t)\| + \|\theta_x(t)\|_{H^2} + \|\eta_x(t)\|_{H^1} + \|v_{xx}(t)\| + \big\|[\eta(S_E)_R]_x\big\|) \\ &\le C_2(\|\eta_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^2} + \|\mathcal{I}_x(t)\|) \end{aligned} \tag{2.6.15}$$

or

$$\|\theta_{xxx}(t)\| \le C_2(\|\eta_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|v_x(t)\|_{H^1} + \|\theta_{xt}(t)\| + \|\mathcal{I}_x(t)\|). \tag{2.6.16}$$

Differentiating (2.1.5) with respect to x twice, using theorems 2.1.1 and 2.1.2 and the embedding theorem, we conclude

$$\begin{aligned}\|\theta_{xxt}(t)\| &\le C_2(\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3} + \big\|[\eta(S_E)_R]_{xx}\big\|) \\ &\le C_2(\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3} + \|\mathcal{I}_x(t)\|_{H^1}), \end{aligned} \tag{2.6.17}$$

or

$$\|\theta_{xxxx}(t)\| \le C_2(\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_{xxt}(t)\| + \|\mathcal{I}_x(t)\|_{H^1}). \tag{2.6.18}$$

Differentiating (2.1.4) with respect to t, using (2.6.6), (2.6.8), (2.6.14) and (2.6.15), we have

$$\begin{aligned}\|v_{tt}(t)\| &\le C_2(\|v_x(t)\|_{H^1} + \|\eta_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| \\ &\quad + \|v_{xt}(t)\| + \|v_{xxt}(t)\|) \\ &\le C_2(\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^2} + \|\mathcal{I}_x(t)\|). \end{aligned} \tag{2.6.19}$$

Analogously, we get

$$\begin{aligned}\|\theta_{tt}(t)\| &\le C_2(\|v_x(t)\| + \|\eta_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| \\ &\quad + \|v_{xt}(t)\| + \|\theta_x(t)\|_{H^2} + \|\theta_{xxt}(t)\| + \big\|[\eta(S_E)_R]_t\big\|) \\ &\le C_2(\|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^2} + \|\mathcal{I}_x(t)\|_{H^1} + \|\mathcal{I}_t(t)\|). \end{aligned} \tag{2.6.20}$$

Thus by theorem 2.1.1, estimates (2.6.1) and (2.6.2) follow from (2.6.6), (2.6.8), (2.6.15), (2.6.17), (2.6.19) and (2.6.20).

Differentiating (2.1.4) with respect to t twice, multiplying the result by v_{tt} in $L^2(0, 1)$, performing an integration by parts and using the Young inequality and using theorems 2.1.1 and 2.1.2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v_{tt}(t)\|^2 + C_2^{-1}\|v_{xtt}\|^2 \le C_2(&\|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|\theta_{xt}(t)\|^2 \\ &+ \|v_{xt}(t)\|^2 + \|\theta_{tt}(t)\|^2).\end{aligned} \tag{2.6.21}$$

Thus, by theorems 2.1.1 and 2.1.2,

$$\|v_{tt}(t)\|^2 + \int_0^t \|v_{xtt}(s)\|^2 ds \le C_4 + C_2\int_0^t \|\theta_{tt}(s)\|^2 ds$$

which, together with (2.6.20), (2.6.13) and lemma 2.2.9, gives (2.6.3).

Differentiating (2.1.6) with respect to t twice, multiplying the resulting equation by θ_{tt} in $L^2(0, 1)$, integrating by parts and using the Young inequality, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 e_\theta\theta_{tt}^2 dx = &-\int_0^1 \Big(\frac{\kappa\theta_x}{\eta}\Big)_{tt}\theta_{xtt}dx - \int_0^1 (e_{\theta tt}\theta_t + e_{\eta tt}v_x)\theta_{tt}dx \\ &- \frac{3}{2}\int_0^1 e_{\theta t}\theta_{tt}^2 dx - \int_0^1 \Big(e_\eta + p - \mu\frac{v_x}{\eta}\Big)v_{xtt}\theta_{tt}dx \\ &- 2\int_0^1 \Big[e_{\eta t} - \Big(-p + \mu\frac{v_x}{\eta}\Big)_t\Big]v_{xt}\theta_{tt}dx \\ &+ \int_0^1 \Big(-p + \mu\frac{v_x}{\eta}\Big)_{tt}v_x\theta_{tt}dx \\ &- \int_0^1 (\eta(S_E)_R)_{tt}\theta_{tt}dx =: \sum_{i=1}^7 A_i.\end{aligned} \tag{2.6.22}$$

Using theorems 2.1.1–2.1.3 and (2.6.1), (2.6.2), and the embedding theorem, we deduce for any $\epsilon \in (0, 1)$,

$$\begin{aligned}A_1 \le &- C_1^{-1}\|\theta_{ttx}(t)\|^2 + C_2\|\theta_{tx}(t)\|_{L^\infty}(\|v_x(t)\| + \|\theta_t(t)\|)\theta_{ttx}(t) \\ &+ C_2\Big\|\Big(\frac{k}{u}\Big)_{tt}(t)\Big\|\|\theta_x(t)\|_{L^\infty}\|\theta_{xx}(t)\| \\ \le &- (2C_1)^{-1}\|\theta_{ttx}(t)\|^2 + C_2(\|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|^2 + \|\theta_{tx}(t)\|^2 \\ &+ \|v_{tx}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{txx}(t)\|^2),\end{aligned} \tag{2.6.23}$$

$$
\begin{aligned}
A_2 \le{}& C_1 \int_0^1 [(|v_x| + |\theta_t|)^2 + |v_{tx}| + |\theta_{tt}|](|\theta_t| + |v_x|)|\theta_{tt}|\,dx \\
\le{}& C_1 \|\theta_{tt}(t)\|_{L^\infty} (\|\theta_t(t)\| + \|v_x(t)\|)\{(\|v_x(t)\|_{L^\infty} + \|\theta_t(t)\|_{L^\infty}) \\
&\times (\|v_x(t)\| + \|\theta_t(t)\|) + \|v_{tx}(t)\| + \|\theta_{tt}(t)\|\} \\
\le{}& C_2(\|\theta_{tt}(t)\| + \|\theta_{ttx}(t)\|)(\|v_x(t)\|_{H^1} + \|\theta_t(t)\| + \|\theta_{tx}(t)\| \\
&+ \|v_{tx}(t)\| + \|\theta_{tt}(t)\|) \\
\le{}& \epsilon\|\theta_{ttx}(t)\|^2 + C_2\epsilon^{-1}(\|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|^2 + \|\theta_{tx}(t)\|^2 \\
&+ \|v_{tx}(t)\|^2 + \|\theta_{tt}(t)\|^2),
\end{aligned} \tag{2.6.24}
$$

$$
\begin{aligned}
A_3 \le{}& C_1 \int_0^1 (|v_x| + |\theta_t|)\theta_{tt}^2\,dx \\
\le{}& C_1(\|\theta_{tt}(t)\| + \|\theta_{ttx}(t)\|)(\|v_x(t)\| + \|\theta_t(t)\|)\|\theta_{tt}(t)\| \\
\le{}& \epsilon\|\theta_{ttx}(t)\|^2 + C_2\epsilon^{-1}\|\theta_{tt}(t)\|^2,
\end{aligned} \tag{2.6.25}
$$

$$
A_4 \le \epsilon\|v_{ttx}(t)\|^2 + C_2\epsilon^{-1}\|\theta_{tt}(t)\|^2, \tag{2.6.26}
$$

$$
\begin{aligned}
A_5 \le{}& C_2\|v_x(t)\|_{L^\infty}\|\theta_{tt}(t)\|\Big\{(\|v_x(t)\|_{L^\infty} + \|\theta_t(t)\|_{L^\infty})(\|v_x(t)\| + \|\theta_t(t)\|) \\
&+ \|v_{tx}(t)\| + \|\theta_{tt}(t)\| + \|v_{ttx}(t)\| + \|v_x(t)\|\Big\} \\
\le{}& C_2\|\theta_{tt}(t)\|(\|v_x(t)\|_{H^1} + \|\theta_t(t)\| + \|\theta_{tx}(t)\| + \|v_{tx}(t)\| \\
&+ \|\theta_{tt}(t)\| + \|v_{ttx}(t)\|) \\
\le{}& \epsilon\|v_{ttx}(t)\|^2 + C_2\epsilon^{-1}(\|\theta_{tt}(t)\|^2 + \|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|^2 \\
&+ \|v_{tx}(t)\|^2 + \|\theta_{tx}(t)\|^2)
\end{aligned} \tag{2.6.27}
$$

and

$$
\begin{aligned}
A_6 \le{}& C_1 \int_0^1 (|v_x| + |\theta_t| + |v_{tx}| + |v_x|^2)|v_{tx}||\theta_{tt}|\,dx \\
\le{}& C_2\|v_{tx}(t)\|^{1/2}\|v_{txx}(t)\|^{1/2}(\|v_x(t)\| + \|\theta_t(t)\| + \|v_{tx}(t)\|)\|\theta_{tt}(t)\|
\end{aligned} \tag{2.6.28}
$$

which implies

$$
\begin{aligned}
\int_0^t A_6\,d\tau \le{}& C_2 \sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|\Big(\int_0^t \|v_{txx}\|^2(\tau)\,d\tau\Big)^{1/4}\Big(\int_0^t \|v_{tx}\|^2(\tau)\,d\tau\Big)^{1/4} \\
&\times \left\{\int_0^t (\|v_x\|^2 + \|\theta_t\|^2 + \|v_{tx}\|^2)(\tau)\,d\tau\right\}^{1/2} \\
\le{}& \epsilon\Big\{\sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|^2 + \int_0^t \|v_{txx}\|^2(\tau)\,d\tau\Big\} + C_2\epsilon^{-3}.
\end{aligned} \tag{2.6.29}
$$

Now we estimate the last term A_7. Using the Young inequality, we arrive at

$$\begin{aligned} A_7 &= -\int_0^1 (\eta(S_E)_R)_{tt}\theta_{tt}dx \\ &= -\int_0^1 (v_{xt} + 2v_x[(S_E)_R]_t + \eta[(S_E)_R]_{tt})\theta_{tt}dx \\ &\le C_1\int_0^1 v_{xt}^2dx + C_1\int_0^1 v_x^2[(S_E)_R]_t^2dx + C_1\int_0^1 \theta_{tt}^2dx + C_1\int_0^1 [(S_E)_R]_{tt}^2dx \\ &\le C_1\int_0^1 v_{xt}^2dx + C_1\|v_x\|_{L^\infty}^2\int_0^1 [(S_E)_R]_t^2dx + C_1\int_0^1 \theta_{tt}^2dx + C_1\int_0^1 [(S_E)_R]_{tt}^2dx, \end{aligned}$$

which, using the interpolation inequality, (2.6.13) and theorems 2.1.1 and 2.1.2, yields

$$\begin{aligned} A_7 \le C_1(&\|v_{xt}(t)\|^2 + \|v_x(t)\|^2 + \|\theta_t(t)\|^2 + \|\theta_{tt}(t)\|^2 \\ &+ \|I_t(t)\|_{L^2(\Omega\times(-1,1)\times\mathbb{R}_+)}^2) + C_1\int_0^1 [(S_E)_R]_{tt}^2dx. \end{aligned} \tag{2.6.30}$$

Performing the Hölder inequality and the interpolation inequality, using (2.1.18) and theorems 2.1.1 and 2.1.2, we have

$$\begin{aligned} \int_0^1 [(S_E)_R]_{tt}^2dx &= \int_0^1 \left(\int_0^{+\infty}\int_{S^1} \sigma_a(B-I) + \sigma_s(\tilde{I}-I)d\omega dv\right)_{tt}^2 dx \\ &= \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} ((\sigma_a)_{\eta\eta}v_x^2 + (\sigma_a)_{\theta\eta}\theta_t v_x + (\sigma_a)_\eta v_{xt} + (\sigma_a)_{\theta\eta}v_x\theta_t \\ &\quad + (\sigma_a)_{\theta\theta}\theta_t^2 + (\sigma_a)_\theta\theta_{tt})(B-I) + 2((\sigma_a)_\eta v_x + (\sigma_a)_\theta\theta_t)(B_\theta\theta_t - I_t) \\ &\quad + \sigma_a(B_{\theta\theta}\theta_t^2 + B_\theta\theta_{tt} - I_{tt}) + ((\sigma_s)_{\eta\eta}v_x^2 + (\sigma_s)_{\theta\eta}\theta_t v_x + (\sigma_s)_\eta v_{xt} \\ &\quad + (\sigma_s)_{\theta\eta}v_x\theta_t + (\sigma_s)_{\theta\theta}\theta_t^2 + (\sigma_s)_\theta\theta_{tt})(\tilde{I}-I)) + ((\sigma_s)_\eta v_x + 2(\sigma_s)_\theta\theta_t) \\ &\quad \times (\tilde{I}-I)_t + \sigma_s(\tilde{I}-I)_{tt}d\omega dv\Big)^2 dx \\ &\le C_1\int_0^1 (v_x^4 + \theta_t^2v_x^2 + v_{xt}^2 + \theta_t^4 + \theta_{tt}^2 + v_x^4)dx \\ &\quad + C_1\int_0^1\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv dx \\ &\le C_1(\|v_x(t)\|_{H^1}^2 + \|v_{xt}(t)\|^2 + \|\theta_t(t)\|_{H^1}^2 + \|\theta_{tt}(t)\|^2\Big) \\ &\quad + C_1\int_0^1\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv dx. \end{aligned} \tag{2.6.31}$$

Now we estimate $\int_0^1\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv dx$. Differentiating (2.2.83) with respect to t, we have

$$
\begin{aligned}
I_{tt} = & \int_0^x e^{\int_x^y \frac{\eta}{\omega}(\sigma_a+\sigma_s)dz}\Big(\int_x^y \Big[\frac{v_x}{\omega} + \frac{\eta}{\omega}(\sigma_a+\sigma_s) + \frac{\eta}{\omega}((\sigma_a+\sigma_s)_\eta v_x + (\sigma_a+\sigma_s)_\theta \theta_t)\Big]dz\Big)^2 \\
& \cdot \frac{\eta}{\omega}(\sigma_a B + \sigma_s \tilde{I})dy + \int_0^x e^{\int_x^y \frac{\eta}{\omega}(\sigma_a+\sigma_s)dz}\Big(\int_x^y \frac{v_{xt}}{\omega}(\sigma_a+\sigma_s) + \frac{v_x}{\omega}((\sigma_a+\sigma_s)_\eta v_x \\
& + (\sigma_a+\sigma_s)_\theta \theta_t) + \frac{v_x}{\omega}((\sigma_a+\sigma_s)_{\eta\eta} v_x + (\sigma_a+\sigma_s)_{\eta\theta}\theta_t v_x + (\sigma_a+\sigma_s)_\eta v_{xt} \\
& + (\sigma_a+\sigma_s)_{\theta\theta}\theta_t^2 + (\sigma_a+\sigma_s)_\theta \theta_{tt})dz\Big)\frac{\eta}{\omega}(\sigma_a B + \sigma_s \tilde{I})dy + \int_0^x e^{\int_x^y \frac{\eta}{\omega}(\sigma_a+\sigma_s)dz} \\
& \cdot \Big(\frac{v_x}{\omega}(\sigma_a+\sigma_s) + \frac{\eta}{\omega}((\sigma_a+\sigma_s)_\eta v_x + (\sigma_a+\sigma_s)_\theta \theta_t)\Big) \cdot \Big(\frac{v_x}{\omega}(\sigma_a B + \sigma_s \tilde{I}) \\
& + \frac{\eta}{\omega}((\sigma_a)_\eta v_x B + (\sigma_a)_\theta \theta_t B + \sigma_a B_\theta \theta_t + (\sigma_s)_\eta v_x \tilde{I} + (\sigma_a)_\theta \theta_t \tilde{I} + \sigma_s \tilde{I}_t)\Big)dy \\
& + \int_0^x e^{\int_x^y \frac{\eta}{\omega}(\sigma_a+\sigma_s)dz}\Big(\frac{v_{xt}}{\omega}(\sigma_a B + \sigma_s \tilde{I}) + 2\frac{v_x}{\omega}((\sigma_a)_\eta v_x B + (\sigma_a)_\theta \theta_t B + \sigma_a B_\theta \theta_t \\
& + (\sigma_s)_\eta v_x \tilde{I} + (\sigma_a)_\theta \theta_t \tilde{I} + \sigma_s \tilde{I}_t) + \frac{\eta}{\omega}((\sigma_a)_{\eta\eta} v_x^2 B + (\sigma_a)_{\eta\theta}\theta_t v_x B + (\sigma_a)_\eta v_{xt} B \\
& + (\sigma_a)_\eta v_x B_\theta \theta_t + (\sigma_a)_{\theta\eta} v_x \theta_t B + (\sigma_a)_{\theta\theta}\theta_t^2 B + 2(\sigma_a)_\theta \theta_t^2 B_\theta + (\sigma_a)_\eta B_\theta \theta_t + \sigma_a B_{\theta\theta}\theta_t^2 \\
& + \sigma_a B_\theta \theta_{tt} + (\sigma_s)_{\eta\eta} v_x^2 \tilde{I} + 2(\sigma_s)_{\eta\theta} v_x \theta_t \tilde{I} + (\sigma_s)_\eta v_{xt}\tilde{I} + (\sigma_s)_{\theta\theta}\theta_t^2 \tilde{I} + 2(\sigma_s)_\eta v_x \tilde{I}_t \\
& + 2(\sigma_s)_\theta \theta_t \tilde{I}_t + \sigma_s \tilde{I}_{tt})\Big)dy =: \sum_{i=1}^4 D_i. \qquad (2.6.32)
\end{aligned}
$$

Using the Hölder and the interpolation inequalities, (2.1.18) and theorems 2.1.1 and 2.1.2, we derive

$$
\begin{aligned}
\int_0^t \int_0^{+\infty} \int_{S^1} D_1^2 d\omega dv dx & \le C_1 \int_0^t \int_0^1 (v_x^4 + \theta_t^4)dxds \\
& \le C_1 \int_0^t \|v_x\|_{L^\infty}^2 \int_0^1 v_x^2 dxds \\
& \quad + C_1 \int_0^t \|\theta_t\|_{L^\infty}^2 \int_0^1 \theta_t^2 dxds \\
& \le C_2, \qquad (2.6.33)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \int_0^{+\infty} \int_{S^1} D_2^2 d\omega dv dx \le & C_1 \int_0^t \int_0^1 (v_{xt}^2 + v_x^4 + v_x^2\theta_t^2 + v_x^6 + v_x^4\theta_t^2 \\
& + v_x^2 + \theta_t^4 + v_x^2\theta_{tt}^2)dxds \\
\le & C_2 + C_2 \int_0^t \int_0^1 \theta_{tt}^2 dxds, \qquad (2.6.34)
\end{aligned}
$$

$$\begin{aligned}\int_0^t\int_0^{+\infty}\int_{S^1} D_3^2 d\omega dv dx &\le C_1\int_0^t\int_0^1 (v_x^4+v_x^2\theta_t^2+\theta_t^4+v_x^2+\theta_t^2)dxds\\ &\quad + C_2\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1} I_t^2 d\omega dv dx ds\\ &\le C_2,\end{aligned} \tag{2.6.35}$$

$$\begin{aligned}\int_0^t\int_0^{+\infty}\int_{S^1} D_4^2 d\omega dv dx &\le C_1\int_0^t\int_0^1 (v_x^4+v_x^2\theta_t^2+v_{xt}^2+\theta_t^4+\theta_{tt}^2)dxds\\ &\quad + C_2\int_0^t\int_0^x\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv dy ds\\ &\le C_2 + C_2\int_0^t\int_0^1 \theta_{tt}^2 dxds\\ &\quad + C_2\int_0^t\int_0^x\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv dy ds.\end{aligned} \tag{2.6.36}$$

Squaring (2.6.32) in both sides and inserting (2.6.33)–(2.6.36) into the resulting equation, we obtain

$$\int_0^t\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv ds \le C_2 + C_2\int_0^t \|\theta_{tt}(s)\|^2 ds + C_2\int_0^x\int_0^t\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv ds dy. \tag{2.6.37}$$

For fixed $t>0$, applying the Gronwall inequality to (2.6.37) implies

$$\begin{aligned}\int_0^t\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv ds &\le \Big(C_2 + C_2\int_0^t \|\theta_{tt}(s)\|^2 ds\Big)e^{C_2 x}\\ &\le C_2 + C_2\int_0^t \|\theta_{tt}(s)\|^2 ds.\end{aligned} \tag{2.6.38}$$

Integrating (2.6.38) over Ω with respect to x, we have

$$\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1} I_{tt}^2 d\omega dv ds \le C_2 + C_2\int_0^t \|\theta_{tt}(s)\|^2 ds. \tag{2.6.39}$$

Integrating (2.6.22) over $(0,\ t)$, using (2.6.22), (2.6.39) and theorems 2.1.1 and 2.1.2, we obtain

$$\begin{aligned}\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{xtt}(s)\|^2 ds &\le C_1\varepsilon\Big\{\sup_{0\le s\le t}\|\theta_{tt}(s)\|^2 + \int_0^t (\|v_{xxt}\|^2 + \|v_{xtt}\|^2)(s)ds\Big\}\\ &\quad + C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t (\|\theta_{tt}(s)\|^2 + \|\theta_{xxt}\|^2)(s)ds,\end{aligned}$$

which, together with (2.2.78), (2.6.20), (2.6.13), lemma 2.2.9 and theorem 2.1.1, implies (2.6.4). Similarly, we can derive the same result for $\omega\in(-1,0)$. The proof is now complete. □

Lemma 2.6.2. *If assumptions in theorem 2.1.3 hold, then for any* $(\eta_0, v_0, \theta_0, \mathcal{I}_0) \in \mathcal{H}$*, the following estimates hold for all* $t > 0$,

$$\|v_{xt}(t)\|^2 + \int_0^t \|v_{xxt}(s)\|^2 ds \leq C_3\varepsilon^{-6} + C_1\varepsilon^2 \int_0^t (\|\theta_{xtt}\|^2 + \|v_{xtt}\|^2)(s)\, ds, \tag{2.6.40}$$

$$\begin{aligned}\|\theta_{xt}(t)\|^2 &+ \int_0^t \|\theta_{xxt}(s)\|^2 ds \\ &\leq C_3\varepsilon^{-6} + C_1\varepsilon^2 \int_0^t \Big(\|v_{xxt}\|^2 + \|\theta_{xtt}\|^2 + \|\theta_{xxx}\|^2\|\theta_{xt}\|^2 \\ &\quad + \|I_x\|^2_{L^2(\mathbb{R}_+\times S^1, L^2(0,1))} + \|I_t\|^2_{L^2(\mathbb{R}_+\times S^1, L^2(0,1))}\Big)(s)\, ds. \end{aligned} \tag{2.6.41}$$

Proof. Differentiating (2.1.4) with respect to x and t, multiplying the resulting equation by v_{tx} in $L^2(0, 1)$, and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|v_{tx}(t)\|^2 = B_0(x, t) + B_1(t) \tag{2.6.42}$$

with

$$B_0(x, t) = \sigma_{tx} v_{tx}|_{x=0}^{x=1}, \quad B_1(t) = -\int_0^1 \sigma_{tx} v_{txx}\, dx.$$

Employing theorems 2.1.1 and 2.1.2, the interpolation inequality and the Poincaré inequality, we get

$$\begin{aligned} B_0 \leq\ & C_1(\|v_x(t)\|_{L^\infty} + \|\theta_t(t)\|_{L^\infty})(\|u_x(t)\|_{L^\infty} + \|\theta_x(t)\|_{L^\infty}) \\ &+ \|v_{xx}(t)\|_{L^\infty} + \|\theta_{tx}(t)\|_{L^\infty} + \|v_{txx}(t)\|_{L^\infty} + \|u_x(t)\|_{L^\infty}\|v_{tx}(t)\|_{L^\infty} \\ &+ \|v_x(t)\|_{L^\infty}\|v_{xx}(t)\|_{L^\infty} + \|v_x(t)\|^2_{L^\infty})\|v_{tx}(t)\|_{L^\infty} \\ \leq\ & C_2(B_{01} + B_{02})\|v_{tx}(t)\|^{1/2}\|v_{txx}(t)\|^{1/2} \end{aligned} \tag{2.6.43}$$

where

$$B_{01} = \|v_x(t)\|_{H^2} + \|\theta_t(t)\| + \|\theta_{tx}(t)\|$$

and

$$\begin{aligned} B_{02} = \|\theta_{tx}(t)\|^{1/2}\|\theta_{txx}(t)\|^{1/2} + \|v_{txx}(t)\|^{1/2}\|v_{txxx}(t)\|^{1/2} \\ + \|v_{txx}(t)\| + \|v_{tx}(t)\|^{1/2}\|v_{txx}(t)\|^{1/2}. \end{aligned}$$

Exploiting the Young inequality several times, we have that for any $\epsilon \in (0, 1)$,

$$\begin{aligned} C_2 B_{01}\|v_{tx}(t)\|^{1/2}\|v_{txx}(t)\|^{1/2} \leq \frac{\epsilon^2}{2}\|v_{txx}(t)\|^2 + C_2\epsilon^{-2/3}(\|v_{tx}(t)\|^2 \\ + \|v_x(t)\|^2_{H^2} + \|\theta_t(t)\|^2 + \|\theta_{tx}(t)\|^2) \end{aligned} \tag{2.6.44}$$

and

$$C_2 B_{02}\|v_{tx}(t)\|^{1/2}\|v_{txx}(t)\|^{1/2} \le \frac{\epsilon^2}{2}\|v_{txx}(t)\|^2 + \epsilon^2(\|\theta_{txx}(t)\|^2 + \|v_{txxx}(t)\|^2) + C_2\epsilon^{-6}(\|\theta_{tx}(t)\|^2 + \|v_{tx}(t)\|^2). \tag{2.6.45}$$

Thus we infer from (2.6.43)–(2.6.45), theorems 2.1.1, 2.1.2 and lemma 2.4.1

$$B_0 \le \epsilon^2(\|v_{txx}(t)\|^2 + \|v_{txxx}(t)\|^2 + \|\theta_{txx}(t)\|^2) + C_2\epsilon^{-6}(\|v_x(t)\|_{H^2}^2 + \|\theta_t(t)\|^2 + \|\theta_{tx}(t)\|^2 + \|v_{tx}(t)\|^2) \tag{2.6.46}$$

which, with theorems 2.1.1 and 2.1.2, further leads to

$$\int_0^t B_0\,d\tau \le \epsilon^2 \int_0^t (\|v_{txx}\|^2 + \|v_{txxx}\|^2 + \|\theta_{txx}\|^2)(\tau)\,d\tau + C_2\epsilon^{-6}, \quad \forall t > 0. \tag{2.6.47}$$

In the same manner, using theorems 2.1.1, 2.1.2, the embedding theorem, we get that for any $\epsilon \in (0,1)$,

$$\begin{aligned} B_1 \le & -\mu_0 \int_0^1 \frac{v_{txx}^2}{u}dx + C_1\Big\{(\|v_x(t)\| + \|\theta_t(t)\|)(\|u_x(t)\|_{L^\infty} + \|\theta_x(t)\|_{L^\infty}) \\ & + \|v_{xx}(t)\| + \|\theta_{tx}(t)\| + \|u_x(t)\|_{L^\infty}\|v_{tx}(t)\| + \|v_x(t)\|_{L^\infty}\|v_{xx}(t)\| \\ & + \|v_x(t)\|_{L^\infty}^2\|u_x(t)\|\Big\}\|v_{txx}(t)\| \\ \le & -(2C_1)^{-1}\|v_{txx}(t)\|^2 + C_2(\|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|_{H^1}^2 \\ & + \|v_{tx}(t)\|^2 + \|u_x(t)\|^2) \end{aligned} \tag{2.6.48}$$

which, with (2.6.42), (2.6.47), theorems 2.1.1, 2.1.2, and lemma 2.2.9, gives that for $\epsilon \in (0,1)$ small enough,

$$\|v_{tx}(t)\|^2 + \int_0^t \|v_{txx}\|^2(\tau)\,d\tau \le C_3\epsilon^{-6} + C_1\epsilon^2\int_0^t (\|\theta_{txx}\|^2 + \|v_{txxx}\|^2)(\tau)\,d\tau. \tag{2.6.49}$$

Differentiating (2.1.4) with respect to x and t, and using theorems 2.1.1 and 2.1.2, we deduce

$$\|v_{xxxt}(t)\| \le C_1\|v_{ttx}(t)\| + C_2(\|v_{xx}(t)\|_{H^1} + \|\theta_x(t)\|_{H^1} + \|\eta_x(t)\|_{H^1} + \|\theta_t(t)\|_{H^2}). \tag{2.6.50}$$

Differentiating (2.1.5) with respect to x and t, multiplying the resulting equation by θ_{xt}, and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 e_\theta\theta_{xt}^2\,dx = D_0 + D_1 + D_2 + D_3 + D_4 + D_5 \tag{2.6.51}$$

where

$$D_0 = \left(\frac{\kappa\theta_x}{\eta}\right)_{xt}\theta_{xt}\Big|_{x=0}^{x=1},\ D_1 = -\int_0^1 \left(\frac{\kappa\theta_x}{\eta}\right)_{xt}\theta_{xxt}dx,$$
$$D_2 = -\int_0^1 (e_\eta v_x + \sigma v_x)_{xt}\theta_{xt}dx,$$
$$D_3 = -\int_0^1 \Big(e_{\theta xt}\theta_t + \frac{1}{2}e_{\theta t} + e_{\theta x}\theta_{tt}\Big)\theta_{xt}dx,$$
$$D_4 = -\int_0^1 (\eta(S_E)_R)_{xt}\theta_{xt}dx = \int_0^1 [\eta(S_E)_R]_t\theta_{xxt}dx - (\eta(S_E)_R)_t\theta_{xt}|_{x=0}^{x=1}.$$

Similarly to (2.6.42)–(2.6.49), we infer

$$\begin{aligned}D_0 &\le C_2(\|v_x(t)\|_{L^\infty} + \|\theta_t(t)\|_{L^\infty} + \|v_{xx}(t)\|_{L^\infty} + \|\theta_{tx}(t)\|_{L^\infty}\\ &\quad + \|\theta_t(t)\|_{L^\infty}\|\theta_{xx}(t)\|_{L^\infty} + \|\theta_{txx}(t)\|_{L^\infty} + \|\theta_{xx}(t)\|_{L^\infty})\|\theta_{tx}(t)\|_{L^\infty}\\ &\le C_2(D_{01}+D_{02})(D_{03}+D_{04})\end{aligned}$$

where

$$C_2D_{01}D_{03} \le \frac{\epsilon^2}{3}\|\theta_{txx}(t)\|^2 + C_2\epsilon^{-2}(\|v_x(t)\|_{H^2}^2 + \|\theta_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2),$$
$$C_2D_{02}D_{03} \le \frac{\epsilon^2}{3}(\|\theta_{txx}(t)\|^2 + \|\theta_{txxx}(t)\|^2) + C_2\epsilon^{-6}\|\theta_{tx}(t)\|^2,$$
$$C_2D_{01}D_{04} \le C_2(\|v_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^2}^2),$$

and

$$C_2D_{02}D_{04} \le \frac{\epsilon^2}{3}(\|\theta_{txx}(t)\|^2 + \|\theta_{txxx}(t)\|^2) + C_2\epsilon^{-2}\|\theta_{tx}(t)\|^2.$$

That is,

$$D_0 \le \epsilon^2(\|\theta_{txx}(t)\|^2 + \|\theta_{txxx}(t)\|^2) + C_2\epsilon^{-6}(\|v_x(t)\|_{H^2}^2 + \|\theta_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2). \tag{2.6.52}$$

Similarly,

$$D_1 \le -(2C_1)^{-1}\|\theta_{txx}(t)\|^2 + C_2(\|v_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2), \tag{2.6.53}$$

$$D_2 \le \epsilon^2\|v_{txx}(t)\|^2 + C_2\epsilon^{-2}(\|v_x(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^1}^2 + \|v_{tx}(t)\|^2), \tag{2.6.54}$$

$$\begin{aligned}D_3 &\le \epsilon^2\|\theta_{txx}(t)\|^2 + C_2\epsilon^{-2}(\|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^2}^2\\ &\quad + \|v_{tx}(t)\|^2 + \|u_x(t)\|^2).\end{aligned} \tag{2.6.55}$$

Using the Young inequality and (2.6.13), (2.6.14), we derive

$$\begin{aligned} D_4 &\le C_1\|\theta\|_{L^\infty}^{1+\alpha}\|\theta_{xt}\|_{L^\infty} + C_1\|I\|_{L^2(\mathbb{R}_+\times S^1,L^\infty)}\|\theta_{xt}\| + \varepsilon^2\|\theta_{xxt}(t)\|^2 \\ &\quad + C_1\varepsilon^{-2}\int_0^1 [\eta(S_E)_R]_t^2\,dx \\ &\le \varepsilon^2\|\theta_{xxt}(t)\|^2 + C_2\varepsilon^{-2}(\|v_x(t)\|^2 + \|\theta_x\|^2 + \|\theta_t(t)\|^2 \\ &\quad + \|I_t(t)\|^2_{L^2(\mathbb{R}_+\times S^1,L^2(0,1))} + \|I_x(t)\|^2_{L^2(\mathbb{R}_+\times S^1,L^2(0,1))} \end{aligned}$$

Exploiting lemma 2.6.1, theorems 2.1.1, 2.1.2 and the embedding theorem result in

$$\left\|\left(\frac{k}{u}\right)_{txx}(t)\right\| \le C_2(\|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|\theta_t(t)\|_{H^2}),$$

$$\left\|\left(\frac{k}{u}\right)_{t}(t)\right\| + \left\|\left(\frac{k}{u}\right)_{tx}(t)\right\| \le C_2(\|v_x(t)\|_{H^1} + \|\theta_t(t)\|_{H^1})$$

and

$$\left\|\left(\frac{k}{u}\right)_{x}(t)\right\|_{L^\infty} + \left\|\left(\frac{k}{u}\right)_{xx}(t)\right\| \le C_2(\|u_x(t)\|_{H^1} + \|\theta_x(t)\|_{H^1}) \le C_2$$

which imply

$$\begin{aligned} \left\|\left(\frac{k}{u}\right)_{txx}\theta_x(t)\right\| &\le C_2\left\|\left(\frac{k}{u}\right)_{txx}(t)\right\| \\ &\le C_2(\|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|\theta_t(t)\|_{H^2}), \end{aligned} \tag{2.6.56}$$

$$\begin{aligned} \left\|\left(\frac{k}{u}\right)_{tx}\theta_{xx}(t)\right\| &\le C_2\left\|\left(\frac{k}{u}\right)_{tx}\right\|_{L^\infty} \\ &\le C_2(\|u_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^1} + \|\theta_t(t)\|_{H^2}), \end{aligned} \tag{2.6.57}$$

$$\begin{aligned} \left\|\left(\frac{k}{u}\right)_{t}\theta_{xxx}(t)\right\| &\le C_1\left(\left\|\left(\frac{k}{u}\right)_{t}\right\|_{L^\infty}\right)\|\theta_{xxx}(t)\| \\ &\le C_2(1 + \|\theta_{tx}(t)\|)\|\theta_{xxx}(t)\|, \end{aligned} \tag{2.6.58}$$

$$\left\|\left(\frac{k}{u}\right)_{xx}\theta_{tx}(t)\right\| + \left\|\left(\frac{k}{u}\right)_{x}\theta_{txx}(t)\right\| \le C_2\|\theta_{tx}(t)\|_{H^1}. \tag{2.6.59}$$

Differentiating (2.1.5) with respect to x and t, using estimates (2.6.57)–(2.6.59) and theorems 2.1.1, 2.1.2, we conclude

$$\begin{aligned}\|\theta_{xxxt}(t)\| &\le C_1\bigg(\Big\|\Big(\frac{\kappa\theta_x}{\eta}\Big)_{xxt}(t)\Big\| + \Big\|\Big(\frac{\kappa}{\eta}\Big)_{xxt}\theta_{xx}(t)\Big\| + \Big\|\Big(\frac{\kappa}{\eta}\Big)_{t}\theta_{xxx}(t)\Big\| \\ &\quad + \Big\|\Big(\frac{\kappa}{\eta}\Big)_{xt}\theta_{xx}(t)\Big\| + \Big\|\Big(\frac{\kappa}{\eta}\Big)_{xx}\theta_{xt}(t)\Big\| + \Big\|\Big(\frac{\kappa}{\eta}\Big)_{xx}\theta_{xt}(t)\Big\| \\ &\quad + \Big\|\Big(\frac{\kappa}{\eta}\Big)_{x}\theta_{xxt}(t)\Big\|\bigg) \\ &\le C_1(\|\eta_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_t(t)\|_{H^2} + \|\theta_{tt}(t)\|_{H^1} \\ &\quad + \|v_{xt}(t)\|_{H^1} + \|\theta_{xt}(t)\|\|\theta_{xxx}(t)\| + \big\|[\eta(S_E)_R]_{xt}(t)\big\|). \end{aligned} \tag{2.6.60}$$

Now we estimate $\big\|[\eta(S_E)_R]_{xt}(t)\big\|$.

$$\begin{aligned}\int_0^1 [\eta(S_E)_R]_{xt}^2 dx &\le \int_0^1 (v_{xx}^2(S_E)_R^2 + v_x^2[(S_E)_R]_x^2 + \eta_x^2[(S_E)_R]_t^2 + [(S_E)_R]_{xt}^2)\,dx \\ &=: \sum_{i=1}^4 F_i. \end{aligned} \tag{2.6.61}$$

Using the Hölder inequality, (2.1.18) and theorems 2.1.1, 2.1.2, we derive

$$\begin{aligned} F_1 &= \int_0^1 v_{xx}^2\bigg(\int_0^{+\infty}\int_{S^1} \sigma_a(B-I) + \sigma_s(\tilde I - I)\,d\omega dv\bigg)^2 dx \\ &\le C_1\|v_x(t)\|_{H^1}^2, \end{aligned} \tag{2.6.62}$$

$$\begin{aligned} F_2 &\le \int_0^1 v_x^2\Big(\int_0^{+\infty}\int_{S^1} ((\sigma_a)_\eta\eta_x + (\sigma_a)_\theta\theta_x)(B-I) + \sigma_a(B_\theta\theta_x - I_x) \\ &\quad + ((\sigma_s)_\eta\eta_x + (\sigma_s)_\theta\theta_x)(\tilde I - I) + \sigma_s(\tilde I - I)_x\,d\omega dv\Big)^2 dx \\ &\le C_1\int_0^1 (v_x^2\eta_x^2 + v_x^2\theta_x^2 + v_x^2)\,dx \le C_1(\|\eta_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^1}^2 + \|v_x(t)\|^2). \end{aligned} \tag{2.6.63}$$

Similarly, we have

$$F_3 \le C_1(\|v_x(t)\|_{H^1}^2 + \|\theta_t(t)\|_{H^1}^2 + \|\eta_x(t)\|^2) \tag{2.6.64}$$

and

$$\begin{aligned} F_4 &\le C_1(\|v_x(t)\|_{H^1}^2 + \|\eta_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^1}^2 + \|\theta_t(t)\|_{H^1}^2) \\ &\quad + C_1\int_0^1\int_0^{+\infty}\int_{S^1} I_{xt}^2\,d\omega dv dx, \end{aligned}$$

which, together with (2.6.62)–(2.6.64), gives

$$\begin{aligned}\|\theta_{xxxt}(t)\| &\le C_1(\|\eta_x(t)\|_{H^1} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_t(t)\|_{H^2} + \|\theta_{tt}(t)\|_{H^1} \\ &\quad + \|v_{xt}(t)\|_{H^1} + \|\theta_{xt}(t)\|\|\theta_{xxx}(t)\| + \|I_{xt}(t)\|_{L^2(\Omega\times(-1,1)\times\mathbb{R}_+)}). \end{aligned} \tag{2.6.65}$$

Differentiating (2.1.6) with respect to t, using (2.1.18), the Hölder inequality and theorem 2.1.1, we obtain

$$\begin{aligned}
&\int_1^t \int_0^1 \int_0^{+\infty} \int_{S^1} I_{xt}^2 d\omega d\nu dx ds \\
&\quad = \int_1^t \int_0^1 \int_0^{+\infty} \int_{S^1} \frac{1}{\omega^2}(v_x^2 S^2 + \eta^2 S_t^2) d\omega d\nu dx ds \\
&\quad \le C_1 \int_1^t \int_0^1 (v_x^2 + \theta_t^2) dx ds + C_1 \int_1^t \int_0^1 \int_0^{+\infty} \int_{S^1} I_t^2 d\omega d\nu dx ds \\
&\quad \le C_1. \qquad (2.6.66)
\end{aligned}$$

Integrating (2.6.32) over $(0,\ t)$, using (2.6.52)–(2.6.56), (2.6.66) and theorems 2.1.1, 2.1.2, we easily get (2.6.41). □

Lemma 2.6.3. *If assumptions in theorem 2.1.3 hold, then for any* $(\eta_0, v_0, \theta_0, I_0) \in \mathcal{H}_4$*, the following estimates hold, for all* $t > 0$,

$$\begin{aligned}
&\|v_{tt}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \int_0^t (\|v_{xtt}\|^2 + \|v_{xtt}\|^2 \\
&\quad + \|\theta_{xtt}\|^2 + \|\theta_{xxt}\|^2)(s)\, ds \le C_4, \qquad (2.6.67)
\end{aligned}$$

$$\|\eta_{xxx}(t)\|_{H^1}^2 + \|\eta_{xx}(t)\|_{W^{1,\infty}}^2 + \int_0^t (\|\eta_{xxx}\|_{H^1}^2 + \|\eta_{xx}\|_{W^{1,\infty}}^2)(s)\, ds \le C_4, \qquad (2.6.68)$$

$$\begin{aligned}
&\|v_{xxx}(t)\|_{H^1}^2 + \|v_{xx}(t)\|_{W^{1,\infty}}^2 + \|\theta_{xxx}(t)\|^2 + \|\theta_{xx}\|_{W^{1,\infty}}^2 + \|\eta_{xxxx}(t)\|^2 \\
&\quad + \|v_{xxt}(t)\|^2 + \|\theta_{xxt}(t)\|^2 + \int_0^t (\|v_{tt}\|^2 + \|\theta_{tt}\|^2 + \|v_{xx}\|_{W^{2,\infty}}^2 + \|\theta_{xx}\|_{W^{2,\infty}}^2 \\
&\quad + \|\theta_{xxt}\|_{H^1}^2 + \|v_{xxt}\|_{H^1}^2 + \|\theta_{xt}\|_{W^{1,\infty}}^2 + \|v_{xt}\|_{W^{1,\infty}}^2 + \|\eta_{xxxt}\|_{H^1}^2)(s)\, ds \le C_4, \qquad (2.6.69)
\end{aligned}$$

$$\int_0^t (\|v_{xxxx}\|_{H^1}^2 + \|\theta_{xxxx}\|_{H^1}^2)(s)\, ds \le C_4. \qquad (2.6.70)$$

Proof. Adding up (2.6.40) and (2.6.41), picking $\epsilon \in (0, 1)$ small enough, we arrive at

$$\begin{aligned}
&\|v_{tx}(t)\|^2 + \|\theta_{tx}(t)\|^2 + \int_0^t (\|v_{txx}\|^2 + \|\theta_{txx}\|^2)(\tau) \\
&\quad \le C_3\epsilon^{-6} + C_2\epsilon^2 \int_0^t (\|v_{ttx}\|^2 + \|\theta_{ttx}\|^2 + \|\theta_{xxx}\|^2 \|\theta_{tx}\|^2)(\tau)\, d\tau. \qquad (2.6.71)
\end{aligned}$$

Now multiplying (2.6.3) and (2.6.4) by ϵ and $\epsilon^{3/2}$, respectively, then adding the resultant to (2.6.72), and choosing $\epsilon \in (0, 1)$ small enough, we obtain

$$\|v_{tx}(t)\|^2+\|\theta_{tx}(t)\|^2+\|v_{tt}(t)\|^2+\|\theta_{tt}(t)\|^2+\int_0^t(\|\theta_{txx}\|^2+\|v_{txx}\|^2$$
$$+\|v_{ttx}\|^2+\|\theta_{ttx}\|^2)(\tau)\,d\tau\le C_4\epsilon^{-6}+C_2\epsilon^2\int_0^t\|\theta_{xxx}\|^2\|\theta_{tx}\|^2(\tau)\,d\tau$$

which, with lemma 2.3.2 and the Gronwall inequality, gives estimate (2.6.68).

Differentiating (2.4.7) with respect to x, and using (2.1.3), we get

$$\mu\frac{\partial}{\partial t}\Big(\frac{\eta_{xxx}}{\eta}\Big)-p_\eta\eta_{xxx}=E_1(x,t) \tag{2.6.72}$$

with

$$E_1(x,t)=v_{txx}+E_x(x,t)+p_{\eta x}\eta_{xx}+\mu\Big(\frac{\eta_{xx}\eta_x}{\eta^2}\Big)_t.$$

Obviously, we can infer from theorems 2.1.1, 2.1.2 and lemmas 2.6.1, 2.6.2 that

$$\|E_1(t)\|\le C_2(\|\eta_x(t)\|_{H^1}+\|v_x(t)\|_{H^2}+\|\theta_x(t)\|_{H^2}+\|v_{txx}(t)\|) \tag{2.6.73}$$

leading to

$$\int_0^t\|E_1\|^2(\tau)\,d\tau\le C_4,\quad\forall t>0. \tag{2.6.74}$$

Multiplying (2.6.72) by $\frac{\eta_{xxx}}{\eta}$ in $L^2(0,\,1)$, we obtain

$$\frac{d}{dt}\Big\|\frac{\eta_{xxx}}{\eta}(t)\Big\|^2+C_1^{-1}\Big\|\frac{\eta_{xxx}}{\eta}(t)\Big\|^2\le C_1\|E_1(t)\|^2 \tag{2.6.75}$$

which, combined with (2.6.74) and theorems 2.1.1, 2.1.2 and lemmas 2.6.1, 2.6.2 gives, for all $t>0$,

$$\|\eta_{xxx}(t)\|^2+\int_0^t\|\eta_{xxx}(s)\|^2\,ds\le C_1. \tag{2.6.76}$$

Using (2.1.18), the Hölder inequality and theorem 2.1.1, we derive from (2.1.6) that for any $t>0$,

$$\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1}I_x^2\,d\omega\,dv\,dx\,ds\le C_1\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1}\frac{1}{\omega^2}S^2\,d\omega\,dv\,dx\,ds\le C_1, \tag{2.6.77}$$

$$\begin{aligned}\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1}I_{xx}^2\,d\omega\,dv\,dx\,ds&\le C_1\int_0^t\int_0^1(\eta_x^2+\theta_x^2)\,dx\,ds\\&\quad+\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1}I_x^2\,d\omega\,dv\,dx\,ds\\&\le C_1,\end{aligned} \tag{2.6.78}$$

$$\begin{aligned}
&\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1} I_{xxx}^2\, d\omega\, dv\, dx ds\\
&\le C_1\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1}\frac{1}{\omega^2}(\eta_{xx}^2S^2+\eta_x^2S_x^2+S_{xx}^2)\, d\omega\, dv\, dx ds\\
&\le C_1\int_0^t\int_0^1(\eta_{xx}^2+\theta_{xx}^2+\eta_x^2+\theta_x^2)\, dx ds\\
&\quad + C_1\int_0^t\int_0^1\int_0^{+\infty}\int_{S^1} I_{xx}^2\, d\omega\, dv\, dx ds\\
&\le C_2. \qquad (2.6.79)
\end{aligned}$$

By (2.6.7), (2.6.9), (2.6.16), (2.6.18), (2.6.72)–(2.6.79) and theorem 2.1.1, and using the interpolation inequality, we obtain for any $t>0$,

$$\begin{aligned}
&\|v_{xxx}(t)\|^2+\|\theta_{xxx}(t)\|^2+\|v_{xx}(t)\|_{L^\infty}^2+\|\theta_{xx}(t)\|_{L^\infty}^2\\
&\quad+\int_0^t(\|v_{xxxx}\|_{H^1}^2+\|\theta_{xxx}\|_{H^1}^2+\|v_{xx}\|_{W^{1,\infty}}^2+\|\theta_{xx}\|_{W^{1,\infty}}^2)(s)\, ds\le C_4. \qquad (2.6.80)
\end{aligned}$$

Differentiating (2.1.4) and (2.1.5) with respect to t, using (2.6.69), (2.6.12) and theorems 2.1.1 and 2.1.2, we derive that for any $t>0$,

$$\|v_{xxt}(t)\|\le C_1\|v_{tt}(t)\|+C_2(\|v_x(t)\|_{H^1}+\|v_{xt}(t)\|+\|\theta_t(t)\|_{H^1})\le C_4, \qquad (2.6.81)$$

$$\begin{aligned}
\|\theta_{xxt}(t)\|\le\ & C_1(\|\theta_{tt}(t)\|+\|\mathcal{I}_t(t)\|)+C_2(\|v_x(t)\|_{H^1}+\|v_{xt}(t)\|+\|\theta_t(t)\|_{H^1}\\
&+\|\theta_x(t)\|_{H^1})\le C_4 \qquad (2.6.82)
\end{aligned}$$

which, together with (2.6.9), (2.6.18), (2.6.80) and theorem 2.1.1, yields for all $t>0$,

$$\begin{aligned}
&\|v_{xxxx}(t)\|^2+\|\theta_{xxxx}(t)\|^2\\
&\quad+\int_0^t(\|\theta_{xxt}\|^2+\|\theta_{xxxx}\|^2+\|v_{xxt}\|^2+\|v_{xxxx}\|^2)(s)\, ds\le C_4. \qquad (2.6.83)
\end{aligned}$$

Employing the interpolation inequality and (2.6.83), we get for any $t>0$,

$$\|v_{xxx}(t)\|_{L^\infty}^2+\|\theta_{xxx}(t)\|_{L^\infty}^2+\int_0^t(\|v_{xxx}\|_{L^\infty}^2+\|\theta_{xxx}\|_{L^\infty}^2)(s)\, ds\le C_4. \qquad (2.6.84)$$

Now differentiating (2.6.72) with respect to x, we find

$$\mu\frac{\partial}{\partial t}\Big(\frac{\eta_{xxxx}}{\eta}\Big)-p_\eta\eta_{xxxx}=E_2(x,t) \qquad (2.6.85)$$

where

$$E_2(x,t)=E_{1x}(x,t)+p_{\eta x}\eta_{xxx}+\mu\frac{\partial}{\partial t}\Big(\frac{\eta_{xxx}\eta_x}{\eta^2}\Big).$$

Using the embedding theorem, lemmas 2.6.1 and 2.6.2 and theorems 2.1.1, 2.1.2, (2.6.68) and (2.6.82)–(2.6.85), we can conclude that

$$\|E_{xx}(t)\| \le C_4(\|\theta_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2}),$$
$$\|E_{1x}(t)\| \le C_1\Big(\|E_{xx}(t)\| + \|v_{txxx}(t)\| + \|(p_{\eta x}\eta_{xx})_x(t)\| + \Big\|\Big(\frac{\eta_{xx}\eta_x}{\eta^2}\Big)_{tx}(t)\Big\|\Big)$$
$$\le C_1\|v_{txxx}(t)\| + C_4\Big(\|\theta_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^3}\Big)$$

which imply

$$\|E_2(t)\| \le C_1\|v_{txxx}(t)\| + C_4(\|\theta_x(t)\|_{H^3} + \|\eta_x(t)\|_{H^2} + \|v_x(t)\|_{H^3}). \tag{2.6.86}$$

We infer from (2.6.19) and (2.6.20) that

$$\int_0^t (\|v_{tt}\|^2 + \|\theta_{tt}\|^2)(\tau)\,d\tau \le C_4, \quad \forall t > 0 \tag{2.6.87}$$

which, together with (2.6.48), (2.6.50) and (2.6.66), gives

$$\int_0^t (\|v_{txxx}\|^2 + \|\theta_{txxx}\|^2)(\tau)\,d\tau \le C_4, \quad \forall t > 0. \tag{2.6.88}$$

Thus it follows from (2.6.68), (2.6.88) and lemmas 2.6.1, 2.6.2 and theorems 2.1.1, 2.1.2 that

$$\int_0^t \|E_2\|^2(\tau)\,d\tau \le C_4, \quad \forall t > 0. \tag{2.6.89}$$

Multiplying (2.6.85) by $\frac{\eta_{xxxx}}{\eta}$ in $L^2(0,1)$, we get

$$\frac{d}{dt}\Big\|\frac{\eta_{xxxx}}{\eta}(t)\Big\|^2 + C_1^{-1}\Big\|\frac{\eta_{xxxx}}{\eta}(t)\Big\| \le C_1\|E_2(t)\|^2. \tag{2.6.90}$$

Thus it follows from (2.6.89) and (2.6.90) that for all $t > 0$,

$$\|\eta_{xxxx}(t)\|^2 + \int_0^t \|\eta_{xxxx}(s)\|^2\,ds \le C_4. \tag{2.6.91}$$

Differentiating (2.1.4) with respect to x three times, using lemmas 2.6.1, 2.6.2 and theorems 2.1.1, 2.1.2 and the Poincaré inequality, we can derive for all $t > 0$, we infer

$$\|v_{xxxxx}(t)\| \le C_1\|v_{txxx}(t)\| + C_2(\|u_x(t)\|_{H^3} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^3}). \tag{2.6.92}$$

Thus it follows from (2.6.89)–(2.6.92) that

$$\int_0^t (\|v_{xxxxx}\|^2 + \|\eta_{xxxt}\|_{H^1}^2)(s)\,ds \le C_4. \tag{2.6.93}$$

Differentiating (2.1.5) with respect to x three times, using (2.6.67), (2.6.68) and theorems 2.1.1, 2.1.2 and the Poincaré inequality, we infer

$$\|\theta_{xxxxx}(t)\| \leq C_4(\|\eta_x(t)\|_{H^3} + \|v_x(t)\|_{H^3} + \|\theta_x(t)\|_{H^3} + \|\theta_{xxt}(t)\|_{H^1} + \|I_{xxx}(t)\|_{L^2(\Omega\times\mathbb{R}_+\times(-1,1))}). \tag{2.6.94}$$

From (2.6.80), (2.6.84), (2.6.92) and (2.6.94), we conclude for all $t > 0$,

$$\int_0^t \|\theta_{xxxxx}(s)\|^2 ds \leq C_4, \tag{2.6.95}$$

which, together with (2.6.81) and (2.6.94), gives for all $t > 0$,

$$\int_0^t (\|v_{xx}\|^2_{W^{2,\infty}} + \|\theta_{xx}\|^2_{W^{2,\infty}})(s)\, ds \leq C_4. \tag{2.6.96}$$

At last, using all previous estimates and the interpolation inequality, we can easily derive desired estimates (2.6.67)–(2.6.70). The proof is complete. □

Lemma 2.6.4. *If assumptions in theorem 2.1.3 hold, then for any* $(\eta_0, v_0, \theta_0, \mathcal{I}_0) \in \mathcal{H}$, *the following estimate holds for all* $t > 0$,

$$\|\mathcal{I}(t)\|_{H^5} \leq C_4. \tag{2.6.97}$$

Proof. By (2.1.6), we have

$$\begin{aligned} \|\mathcal{I}_{xxxx}(t)\|^2 &= \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} \frac{1}{\omega}(\eta S)_{xxx}\, d\omega dv\Big)^2 dx \\ &\leq C_1 \int_0^1 \Big(\int_0^{+\infty}\int_{S^1} \frac{1}{\omega}\eta_{xxx} S d\omega dv\Big)^2 + \Big(\int_0^{+\infty}\int_{S^1} \frac{1}{\omega}\eta_{xx} S_x d\omega dv\Big)^2 dx \\ &\quad + \Big(\int_0^{+\infty}\int_{S^1} \frac{1}{\omega}\eta_x S_{xx} d\omega dv\Big)^2 + \Big(\int_0^{+\infty}\int_{S^1} \frac{1}{\omega} S_{xxx} d\omega dv\Big)^2 dx \\ &=: \sum_{i=1}^4 G_i. \end{aligned} \tag{2.6.98}$$

Applying the Hölder inequality and the interpolation inequality, (2.1.18) and theorems 2.1.1, 2.1.2 and lemma 2.6.3, we get

$$\begin{aligned} G_1 &\leq C_1 \int_0^1 \eta^2_{xxx}\Big(\int_0^{+\infty}\int_{S^1} \frac{1}{\omega^2}\sigma_a + \sigma_a(B^2 + I^2) + \frac{1}{\omega^2}\sigma_s + \sigma_s(\tilde{I} - I)^2 d\omega dv\Big) dx \\ &\leq C_1 \int_0^1 \eta^2_{xxx} dx \leq C_4, \end{aligned} \tag{2.6.99}$$

$$\begin{aligned} G_2 &\leq C_1 \int_0^1 \eta^2_{xx}(\eta^2_x + \theta^2_x)\, dx + C_1 \int_0^1 \eta^2_{xx}\Big(\int_0^{+\infty}\int_{S^1} (\sigma_a + \sigma_s) I_x d\omega dv\Big)^2 dx \\ &\quad + C_1 \int_0^1 (\eta^2_x + \theta^2_x)\, dx \\ &\leq C_1(\|\eta_{xx}\|^2_{H^1} + \|\theta_x\|^2) \leq C_4. \end{aligned} \tag{2.6.100}$$

Analogously, we have

$$G_3 \leq C_1(\|\eta_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2) + C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} (\sigma_a + \sigma_s) I_{xx} d\omega dv \Big)^2 dx$$
$$\leq C_4, \tag{2.6.101}$$

$$G_4 \leq C_1(\|\eta_{xx}\|_{H^1}^2 + \|\theta_{xx}\|_{H^1}^2) + C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} (\sigma_a + \sigma_s) I_{xxx} d\omega dv \Big)^2 dx$$
$$\leq C_4. \tag{2.6.102}$$

Inserting (2.6.99)–(2.6.102) into (2.6.98), we obtain for all $t > 0$,

$$\|\mathcal{I}_{xxxx}(t)\|^2 \leq C_4. \tag{2.6.103}$$

Differentiating (2.1.6) with respect to x four times, we arrive at

$$\begin{aligned} \|\mathcal{I}_{xxxxx}(t)\|^2 \leq{}& C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_{xxxx} S d\omega dv \Big)^2 + C_1 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_{xxx} S_x d\omega dv \Big)^2 \\ &+ C_1 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_{xx} S_{xx} d\omega dv \Big)^2 + C_1 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_x S_{xxx} d\omega dv \Big)^2 \\ &+ C_1 \Big(\int_0^{+\infty} \int_{S^1} \frac{1}{\omega} S_{xxxx} d\omega dv \Big)^2 dx =: \sum_{i=1}^5 H_i. \end{aligned} \tag{2.6.104}$$

Similarly, we can deduce from lemmas 2.2.1–2.2.9, 2.3.1 and 2.3.2,

$$H_1 \leq C_1 \int_0^1 \eta_{xxxx}^2 dx \leq C_4, \tag{2.6.105}$$

$$\begin{aligned} H_2 \leq{}& C_1 \int_0^1 \eta_{xxx}^2 (\eta_x^2 + \theta_x^2) + C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} I_x d\omega dv \Big)^2 dx \\ \leq{}& C_1(\|\eta_x\|_{H^3}^2 + \|\theta_x\|^2) + C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} I_x d\omega dv \Big)^2 dx \leq C_4, \end{aligned} \tag{2.6.106}$$

$$H_3 \leq C_1(\|\eta_{xx}\|_{H^1}^2 + \|\theta_{xx}\|_{H^1}^2) + C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} I_{xx} d\omega dv \Big)^2 dx \leq C_4, \tag{2.6.107}$$

$$H_4 \leq C_1(\|\eta_{xx}\|_{H^1}^2 + \|\theta_{xx}\|_{H^1}^2) + C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} I_{xxx} d\omega dv \Big)^2 dx \leq C_4, \tag{2.6.108}$$

$$H_4 \leq C_1(\|\eta_{xx}\|_{H^2}^2 + \|\theta_{xx}\|_{H^2}^2) + C_1 \int_0^1 \Big(\int_0^{+\infty} \int_{S^1} I_{xxxx} d\omega dv \Big)^2 dx \leq C_4. \tag{2.6.109}$$

Inserting (2.6.105)–(2.6.109) into (2.6.104), we get for all $t > 0$,

$$\|\mathcal{I}_{xxxxx}(t)\|^2 \le C_4,$$

which, together with (2.6.103) and theorems 2.1.1, 2.1.2, implies (2.6.97). The proof is now complete. □

2.7 Asymptotic Behavior of Solutions in $\mathcal{H}_4$

This section will be devoted to the study of the asymptotic behavior in $\mathcal{H}_4$.

Lemma 2.7.1. *If assumptions in theorem 2.1.3 hold, then we have*

$$\lim_{t\to+\infty}\|\eta(t)-\overline{\eta}\|_{H^4}=0, \tag{2.7.1}$$

where $\overline{\eta}=\int_0^1\eta(y,t)dy$.

Proof. Using (2.6.68), (2.6.75) and lemma 1.1.2, we get

$$\lim_{t\to+\infty}\|\eta_{xxx}(t)\|=0. \tag{2.7.2}$$

Recalling (2.6.91) and (2.6.92) and lemma 1.1.2, we obtain

$$\lim_{t\to+\infty}\|\eta_{xxxx}(t)\|=0,$$

which, together with (2.7.2), (2.5.1) in lemma 2.5.1 and the Poincaré inequality, yields (2.7.1). The proof is now complete. □

Lemma 2.7.2. *Under assumptions in theorem 2.1.3, we have*

$$\lim_{t\to+\infty}\|v(t)\|_{H^4}=0. \tag{2.7.3}$$

Proof. From (2.6.42)–(2.6.48), we can estimate for any $\varepsilon>0$,

$$\begin{aligned}\frac{d}{dt}\|v_{xt}(t)\|^2+C_1^{-1}\|v_{xxt}(t)\|^2\le{}&\varepsilon(\|v_{xxxt}(t)\|^2+\|\theta_{xxt}(t)\|^2)+C_4(\varepsilon)(\|v_x(t)\|_{H^2}^2\\&+\|\theta_t(t)\|_{H^1}^2+\|\theta_{xt}(t)\|^2+\|v_{xt}(t)\|^2\\&+\|\eta_x(t)\|^2).\end{aligned} \tag{2.7.4}$$

Using (2.7.4), theorem 2.1.2, lemmas 2.6.1–2.6.3, we obtain

$$\lim_{t\to+\infty}\|v_{xt}(t)\|^2=0, \tag{2.7.5}$$

which, along with (2.6.7) and theorem 2.1.2, implies

$$\lim_{t\to+\infty}\|v_{xxx}(t)\|=0. \tag{2.7.6}$$

By theorems 2.1.1, 2.1.2 and lemma 2.6.3, and using the interpolation inequality, we obtain

$$\|p_{xxt}(t)\| \leq C_2(\|v_x(t)\|_{H^2} + \|\eta_x(t)\|_{H^2} + \|\theta_t(t)\|_{H^2} + \|\theta_x(t)\|_{H^2}). \tag{2.7.7}$$

Differentiating (2.1.4) with respect to t once and x twice, multiplying the resulting by v_{xxt} in $L^2(0,1)$ and using the Young inequality and theorems 2.1.1, 2.1.2, we dedive

$$\frac{d}{dt}\|v_{xxt}(t)\|^2 + \|v_{xxxt}(t)\|^2 \leq C_1\|p_{xxt}(t)\|^2 + C_2(\|v_{xt}(t)\|^2_{H^2} + \|v_x(t)\|^2_{H^2} + \|\eta_x(t)\|^2_{H^2}),$$

which, together with (2.7.7), theorems 2.1.1, 2.1.2, lemmas 2.6.3 and 1.1.2, gives

$$\lim_{t\to+\infty}\|v_{xxt}(t)\|^2 = 0. \tag{2.7.8}$$

From (2.6.51)–(2.6.55), we have derived for small $\varepsilon > 0$,

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{e_\theta}\theta_{xt}(t)\|^2 + C_2^{-1}\|\theta_{xxt}(t)\|^2\\
&\quad \leq \varepsilon(\|\theta_{xxxt}(t)\|^2 + \|v_{xxt}(t)\|^2) + C_2(\varepsilon)(\|v_x(t)\|^2_{H^2} + \|\theta_x(t)\|^2_{H^2}\\
&\qquad + \|\theta_t(t)\|^2_{H^1} + \|\eta_x(t)\|^2 + \|v_{xt}(t)\|^2 + \|I_t(t)\|^2_{L^2(\Omega\times S^1\times\mathbb{R}_+)} + \|I_x\|^2_{L^2(\Omega\times S^1\times\mathbb{R}_+)}),
\end{aligned}$$

which, combined with lemmas 2.2.9, 2.6.3 and 1.1.2, implies

$$\lim_{t\to+\infty}\|\theta_{xt}(t)\| = 0. \tag{2.7.9}$$

By (2.7.9), (2.6.16), theorems 2.1.1, 2.1.2 and the fact $\lim_{t\to+\infty}\|\mathcal{I}_x(t)\|^2 = 0$, we arrive at

$$\lim_{t\to+\infty}\|\theta_{xxx}(t)\| = 0. \tag{2.7.10}$$

Thus, by (2.7.2), (2.7.6), (2.7.8), (2.7.10) and theorems 2.1.1, 2.1.2, we obtain

$$\lim_{t\to+\infty}\|v_{xxxx}(t)\| = 0,$$

which, together with (2.7.6) and theorems 2.1.1, 2.1.2, yields (2.7.3). This proves the proof. □

Lemma 2.7.3. *If assumptions in theorem 2.1.3 hold, then we have*

$$\lim_{t\to+\infty}\left\|\theta(t) - \overline{\theta}\right\|_{H^4} = 0, \tag{2.7.11}$$

where $\overline{\theta} > 0$ *is determined by* $e(\overline{\eta},\overline{\theta}) = \int_0^1 (\frac{1}{2}v_0^2 + e(\eta_0,\theta_0) + F_R(0))\,dx$.

Proof. Obviously, the Young inequality gives

$$\begin{aligned}
\frac{d}{dt}\|I_t(t)\|^2_{L^2(\Omega\times(-1,1)\times\mathbb{R}_+)} &= 2\int_0^1\int_0^{+\infty}\int_{S^1} I_t\cdot I_{tt}\,d\omega\,dv\,dx\\
&\leq \int_0^1\int_0^{+\infty}\int_{S^1} I_t^2\,d\omega\,dv\,dx + \int_0^1\int_0^{+\infty}\int_{S^1} I_{tt}^2\,d\omega\,dv\,dx,
\end{aligned}$$

which, together with (2.6.39), (2.6.69) and lemma 1.1.2, gives

$$\lim_{t\to+\infty}\|I_t(t)\|^2_{L^2(\Omega\times(-1,1)\times\mathbb{R}_+)}=0. \tag{2.7.12}$$

Thus it follows from (2.6.13), (2.7.12) and theorems 2.1.1 and 2.1.2 that

$$\lim_{t\to+\infty}\big\|[\eta(S_E)_R]_t\big\|=0. \tag{2.7.13}$$

Similarly, from theorems 2.1.1 and 2.1.2, we derive for small $\epsilon>0$,

$$\begin{aligned}\frac{d}{dt}\|\theta_{tt}(t)\|^2+C_1\|\theta_{xxt}(t)\|^2\le{}&\varepsilon(\|\theta_{xtt}(t)\|^2+\|v_{xtt}(t)\|^2)+C_2(\|v_x(t)\|^2_{H^1}+\|\theta_{tt}(t)\|^2\\&+\|\theta_t(t)\|^2_{H^1}+\|v_{xt}(t)\|^2+\|\theta_{xt}(t)\|^2+\|I_t(t)\|_{L^2(\Omega\times(-1,1)\times\mathbb{R}_+)}\\&+\|I_{tt}(t)\|_{L^2(\Omega\times(-1,1)\times\mathbb{R}_+)}),\end{aligned}$$

which, together with (2.6.39), theorems 2.1.1, 2.1.2, lemmas 2.6.3 and 1.1.2, implies

$$\lim_{t\to+\infty}\|\theta_{tt}(t)\|=0. \tag{2.7.14}$$

We can derive from the similar estimate as (2.6.21)

$$\begin{aligned}\|\theta_{xxt}(t)\|\le C_2(&\|\theta_{tt}(t)\|+\|v_x(t)\|+\|\eta_x(t)\|+\|\theta_t(t)\|+\|\theta_{xt}(t)\|+\|\theta_x(t)\|_{H^2}\\&+\big\|[\eta(S_E)_R]_t\big\|)\end{aligned}$$

whence, by (2.7.9), (2.7.10) and (2.7.12)–(2.7.14),

$$\lim_{t\to+\infty}\|\theta_{xxt}(t)\|=0,$$

which, combined with (2.1.25), (2.7.2), (2.7.6), (2.7.10) and the fact $\lim_{t\to+\infty}\|\mathcal{I}_{xx}(t)\|^2=0$, yields

$$\lim_{t\to+\infty}\|\theta_{xxxx}(t)\|^2=0. \tag{2.7.15}$$

Thus (2.7.11) follows from (2.7.10) and (2.7.15). The proof is thus complete. □

Lemma 2.7.4. *If assumptions in theorem 2.1.3 hold, then we have*

$$\lim_{t\to+\infty}\|\mathcal{I}(t)\|_{H^5}=0. \tag{2.7.16}$$

Proof. From (2.1.6), it follows

$$\begin{aligned}\|\mathcal{I}_{xxxx}(t)\|^2\le{}&C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_{xxx}S\,d\omega\,dv\right\|^2+C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_{xx}S_x\,d\omega\,dv\right\|^2\\&+C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta_xS_{xx}\,d\omega\,dv\right\|^2+C_1\left\|\int_0^{+\infty}\int_{S^1}\frac{1}{\omega}\eta S_{xxx}\,d\omega\,dv\right\|^2\\=:{}&\sum_{i=1}^4M_i.\end{aligned} \tag{2.7.17}$$

Using (2.1.18), theorems 2.1.1, 2.1.2 and lemma 2.6.3, we deduce

$$\begin{aligned} M_1 &\le C_1 \int_0^1 \eta_{xxx}^2 \Big(\int_0^{+\infty} \int_{S^1} (\sigma_1(B-I) + \sigma_s(\tilde{I} - I)) d\omega dv \Big)^2 dx \\ &\le C_1 \|\eta_{xxx}(t)\|^2, \end{aligned} \tag{2.7.18}$$

$$M_2 \le C_1 \int_0^1 \eta_{xx}^2 (\eta_x^2 + \theta_x^2 + 1)\, dx \le C_1(\|\eta_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^1}^2). \tag{2.7.19}$$

Analogously,

$$M_3 \le C_1(\|\eta_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^1}^2 + \|\mathcal{I}_{xx}(t)\|^2), \tag{2.7.20}$$

$$M_4 \le C_1(\|\eta_x(t)\|_{H^2}^2 + \|\theta_x(t)\|_{H^2}^2 + \|\mathcal{I}_{xx}(t)\|^2 + \|\mathcal{I}_{xxx}(t)\|^2). \tag{2.7.21}$$

Inserting (2.7.18)–(2.7.21) into (2.7.17) and using theorems 2.1.1, 2.1.2, (2.7.2) and (2.7.15), we have

$$\lim_{t\to +\infty} \|\mathcal{I}_{xxxx}(t)\|^2 = 0. \tag{2.7.22}$$

Similarly, we get

$$\begin{aligned} \|\mathcal{I}_{xxxxx}(t)\|^2 \le{}& C_1 \Big\| \int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_{xxxx} S d\omega dv \Big\|^2 + C_1 \Big\| \int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_{xxx} S_x d\omega dv \Big\|^2 \\ &+ C_1 \Big\| \int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_{xx} S_{xx} d\omega dv \Big\|^2 + C_1 \Big\| \int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta_x S_{xxx} d\omega dv \Big\|^2 \\ &+ C_1 \Big\| \int_0^{+\infty} \int_{S^1} \frac{1}{\omega} \eta S_{xxxx} d\omega dv \Big\|^2 =: \sum_{i=1}^5 N_i. \end{aligned} \tag{2.7.23}$$

Similarly to (2.7.17) and by a more delicated computation, we can derive

$$N_1 \le C_1 \|\eta_x(t)\|_{H^3}^2, \tag{2.7.24}$$

$$N_2 \le C_4(\|\eta_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^1}^2), \tag{2.7.25}$$

$$N_3 \le C_3(\|\eta_x(t)\|_{H^1}^2 + \|\theta_x(t)\|_{H^1}^2 + \|\mathcal{I}_{xx}(t)\|^2), \tag{2.7.26}$$

$$N_4 \le C_1(\|\eta_x(t)\|_{H^2}^2 + \|\theta_x(t)\|_{H^2}^2 + \|\mathcal{I}_{xx}(t)\|^2 + \|\mathcal{I}_{xxx}(t)\|^2), \tag{2.7.27}$$

$$N_5 \le C_1(\|\eta_x(t)\|_{H^3}^2 + \|\theta_x(t)\|_{H^3}^2 + \|\mathcal{I}_{xx}(t)\|^2 + \|\mathcal{I}_{xxx}(t)\|^2 + \|\mathcal{I}_{xxxx}(t)\|^2). \tag{2.7.28}$$

Plugging (2.7.24)–(2.7.28) into (2.7.23) and using theorems 2.1.1, 2.1.2, (2.7.1), (2.7.11) and (2.7.22), we get

$$\lim_{t\to+\infty}\|\mathcal{I}_{xxxxx}(t)\|^2=0,$$

which gives (2.7.16). Thus this completes the proof. □

Proof of Theorem 2.1.3. Combining lemmas 2.6.1–2.6.4 and 2.7.1–2.7.4, we can complete the proof of theorem 2.1.3. □

2.8 Bibliographic Comments

It is well-known that radiation dynamics includes the radiative effects into the hydrodynamical framework. When equilibrium holds between matter and radiation, a simple way to do that is to include local radiation terms into state functions and transport coefficients. From quantum mechanics, we know that radiation can be described by its quanta, the photons, which are massless particles traveling at the speed c of light, characterized by their frequency ν, their energy $E = h\nu$ (where h is Planck's constant), and their momentum $\vec{p} = \frac{h\nu}{c}\vec{\Omega}$ with $\vec{\Omega}$ is a vector of the 2-unit sphere. Moreover, from statistical mechanics, we can describe macroscopically an assembly of massless photons of energy E and momentum $\vec{p}$ using a distribution function: the radiative density $I(r, t, \vec{\Omega}, \nu)$. Using this fundamental quantity, we can derive global quantities by integrating with respect to the angular and frequency variables: the spectral radiative energy density $E_R(r,\ t)$ per unit volume is then $E_R(r,t) := \frac{1}{c}\int\int I(r,t,\vec{\Omega},\nu)d\Omega d\nu$, and the spectral radiative flux $\overrightarrow{F_R} = \int\int\vec{\Omega}I(r,t,\vec{\Omega},\nu)\ d\Omega d\nu$. If the matter is in thermodynamic equilibrium at constant temperature T and if radiation is also in thermodynamic equilibrium at matter, its temperature is also T and statistical mechanics tells us that the distribution function for photons is given by the Bose–Einstein statistics with zero chemical potential.

When there are no radiative effects, the complete hydrodynamical system can be derived from the standard conservation laws of mass, momentum and energy using Boltzmann's equation satisfied by the $f_m(r,\ \vec{v},\ t)$ and the Chapman-Enskog expansion Gallavotti [43]. Then this reduces to the compressible Navier–Stokes system

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0, \\ (\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) = -\nabla\cdot\vec{\Pi} + \vec{f}, \\ (\rho\varepsilon)_t + \nabla\cdot(\rho\varepsilon\vec{u}) = -\nabla\vec{q} - \vec{D}:\vec{\Pi} + g, \end{cases} \tag{2.8.1}$$

where $\vec{\Pi} = -p(\rho,\ T)\vec{I} + \vec{\pi}$ is the material stress tensor for a newtonian fluid with the viscous contribution $\vec{\pi} = 2\mu\vec{D} + \lambda\nabla\cdot\vec{u}\ \vec{I}$ with $3\lambda + 2\mu \geq 0$ and $\mu > 0$, and the strain tensor $\vec{D}$ such that $\vec{D}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. $\vec{q}$ is the thermal heat flux and $\vec{f}$ and g are external force and source terms. There are many mathematical researchers who studied such models and related models. We can refer to Antontsev *et al.* [3], Batchelor [4], Choe and Kim [13, 14], Constantin *et al.* [15], Ducomet and Zlotnik [22], Feireisl [28], Feireisl and Novotný [29], Feireisl and Petzeltova [31–33], Feireisl *et al.* [30], Fang and Zhang [26, 27], Foias and Temam [34, 35], Frid and Shelukhin [36], Fujita and Kato [39], Galdi [42], Hoff [46–49], Hoff and Serre [51], Hoff and Smoller [52], Hoff and Ziane [53, 54], Huang *et al.* [58], Jiang [61–67], Kawashima [68], Kawashima and Nishida [69], Nishibata and Zhu [70], Kazhikhov [71, 72], Kazhikov and Shelukhin [73], Lions [84], Matsumura and Nishida [90–93], Okada [97], Paicu and Zhang [98], Qin [101–105], Qin and Hu [113, 114], Qin *et al.* [115, 116, 118–120, 126, 127, 131], Qin and Huang [121, 122], Qin and Jiang [125], Qin and Rivera [129], Qin and Song [130], Qin and Wen [132], Qin and Zhao [134], Serrin [136], Temam [139, 140], Xin [146], Xin and Yan [147], Zhang and Fang [150–154], Zheng and Qin [156], and references therein.

In the framework of special relativity, the foundations of radiative fluids have been described by Pomraning [99, 100] and Mihalas *et al.* [94]. Later on, Buet *et al.* [10] and Lowrie *et al.* [89] studied in the inviscid case. Dubroca *et al.* [17], Lin [79] and Lin *et al.* [80] investigated for numerical aspects. For more results, we can refer to Chandrasekhar [12], Gallavotti [43], Jiang [61], Lowrie *et al.* [89] and Zhong and Jiang [157].

When radiation is present, Chandrasekhar [12] investigated the radiation integro-differential equation: terms $\vec{f}$ and g include the terms for the coupling between the matter and the radiation, depending on I, and I is driven by a transport equation.

If the matter is at local thermodynamics equilibrium (LTE), the coupled system reads (see, *e.g.*, Mihalas and Weibel-Mihalas [94] and Pomraning [99] for details)

$$\begin{cases} \rho_t + \nabla\cdot(\rho\vec{u}) = 0, \\ (\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) = -\nabla\cdot\vec{\Pi} + \overrightarrow{S_F}, \\ (\rho\varepsilon)_t + \nabla\cdot(\rho\varepsilon\vec{u}) = -\nabla\vec{q} - \vec{D}:\vec{\Pi} + S_E, \\ \frac{1}{c}\frac{\partial}{\partial t} I(r,t,\vec{\Omega},\nu) + \vec{\Omega}\cdot\nabla I(r,t,\vec{\Omega},\nu) = S_t(r,t,\vec{\Omega},\nu), \end{cases} \tag{2.8.2}$$

where $\rho(x,t)$, $\vec{u}(x,t)$, $\theta(x,t)$ represent the density, velocity and temperature, respectively, the coupling terms are

$$\begin{aligned} S_t(r,\ t,\ \vec{\Omega},\ \nu) = {} & \sigma_a\Big(\nu,\ \vec{\Omega},\ \rho,\ T,\ \frac{\vec{\Omega}\cdot\vec{u}}{c}\Big)\Big[B(\nu,\ T) - I(r,\ t,\ \vec{\Omega},\ \nu)\Big] \\ & + \int\int \sigma_s(r,\ t,\ \rho,\ \vec{\Omega}'\cdot\vec{\Omega},\ \nu'\to\nu) \\ & \times\Big\{\frac{\nu}{\nu'} I(r,\ t,\ \vec{\Omega}',\ \nu')I(r,\ t,\ \vec{\Omega},\ \nu) - \sigma_s(r,\ t,\ \rho,\ \vec{\Omega}'\cdot\vec{\Omega},\ \nu'\to\nu) \\ & \times I(r,\ t,\ \vec{\Omega}',\ \nu')I(r,\ t,\ \vec{\Omega},\ \nu)\Big\} d\Omega' d\nu', \end{aligned}$$

the radiative energy source

$$S_E(r,t) := \int\int S_t(r,\ t,\ \vec{\Omega},\ \nu)\, d\Omega d\nu,$$

the radiative flux

$$\overrightarrow{S_F}(r,\ t) := \frac{1}{c}\int\int \vec{\Omega} S_t(r,\ t,\ \vec{\Omega},\ \nu)\, d\Omega d\nu,$$

the functions σ_a and σ_s describe in a phenomenological way the absorption–emission and scattering properties of the photon-matter interaction, and Planck function $B(\nu,\ \theta)$ describes the frequency-temperature black body distribution. We also note some results in Buet and Després [10], Dubroca and Feugeas [17], Jiang [61], Lin [79], Lin, Coulombel and Goudon [80], Lowrie, Morel and Hittinger [89] and Zhong and Jiang [157].

Let us now recall some previous works concerning the one-dimensional radiative fluids. In Ducomet and Nečasová [19], Ducomet and Nečasová considered the following system

$$\begin{cases} \eta_t = v_x, \\ v_t = \sigma_x - \eta (S_F)_R, \\ (e + \frac{1}{2}v^2) = (\sigma v - Q)_x - \eta (S_E)_R, \\ I_t + \eta^{-1}(c\omega - v) I_x = cS \end{cases} \tag{2.8.3}$$

with $(S_F)_R = \frac{1}{c}\int_{-1}^{1}\int_0^{+\infty} \omega S(x, t; \nu, \omega)\, d\nu d\omega$ in the domain $(0, M) \times \mathbb{R}_+$ subjected to the Dirichlet–Neumann boundary conditions

$$v|_{x=0,M} = 0, \quad Q|_{x=0,M} = 0, \tag{2.8.4}$$

and

$$I|_{x=0} = 0 \text{ for } \omega \in (0, 1), \quad I|_{x=M} = 0 \text{ for } \omega \in (-1, 0). \tag{2.8.5}$$

For $q \geq r + 1$ with some suitable assumptions, they proved the existence and uniqueness of weak solutions. However, all estimates depended on any given time $T > 0$. So they could not study the large-time behavior of problem (2.8.3)–(2.8.5) based on their estimates. Ducomet and Nečasová [20] investigated the problem (2.8.3) with the different boundary conditions from Ducomet and Nečasová [19]

$$v|_{x=0,M} = 0, \quad Q|_{x=0,M} = 0, \tag{2.8.6}$$

and

$$I|_{x=0} = I_b(\nu) \text{ for } \omega \in (0, 1), \quad I|_{x=M} = I_b(\nu) \text{ for } \omega \in (-1, 0) \tag{2.8.7}$$

and proved that the unique strong solutions of problem (2.8.3)–(2.8.6) converge to a well-determined equilibrium state at exponential rate in $H^1(0,M)$ for the fluid variables η, v, θ and in $L^2(0,\ M)$ for the radiative intensity $\mathcal{I} = \int_0^{+\infty}\int_{S^1} I(x, t; \nu, \omega)\, d\omega d\nu$.

Ducomet and Nečasová [18] established the global existence of solutions to the system (2.1.3) and (2.1.14) in $\mathcal{H}_i$ $(i = 1,\ 2)$. However, estimates obtained there depend on any given time T, so they also could not investigate the large-time behavior of global solutions in $\mathcal{H}_i$ $(i = 1,\ 2)$ based on their estimates. Moreover, in

Ducomet and Nečasová [18], all estimates hold only for $q \geq 2r + 1$. In [109], we have established uniform-in-time estimates of $(\eta(t), v(t), \theta(t), \mathcal{I}(t))$ in $\mathcal{H}_i$ $(i = 1, 2, 4)$, which hold for $q \geq r + 1$. For $q \geq r + 1$, $\alpha \geq 0$, Qin *et al.* [111] established the gobal existence of solutions in $\mathcal{H}_i$ $(i = 1, 2)$. Hence our results in this chapter have improved those in Ducomet and Nečasová [18] and Qin *et al.* [111]. Furthermore, the system considered here is quite different from that in Qin [104], so our uniform-in-time estimates are also quite different from those in Qin [104].

Remark 2.8.1. *The multi-dimensional viscous situation has been poorly understood even at the formal level. Since the one-dimensional model possesses the special constitutive state equations, which the multi-dimensional model do not have, to our knowledge, we have not found any results on the global existence and asymptotic behavior of solutions to system* (2.8.2)*, i.e., the multi-dimensional case of* (2.1.3)–(2.1.6)*. Moreover, some Sobolev embedding inequalities and interpolation inequalities involved in our arguments heavily depend on the dimension, hence this may bring about some difficulties in deriving uniform-in-time estimates. In a word, the method we deal with the one-dimensional case can not be applied directly to the multi-dimensional case, which depends on the special constitutive relations of state functions, and so on. However, we can refer to Kippenhahn and Weigert* [74] *for a macroscopic treatment of radiation in the astrophysical context, and Feireisl* [28] *and Qin* [104] *for the associated mathematical treatment.*

Chapter 3

Global Existence and Regularity of a One-Dimensional Liquid Crystal System

3.1 Main Results

This chapter will establish the global existence and regularity of solutions to the following system

$$\begin{cases} \rho_t + (\rho u)_x = 0, & (3.1.1)\\ (\rho u)_t + (\rho u^2)_x + (P(\rho))_x = \mu u_{xx} - \lambda(|n_x|^2)_x, & (3.1.2)\\ n_t + un_x = \theta(n_{xx} + |n_x|^2 n) & (3.1.3)\end{cases}$$

where $x \in [0,\ 1]$. Here, $\rho > 0$ is the density function, u denotes the velocity, n represents the optical director of the molecules, $P(\rho) = \rho^\gamma(\gamma \geq 1)$ is the pressure, and λ, μ, θ are positive constants. For simplicity, we assume $\lambda = \mu = \theta = 1$ in this chapter. The results of this chapter are chosen from [117].

We consider the following initial-boundary value problem for (3.1.1)–(3.1.3) in the reference domain $\{(x,\ t)\colon 0 < x < 1,\ t \in [0,T]\}$. for any given $T > 0$ under the initial conditions and boundary conditions

$$\rho(x,0) = \rho_0(x),\ u(x,0) = u_0(x),\ n(x,0) = n_0(x), \tag{3.1.4}$$

$$u(0,t) = u(1,t) = 0,\ n_x(0,t) = n_x(1,t) = 0. \tag{3.1.5}$$

Equations (3.1.1)–(3.1.3) reveal system modeling the nematic liquid crystal flow which consists of subsystem of the compressible Navier–Stokes equations coupling with a subsystem including the heat flow equation for harmonic maps. It was derived from the theory of hydrodynamics motivated by the Ericksen–Leslie system (Ericksen [23] and Lesile [77]) for the nematic liquid crystal flow. Equation (3.1.1) represents the transporting relation (conservation of mass), equation (3.1.2) is the conservation of the linear momentum and equation (3.1.3) is the heat flow of harmonic map equation.

DOI: 10.1051/978-2-7598-2903-3.c003

The notation in this chapter is standard. We put $\|\cdot\| = \|\cdot\|_{L^2[0,1]}$. Subscripts t and x denote the (partial) derivatives with respect to t and x, respectively. We use C_i $(i=1,\ 2)$ to denote the generic positive constant depending on the $\|(\rho_0, u_0, n_0)\|_{H^i\times H^i\times H^{i+1}}$ $(i=1,2)$, $\min_{x\in[0,1]} u_0(x)$, $\min_{x\in[0,1]} n_0(x)$ and time T, and C_j $(j=3,\ 4)$ depending on $H^j[0,\ 1]$ norm of initial data $(\rho_0,\ u_0,\ n_0)$, $\min_{x\in[0,1]} u_0(x)$, $\min_{x\in[0,1]} n_0(x)$ and time T.

Without loss of generality, we may assume $\int_0^1 \rho_0(x)\,dx = 1$. Under the Lagrangian coordinates, *a.e.*,

$$y = \int_0^x \rho_0(\xi,\tau)\ d\xi, \qquad t=\tau,$$

system (3.1.1)–(3.1.5) is transformed into the following system

$$v_t = u_y, \tag{3.1.6}$$

$$u_t = \left(-P + \frac{u_y}{v} - \frac{|n_y|^2}{v^2}\right)_y, \tag{3.1.7}$$

$$n_t = \frac{1}{v}\left(\frac{n_y}{v}\right)_y + \frac{|n_y|^2}{v^2}\, n, \tag{3.1.8}$$

$$(v,u,n)|_{t=0} = (v_0,u_0,n_0), \tag{3.1.9}$$

$$u|_{y=0,1} = 0, \quad n_y|_{y=0,1} = 0 \tag{3.1.10}$$

where $v = \frac{1}{\rho}$ and $v_0 = \frac{1}{\rho_0}$.

In the sequel, we shall only consider the following case:

$$|n|^2 = n_i n_i = 1. \tag{3.1.11}$$

Our main results in this chapter will read as follows (see also Qin and Huang [124]).

Theorem 3.1.1. *Suppose that* $(v_0,\ u_0,\ n_0) \in H^1[0,\ 1]\times H_0^1[0,\ 1]\times H^2[0,\ 1]$ *and the compatibility conditions hold. Then there exists a unique global solution* $(v(t),\ u(t),\ n(t)) \in H^1[0,\ 1]\times H_0^1[0,\ 1]\times H^2[0,\ 1]$ *to the problem* (3.1.6)–(3.1.10) *such that for any* $(y,\ t)\in[0,\ 1]\times[0,\ T]$ *(for all* $T>0$*),*

$$0 < C_1^{-1} \le v(y,t) \le C_1, \tag{3.1.12}$$

and

$$\|v(t)\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \|n(t)\|_{H^2}^2 + \|n_t(t)\|^2 + \int_0^t \left(\|u\|_{H^2}^2 + \left\|n_{ty}\right\|^2 + \|u_t\|^2\right)(\tau)\ d\tau \le C_1. \tag{3.1.13}$$

Theorem 3.1.2. *Suppose that* $(v_0,\ u_0,\ n_0) \in H^2[0,\ 1] \times H_0^2[0,\ 1] \times H^3[0,\ 1]$ *and the compatibility conditions hold. Then there exists a unique global solution* $(v(t),\ u(t),\ n(t)) \in H^2[0,\ 1] \times H_0^2[0,\ 1] \times H^3[0,\ 1]$ *to the problem* (3.1.6)–(3.1.10) *such that for any* $(y,\ t) \in [0,\ 1] \times [0,\ T]$ *(for all* $T > 0$*),*

$$\|v(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|n(t)\|_{H^3}^2 + \|n_t(t)\|_{H^1}^2 + \|u_t(t)\|^2 + \int_0^t (\left\|u_{ty}\right\|^2 + \left\|n_{tyy}\right\|^2 + \left\|u_{yyy}\right\|^2)(\tau)\ d\tau \le C_2. \tag{3.1.14}$$

Theorem 3.1.3. *Suppose that* $(v_0,\ u_0,\ n_0) \in H^4[0,\ 1] \times H_0^4[0,\ 1] \times H^4[0,\ 1]$ *and the compatibility conditions hold. Then there exists a unique global solution* $(v(t),\ u(t),\ n(t)) \in H^4[0,\ 1] \times H_0^4[0,\ 1] \times H^4[0,\ 1])$ *to the problem* (3.1.6)–(3.1.10) *such that for any* $(y,\ t) \in [0,\ 1] \times [0,\ T]$ *(for all* $T > 0$*),*

$$\|v(t)\|_{H^4}^2 + \|u(t)\|_{H^4}^2 + \|n(t)\|_{H^4}^2 + \|n_t(t)\|_{H^2}^2 + \|u_t(t)\|_{H^2}^2 + \|n_{tt}(t)\|^2 + \|u_{tt}(t)\|^2 + \int_0^t (\left\|u_y\right\|_{H^4}^2 + \left\|n_y\right\|_{H^4}^2 + \left\|u_{ty}\right\|_{H^2}^2 + \left\|n_{ty}\right\|_{H^2}^2 + \left\|u_{tty}\right\|^2 + \left\|n_{tty}\right\|^2)(\tau)\ d\tau \le C_4. \tag{3.1.15}$$

Remark 3.1.1. *It is worthy to point out here that the solution* $(v(t),\ u(t),\ n(t))$ *obtained in theorem 3.1.3 is, in fact, a classical solution such that*

$$\|(v(t), u(t), n(t))\|_{C^{3+\frac{1}{2}}(0,1)\times C^{3+\frac{1}{2}}(0,1)\times C^{3+\frac{1}{2}}(0,1)} \le C_4. \tag{3.1.16}$$

3.2 Global Existence in $H^1 \times H_0^1 \times H^2$

In this section, we shall prove theorem 3.1.1 by establishing a series of lemmas.

Lemma 3.2.1. *If assumptions in theorem 3.1.1 hold, then the following estimates are valid in the Euler coordinates,*

$$\int_0^1 (\rho u^2 + \pi(\rho) + 2|n_x|^2)\, dx + \int_0^T \int_0^1 (u_x^2 + |n \times n_{xx}|^2)\, dxdt \le C_1, \tag{3.2.1}$$

$$\int_0^1 |n_x|^2 dx + \int_0^T \int_0^1 |n_{xx}|^2 dxdt \le C_1 \tag{3.2.2}$$

where $\pi(\rho) = \begin{cases} \frac{\rho^\gamma}{\gamma-1}, & \gamma > 1, \\ \rho \log \rho - \rho, & \gamma = 1. \end{cases}$

Proof. Multiplying (3.1.2) by u, using (3.1.1) and integrating the resulting equality by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\rho u^2 + \pi(\rho))dx + \int_0^1 u_x^2 dx = \int_0^1 |n_x|^2 u_x dx. \tag{3.2.3}$$

Thus we derive from (3.1.11) that

$$n_{xx} + |n_x|^2 n = -n \times (n \times n_{xx}). \tag{3.2.4}$$

Multiplying (3.2.4) by n_{xx}, we obtain

$$\frac{d}{dt}\int_0^1 |n_x|^2 dx + \int_0^1 |n_x|^2 u_x dx + 2\int_0^1 |n \times n_{xx}|^2 dx = 0$$

which, along with (3.2.3), gives (3.2.1).

Multiplying (3.1.3) by n_{xx} in $L^2[0, 1]$, integrating the resultant by parts and using the Gagliardo–Nirenberg interpolation inequality, we derive

$$\begin{aligned} \frac{d}{dt}\int_0^1 |n_x|^2 dx + \int_0^1 |n_{xx}|^2 dx &= \int_0^1 un_x \cdot n_{xx} dx + \int_0^1 |n_x|^4 dx \\ &= \int_0^1 |n_{xx}|^4 dx - \frac{1}{2}\int_0^1 |n_x|^2 u_x dx \\ &\le C_1 \int_0^1 |n_x|^4 dx + C_1 \int_0^1 u_x^2 dx \\ &\le C_1 \|n_x\|^3 \|n_{xx}\| + C_1 \|u_x\|^2 \\ &\le \frac{1}{2}\|n_{xx}\|^2 + C_1(\|n_x\|^6 + \|u_x\|^2) \end{aligned}$$

which, together with (3.2.1), gives (3.2.2). Thus this proves the lemma. □

Lemma 3.2.2. *If assumptions in theorem 3.1.1 hold, then the following estimates are valid for any $T > 0$ in the Lagrangian coordinates, and for all $t \in [0, T]$,*

$$\int_0^1 \left(u^2 + \pi_+(v) + \frac{|n_y|^2}{v}\right) dy + \int_0^T \int_0^1 \left(\frac{|u_y|^2}{v} + \frac{1}{v}\left|n \times \left(\frac{n_y}{v}\right)_y\right|^2\right) dydt \le C_1, \tag{3.2.5}$$

$$\int_0^1 \frac{|n_y|^2}{v} dy + \int_0^T \int_0^1 \frac{1}{v}\left[\left(\frac{n_y}{v}\right)_y\right]^2 dydt \le C_1 \tag{3.2.6}$$

where $\pi_+(v) = \begin{cases} \frac{v^{1-\gamma}}{\gamma-1}, & \gamma > 1, \\ v - \log v - 1, & \gamma = 1. \end{cases}$

Proof. Using lemma 3.2.1 and the total mass conversation $\int_0^1 v dy = \int_0^1 v_0 dy$, we easily derive (3.2.5) and (3.2.6). □

Lemma 3.2.3. *For any* $t \in [0,\ T]$ *(for all* $T > 0$*), there exists one point* $y_1 = y_1(t) \in [0,\ 1]$ *such that the solution* $v(y,\ t)$ *to problem* (3.1.6)–(3.1.10) *possesses the following expression*

$$v(y,t) = D(y,t)Z(t)\left[1 + \int_0^t \left(P + \frac{|n_y|^2}{v^2}\right) vD^{-1}(y,s)Z^{-1}(s)ds\right] \tag{3.2.7}$$

where

$$\begin{cases} D(y,t) = v_0(y)\exp\left(\int_{y_1}^y ud\xi - \int_0^y u_0(\xi)d\xi + \frac{1}{\overline{v}_0}\int_0^1 \int_0^y u_0(z)dzdy\right), & (3.2.8) \\ Z(t) = \exp\left(-\frac{1}{\overline{v}_0}\int_0^t \int_0^1 \left(vP + u^2 + \frac{|n_y|^2}{v}\right) dyds\right),\ \overline{v}_0 = \int_0^1 v_0 dy. & (3.2.9) \end{cases}$$

Proof. The proof is standard, we refer to lemma 2.2.2. □

Lemma 3.2.4. *For any* $T > 0$*, we have*

$$0 < C_1^{-1} \le v(y,t) \le C_1, \quad \text{for all } (y,t) \in [0,1] \times [0,T], \tag{3.2.10}$$

$$\int_0^t \left\|n_y(s)\right\|_{L^\infty}^2 ds \le C_1, \quad \text{for all } t \in [0,\ T]. \tag{3.2.11}$$

Proof. The present proof is more delicate than that of lemma 4.2.5 of this book. Obviously, we derive from (3.2.3) that

$$0 < C^{-1} \le D(y,t) \le C_1, \quad 0 < C_1^{-1} \le Z(t) \le 1. \tag{3.2.12}$$

Noting that $v\left(P + \frac{|n_y|^2}{v^2}\right) > 0$, using (3.2.7) and (3.2.12), we have

$$v(y,t) \ge D(y,t)Z(t) \ge C_1^{-1} > 0. \tag{3.2.13}$$

Employing $W^{1,1}\hookrightarrow L^\infty$, lemmas 3.2.2, 3.2.3 and (3.2.12), (3.2.13), we deduce

$$\begin{aligned}
v(y,t) &\le C_1 + C_1\int_0^t \left(vP + \frac{|n_y|^2}{v}\right) ds \\
&\le C_1 + C_1\int_0^t \left\|\frac{|n_y|}{v}\right\|_{L^\infty}^2 v ds \\
&\le C_1 + C_1\int_0^t \left[\int_0^1 \frac{|n_y|}{v} dy + \int_0^1 \left|\left(\frac{|n_y|}{v}\right)_y\right| dy\right]^2 v(y,s)\, ds \\
&\le C_1 + C_1\int_0^t \left[\left(\int_0^1 \frac{|n_y|^2}{v} dy\right)^{1/2}\left(\int_0^1 \frac{1}{v}\, dy\right)^{1/2}\right. \\
&\quad \left. + \left(\int_0^1 \frac{1}{v}\left|\left(\frac{|n_y|}{v}\right)_y\right|^2 dy\right)^{1/2}\left(\int_0^1 v dy\right)^{1/2}\right] v(y,s)\, ds \\
&\le C_1 + C_1\int_0^t \left(\int_0^1 \frac{1}{v}\, dy + \int_0^1 \frac{|n_y|^2}{v} dy\right. \\
&\quad \left. + \int_0^1 \frac{1}{v}\left|\left(\frac{|n_y|}{v}\right)_y\right|^2 dy + \int_0^1 v dy dy\right) v(y,s)\, ds \\
&\le C_1 + C_1\int_0^t \left(1 + \int_0^1 \frac{1}{v}\left|\left(\frac{|n_y|}{v}\right)_y\right|^2 dy\right) v(y,s)\, ds
\end{aligned} \tag{3.2.14}$$

which, using the Gronwall inequality and (3.2.6), (3.2.13), gives (3.2.10). The estimate (3.2.11) hence follows from (3.2.10) and the proof of (3.2.14). Thus this completes the proof. □

Lemma 3.2.5. *If assumptions in theorem 3.1.1 hold, then the following estimate is valid for any $T > 0$, and for all $t \in [0,\ T]$,*

$$\int_0^1 v_y^2 dy + \int_0^t \int_0^1 (n_y^4 + |n_{yy}|^2 + |n_t|^2) dy ds \le C_1. \tag{3.2.15}$$

Proof. It clearly follows from (3.1.7) that

$$\left(u - \frac{v_y}{v}\right)_t = -\frac{2n_y \cdot n_{yy}}{v^2} + \frac{2|n_y|^2 v_y}{v^3} + \frac{\gamma v_y}{v^{\gamma+1}}. \tag{3.2.16}$$

Multiplying (3.2.16) by $u - \frac{v_y}{v}$ and then integrating the resulting equation over $Q_t = [0,\ 1] \times [0,\ t], t \in [0, T]$ (for all $T > 0$), we have for any $\varepsilon > 0$,

$$
\begin{aligned}
&\left\|u-\frac{v_y}{v}\right\|^2+\int_0^t\int_0^1 (v_y^2+|n_y|^2v_y^2)\,dyds\\
&\quad\le\left\|u_0-\frac{v_{0y}}{v_0}\right\|^2+C_1\int_0^t\int_0^1\Big(|v_yu|+|n_y\cdot n_{yy}u|+|n_y\cdot n_{yy}v_y|+|n_y|^2|v_yu|\Big)\,dyds\\
&\quad\le C_1+\varepsilon\int_0^t\int_0^1 (v_y^2+|n_y|^2v_y^2)\,dyds\\
&\qquad+C_1\int_0^t(1+\|n_y\|_{L^\infty}^2)\int_0^1 u^2\,dyds+C_1\int_0^t\int_0^1|n_{yy}|^2\,dyds
\end{aligned}
$$

which, by taking $\varepsilon>0$ small enough and using (3.2.5) and (3.2.11), gives

$$
\left\|u-\frac{v_y}{v}\right\|^2+\int_0^t\int_0^1 (v_y^2+|n_y|^2v_y^2)\,dyds\le C_1+C_1\int_0^t\int_0^1|n_{yy}|^2\,dyds. \tag{3.2.17}
$$

By (3.2.6), we derive

$$
\begin{aligned}
\int_0^t\int_0^1\left(\frac{|n_{yy}|^2}{v^3}+\frac{|n_y|^2v_y^2}{v^5}\right)dyds&=\int_0^t\int_0^1\frac{1}{v}\left[\left(\frac{n_y}{v}\right)_y\right]^2dydt+\int_0^t\int_0^1 2\,\frac{n_y\cdot n_{yy}v_y}{v^4}\,dyds\\
&\le C_1+\frac{1}{2}\int_0^t\int_0^1\frac{|n_{yy}|^2}{v^3}dyds+C_1\int_0^t\int_0^1\frac{|n_y|^2v_y^2}{v^5}dyds.
\end{aligned}
$$

Thus

$$
\int_0^t\int_0^1|n_{yy}|^2\,dyds\le C_1+C_1\int_0^t\|n_y(s)\|_{L^\infty}^2\int_0^1 v_y^2\,dyds
$$

which, with (3.2.17) and $\|u\|^2\le C_1$ in (3.2.5), gives

$$
\|v_y(t)\|^2+\int_0^t\int_0^1 (v_y^2+|n_y|^2v_y^2)\,dyds\le C_1+C_1\int_0^t\|n_y(s)\|_{L^\infty}^2\int_0^1 v_y^2\,dyds. \tag{3.2.18}
$$

Applying the Gronwall inequality to (3.2.18), and using (3.2.6) and (3.2.11), we conclude

$$
\|v_y(t)\|^2+\int_0^t\int_0^1(|n_y|^4+|n_y|^2v_y^2+|n_{yy}|^2+|v_y|^2)\,dyds\le C_1. \tag{3.2.19}
$$

From (3.1.8) and (3.2.19) it follows that

$$
\|n_t(t)\|\le C_1(\|n_y(t)\|+\|n_{yy}(t)\|) \tag{3.2.20}
$$

or

$$
\|n_{yy}(t)\|\le C_1(\|n_y(t)\|+\|n_t(t)\|). \tag{3.2.21}
$$

Thus we deduce from (3.2.11), (3.2.19) and (3.2.20) that for all $t\in[0,\ T]$,

$$
\int_0^t\int_0^1|n_t|^2\,dyds\le C_1,
$$

which, with (3.2.19) and (3.2.21), gives us (3.2.15). Hence this proves the lemma. □

Lemma 3.2.6. *If assumptions in theorem 3.1.1 hold, then the following estimates are valid for any $T > 0$ and for all $t \in [0,\ T]$,*

$$\|n_t(t)\|^2 + \|n_{yy}(t)\|^2 + \|u_y(t)\|^2 + \int_0^t (\|u_t\|^2 + \|u_{yy}\|^2 + \|n_{ty}\|^2)(s)\,ds \le C_1. \tag{3.2.22}$$

Proof. Multiplying (3.1.7) by u_{yy} and then integrating the resulting equation over Q_t, using the Poincaré inequality and (3.2.15), we obtain

$$\begin{aligned}
&\|u_y(t)\|^2 + \int_0^t \|u_{yy}(s)\|^2 ds \\
&\le C_1 + C_1 \int_0^t \int_0^1 \Big(|v_y| + |u_y v_y| + |n_y \cdot n_{yy}| + |n_y|^2 |v_y|\Big)|u_{yy}|\,dyds \\
&\le C_1 + \frac{1}{2}\int_0^t \|u_{yy}(s)\|^2 ds + C_1 \int_0^t \Big[(\|u_y\|_{L^\infty}^2 + \|n_y\|_{L^\infty}^4)\|v_y\|^2 + \|n_y\|_{L^\infty}^2 \|n_{yy}\|^2\Big](s)\,ds \\
&\le C_1 + \frac{1}{2}\int_0^t \|u_{yy}(s)\|^2 ds + C_1 \int_0^t (\|u_y\|\|u_{yy}\| + \|n_y\|_{L^\infty}^2 \|n_{yy}\|^2)(s)\,ds \\
&\le C_1 + \frac{3}{4}\int_0^t \|u_{yy}(s)\|^2 ds + C_1 \int_0^t \|n_y(s)\|_{L^\infty}^2 \|n_{yy}(s)\|^2 ds,
\end{aligned}$$

which implies

$$\|u_y(t)\|^2 + \int_0^t \|u_{yy}(s)\|^2 ds \le C_1 + C_1 \int_0^t \|n_y(s)\|_{L^\infty}^2 \|n_{yy}(s)\|^2 ds. \tag{3.2.23}$$

Differentiating (3.1.8) with respect to t, multiplying the resulting equation by n_t, integrating it by parts and using lemmas 3.2.2–3.2.5, (3.2.20), (3.2.21) and (3.2.23), we arrive at

$$\begin{aligned}
\|n_t(t)\|^2 + 2\int_0^t \int_0^1 \frac{|n_{ty}|^2}{v^2} dyds &= \|n_t(y,0)\|^2 + 2\int_0^t \sum_{i=1}^5 A_i\,ds \\
&\le C_1 + 2\int_0^t \sum_{i=1}^5 A_i\,ds
\end{aligned} \tag{3.2.24}$$

where

$$\begin{aligned}
&A_1 = \int_0^1 \frac{n_y(u_{yy} n_t + u_y n_{ty})}{v^3} dy, \quad A_2 = -3\int_0^1 \frac{n_y \cdot n_t u_y v_y}{v^4} dy, \\
&A_3 = \int_0^1 \frac{n_{ty} \cdot n_y u_y + n_{ty} \cdot n_t v_y}{v^3} dy, \quad A_4 = \int_0^1 \frac{|n_y|^2 |n_t|^2 + 2 n_y \cdot n_{ty} n \cdot n_t}{v^2} dy, \\
&A_5 = -\int_0^1 \frac{2|n_y|^2 n \cdot n_t u_y}{v^3} dy.
\end{aligned}$$

Using lemmas 3.2.2–3.2.5, the embedding theorem and (3.2.23), we derive that for any $\varepsilon \in (0, 1)$,

$$
\begin{aligned}
&\int_0^t \int_0^1 A_1 \, dy ds \\
&\quad = \int_0^t \int_0^1 \frac{n_y(u_{yy} n_t + u_y n_{ty})}{v^3} dy ds \\
&\quad \leq \varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1(\varepsilon) \int_0^t (\left\|u_y\right\|_{L^\infty}^2 \left\|n_y\right\|^2 + \|n_t\|_{L^\infty} \left\|n_y\right\| \left\|u_{yy}\right\|)(s) \, ds \\
&\quad \leq \varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1 \int_0^t \left(\|n_t(s)\|^{1/2} \left\|n_{ty}(s)\right\|^{1/2} + \|n_t(s)\| \right) \left\|u_{yy}(s)\right\| ds \\
&\qquad + C_1(\varepsilon) \left[\left(\int_0^t \left\|u_y(s)\right\|^2 ds \right)^{1/2} \left(\int_0^t \left\|u_{yy}(s)\right\|^2 ds \right)^{1/2} + \int_0^t \left\|u_y(s)\right\|^2 ds \right] \\
&\quad \leq \varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1 \left(\int_0^t \left\|n_{ty}(s)\right\|^2 ds \right)^{1/4} \left(\int_0^t (\|n_t(s)\|^{\frac{1}{2}} \left\|u_{yy}(s)\right\|)^{\frac{4}{3}} ds \right)^{3/4} \\
&\qquad + C_1 \left(\left\| \int_0^t n_t(s) \right\|^2 ds \right)^{1/2} \left(\int_0^t \left\|u_{yy}(s)\right\|^2 ds \right)^{1/2} \\
&\qquad + C_1 \left(\int_0^t \left\|u_{yy}(s)\right\|^2 ds \right)^{1/2} + C_1(\varepsilon) \\
&\quad \leq 2\varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1(\varepsilon) \left(\int_0^t \left\|u_{yy}(s)\right\|^2 ds \right)^{1/2} \\
&\qquad + C_1 \int_0^t \|n_t(s)\|^{\frac{2}{3}} \left\|u_{yy}(s)\right\|^{\frac{4}{3}} ds + C_1(\varepsilon) \\
&\quad \leq 2\varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1(\varepsilon) \left(\int_0^t \left\|u_{yy}\right\|^2 ds \right)^{1/2} \\
&\qquad + C_1 \left(\int_0^t \|n_t\|^2 ds \right)^{1/3} \left(\int_0^t \left\|u_{yy}(s)\right\|^2 ds \right)^{2/3} + C_1(\varepsilon) \\
&\quad \leq C_1 + 2\varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1(\varepsilon) \left(\int_0^t \left\|u_{yy}(s)\right\|^2 ds \right)^{2/3} \\
&\quad \leq 2\varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1(\varepsilon) \left[1 + \int_0^t \left\|n_y(s)\right\|_{L^\infty}^2 \left\|n_{yy}(s)\right\|^2 ds \right]^{2/3} \\
&\quad \leq 2\varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1(\varepsilon) \left[1 + \int_0^t \left\|n_y(s)\right\|_{L^\infty}^2 (1 + \|n_t(s)\|^2) \, ds \right]^{2/3} \\
&\quad \leq 2\varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds + C_1(\varepsilon) + C_1(\varepsilon) \sup_{0 \leq s \leq t} \|n_t(s)\|^{\frac{4}{3}} \left(\int_0^t \left\|n_y(s)\right\|_{L^\infty}^2 ds \right)^{2/3} \\
&\quad \leq C_1(\varepsilon) + \varepsilon \sup_{0 \leq s \leq t} \|n_t(s)\|^2 + 2\varepsilon \int_0^t \left\|n_{ty}(s)\right\|^2 ds,
\end{aligned}
\tag{3.2.25}
$$

and

$$\begin{aligned}
\int_0^t A_2\,ds &\le C_1 \int_0^t \int_0^1 |n_y \cdot n_t u_y v_y|\,dyds \\
&\le C_1 \left(\int_0^t \|n_t(s)\|^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|n_y(s)\|_{L^\infty}^2 \|u_y(s)\|_{L^\infty}^2 \|v_y(s)\|^2 ds\right)^{\frac{1}{2}} \\
&\le C_1 \sup_{0\le s\le t} \|n_y(s)\|_{L^\infty} \left[\left(\int_0^t \|u_y(s)\|^2 ds\right)^{\frac{1}{4}} \left(\int_0^t \|u_{yy}(s)\|^2 ds\right)^{\frac{1}{4}} \right. \\
&\quad \left. + \left(\int_0^t \|u_y(s)\|^2 ds\right)^{\frac{1}{2}}\right] \\
&\le C_1 \sup_{0\le s\le t} \left(\|n_y(s)\|^{\frac{1}{2}} \|n_{yy}(s)\|^{\frac{1}{2}}\right) \left[1 + \int_0^t \|u_{yy}(s)\|^2 ds\right]^{\frac{1}{4}} \\
&\le C_1 \sup_{0\le s\le t} (1 + \|n_t(s)\|^{\frac{1}{2}}) \left[1 + \int_0^t \|u_{yy}(s)\|^2 ds\right]^{\frac{1}{4}} \\
&\le 2\varepsilon \sup_{0\le s\le t} \|n_t(s)\|^2 + C_1 + C_1 \left(\int_0^t \|u_{yy}(s)\|^2 ds\right)^{\frac{1}{3}} \\
&\le 2\varepsilon \sup_{0\le s\le t} \|n_t(s)\|^2 + C_1 \left[1 + \int_0^t \|n_y(s)\|_{L^\infty}^2 \|n_{yy}(s)\|^2 ds\right]^{\frac{1}{3}} \\
&\le 2\varepsilon \sup_{0\le s\le t} \|n_t(s)\|^2 + C_1 + C_1 \sup_{0\le s\le t} \|n_t(s)\|^{\frac{2}{3}} \left(\int_0^t \|n_y(s)\|_{L^\infty}^2 ds\right)^{1/3} \\
&\le 3\varepsilon \sup_{0\le s\le t} \|n_t(s)\|^2 + C_1. \qquad (3.2.26)
\end{aligned}$$

Similarly, we conclude that for any $\varepsilon > 0$,

$$\begin{aligned}
\int_0^t A_3\,ds &\le C_1 \int_0^t \int_0^1 \big(|n_{ty} \cdot n_y u_y| + |n_{ty} \cdot n_t v_y|\big)\,dyds \\
&\le \varepsilon \int_0^t \|n_{ty}(s)\|^2 ds + C_1(\varepsilon) \int_0^t (\|n_y\|_{L^\infty}^2 \|u_y\|^2 + \|n_t\|_{L^\infty}^2 \|v_y\|^2)(s)\,ds \\
&\le 2\varepsilon \int_0^t \|n_{ty}(s)\|^2 ds + C_1(\varepsilon) + C_1(\varepsilon) \int_0^t \|n_y(s)\|_{L^\infty}^2 \|u_y(s)\|^2 ds, \qquad (3.2.27)
\end{aligned}$$

$$\begin{aligned}
\int_0^t A_4\,ds &\le C_1 \int_0^t \int_0^1 (|n_t|^2 |n_y|^2 + |n_y \cdot n_{ty} n \cdot n_t|)\,dyds \\
&\le \varepsilon \int_0^t \|n_{ty}(s)\|^2 ds + C_1 \int_0^t \|n_t(s)\|_{L^\infty}^2 \|n_y(s)\|^2 ds \\
&\le 2\varepsilon \int_0^t \|n_{ty}(s)\|^2 ds + C_1(\varepsilon), \qquad (3.2.28)
\end{aligned}$$

$$\begin{aligned}\int_0^t A_5\,ds &\le C_1\int_0^t\int_0^1 |n_y|^2|n\cdot n_t u_y|\,dyds\\ &\le C_1\left(\int_0^t \|n_y(s)\|_{L^\infty}^2\|u_y(s)\|^2 ds\right)^{\frac12}\left(\int_0^t \|n_t(s)\|_{L^\infty}^2\|n_y(s)\|^2 ds\right)^{\frac12}\\ &\le \varepsilon\int_0^t \|n_{ty}(s)\|^2 ds + C_1(\varepsilon) + C_1(\varepsilon)\int_0^t \|n_y(s)\|_{L^\infty}^2\|u_y(s)\|^2 ds. \end{aligned} \tag{3.2.29}$$

Inserting (3.2.25)–(3.2.29) into (3.2.24), then taking supremum in t on the left-hand side of (3.2.24), picking $\varepsilon > 0$ small enough, we finally derive

$$\|n_t(t)\|^2 + \int_0^t \|n_{ty}(s)\|^2 ds \le C_1 + C_1\int_0^t \|n_y(s)\|_{L^\infty}^2\|u_y(s)\|^2 ds. \tag{3.2.30}$$

Using (3.2.21), (3.2.23) and (3.2.30), we deduce

$$\begin{aligned}\|u_y(t)\|^2 + \int_0^t \|u_{yy}(s)\|^2 ds &\le C_1 + C_1\sup_{0\le s\le t}\|n_{yy}(s)\|^2\int_0^t \|n_y(s)\|_{L^\infty}^2 ds\\ &\le C_1 + C_1\sup_{0\le s\le t}(\|n_t(t)\|^2+1)\\ &\le C_1 + C_1\int_0^t \|n_y(s)\|_{L^\infty}^2\|u_y(s)\|^2 ds\end{aligned}$$

which, using the Gronwall inequality and (3.2.11), implies

$$\|u_y(t)\|^2 + \int_0^t \|u_{yy}(s)\|^2 ds \le C_1.$$

Thus it follows from (3.2.21) and (3.2.30) that

$$\|n_t(t)\|^2 + \|u_y(t)\|^2 + \|n_{yy}(t)\|^2 + \int_0^t (\|u_{yy}\|^2 + \|n_{ty}\|^2)(s)\,ds \le C_1. \tag{3.2.31}$$

Moreover, we can derive from (3.1.7) that

$$\|u_t(t)\| \le C_1(\|u_y(t)\| + \|u_{yy}(t)\| + \|v_y(t)\| + \|n_y(t)\| + \|n_{yy}(t)\|) \tag{3.2.32}$$

or

$$\|u_{yy}(t)\| \le C_1(\|u_y(t)\| + \|u_t(t)\| + \|v_y(t)\| + \|n_y(t)\| + \|n_{yy}(t)\|). \tag{3.2.33}$$

Thus we deduce from (3.2.31) and (3.2.32) that

$$\int_0^t \|u_t(s)\|^2 ds \le C_1$$

which, along with (3.2.31), gives us (3.2.22). The proof is complete. □

Proof of Theorem 3.1.1. Using lemmas 3.2.2–3.2.6, we readily complete the proof of theorem 3.1.1. □

3.3 Proof of Theorem 3.1.2

This section will establish the regularity in $H^2 \times H_0^2 \times H^3$.

Lemma 3.3.1. *Suppose assumptions in theorem 3.1.2 are valid, then the following estimates hold for any $T > 0$ and for all $t \in [0, T]$,*

$$\|u_t(t)\|^2 + \|u_{yy}(t)\|^2 + \int_0^t \|u_{ty}(s)\|^2 ds \leq C_1, \tag{3.3.1}$$

$$\|v_{yy}(t)\|^2 + \int_0^t (\|n_{yyy}\|^2 + \|u_{yyy}\|^2)(s) ds \leq C_1. \tag{3.3.2}$$

Proof. Differentiating (3.1.7) with respect to t, multiplying the resulting equation by u_t, integrating it by parts and using lemmas 3.2.2–3.2.6, (3.2.32), we deduce that for any $\varepsilon > 0$,

$$\begin{aligned}
&\|u_t(t)\|^2 + \int_0^t \|u_{ty}(s)\|^2 ds \\
&\leq \|u_t(y,0)\|^2 + C_1 \int_0^t \int_0^1 \Big[|u_y| + |u_y|^2 + |n_y \cdot n_{ty}| + |n_y|^2 |u_y|\Big] |u_{ty}| dydt \\
&\leq C_2 + \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds + C_1 \int_0^t \Big[\|u_y\|^2 + \|u_y\|_{L^\infty}^2 \|u_y\|^2 \\
&\quad + \|n_y\|_{L^\infty}^4 \|u_y\|^2 + \|n_y\|_{L^\infty}^2 \|n_{ty}\|^2\Big](s) ds \\
&\leq C_2 + \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds + C_1 \int_0^t \Big[\|u_y\|^2 + \|u_{yy}\|^2 + \|n_{ty}\|^2\Big](s) ds \\
&\leq C_2 + \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds.
\end{aligned}$$

Now taking $\varepsilon \in (0, 1)$ small enough and using (3.2.33), we obtain (3.3.1).

Differentiating (3.1.8) with respect to y, using lemmas 3.2.2–3.2.6 and the embedding theorem, we deduce

$$\|n_{ty}(t)\| \leq C_1(\|n_y(t)\|_{H^2} + \|v_y(t)\|_{H^1}) \tag{3.3.3}$$

or

$$\|n_{yyy}(t)\| \leq C_1(\|n_{ty}(t)\| + \|n_y(t)\|_{H^1} + \|v_y(t)\|_{H^1}). \tag{3.3.4}$$

Similarly, we infer from (3.1.7),

$$\|u_{ty}(t)\| \leq C_1(\|v_y(t)\|_{H^1} + \|u_y(t)\|_{H^2} + \|n_y(t)\|_{H^2}) \tag{3.3.5}$$

or

$$\|u_{yyy}(t)\| \leq C_1(\|v_y(t)\|_{H^1} + \|u_y(t)\|_{H^1} + \|n_y(t)\|_{H^2} + \|u_{ty}(t)\|). \tag{3.3.6}$$

Now differentiating (3.1.7) with respect to y, using (3.1.6) ($v_{tyy} = u_{yyy}$), we derive

$$\left(\frac{v_{yy}}{v}\right)_t + \gamma \frac{v_{yy}}{v^{\gamma+1}} = u_{ty} + E(y,t) \tag{3.3.7}$$

where

$$E(y,t) = \gamma \frac{(\gamma+1)v_y^2}{v^{\gamma+2}} + \frac{2v_y u_{yy}}{v^2} - \frac{2v_y^2 u_y}{v^3} + \frac{2|n_{yy}|^2 + 2n_y \cdot n_{yyy}}{v^2} - \frac{8n_y \cdot n_{yy} v_y + 2|n_y|^2 v_{yy}}{v^3} + \frac{6|n_y|^2 v_y^2}{v^4}.$$

Multiplying (3.3.7) by $\frac{v_{yy}}{v}$, integrating the resulting equation over $Q_t = [0,1] \times [0,t]$ and using the Young inequality, lemmas 3.2.2–3.2.6 and (3.3.1), we conclude

$$\|v_{yy}(t)\|^2 + \int_0^t \|v_{yy}(s)\|^2 ds \le C_2 + \varepsilon \int_0^t \|v_{yy}(s)\|^2 ds + C_1(\varepsilon) \int_0^t \left(\|u_{ty}\|^2 + \|E\|^2\right)(s)\, ds \tag{3.3.8}$$

where

$$\begin{aligned}\int_0^t \|E(s)\|^2 ds \le{}& C_1 \int_0^t \Big[\|v_y\|_{L^\infty}^2 \|v_y\|^2 + \|v_y\|_{L^\infty}^2 \|u_{yy}\|^2 + \|u_y\|_{L^\infty}^2 \|v_y\|_{L^4}^4 \\ &+ \|n_{yy}\|_{L^\infty}^2 \|n_{yy}\|^2 + \|n_y\|_{L^\infty}^2 \|n_{yyy}\|^2 + \|n_y\|_{L^\infty}^2 \|v_y\|_{L^\infty}^2 \|n_{yy}\|^2 \\ &+ \|n_y\|_{L^\infty}^4 \|v_{yy}\|^2 + \|n_y\|_{L^\infty}^4 \|v_y\|_{L^4}^4\Big](s)\, ds \\ \le{}& C_1 \int_0^t \left(\|v_y\|_{H^1}^2 + \|n_y\|_{H^2}^2\right)(s)\, ds.\end{aligned} \tag{3.3.9}$$

Inserting (3.3.9) into (3.3.8), picking $\varepsilon \in (0,1)$ small enough, and using lemmas 3.2.2–3.2.6, (3.3.1) and (3.3.4), we conclude

$$\begin{aligned}\|v_{yy}(t)\|^2 + \int_0^t \|v_{yy}(s)\|^2 ds &\le C_2 + C_1(\varepsilon) \int_0^t \left(\|n_{yyy}\|^2 + \|v_{yy}\|^2\right)(s)\, ds \\ &\le C_2 + C_1(\varepsilon) \int_0^t \|v_{yy}(s)\|^2 ds\end{aligned}$$

which, using the Gronwall inequality and estimates (3.3.1), (3.3.4) and (3.3.6), gives us (3.3.2). The proof is complete. □

Lemma 3.3.2. *Under assumptions in theorem 3.1.2, the following estimate holds for any $T > 0$ and for all $t \in [0, T]$,*

$$\|n_{ty}(t)\|^2 + \|n_{yyy}(t)\|^2 + \int_0^t \|n_{tyy}(s)\|^2 ds \le C_1. \tag{3.3.10}$$

Proof. Differentiating (3.1.8) with respect to t and y, multiplying the resulting equation by n_{ty} in $L^2(0, 1)$ and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|n_{ty}(t)\|^2 + \int_0^1 \frac{|n_{tyy}|^2}{v^2}dy = B_0(t) + B_1(t) \tag{3.3.11}$$

where

$$\begin{cases} B_0(t) = \int_0^1 \left(\frac{2n_{yy}u_y}{v^3} + \frac{n_{ty}v_y + n_y u_{yy}}{v^3} - 3\frac{n_y v_y u_y}{v^4}\right) n_{tyy}\,dy, \\ B_1(t) = \int_0^1 \left(\frac{|n_y|^2 n}{v^2}\right)_{ty} n_{ty}\,dy. \end{cases}$$

We now employ lemmas 3.2.2–3.2.6, the Gagliardo–Nirenberg interpolation inequality and the Poincaré inequality to get

$$\begin{aligned} B_0 &\le \varepsilon \int_0^1 \frac{|n_{tyy}|^2}{v^2}dy + C_1(\varepsilon)(\|u_y(t)\|_{L^\infty}^2 \|n_{yy}(t)\|^2 + \|n_y(t)\|_{L^\infty}^2 \|u_{yy}(t)\|^2 \\ &\quad + \|v_y(t)\|_{L^\infty}^2 \|n_{ty}(t)\|^2 + \|n_y(t)\|_{L^\infty}^2 \|v_y(t)\|_{L^\infty}^2 \|u_y(t)\|^2) \\ &\le \varepsilon \int_0^1 \frac{|n_{tyy}|^2}{v^2}dy + C_1(\varepsilon)\left(\|u_y(t)\|_{H^1}^2 + \|n_{ty}(t)\|^2\right). \end{aligned} \tag{3.3.12}$$

Similarly, using lemmas 3.2.2–3.2.6, 3.3.1 and the embedding theorem, we derive that for any small $\varepsilon \in (0, 1)$,

$$\begin{aligned} B_1 &\le C_1 \int_0^1 [(|n_{ty}\cdot n_{yy}| + |n_y \cdot n_{tyy}| + |n_y \cdot n_{yy}||n_t| + |n_y \cdot n_{yy}u_y| + |n_{ty}||n_y|^2 + |n_y||u_y||n_y|^2 \\ &\quad + |n_{ty}\cdot n_y v_y| + |n_t||v_y||n_y|^2 + |u_{yy}||n_y|^2 + |v_y||u_y||n_y|^2)|n_{ty}|]\,dy \\ &\le \varepsilon \int_0^1 \frac{|n_{tyy}|^2}{v^2}dy + C_2\left(\|n_t(t)\|_{H^1}^2 + \|n_{ty}(t)\|_{L^\infty}^2 \|n_{yy}(t)\|^2 + \|u_y(t)\|_{H^1}^2 + \|n_y(t)\|_{H^1}^2\right) \\ &\le 2\varepsilon \int_0^1 \frac{|n_{tyy}|^2}{v^2}dy + C_2\left(\|n_t(t)\|_{H^1}^2 + \|u_y(t)\|_{H^1}^2 + \|n_y(t)\|_{H^1}^2\right) \end{aligned}$$

which, combined with (3.3.11), (3.3.12), (3.3.1)–(3.3.3) and lemmas 3.2.2–3.2.6, gives that for $\varepsilon \in (0, 1)$ small enough,

$$\|n_{ty}(t)\|^2 + \int_0^t \|n_{tyy}(s)\|^2 ds \le C_2. \tag{3.3.13}$$

By (3.3.3) and (3.3.13), we deduce

$$\|n_{yyy}(t)\| \le C_2$$

which, with (3.3.13), gives (3.3.10). The proof is complete. □

Proof of Theorem 3.1.2. Using lemmas 3.3.1 and 3.3.2, we readily complete the proof of theorem 3.1.2. □

3.4 Proof of Theorem 3.1.3

This section will establish the regularity in $H^4 \times H_0^4 \times H^4$.

Lemma 3.4.1. *If assumptions in theorem 3.1.3 are valid, then the following estimates hold for all* $t \in [0,\ T]$,

$$\begin{aligned} &\left\|n_{tyy}(y,0)\right\|^2 + \left\|u_{ty}(y,0)\right\|^2 + \left\|u_{tyy}(y,0)\right\|^2 \\ &\quad + \left\|n_{tt}(y,0)\right\|^2 + \left\|u_{tt}(y,0)\right\|^2 \le C_4, \end{aligned} \tag{3.4.1}$$

$$\begin{aligned} &\left\|u_{tt}(t)\right\|^2 + \left\|n_{tt}(t)\right\|^2 + \left\|n_{tyy}(t)\right\|^2 + \int_0^t (\left\|u_{tty}\right\|^2 + \left\|u_{tyy}\right\|^2 + \left\|n_{tt}\right\|^2 \\ &\quad + \left\|n_{tty}\right\|^2 + \left\|n_{tyyy}\right\|^2)(s)\,ds \le C_4, \end{aligned} \tag{3.4.2}$$

$$\left\|u_{ty}(t)\right\|^2 + \left\|u_{yyy}(t)\right\|^2 + \left\|u_{tyy}(t)\right\|^2 + \int_0^t \left\|u_{tyyy}(s)\right\|^2 ds \le C_4. \tag{3.4.3}$$

Proof. Differentiating (3.1.7) and (3.1.8) with respect to y twice, using theorems 3.1.1, 3.1.2 and the embedding theorem, we deduce

$$\left\|u_{tyy}(t)\right\| \le C_2(\left\|u_y(t)\right\|_{H^3} + \left\|v_y(t)\right\|_{H^2} + \left\|n_y(t)\right\|_{H^3}), \tag{3.4.4}$$

$$\left\|n_{tyy}(t)\right\| \le C_2(\left\|n_y(t)\right\|_{H^3} + \left\|v_y(t)\right\|_{H^2}) \tag{3.4.5}$$

or

$$\left\|u_{yyyy}(t)\right\| \le C_2(\left\|u_y(t)\right\|_{H^2} + \left\|v_y(t)\right\|_{H^2} + \left\|n_y(t)\right\|_{H^3} + \left\|u_{tyy}(t)\right\|), \tag{3.4.6}$$

$$\left\|n_{yyyy}(t)\right\| \le C_2(\left\|n_y(t)\right\|_{H^2} + \left\|v_y(t)\right\|_{H^2} + \left\|n_{tyy}(t)\right\|). \tag{3.4.7}$$

Differentiating (3.1.7), (3.1.8) with respect to t, respectively, using theorems 3.1.1, 3.1.2, (3.3.3), (3.3.5), (3.4.4) and (3.4.5), we obtain

$$\left\|u_{tt}(t)\right\| \le C_2(\left\|u_{tyy}(t)\right\| + \left\|u_{ty}(t)\right\| + \left\|n_{ty}(t)\right\| + \left\|n_{tyy}(t)\right\| + \left\|u_y(t)\right\|_{H^1}) \tag{3.4.8}$$

$$\le C_2(\left\|u_y(t)\right\|_{H^3} + \left\|n_y(t)\right\|_{H^3} + \left\|v_y(t)\right\|_{H^2}), \tag{3.4.9}$$

$$\left\|n_{tt}(t)\right\| \le C_2(\left\|n_t(t)\right\| + \left\|n_{ty}(t)\right\| + \left\|n_{tyy}(t)\right\| + \left\|u_y(t)\right\|_{H^1}) \tag{3.4.10}$$

$$\le C_2(\left\|n_y(t)\right\|_{H^3} + \left\|v_y(t)\right\|_{H^2} + \left\|u_y(t)\right\|_{H^1}) \tag{3.4.11}$$

or

$$\|u_{tyy}(t)\| \le C_1\|u_{tt}(t)\| + C_2(\|u_{ty}(t)\| + \|n_{ty}(t)\| + \|n_{tyy}(t)\| + \|u_y(t)\|_{H^1}), \quad (3.4.12)$$

$$\|n_{tyy}(t)\| \le C_1\|n_{tt}(t)\| + C_2(\|n_t(t)\| + \|n_{ty}(t)\| + \|u_y(t)\|_{H^1}). \quad (3.4.13)$$

Differentiating (3.1.7) with respect to y and t, using theorems 3.1.1 and 3.1.2, we derive

$$\|u_{tyyy}(t)\| \le C_2(\|u_{tty}(t)\| + \|u_{ty}(t)\|_{H^1} + \|n_{ty}(t)\|_{H^2} + \|n_y(t)\|_{H^2} + \|u_y(t)\|_{H^2}). \quad (3.4.14)$$

Similarly, we get

$$\|n_{tyyy}(t)\| \le C_2(\|n_{tty}(t)\| + \|u_y(t)\|_{H^2} + \|v_y(t)\|_{H^1} + \|n_{ty}(t)\|_{H^1} + \|n_y(t)\|_{H^2}). \quad (3.4.15)$$

Thus estimate (3.4.1) follows from (3.3.5), (3.4.4), (3.4.5), (3.4.9) and (3.4.11).

Differentiating (3.1.7) with respect to t twice, multiplying the resulting equation by u_{tt} in $L^2(0,1)$, performing an integration by parts, using theorems 3.1.1, 3.1.2 and (3.4.1), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{tt}(t)\|^2 &= -\int_0^1 \left(-\frac{1}{v^\gamma} + \frac{u_y}{v} - \frac{|n_y|^2}{v^2}\right)_{tt} u_{tty}\,ds\\ &\le -\int_0^1 \frac{u_{tty}^2}{v}dy + C_2\left[\left\|\left(\frac{1}{v^\gamma}\right)_{tt}\right\| + \left\|\left(\frac{|n_y|^2}{v^2}\right)_{tt}\right\| + \|u_{ty}u_y\| + \|u_y u_{ty}\| + \|u_y\|_{L^6}^3\right]\|u_{tty}\|\\ &\le -C_1^{-1}\|u_{tty}(t)\|^2 + C_2(\|u_y(t)\|_{H^1}^2 + \|u_{ty}(t)\|^2 + \|n_{ty}(t)\|_{H^1}^2 + \|n_{tty}(t)\|^2)\end{aligned}$$

which thus, by theorems 3.1.1 and 3.1.2, implies

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{tty}(s)\|^2 ds \le C_4 + C_2\int_0^t \|n_{tty}(s)\|^2 ds. \quad (3.4.16)$$

By (3.3.10) and (3.4.10), we further have

$$\int_0^t \|n_{tt}(s)\|^2 ds \le C_2. \quad (3.4.17)$$

Similarly, differentiating (3.1.8) with respect to t twice, multiplying the resulting equation by n_{tt} in $L^2(0,1)$ and integrating it by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|n_{tt}(t)\|^2 = D_0 + D_1 + D_2 \quad (3.4.18)$$

where

$$D_0 = -\int_0^1 \left(\frac{n_y}{v}\right)_{tt}\left(\frac{n_{tt}}{v}\right)_y dy, \quad D_1 = \int_0^1 \left[\left(\frac{1}{v}\right)_{tt}\left(\frac{n_y}{v}\right)_y + \left(\frac{2}{v}\right)_t\left(\frac{n_y}{v}\right)_{ty}\right] n_{tt}\, dy,$$
$$D_2 = \int_0^1 \left(\frac{|n_y|^2}{v^2} n\right)_{tt} n_{tt}\, dy.$$

Employing theorems 3.1.1 and 3.1.2, the Gagliardo–Nirenberg interpolation inequality and the Poincaré inequality, we conclude that for any $\varepsilon > 0$,

$$\begin{aligned} D_0 \leq & -\int_0^1 \frac{|n_{tty}|^2}{v^2} dy + C_2[\|n_{tt}(t)\|^2 + \|n_{ty}(t)\|^2 + \|u_{ty}(t)\|^2 + \|u_y(t)\|_{L^4}^4 \\ & + (\|n_{ty}(t)\| + \|u_{ty}(t)\| + \|u_y(t)\|_{L^4}^2 + \|n_{tt}(t)\|)\|n_{tty}(t)\|] \\ \leq & -C_1^{-1}\|n_{tty}(t)\|^2 + C_2(\|n_{ty}(t)\|^2 + \|u_{ty}(t)\|^2 + \|u_y(t)\|_{H^1}^2 + \|n_{tt}(t)\|^2), \end{aligned} \tag{3.4.19}$$

$$\begin{aligned} D_1 \leq & C_1 \int_0^1 \Bigg\{ \Big[(|u_{ty}| + |u_y|^2)(|n_{yy}| + |n_y v_y|)\Big]|n_{tt}| \\ & + \big(|n_{tyy}| + |n_{yy} u_y| + |n_{ty} v_y| + |n_y u_{yy}| + |n_y v_y u_y|\big)|u_y||n_{tt}|\Bigg\} dy \\ \leq & C_2(\|n_{tt}(t)\|^2 + \|n_{ty}(t)\|_{H^1}^2 + \|u_y(t)\|_{H^1}^2 + \|n_y(t)\|_{H^2}^2 + \|u_{ty}(t)\|^2), \end{aligned} \tag{3.4.20}$$

$$\begin{aligned} D_2 \leq & C_1 \int_0^1 (|n_{ty}|^2 + |n_y \cdot n_{tty}| + |n_y \cdot n_{ty}||n_t| + |n_y|^2|n_{tt}| \\ & + |n_y \cdot n_{ty} u_y| + |n_y|^2|n_t u_y| + |n_y|^2|u_{ty}| + |n_y|^2|u_y|^2)|n_{tt}|dy \\ \leq & \varepsilon\|n_{tty}(t)\|^2 + C_2(\|n_{tt}(t)\|^2 + \|n_{ty}(t)\|_{H^1}^2 + \|u_{ty}(t)\|^2 + \|u_y(t)\|_{H^1}^2) \end{aligned}$$

which, along with (3.4.17)–(3.4.20), (3.4.1) and theorem 3.1.2, leads to that for $\varepsilon \in (0, 1)$ small enough,

$$\|n_{tt}(t)\|^2 + \int_0^t \|n_{tty}(s)\|^2 ds \leq C_4. \tag{3.4.21}$$

On the other hand, we derive from (3.4.16) and (3.4.21) that

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{tty}(s)\|^2 ds \leq C_4. \tag{3.4.22}$$

Exploiting theorems 3.1.1, 3.1.2, (3.4.12), (3.4.13), (3.4.15), (3.4.21) and (3.4.22), we obtain (3.4.2).

Differentiating (3.1.7) with respect to t and y, multiplying the resulting equation by u_{ty} in $L^2(0,1)$ and integrating by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|u_{ty}(t)\|^2 = F_0(t) + F_1(t) \tag{3.4.23}$$

where

$$F_0(y,t) = \left(-P + \frac{u_y}{v} - \frac{|n_y|^2}{v^2}\right)_{ty} u_{ty}\Big|_0^1, \quad F_1(t) = \int_0^1 \left(-P + \frac{u_y}{v} - \frac{|n_y|^2}{v^2}\right)_{ty} u_{tyy}\, dy.$$

Employing (3.4.1) and (3.4.2), theorems 3.1.1 and 3.1.2, the Gagliardo–Nirenberg interpolation inequality and the Poincaré inequality and the Young inequality, we get

$$\begin{aligned} F_0 &\le C_1(\|u_{yy}(t)\|_{L^\infty} + \|v_y(t)\|_{L^\infty}\|u_y(t)\|_{L^\infty} + \|u_{tyy}(t)\|_{L^\infty} \\ &\quad + \|u_{yy}(t)\|_{L^\infty}\|u_y(t)\|_{L^\infty} + \|u_{ty}(t)\|_{L^\infty}\|v_y(t)\|_{L^\infty} \\ &\quad + \|u_y(t)\|_{L^\infty}^2\|v_y(t)\|_{L^\infty})\|u_{ty}(t)\|_{L^\infty} \\ &\le C_1(\|u_y(t)\|_{H^2} + \|u_{tyy}(t)\|^{\frac{1}{2}}\|u_{tyyy}(t)\|^{\frac{1}{2}} + \|u_{tyy}(t)\| \\ &\quad + \|u_{ty}(t)\|^{\frac{1}{2}}\|u_{tyy}(t)\|^{\frac{1}{2}})\|u_{ty}(t)\|^{\frac{1}{2}}\|u_{tyy}(t)\|^{\frac{1}{2}} \\ &\le C_2(\|u_{tyy}(t)\|^2 + \|u_{tyyy}(t)\|^2 + \|u_y(t)\|_{H^2}^2 + \|u_{ty}(t)\|^2) \end{aligned} \tag{3.4.24}$$

which, together with (3.4.2), (3.4.14) and theorem 3.1.2, further leads to

$$\int_0^t F_0\, ds \le C_2\int_0^t (\|u_{tyy}\|^2 + \|u_{tyyy}\|^2)(s)\, ds \le C_4. \tag{3.4.25}$$

Similarly, using theorems 3.1.1 and 3.1.2, (3.4.1), (3.4.2) and the embedding theorem, we conclude that for any small $\varepsilon \in (0,1)$,

$$\begin{aligned} F_1 &\le C_1\int_0^1 \Big[(|u_{yy}| + |v_y u_y| + |u_{yy}u_y| + |u_{ty}v_y| + |u_y^2 v_y| + |n_{ty}\cdot n_{yy}| + |n_y\cdot n_{tyy}| \\ &\quad + |n_y\cdot n_{yy}u_y| + |n_y\cdot n_{ty}v_y| + |u_{yy}||n_y|^2 + |n_y|^2|u_y||v_y|\Big]|u_{tyy}|\, dy \\ &\le C_2(\|u_{ty}(t)\|_{H^1}^2 + \|n_{ty}(t)\|_{H^1}^2 + \|u_y(t)\|_{H^1}^2 + \|n_y(t)\|_{H^1}^2 + \|v_y(t)\|_{H^1}^2) \end{aligned}$$

which, combined with (3.4.23), (3.4.25), (3.4.1), (3.4.2) and theorems 3.1.1 and 3.1.2, gives us

$$\|u_{ty}(t)\|^2 + \int_0^t \|u_{tyy}(s)\|^2 ds \le C_4. \tag{3.4.26}$$

Therefore, by (3.3.6), (3.4.1), (3.4.2), (3.4.12), (3.4.14) and (3.4.26), we can obtain (3.4.3). This completes the proof. □

Lemma 3.4.2. *If assumptions in theorem 3.1.3 are valid, then the following estimates hold for any* $t \in [0, T](T > 0)$,

$$\left\|v_{yyy}(t)\right\|^2 + \left\|n_{yyyy}(t)\right\|^2 + \left\|u_{yyyy}(t)\right\|^2 \le C_4, \tag{3.4.27}$$

$$\left\|v_{yyyy}(t)\right\|^2 + \int_0^t (\left\|n_{yyyyy}\right\|^2 + \left\|u_{yyyyy}\right\|^2)(s)\,ds \le C_4. \tag{3.4.28}$$

Proof. Differentiating (3.3.7) with respect to y, we arrive at

$$\left(\frac{v_{yyy}}{v}\right)_t + \gamma \frac{v_{yyy}}{v^{\gamma+1}} = E_1(y, t) \tag{3.4.29}$$

with

$$E_1(y, t) = u_{tyy} + E_y(y, t) + \left(\frac{v_{yy} v_y}{v^2}\right)_t + \gamma \frac{(\gamma+1) v_{yy} v_y}{v^{\gamma+2}}.$$

Using theorems 3.1.1 and 3.1.2 and lemma 3.4.1, we have

$$\|E_1(t)\|^2 \le C_2(\left\|u_{tyy}(t)\right\|^2 + \left\|v_y(t)\right\|_{H^2}^2 + \left\|u_y(t)\right\|_{H^2}^2 + \left\|n_y(t)\right\|_{H^3}^2)$$

leading to

$$\int_0^t \|E_1(s)\|^2 ds \le C_4 + C_2 \int_0^t (\left\|v_{yyy}\right\|^2 + \left\|n_{yyyy}\right\|^2)(s)\,ds. \tag{3.4.30}$$

Thus it follows from (3.4.2), (3.4.7) and (3.4.30) that for all $t \in [0,\ T]$,

$$\int_0^t \|E_1(s)\|^2 ds \le C_4 + C_2 \int_0^t \left\|v_{yyy}(s)\right\|^2 ds. \tag{3.4.31}$$

Multiplying (3.4.29) by $\frac{v_{yyy}}{v}$, integrating the result over Q_t, and using the Young inequality, (3.4.1) and (3.4.31), we infer that

$$\left\|v_{yyy}(t)\right\|^2 + \int_0^t \left\|v_{yyy}(s)\right\|^2 ds \le C_4 + C_2 \int_0^t \|E_1(s)\|^2 ds \le C_4 + C_2 \int_0^t \left\|v_{yyy}(s)\right\|^2 ds$$

which, using the Gronwall inequality, gives us for all $t \in [0,\ T]$,

$$\left\|v_{yyy}(t)\right\|^2 + \int_0^t \left\|v_{yyy}(s)\right\|^2 ds \le C_4. \tag{3.4.32}$$

Thus from (3.4.2), (3.4.3), (3.4.6), (3.4.7), (3.4.32) and theorem 3.1.2 it follows that

$$\left\|n_{yyyy}(t)\right\| + \left\|u_{yyyy}(t)\right\| \le C_4$$

which, together with (3.4.32), gives us estimate (3.4.27).

Now differentiating (3.4.29) with respect to y, we arrive at

$$\left(\frac{v_{yyyy}}{v}\right)_t + \gamma \frac{v_{yyyy}}{v^{\gamma+1}} = E_2(y, t) \tag{3.4.33}$$

with

$$E_2(y,t) = E_{1y}(y,t) + \left(\frac{v_{yyy}v_y}{v^2}\right)_t + \gamma(\gamma+1)\frac{v_{yyy}v_y}{v^{\gamma+2}}.$$

Exploiting the embedding theorem, lemma 3.4.1 and theorem 3.1.2 and (3.4.27), we can derive

$$\|E_2(t)\| \le C_2(\|u_{tyyy}(t)\| + \|u_y(t)\|_{H^3} + \|v_y(t)\|_{H^3} + \|n_y(t)\|_{H^4}). \tag{3.4.34}$$

Differentiating (3.1.7), (3.1.8) with respect to y three times, respectively, using theorems 3.1.1 and 3.1.2, the Poincaré inequality and the Young inequality, we derive

$$\|u_{yyyyy}(t)\| \le C_1\|u_{tyyy}(t)\| + C_2(\|v_y(t)\|_{H^3} + \|u_y(t)\|_{H^3} + \|n_y(t)\|_{H^4}), \tag{3.4.35}$$

$$\|n_{yyyyy}(t)\| \le C_1\|n_{tyyy}(t)\| + C_2(\|n_y(t)\|_{H^3} + \|v_y(t)\|_{H^3}) \tag{3.4.36}$$

which, along with (3.4.2), (3.4.3), (3.4.27), (3.4.34) and theorem 3.1.2, implies

$$\int_0^t \|E_2(s)\|^2 ds \le C_4 + C_2\int_0^t \|v_{yyyy}(s)\|^2 ds. \tag{3.4.37}$$

Multiplying (3.4.33) by $\frac{v_{yyyy}}{v}$, integrating the resulting equation over Q_t, and using the Young inequality, (3.4.1) and (3.4.37), we derive

$$\begin{aligned}\|v_{yyyy}(t)\|^2 + \int_0^t \|v_{yyyy}(s)\|^2 ds &\le C_4 + C_2\int_0^t \|E_2(s)\|^2 ds\\ &\le C_4 + C_2\int_0^t \|v_{yyyy}(s)\|^2 ds\end{aligned}$$

which, using the Gronwall inequality, gives us for all $t \in [0,\ T]$,

$$\|v_{yyyy}(t)\|^2 + \int_0^t \|v_{yyyy}(s)\|^2 ds \le C_4. \tag{3.4.38}$$

Therefore, it follows from (3.4.2), (3.4.3), (3.4.27), (3.4.35), (3.4.36), (3.4.38) and theorem 3.1.2 that for all $t \in [0,\ T]$,

$$\int_0^t (\|n_{yyyyy}\|^2 + \|u_{yyyyy}\|^2)(s)\, ds \le C_4,$$

which, together with (3.4.38), gives us estimate (3.4.28). This readily completes the proof. □

Proof of Theorem 3.1.3. Using lemmas 3.4.1, 3.4.2, we can complete the proof of theorem 3.1.3. □

3.5 Bibliographic Comments

In this section, we briefly review some related literature. Note that the dynamic theory of nematic liquid crystals was established by Ericksen [23] and Leslie [78] in the 1960s. This theory is in fact derived from the macroscopic point of view; which helps in understanding the coupling between the director field and the velocity field, and also gives a tool to describe the motion of the defects in the molecule configurations under the influence of the flow velocity. In order to describe the dynamic property of nematic materials, Ericksen [24] and Leslie [77] established a system consisting of equations for the conversation of the mass, the linear momentum and an extra equation for the conversation of the momentum due to vector field n. The Ericksen–Leslie system is well studied for describing many special flows for the materials, especially for those with small molecules, and is widely accepted in engineering and mathematical communities studying liquid crystals.

For the following simplified Ericksen–Leslie equation

$$\begin{cases} v_t + (v\cdot\nabla)v - \nu\Delta v + \nabla P = -\lambda\nabla\cdot(\nabla d\odot\nabla d), \\ d_t + (v\cdot\nabla)d = \gamma(\Delta d - f(d)), \\ \nabla\cdot v = 0, \end{cases} \tag{3.5.1}$$

where v is the flow velocity and d is the relaxation of the molecule direction. The term $\lambda\nabla\cdot(\nabla d\odot\nabla d) = \lambda\nabla_j(\nabla_i d^k\nabla_j d^k) = \lambda\nabla_i d^k\nabla_j d^k + \frac{\lambda}{2}\nabla_i\frac{|\nabla d|^2}{2}$ in the stress tensor represents the anisotropic feature of the system. When the system (3.5.1) is subjected to Dirichlet boundary conditions, Lin and Liu [81] proved the global existence of weak solutions and classical solutions, and discussed the uniqueness and some stability properties of the system. Later on, they established the partial regularity results in [82], and most results were extended to the general Ericksen–Leslie equations in [83]. When the system (3.5.1) is subjected to free-slip boundary condition for v and Neumann boundary for d (*i.e.*, $v\times\nu = 0$, $(\nabla\times v)\times\nu = 0$, $\frac{\partial d}{\partial\nu} = 0$ on $\partial\Omega$), Liu and Shen [86] proved the local classical solutions and global weak solutions in 2D and 3D cases. When the system (3.5.1) is subjected to the Dirichlet boundary condition and reproductivity conditions (*i.e.*, $v(x,0) = v(x,T)$, $d(x,0) = d(x,T)$), Blanca *et al.* [9] showed the existence of weak solutions with the reproductivity in time property in 2D and 3D cases. Fan and Ozawa [25] proved some regularity criteria for this simplified Ericksen–Leslie system and also obtained the existence and uniqueness of global smooth solutions for a regularization model of this simplified system. Hu and Wang [57] studied three-dimensional case of (3.5.1) in a smooth bounded domain, and obtained the existence and uniqueness of the global strong solutions with small initial data and also proved that when the strong solution exists, all the global weak solutions constructed in Lin and Liu [81] must be equal to the unique strong solution. Hong [56] studied the system in 2D, and proved global existence of solutions with initial data, where the solutions are regular except for at a finite number of singular times. Some of the numerical experiments to the system (3.5.1) were performed in Liu and Walkington [87, 88] who also demonstrated the coupling between the fluid field and the director field. In this direction, we also mention the works by Calderer *et al.* [11] and Liu [85] for the nematic liquid crystal model. Wen and Ding [143] studied the incompressible hydrodynamic flow of the nematic liquid crystals in dimension N ($N = 2$ or 3):

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \odot u) + \nabla P = \gamma \Delta u - \lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ d_t + (u \cdot \nabla) d = \theta(\Delta d + |\nabla n|^2 n), \end{cases} \tag{3.5.2}$$

where ρ denotes density, u is the flow velocity and d is the relaxation of the molecule direction. Under the assumption $\rho_0 > 0$, the authors obtained the local existence and uniqueness of the solutions. In addition, if ρ_0 had a positive bound from below, and $N = 2$, they obtained the global existence and uniqueness of solutions with small initial data.

We also note that for the system (3.1.1)–(3.1.5), Ding [16] proved the existence of global strong solutions under assumptions for initial data $(\rho_0, u_0, n_0) \in H^1(0,1) \times H_0^1(0,1) \times H^2(0,1)$ with $0 < c_0^{-1} \le \rho_0 \le c_0$, and global smooth (classical) solutions under assumptions for initial data $(\rho_0, u_0, n_0) \in C^{1+\alpha}[0,1] \times C^{2+\alpha}[0,1] \times C^{2+\alpha}[0,1]$ with $0 < \alpha < 1$, $0 < c_0^{-1} \le \rho_0 \le c_0$. It is different from our result. First, Ding [16] studied the existence of global strong solutions to problem (3.1.1)–(3.1.5) in Euler coordinates, while we have established the existence of global solutions in H^i ($i = 1$, 2, 4) in Lagrangian coordinates in this chapter. Second, we have established the regularity in $H^2 \times H_0^2 \times H^3$ and $H^4 \times H_0^4 \times H^4$, while Ding [16] proved the existence smooth solutions in $C^{1+\alpha} \times C^{2+\alpha} \times C^{2+\alpha}$, which is a direct consequence of our results.

Chapter 4

Large-Time Behavior of Solutions to a One-Dimensional Liquid Crystal System

4.1 Introduction

In this chapter, we shall continue to study the large-time behavior of solutions in $H^i \times H_0^i \times H^{i+1}(i=1,2)$ and $H^4 \times H_0^4 \times H^4$ to the following 1D liquid crystal system based on the results in chapter 3:

$$\begin{cases} v_t = u_y, & (4.1.1) \\ u_t = \left(-P + \dfrac{u_y}{v} - \dfrac{|n_y|^2}{v^2}\right)_y, & (4.1.2) \\ n_t = \dfrac{1}{v}\left(\dfrac{n_y}{v}\right)_y + \dfrac{|n_y|^2}{v^2} n, & (4.1.3) \\ (v, u, n)|_{t=0} = (v_0, u_0, n_0), & (4.1.4) \\ u|_{y=0,1} = 0, \quad n_y|_{y=0,1} = 0 & (4.1.5) \end{cases}$$

where $v = \frac{1}{\rho}$, $v_0 = \frac{1}{\rho_0}$ and $P = \rho = \frac{1}{v}$ ($i.e.$, $\gamma = 1$ in chapter 3).

In this chapter, we still restrict ourselves to the following case:

$$|n|^2 = n_i n_i = 1. \tag{4.1.6}$$

In this chapter, we use C_i $(i = 1,\ 2)$ to denote the generic positive constant depending on the $\|(\rho_0, u_0, n_0)\|_{H^i \times H^i \times H^{i+1}}$ $(i = 1, 2)$, $\min_{x\in[0,1]} u_0(x)$, $\min_{x\in[0,1]} n_0(x)$, but not depending on time T, and C_j $(j = 3, 4)$ depending on $H^j[0, 1]$ norm of initial data $(\rho_0,\ u_0,\ n_0)$, $\min_{x\in[0,1]} u_0(x)$, $\min_{x\in[0,1]} n_0(x)$, but not depending on T.

The results of this chapter are selected from Qin and Feng [110].

DOI: 10.1051/978-2-7598-2903-3.c004

Theorem 4.1.1. *Suppose that* $(v_0, u_0, n_0) \in H^1(0,1) \times H_0^1(0,1) \times H^2(0,1)$ *and the compatibility conditions are valid. Then there exists a unique global solution* $(v(t), u(t), n(t)) \in H^1(0,1) \times H_0^1(0,1) \times H^2(0,1)$ *to the problem* (4.1.1)–(4.1.5) *such that for all* $(y,t) \in [0,1] \times [0,+\infty)$,

$$0 < C_1^{-1} \le v(y,t) \le C_1, \tag{4.1.7}$$

and for all $t > 0$,

$$\|v(t)\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \|n(t)\|_{H^2}^2 + \|n_t(t)\|^2 + \int_0^t \left(\|u\|_{H^2}^2 + \left\|n_{ty}\right\|^2 + \|u_t\|^2\right)(\tau)\, d\tau \le C_1. \tag{4.1.8}$$

Moreover, we have as $t \to +\infty$,

$$\|v(t) - \bar{v}\|_{H^1} \to 0, \quad \|u(t)\|_{H^1} \to 0, \quad \|n(t) - \bar{n}\|_{H^2} \to 0 \tag{4.1.9}$$

where $\bar{v} = \int_0^1 v(y,t)\,dy = \int_0^1 v_0\,dy$, $\bar{n} = \int_0^1 n(y,t)\,dy$.

Theorem 4.1.2. *Suppose that* $(v_0, u_0, n_0) \in H^2(0,1) \times H_0^2(0,1) \times H^3(0,1)$ *and the compatibility conditions are valid. Then there exists a unique global solution* $(v(t), u(t), n(t)) \in H^2(0,1) \times H_0^2(0,1) \times H^3(0,1)$ *to the problem* (4.1.1)–(4.1.5) *such that for all* $t > 0$,

$$\|v(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|n(t)\|_{H^3}^2 + \|n_t(t)\|_{H^1}^2 + \|u_t\|^2 + \int_0^t \left(\left\|u_{ty}\right\|_{H^2}^2 + \left\|n_{tyy}\right\|^2 + \left\|u_{yyy}\right\|^2\right)(\tau)\, d\tau \le C_2. \tag{4.1.10}$$

Moreover, we have as $t \to +\infty$,

$$\|v(t) - \bar{v}\|_{H^2} \to 0, \quad \|u(t)\|_{H^2} \to 0, \quad \|n - \bar{n}\|_{H^3} \to 0. \tag{4.1.11}$$

Theorem 4.1.3. *Suppose that* $(v_0, u_0, n_0) \in H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$ *and the compatibility conditions are valid. Then there exists a unique global solution* $(v(t), u(t), n(t)) \in H^4(0,1) \times H_0^4(0,1) \times H^4(0,1)$ *to the problem* (4.1.1)–(4.1.5) *such that for any* $t > 0$,

$$\|v(t)\|_{H^4}^2 + \|u(t)\|_{H^4}^2 + \|n(t)\|_{H^4}^2 + \|n_t(t)\|_{H^2}^2 + \|u_t(t)\|_{H^2}^2 + \|n_{tt}(t)\|^2 + \|u_{tt}(t)\|^2 + \int_0^t \left(\left\|u_y\right\|_{H^4}^2 + \left\|n_y\right\|_{H^4}^2 + \left\|u_{ty}\right\|_{H^2}^2 + \left\|n_{ty}\right\|_{H^2}^2 + \left\|u_{tty}\right\|^2 + \left\|n_{tty}\right\|^2\right)(\tau)\ d\tau \le C_4. \tag{4.1.12}$$

Moreover, we have, as $t \to +\infty$,

$$\|v(t) - \bar{v}\|_{H^4} \to 0, \quad \|u(t)\|_{H^4} \to 0, \quad \|n(t) - \bar{n}\|_{H^4} \to 0. \tag{4.1.13}$$

4.2 Uniform Estimates in $H^i \times H_0^i \times H^{i+1}(i=1,2)$ and $H^4 \times H_0^4 \times H^4$

The global existence of solutions in $H^i \times H_0^i \times H^{i+1}(i=1,2)$ and $H^4 \times H_0^4 \times H^4$ has been established in Qin and Huang [124] and also in chapter 3. In this section, we shall derive some uniform estimates in $H^i \times H_0^i \times H^{i+1}(i=1,2)$ and $H^4 \times H_0^4 \times H^4$ by establishing a series of lemmas. First, we shall establish some uniform estimates in $H^1 \times H_0^1 \times H^2$.

Lemma 4.2.1. *If assumptions in theorem 4.1.1 are valid, then the following estimates hold in Euler coordinates for any $t > 0$,*

$$\int_0^1 (\rho u^2 + \pi(\rho) + 2|n_x|^2)\,dx + \int_0^t \int_0^1 (u_x^2 + |n \times n_{xx}|^2)\,dxds \le C_1, \tag{4.2.1}$$

$$\int_0^1 |n_x|^2 dx + \int_0^t \int_0^1 |n_t|^2 dxds \le C_1 \tag{4.2.2}$$

where $\pi(\rho) = \rho \log \rho - \rho$.

Proof. Multiplying (4.1.2) by u, using (4.1.1) and integrating the result by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\rho u^2 + \pi(\rho))\,dx + \int_0^1 u_x^2 dx = \int_0^1 |n_x|^2 u_x dx. \tag{4.2.3}$$

We derive from $|n|^2 = 1$ that

$$n_{xx} + |n_x|^2 n = -n \times (n \times n_{xx}).$$

Multiplying the above equation by n_{xx}, we obtain

$$\frac{d}{dt}\int_0^1 |n_x|^2 dx + \int_0^1 |n_x|^2 u_x dx + 2\int_0^1 |n \times n_{xx}|^2 dx = 0,$$

which, along with (4.2.3), gives (4.2.1).

Using (4.1.6), we easily get

$$n \cdot n_t = 0.$$

Multiplying (4.1.3) by n_t, integrating the result over $[0,\ 1] \times [0,\ t]$, applying the Young inequality and the Poincaré inequality, we have for any $\varepsilon > 0$,

$$\begin{aligned}
\int_0^1 |n_x|^2 dx + \int_0^t \int_0^1 |n_t|^2 dxds &= -\int_0^t \int_0^1 u n_x n_t dxds \\
&\le \varepsilon \int_0^t \int_0^1 |n_t|^2 dxds + C_1 \int_0^t \int_0^1 u^2 |n_x|^2 dxds \\
&\le \varepsilon \int_0^t \int_0^1 |n_t|^2 dxds + C_1 \int_0^t \|u\|_{L^\infty}^2 \int_0^1 |n_x|^2 dxds \\
&\le \varepsilon \int_0^t \int_0^1 |n_t|^2 dxds + C_1 \int_0^t \|u_x\|^2 \int_0^1 |n_x|^2 dxds.
\end{aligned}$$

Choosing $\varepsilon > 0$ small enough, performing the Gronwall inequality, and using (4.2.1), we obtain

$$\int_0^1 |n_x|^2 dx + \int_0^t \int_0^1 |n_t|^2 dxds \le C_1.$$

This proves the lemma. □

Lemma 4.2.2. *If assumptions in theorem 4.1.1 are valid, then the following estimates hold in Euler coordinates for any $t > 0$,*

$$\int_0^t \int_0^1 (|n_x|^4 + |n_{xx}|^2) dxds \le C_1. \tag{4.2.4}$$

Proof. Squaring (4.1.3) in both sides, integrating the result over $[0,\ 1] \times [0,\ t]$, employing integration by parts, the Young inequality and the Poincaré inequality, and using (4.2.1) and (4.2.2), we conclude for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_0^t \int_0^1 (|n_{xx}|^2 + |n_x|^4) dxds \\
&\quad = \int_0^t \int_0^1 (|n_t|^2 + u^2|n_x|^2 + 2n_t u n_x) dxds - 2\int_0^t \int_0^1 n_{xx}|n_x|^2 n dxds \\
&\quad = \int_0^t \int_0^1 (|n_t|^2 + u^2|n_x|^2 + 2n_t u n_x) dxds + \frac{2}{3}\int_0^t \int_0^1 |n_x|^4 dxds \\
&\quad \le 2\int_0^t \int_0^1 (|n_t|^2 + u^2|n_x|^2) dxds + \frac{2}{3}\int_0^t \int_0^1 |n_x|^4 dxds \\
&\quad \le C_1 + \frac{2}{3}\int_0^t \int_0^1 |n_x|^4 dxds + C_1\int_0^t \|u\|_{L^\infty}^2 \|n_x\|^2 dxds \\
&\quad \le C_1 + \frac{2}{3}\int_0^t \int_0^1 |n_x|^4 dxds + C_1\int_0^t \|u_x\|^2 \|n_x\|^2 dxds \\
&\quad \le C_1 + \frac{2}{3}\int_0^t \int_0^1 |n_x|^4 dxds + C_1\int_0^t \|u_x\|^2 dxds \\
&\quad \le C_1 + \frac{2}{3}\int_0^t \int_0^1 |n_x|^4 dxds.
\end{aligned} \tag{4.2.5}$$

Therefore (4.2.4) follows from (4.2.5). This completes the proof. □

Lemma 4.2.3. *If assumptions in theorem 4.1.1 are valid, then the following estimates hold in Lagrangian coordinates, for any $t > 0$,*

$$\int_0^1 \left[u^2 + (v - \log v - 1) + \frac{|n_y|^2}{v}\right](y,t) dy + \int_0^t \int_0^1 \left(\frac{|u_y|^2}{v} + \frac{1}{v}\left|n \times \left(\frac{n_y}{v}\right)_y\right|^2\right) dyds \le C_1, \tag{4.2.6}$$

$$\int_0^1 \frac{|n_y|^2}{v}dy + \int_0^t\int_0^1 \frac{1}{v}\left[\left(\frac{n_y}{v}\right)_y\right]^2 dyds \le C_1. \tag{4.2.7}$$

Proof. Using lemmas 4.2.1–4.2.2 and the total mass conservation $\int_0^1 v dy = \int_0^1 v_0 dy$, we easily derive (4.2.6) and (4.2.7). □

Lemma 4.2.4. *For any $t \geq 0$, there exists one point $y_1 = y_1(t) \in [0, 1]$ such that the solution $v(y, t)$ to problem (4.1.1)–(4.1.5) possesses the following expression:*

$$v(y,t) = D(y,t)Z(t)\left[1 + \int_0^t \left(v^{-1} + \frac{|n_y|^2}{v^2}\right) vD^{-1}(y,s)Z^{-1}(s)ds\right] \tag{4.2.8}$$

where

$$\begin{cases} D(y,t) = v_0(y)\exp\left[\int_{y_1}^y ud\xi - \int_0^y u_0(\xi)d\xi + \frac{1}{\bar{v}_0}\int_0^1\int_0^y u(z)dzdy\right], & (4.2.9)\\ Z(t) = \exp\left(-\frac{1}{\bar{v}_0}\int_0^t\int_0^1\left(1+u^2+\frac{|n_y|^2}{v}\right)dyds\right),\ \bar{v}_0 = \int_0^1 v_0 dy. & (4.2.10)\end{cases}$$

Proof. See, *e.g.*, lemma 2.2.2 or lemma 3.2.3. □

Lemma 4.2.5. *There holds*

$$0 < C_1^{-1} \le v(y,t) \le C_1, \quad \textit{for all } (y,t) \in [0,1]\times[0,+\infty), \tag{4.2.11}$$

$$\int_0^t \|n_y(s)\|_{L^\infty}^2 ds \le C_1, \qquad \textit{for all } t > 0. \tag{4.2.12}$$

Proof. Obviously, using the Young inequality, the Hölder inequality and lemma 4.2.1, we have

$$\begin{aligned} &\left|\int_{y_1}^y ud\xi - \int_0^y u_0(\xi)d\xi + \frac{1}{v_0}\int_0^1\int_0^y u_0(z)dzdy\right| \\ &\quad \le \left(\int_0^1 u^2dy\right)^{\frac{1}{2}} + \left(\int_0^1 u_0^2dy\right)^{\frac{1}{2}} + \frac{1}{\bar{v}_0}\int_0^1\left(\int_0^1 u_0^2dy\right)^{\frac{1}{2}}dy \\ &\quad \le C_1\|u\|^2 + C_1 \le C_1. \end{aligned} \tag{4.2.13}$$

Equations (4.2.9) and (4.2.6) imply that there exists some positive constant $C_1 > 0$ such that for all $(y, t) \in [0, 1] \times [0, +\infty)$,

$$0 < C_1^{-1} \le D(y,t) \le C_1.$$

Noting that for all $t > 0$,

$$1 \le \int_0^1 \left(1 + u^2 + \frac{|n_y|^2}{v}\right)(y,t)\,dy \le C_1, \tag{4.2.14}$$

then for $0 \le s \le t$, we get

$$t - s \le \int_s^t \int_0^1 \left(1 + u^2 + \frac{|n_y|^2}{v}\right) dy \le C_1(t-s). \tag{4.2.15}$$

Therefore, (4.2.10) and (4.2.15) imply that for any $0 \le s \le t$,

$$e^{-C_1(t-s)} \le Z(t)Z^{-1}(s) = \exp\left(-\frac{1}{\bar{v}_0}\int_s^t \int_0^1 \left(1 + u^2 + \frac{|n_y|^2}{v}\right) dyds\right) \le e^{-C_1^{-1}(t-s)}. \tag{4.2.16}$$

Using (4.2.8) and (4.2.16), we derive that there exists a large time t_0 such that as $t \ge t_0$, $y \in [0,1]$,

$$\begin{aligned} v(y,t) &= D(y,t)Z(t)\left[1 + \int_0^t \left(v^{-1} + \frac{|n_y|^2}{v^2}\right) vD^{-1}(y,s)Z^{-1}(s)\,ds\right] \\ &\ge C_1^{-1}\left[e^{-C_1 t} + \int_0^t \left(1 + \frac{|n_y|^2}{v}\right) e^{-C_1(t-s)}\,ds\right] \\ &\ge \int_0^t C_1^{-1} e^{-C_1(t-s)}\,ds \\ &\ge (2C_1)^{-1}. \end{aligned} \tag{4.2.17}$$

Noting that $D(y,t) \ge C_1^{-1}$, $Z(t) \ge \exp(-C_1 t)$, we infer that for any $(y,t) \in [0,1] \times [0,t_0]$,

$$v(y,t) \ge D(y,t)Z(t) \ge C_1^{-1}\exp(-C_1 t) \ge C_1^{-1}\exp(-C_1 t_0),$$

which, together with (4.2.17), implies that for any $(y, t) \in [0, 1] \times [0, +\infty)$,

$$v(y,t) \ge C_1^{-1}. \tag{4.2.18}$$

Employing $W^{1,1} \hookrightarrow L^\infty$, by the Hölder inequality, lemmas 4.2.3–4.2.4 and (4.2.18), we deduce

$$
\begin{aligned}
v(y,t) &\le C_1\left[e^{-C_1 t}+\int_0^t\left(1+\frac{|n_y|^2}{v}\right)e^{-C_1(t-s)}\right]\\
&\le C_1+C_1\int_0^t e^{-C_1(t-s)}ds+C_1\int_0^t\left\|\frac{|n_y|}{v}\right\|_{L^\infty}^2 ve^{-C_1(t-s)}ds\\
&\le C_1+C_1\int_0^t\left[\left(\int_0^1\frac{1}{v}\,dy\right)^{\frac12}\left(\int_0^1\frac{|n_y|^2}{v}dy\right)^{\frac12}\right.\\
&\quad\left.+\left(\int_0^1\frac{1}{v}\left|\left(\frac{n_y}{v}\right)_y\right|^2dy\right)^{\frac12}\left(\int_0^1 vdy\right)^{\frac12}\right]^2 ve^{-C_1(t-s)}ds\\
&\le C_1+C_1\int_0^t\left[\left(\int_0^1\frac{1}{v}\,dy\right)\left(\int_0^1\frac{|n_y|^2}{v}dy\right)\right.\\
&\quad\left.+\left(\int_0^1\frac{1}{v}\left|\left(\frac{n_y}{v}\right)_y\right|^2dy\right)\left(\int_0^1 vdy\right)\right]ve^{-C_1(t-s)}ds\\
&\le C_1+C_1\int_0^t\left[e^{-C_1(t-s)}+\int_0^1\frac{1}{v}\left|\left(\frac{n_y}{v}\right)_y\right|^2dy\right]vds \qquad (4.2.19)
\end{aligned}
$$

which, using the Gronwall inequality and (4.2.7), (4.2.18), gives (4.2.11).

Using (4.2.7) and the Poincaré inequality, we obtain

$$
\begin{aligned}
\int_0^t\|n_y(s)\|_{L^\infty}^2 ds &\le C_1\int_0^t\left\|\frac{|n_y|}{v}\right\|_{L^\infty}^2 ds\\
&\le C_1\int_0^t\left\|\left(\frac{|n_y|}{v}\right)_y\right\|^2 ds\\
&\le C_1. \qquad (4.2.20)
\end{aligned}
$$

This proves the lemma. □

Lemma 4.2.6. *If assumptions in theorem 4.1.1 are valid, then the following estimates hold for any $t>0$,*

$$\int_0^1 v_y^2dy+\int_0^t\int_0^1(|n_y|^4+|n_{yy}|^2+|n_t|^2+v_y^2)dyds\le C_1, \qquad (4.2.21)$$

$$\|n_t(y,0)\|\le C_1. \qquad (4.2.22)$$

Proof. Using (4.1.1), we may rewrite (4.1.2) as

$$\left(u-\frac{v_y}{v}\right)_t=\frac{v_y}{v^2}+\frac{2|n_y|^2v_y}{v^3}-\frac{2n_y\cdot n_{yy}}{v^2}. \qquad (4.2.23)$$

Multiplying (4.2.23) by $u - \frac{v_y}{v}$, integrating the resulting equation over $[0,1] \times [0,t]$, and using the Young and the Poincaré inequalities, we deduce for any $\varepsilon > 0$,

$$\begin{aligned}
&\left\|u - \frac{v_y}{v}\right\|^2 + \int_0^t \int_0^1 (v_y^2 + |n_y|^2 v_y^2)\,dyds \\
&\quad \le \left\|u_0 - \frac{v_{0y}}{v_0}\right\|^2 + C_1 \int_0^t \int_0^1 \Big(|v_y u| + |n_y \cdot n_{yy} u| + |n_y \cdot n_{yy} v_y| + |n_y|^2 |v_y u|\Big)\,dyds \\
&\quad \le C_1 + \varepsilon \int_0^t \int_0^1 \Big(v_y^2 + |n_y|^2 v_y^2\Big)\,dyds + C_1 \int_0^t \|u_y(s)\|^2\,ds \\
&\qquad + C_1 \int_0^t \|n_y\|_{L^\infty}^2 \int_0^1 u^2\,dyds + C_1 \int_0^t \int_0^1 |n_{yy}|^2\,dyds. \qquad (4.2.24)
\end{aligned}$$

Taking ε small enough in (4.2.24), and using (4.2.6), (4.2.12) and (4.2.24), we can obtain

$$\left\|u - \frac{v_y}{v}\right\|^2 + \int_0^t \int_0^1 (v_y^2 + |n_y|^2 v_y^2)\,dyds \le C_1 + C_1 \int_0^t \int_0^1 |n_{yy}|^2\,dyds. \qquad (4.2.25)$$

By (4.2.7), we have

$$\begin{aligned}
\int_0^t \int_0^1 \left(\frac{|n_{yy}|^2}{v^3} + \frac{|n_y|^2 v_y^2}{v^5}\right) dyds &= \int_0^t \int_0^1 \frac{1}{v}\left[\left(\frac{n_y}{v}\right)_y\right]^2 dydt + 2\int_0^t \int_0^1 \frac{n_y \cdot n_{yy} v_y}{v^4}\,dyds \\
&\le C_1 + \frac{1}{2}\int_0^t \int_0^1 \frac{|n_{yy}|^2}{v^3}\,dyds + C_1 \int_0^t \int_0^1 \frac{|n_y|^2 v_y^2}{v^5}\,dyds.
\end{aligned}$$

Thus

$$\int_0^t \int_0^1 |n_{yy}|^2\,dyds \le C_1 + C_1 \int_0^t \|n_y\|_{L^\infty}^2 \int_0^1 v_y^2\,dyds$$

which, with (4.2.25) and the fact $\|u\|^2 \le C_1 \|u_y\|^2 \le C_1$, gives

$$\|v_y(t)\|^2 + \int_0^t \int_0^1 (v_y^2 + |n_y|^2 v_y^2)\,dyds \le C_1 + C_1 \int_0^t \|n_y\|_{L^\infty}^2 \int_0^1 v_y^2\,dyds. \qquad (4.2.26)$$

Applying the Gronwall inequality to (4.2.26), and using (4.2.7) and (4.2.12), we obtain

$$\|v_y(t)\|^2 + \int_0^t \int_0^1 (|n_y|^2 v_y^2 + |n_{yy}|^2 + |v_y|^2)\,dyds \le C_1. \qquad (4.2.27)$$

Thus from (4.2.7) and (4.2.12) it follows that

$$\int_0^t \int_0^1 |n_y|^4\,dyds \le C_1 \int_0^t \|n_y\|_{L^\infty}^2 \int_0^1 |n_y|^2\,dyds \le C_1. \qquad (4.2.28)$$

By (4.1.3) and (4.2.27), we can derive

$$\|n_{yy}(t)\| \le C_1\big(\|n_y(t)\| + \|n_t(t)\|\big), \tag{4.2.29}$$

or

$$\|n_t(t)\| \le C_1\big(\|n_y(t)\| + \|n_{yy}(t)\|\big) \tag{4.2.30}$$

which implies (4.2.22).

Thus, using the Poincaré inequality and (4.2.27), we can infer

$$\begin{aligned}\int_0^t \|n_t(s)\|^2 ds &\le C_1 \int_0^t \Big(\|n_y\|^2 + \|n_{yy}\|^2\Big)(s)\,ds \\ &\le C_1 \int_0^t \|n_{yy}(s)\|^2 ds \le C_1,\end{aligned}$$

which, with (4.2.27) and (4.2.28), yields (4.2.21). This proves the lemma. □

Lemma 4.2.7. *If assumptions in theorem 4.1.1 hold, then the following estimates are valid for any $t > 0$,*

$$\begin{aligned}&\|n_t(t)\|^2 + \|n_{yy}(t)\|^2 + \|u_y(t)\|^2 \\ &\quad + \int_0^t \Big(\|u_t\|^2 + \|u_{yy}\|^2 + \|n_{ty}\|^2\Big)(s)\,ds \le C_1.\end{aligned} \tag{4.2.31}$$

Proof. Obviously, using lemma 3.2.6 in chapter 3 and the uniform estimate of v, we may complete the proof. □

In what follows, we shall derive some uniform estimates in $H^2 \times H_0^2 \times H^3$. The proof of the following is different from lemma 3.3.1 in chapter 3 to some extent.

Lemma 4.2.8. *If assumptions in theorems 4.1.2 hold, the following estimates are valid for any $t > 0$,*

$$\|u_t(t)\|^2 + \|u_{yy}(t)\|^2 + \int_0^t \|u_{ty}(s)\|^2 ds \le C_1, \tag{4.2.32}$$

$$\|v_{yy}(t)\|^2 + \int_0^t \Big(\|v_{yy}\|^2 + \|n_{yyy}\|^2 + \|u_{yyy}\|^2\Big)(s)\,ds \le C_1. \tag{4.2.33}$$

Proof. Differentiating (4.1.2) with respect to t, multiplying the resulting equation by u_t, integrating it by parts, and using (3.2.32) and lemmas 4.2.3–4.2.7, we deduce for any $\varepsilon > 0$,

$$
\begin{aligned}
&\|u_t(t)\|^2 + \int_0^t \|u_{ty}(s)\|^2 ds \\
&\quad \le \|u_t(y,0)\|^2 + C_1 \int_0^t \int_0^1 \left[|u_y| + |u_y|^2 + |n_y \cdot n_{ty}| + |n_y|^2 |u_y|\right] |u_{ty}| dydt \\
&\quad \le C_2 + \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds + C_1 \int_0^t \Big[\|u_y\|^2 + \|u_y\|_{L^\infty}^2 \|u_y\|^2 \\
&\qquad + \|n_y\|_{L^\infty}^4 \|u_y\|^2 + \|n_y\|_{L^\infty}^2 \|n_{ty}\|^2\Big](s)\, ds \\
&\quad \le C_2 + \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds + C_1 \int_0^t \left[\|u_y\|^2 + \|u_{yy}\|^2 + \|n_{ty}\|^2\right](s)\, ds \\
&\quad \le C_2 + \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds
\end{aligned}
$$

which, by taking $\varepsilon \in (0, 1)$ small enough and using (4.2.31), yields (4.2.32).

Differentiating (4.1.2) with respect to y, using (4.1.1) ($v_{tyy} = u_{yyy}$), we arrive at

$$
\left(\frac{v_{yy}}{v}\right)_t + \frac{v_{yy}}{v^2} = u_{ty} + E(y, t) \tag{4.2.34}
$$

where

$$
\begin{aligned}
E(y, t) = &\frac{2v_y^2}{v^3} + \frac{2v_y u_{yy}}{v^2} - \frac{2v_y^2 u_y}{v^3} + \frac{2|n_{yy}|^2 + 2n_y \cdot n_{yyy}}{v^2} \\
&- \frac{8n_y \cdot n_{yy} v_y + 2|n_y|^2 v_{yy}}{v^3} + \frac{6|n_y|^2 v_y^2}{v^4}.
\end{aligned}
$$

Multiplying (4.2.34) by $\frac{v_{yy}}{v}$, integrating the result over $[0, 1] \times [0, t]$ and using the Young inequality, we have for any $\varepsilon > 0$,

$$
\|v_{yy}(t)\|^2 + \int_0^t \|v_{yy}(s)\|^2 ds \le C_1 + \varepsilon \int_0^t \|v_{yy}(s)\|^2 ds + C_1 \int_0^t \left[\|u_{ty}\|^2 + \|E\|^2\right](s)\, ds \tag{4.2.35}
$$

where

$$
\begin{aligned}
\int_0^t \|E(s)\|^2 ds \le\, &C_1 \int_0^t \int_0^1 \Big(v_y^4 + v_y^2 u_{yy}^2 + v_y^4 u_y^2 + |n_{yy}|^4 + |n_y|^2 |n_{yyy}|^2 \\
&+ |n_y|^2 |n_{yy}|^2 v_y^2 + |n_y|^4 v_{yy}^2 + |n_y|^4 v_y^4\Big)\, dyds \\
\le\, &C_1 \int_0^t \Big(\|v_y\|_{L^\infty}^2 \|v_y\|^2 + \|v_y\|_{L^\infty}^2 \|u_{yy}\|^2 + \|u_y\|_{L^\infty}^2 \|v_y\|_{L^4}^4 \\
&+ \|n_{yy}\|_{L^\infty}^2 \|n_{yy}\|^2 + \|n_y\|_{L^\infty}^2 \|n_{yyy}\|^2 + \|n_y\|_{L^\infty}^2 \|v_y\|_{L^\infty}^2 \|n_{yy}\|^2 \\
&+ \|n_y\|_{L^\infty}^4 \|v_{yy}\|^2 + \|n_y\|_{L^\infty}^2 \|v_y\|_{L^4}^4\Big)\, ds.
\end{aligned}
$$

Using the interpolation inequality, the Young inequality and lemmas 4.2.3–4.2.7, we conclude

$$\begin{aligned}\int_0^t \|E(s)\|^2 ds &\le C_1 \int_0^t \left(\|v_y\|_{L^\infty}^2 + \|v_y\|_{L^4}^4 + \|n_y\|_{L^\infty}^2 \|v_{yy}\|^2 + \|n_{yyy}\|^2\right) ds \\ &\le C_1 \int_0^t (\|v_y\| \|v_{yy}\| + \|v_y\|^2 + \|v_y\|^3 \|v_{yy}\| + \|v_y\|^4 \\ &\quad + \|n_y\|_{L^\infty}^2 \|v_{yy}\|^2 + \|n_{yyy}\|^2) ds \\ &\le C_1 + \varepsilon \int_0^t \|v_{yy}(s)\|^2 ds + C_1 \int_0^t (\|n_y(s)\|_{L^\infty}^2 \|v_{yy}(s)\|^2 \\ &\quad + \|n_{yyy}(s)\|^2) ds. \end{aligned} \tag{4.2.36}$$

Differentiating (4.1.3) with respect to y, using the interpolation inequality, the Young inequality and lemmas 4.2.3–4.2.7, we can conclude

$$\begin{aligned}\int_0^t \|n_{yyy}(s)\|^2 ds &\le C_1 \int_0^t \Big(\|n_{ty}\|^2 + \|v_y n_{yy}\|^2 + \left\|n_y v_y^2\right\|^2 + \|n_y v_{yy}\|^2 \\ &\quad + \|n_y \cdot n_{yy}\|^2 + \left\|n_y^2 v_y\right\|^2 + \left\|n_y^3\right\|^2\Big) ds \\ &\le C_1 + C_1 \int_0^t \Big(\|n_{yy}\|_{L^\infty}^2 \|v_y\|^2 + \|n_y\|_{L^\infty}^2 \|v_y\|_{L^\infty}^2 \|v_y\|^2 \\ &\quad + \|n_y\|_{L^\infty}^2 \|v_{yy}\|^2 + \|n_y\|_{L^\infty}^2 \|n_{yy}\|^2 + \|n_y\|_{L^\infty}^4 \|v_y\|^2 \\ &\quad + \|n_y\|_{L^\infty}^4 \|n_y\|^2\Big) ds \\ &\le C_1 + \varepsilon \int_0^t \|n_{yyy}(s)\|^2 ds + C_1 \int_0^t \|n_y(s)\|_{L^\infty}^2 \|v_{yy}(s)\|^2 ds. \end{aligned}$$

Picking $\varepsilon \in (0, 1)$ small enough, we obtain

$$\int_0^t \|n_{yyy}(s)\|^2 ds \le C_1 + C_1 \int_0^t \|n_y(s)\|_{L^\infty}^2 \|v_{yy}(s)\|^2 ds. \tag{4.2.37}$$

Inserting (4.2.36) and (4.2.37) into (4.2.35), and taking ε small enough, we can obtain

$$\|v_{yy}(t)\|^2 + \int_0^t \|v_{yy}(s)\|^2 ds \le C_1 + C_1 \int_0^t \|n_y(s)\|_{L^\infty}^2 \|v_{yy}(s)\|^2 ds.$$

Performing the Gronwall inequality, and using (4.2.12), we conclude that for all $t > 0$,

$$\|v_{yy}(t)\|^2 + \int_0^t \left(\|v_{yy}\|^2 + \|n_{yyy}\|^2\right)(s) ds \le C_1. \tag{4.2.38}$$

Differentiating (4.1.2) with respect to y, by the imbedding theorem and lemmas 4.2.3–4.2.7, we can obtain

$$\|u_{ty}(t)\| \le C_1\Big(\|v_y(t)\|_{H^1} + \|u_y(t)\|_{H^2} + \|n_y(t)\|_{H^2}\Big) \tag{4.2.39}$$

or

$$\|u_{yyy}(t)\| \le C_1\Big(\|v_y(t)\|_{H^1} + \|u_y(t)\|_{H^1} + \|n_y(t)\|_{H^2} + \|u_{ty}\|\Big) \tag{4.2.40}$$

which, with (4.2.31), (4.2.32) and (4.2.38), gives (4.2.33). This proves the lemma. □

Using the same estimate as in chapter 3, we can obtain the following two lemmas concerning uniform estimates in $H^i \times H_0^i \times H^{i+1} (i = 1, 2)$ and $H^4 \times H_0^4 \times H^4$ using uniform estimates of specific volume v in lemma 4.2.5.

Lemma 4.2.9. *If assumptions in theorem 4.1.2 are valid, then the following estimate holds for any $t > 0$,*

$$\|n_{ty}(t)\|^2 + \|n_{yyy}(t)\|^2 + \int_0^t \|n_{tyy}(s)\|^2 \le C_2. \tag{4.2.41}$$

Lemma 4.2.10. *If assumptions in theorem 4.1.3 are valid, then the following estimates hold for any $t > 0$,*

$$\begin{aligned} &\|u_{tt}(t)\|^2 + \|n_{tt}(t)\|^2 + \|n_{tyy}(t)\|^2 + \int_0^t \Big(\|u_{tty}\|^2 + \|u_{tyy}\|^2 \\ &\quad + \|n_{tt}\|^2 + \|n_{tty}\|^2 + \|n_{tyyy}\|^2\Big)(s)\,ds \le C_4, \end{aligned} \tag{4.2.42}$$

$$\|u_{ty}(t)\|^2 + \|u_{yyy}(t)\|^2 + \|u_{tyy}(t)\|^2 + \int_0^t \|u_{tyyy}(s)\|^2\,ds \le C_4, \tag{4.2.43}$$

$$\|v_{yyy}(t)\|^2 + \|n_{yyyy}(t)\|^2 + \|u_{yyyy}(t)\|^2 + \int_0^t \|v_{yyy}(s)\|^2\,ds \le C_4, \tag{4.2.44}$$

$$\|v_{yyyy}(t)\|^2 + \int_0^t \Big(\|v_{yyyy}\|^2 + \|n_{yyyyy}\|^2 + \|u_{yyyyy}\|^2\Big)(s)\,ds \le C_4. \tag{4.2.45}$$

4.3 Large-Time Behavior in $H^i \times H_0^i \times H^{i+1}$ $(i = 1, 2)$ and $H^4 \times H_0^4 \times H^4$

In this section, we shall complete the proofs of theorems 4.1.1–4.1.3.

Lemma 4.3.1. *If assumptions in theorem 4.1.1 are valid, then we have*

$$\lim_{t\to+\infty}\|v(t)-\bar{v}\|_{H^1}=0 \tag{4.3.1}$$

where $\bar{v}=\int_0^1 v(y,t)\,dy=\int_0^1 v_0\,dy$.

Proof. Differentiating (4.1.1) with respect to y, multiplying the result by v_y, then integrating the result over [0, 1], using the Young inequality, we can deduce

$$\frac{d}{dt}\left\|v_y(t)\right\|^2\le\left\|v_y(t)\right\|^2+\left\|u_{yy}(t)\right\|^2\le\frac{1}{2}+\frac{1}{2}\left\|v_y(t)\right\|^4+\left\|u_{yy}(t)\right\|^2$$

which, along with (4.2.21), (4.2.31) and lemma 1.1.2, leads to

$$\lim_{t\to+\infty}\left\|v_y(t)\right\|^2=0. \tag{4.3.2}$$

Moreover, using the embedding theorem, we can deduce

$$\|v-\bar{v}\|\le C_1\left\|v_y\right\|$$

which, together with (4.3.2), gives (4.3.1). This proves the lemma. □

Lemma 4.3.2. *If assumptions in theorem 4.1.1 are valid, then we have*

$$\lim_{t\to+\infty}\|u(t)\|_{H^1}=0. \tag{4.3.3}$$

Proof. Equation (4.1.1) can be rewritten as

$$u_t=\left(-v^{-1}-\frac{|n_y|^2}{v^2}\right)_y+\left(\frac{u_y}{v}\right)_y. \tag{4.3.4}$$

Let

$$\widehat{p}=\widehat{p}(y,t)=\frac{1}{v}+\frac{|n_y|^2}{v^2},\qquad \widehat{\sigma}=\widehat{\sigma}(y,t)=-\frac{1}{v}-\frac{|n_y|^2}{v^2}+\frac{u_y}{v}.$$

Then

$$u_t=\left(-\widehat{p}+\frac{u_y}{v}\right)_y=\widehat{\sigma}_y. \tag{4.3.5}$$

Put

$$\widehat{p}^*=\widehat{p}^*(y,t)=\widehat{p}(y,t)-\int_0^1\widehat{p}(y,t)\,dy, \tag{4.3.6}$$

$$\widehat{\sigma}^*=\widehat{\sigma}^*(y,t)=\widehat{\sigma}(y,t)-\int_0^1\widehat{\sigma}(y,t)\,dy. \tag{4.3.7}$$

Then

$$\int_0^1 \widehat{p}^*(y,t)\,dy = 0, \quad \int_0^1 \widehat{\sigma}^*(y,t)\,dy = 0, \tag{4.3.8}$$

$$(\widehat{p}^*)_y = \widehat{p}_y, \quad (\widehat{\sigma}^*)_y = \widehat{\sigma}_y. \tag{4.3.9}$$

Integrating (4.3.9) by parts, using (4.3.5), lemma 4.2.3 and the Young inequality, we can infer that for any $\varepsilon > 0$,

$$\begin{aligned}
\|\widehat{p}^*\|^2 &= (\widehat{p}^*, \widehat{p}^*) = \left(-\widehat{p}^*_y, \int_0^y \widehat{p}^*\,dx\right)\\
&= \left(-\widehat{p}_y, \int_0^y \widehat{p}^*\,dx\right) = \left(u_t - \left(\frac{u_y}{v}\right)_y, \int_0^y \widehat{p}^*\,dx\right)\\
&= \left(u_t, \int_0^y \widehat{p}^*\,dx\right) - \left(\left(\frac{u_y}{v}\right)_y, \int_0^y \widehat{p}^*\,dx\right)\\
&\le \left(\int_0^1 u_t^2\,dy\right)^{\frac{1}{2}} \left(\int_0^1 \left(\int_0^y \widehat{p}^*\,dx\right)^2 dy\right)^{\frac{1}{2}} + \int_0^1 \frac{u_y}{v}\widehat{p}^*\,dy\\
&\le \varepsilon\|\widehat{p}^*\|^2 + C(\varepsilon)\|u_t\|^2 + C_1\left(\varepsilon\|\widehat{p}^*\|^2 + C(\varepsilon)\|u_y\|^2\right)\\
&\le (C_1+1)\varepsilon\|\widehat{p}^*\|^2 + C_1\left(\|u_t\|^2 + \|u_y\|^2\right).
\end{aligned} \tag{4.3.10}$$

Choosing $\varepsilon > 0$ so small that $(C_1 + 1)\varepsilon < 1$, integrating the result over $(0, t)$, we obtain

$$\int_0^t \|\widehat{p}^*(s)\|^2\,ds \le C_1. \tag{4.3.11}$$

Note that

$$\begin{aligned}
\frac{d}{dt}\|\widehat{p}^*(t)\|^2 &= 2(\widehat{p}^*, \widehat{p}^*_t) = 2\left(\widehat{p}^*_y, -\int_0^y \widehat{p}^*_t\,dy\right)\\
&= 2\left(\left(\frac{u_y}{v}\right)_y - u_t, -\int_0^y \widehat{p}^*_t\,dy\right)\\
&= 2\left(u_t, \int_0^y \widehat{p}^*_t\,dy\right) - 2\left(\left(\frac{u_y}{v}\right)_y, \int_0^y \widehat{p}^*_t\,dy\right)\\
&\le C_1\left(\|u_t(t)\|^2 + \|u_y(t)\|^2 + \|\widehat{p}^*_t(t)\|^2\right).
\end{aligned} \tag{4.3.12}$$

From (4.2.21) and (4.2.30), it follows that

$$\begin{aligned}\left\|\widehat{p}_t^*(t)\right\|^2 &\le C_1\Big(\|v_t(t)\|^2+\left\|n_y(t)\cdot n_{ty}(t)\right\|^2+\left\|n_y^2(t)\cdot v_t(t)\right\|^2\Big)\\ &\le C_1\Big(\left\|u_y(t)\right\|^2+\left\|n_y(t)\right\|_{L^\infty}^2\left\|n_{ty}(t)\right\|^2+\left\|n_y(t)\right\|_{L^\infty}^4\left\|u_y(t)\right\|^2\Big)\\ &\le C_1\Big(\left\|u_y(t)\right\|^2+\left\|n_{yy}(t)\right\|^2\left\|n_{ty}(t)\right\|^2+\left\|n_{yy}(t)\right\|^4\left\|u_y(t)\right\|^2\Big)\\ &\le C_1\Big(\left\|u_y(t)\right\|^2+\left\|n_{ty}(t)\right\|^2\Big).\end{aligned}\tag{4.3.13}$$

Combining (4.3.12) and (4.3.13), we get

$$\frac{d}{dt}\|\widehat{p}^*(t)\|^2\le C_1\Big(\|u_t(t)\|^2+\left\|u_y(t)\right\|^2+\left\|n_{ty}(t)\right\|^2\Big).\tag{4.3.14}$$

Using lemmas 4.2.7 and 1.1.2, we can derive

$$\lim_{t\to+\infty}\|\widehat{p}^*(t)\|^2=0.\tag{4.3.15}$$

Hence from (4.3.4), (4.3.5), (4.3.7) and (4.3.11), it follows

$$\int_0^t\|\widehat{\sigma}^*(s)\|^2ds\le C_1\int_0^t\Big(\|\widehat{p}^*\|+\left\|u_y\right\|^2\Big)(s)\,ds\le C_1.\tag{4.3.16}$$

By (4.3.4), (4.3.5), and (4.3.7), and integrating by parts, we derive

$$\begin{aligned}\frac{d}{dt}\|\widehat{\sigma}^*(t)\|^2&=2(\widehat{\sigma}^*,\widehat{\sigma}_t^*)=2\left(\widehat{\sigma}_y^*,-\int_0^y\widehat{\sigma}_t^*\,dx\right)\\ &=2\left(u_t,-\int_0^y\widehat{\sigma}_t^*\,dx\right)\\ &\le C_1\left(\|u_t(t)\|^2+\left\|\int_0^y\widehat{\sigma}_t^*\,dx\right\|^2\right)\end{aligned}\tag{4.3.17}$$

where

$$\int_0^y\widehat{\sigma}_t^*\,dy=\int_0^y\left(\widehat{\sigma}_t-\int_0^1\widehat{\sigma}_t(\xi,t)\,d\xi\right)dx\tag{4.3.18}$$

and

$$\begin{aligned}\widehat{\sigma}_t(y,t)&=\left(-\frac{1}{v}-\frac{|n_y|^2}{v^2}+\frac{u_y}{v}\right)_t\\ &=\frac{v_t}{v^2}-\left(\frac{|n_y|^2}{v^2}\right)_t+\left(\frac{u_t}{v}\right)_y+\frac{u_tv_y-u_y^2}{v^2}.\end{aligned}$$

Noting that for all $t>0$,

$$u_t(0,t)=u_t(1,t),$$

we derive that for any $y\in[0,1]$,

$$\begin{aligned}\left|\int_0^y \widehat{\sigma}_t(x,t)\,dx\right| &\le C_1\left(\int_0^1 |u_y+n_yn_{ty}+n_y^2u_y+u_tv_y+u_y^2|\,dy+|u_t|\right)\\ &\le C_1\Big[(\|u_y\|+\|n_y\|\cdot\|n_{ty}\|+\left\|n_y^2\right\|\cdot\|u_y\|\\ &\quad+\|u_t\|\cdot\|v_y\|+\|u_y\|^2)+|u_t|\Big]\\ &\le C_1\big(\|u_y\|+\|n_{yt}\|+\|u_t\|+|u_t|\big).\end{aligned}\tag{4.3.19}$$

Analogously,

$$\left|\int_y^1 \widehat{\sigma}_t(x,t)\,dx\right|\le C_1\big(\|u_y\|+\|n_{yt}\|+\|u_t\|+|u_t|\big).\tag{4.3.20}$$

Combining (4.3.18)–(4.3.20), we obtain

$$\begin{aligned}\int_0^y \widehat{\sigma}_t^*\,dy &= \int_0^y\left(\widehat{\sigma}_t-\int_0^1\widehat{\sigma}_t(\xi,t)\,d\xi\right)dx\\ &=\int_0^y\widehat{\sigma}_t\,dx-\int_0^1\int_y^1\widehat{\sigma}_t\,dxd\xi\\ &\le C_1\big(\|u_y\|+\|n_{yt}\|+\|u_t\|+|u_t|\big)\end{aligned}\tag{4.3.21}$$

which gives

$$\left\|\int_0^y\widehat{\sigma}_t^*\,dx\right\|^2\le C_1\Big(\|u_y(t)\|^2+\|n_{yt}(t)\|^2+\|u_t(t)\|^2\Big).\tag{4.3.22}$$

By (4.3.17) and (4.3.22), we can infer

$$\begin{aligned}\frac{d}{dt}\|\widehat{\sigma}^*(t)\|^2 &\le C_1\Big(\|u_y(t)\|^2+\|n_{yt}(t)\|^2+\|u_t(t)\|^2\Big)\\ &\le C_1\Big(1+\|n_{yt}(t)\|^2+\|u_t(t)\|^2\Big),\end{aligned}$$

which, along with (4.3.16) and lemma 1.1.2, yields

$$\lim_{t\to+\infty}\|\widehat{\sigma}^*(t)\|^2=0.\tag{4.3.23}$$

Noting that

$$\int_0^1\left(\frac{u}{v}\right)_y(y,t)\,dy=0,$$

we arrive at

$$\frac{u_y}{v} = \left(\frac{u_y}{v}\right)^* + \int_0^1 \frac{u_y}{v}dy = (\widehat{\sigma}^* + \widehat{p}^*) + \int_0^1 \frac{uv_y}{v^2}dy$$

which, together with lemma 4.2.3, gives

$$\|u_y(t)\| \le C_1 \left\|\frac{u_y}{v}\right\| \le C_1\big(\|\widehat{\sigma}^*(t)\| + \|\widehat{p}^*(t)\| + \|u(t)\|\|v_y(t)\|\big). \tag{4.3.24}$$

Thus from (4.3.15), (4.3.23), (4.3.2) and (4.3.24) it follows that

$$\lim_{t\to+\infty} \|u_y(t)\| = 0. \tag{4.3.25}$$

By the Poincaré inequality, we get

$$\|u(t)\|_{H^1} \le C_1\|u_y(t)\|$$

which, together with (4.3.25), gives (4.3.3). This completes the proof. □

Lemma 4.3.3. *If assumptions in theorem 4.1.1 are valid, then we have*

$$\lim_{t\to+\infty} \|n(t) - \bar{n}\|_{H^2} = 0 \tag{4.3.26}$$

where $\bar{n} = \int_0^1 n(y,t)\,dy$.

Proof. Multiplying equation (4.1.3) by n_t, integrating the result over [0, 1] with respect to y and using (4.1.6), we obtain

$$\begin{aligned}\int_0^1 |n_t|^2 dy &= \int_0^1 \frac{n_t}{v}\left(\frac{n_y}{v}\right)_y dy\\ &= -\int_0^1 \frac{n_y}{v}\left(\frac{n_{ty}}{v} - \frac{n_t v_y}{v^2}\right) dy.\end{aligned} \tag{4.3.27}$$

By the Young inequality, we deduce that for any $\varepsilon > 0$,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |n_y|^2 dy + \int_0^1 |n_t|^2 dy \le \varepsilon \int_0^1 |n_t|^2 dy + C_1 \int_0^1 |n_y|^2 v_y^2 dy.$$

Taking ε small enough and using (4.2.21), we have

$$\frac{d}{dt}\|n_y(t)\|^2 + \|n_t(t)\|^2 \le C_1\|n_y(t)\|^2_{L^\infty} \tag{4.3.28}$$

which, together with lemma 1.1.2 and (4.2.12), results in

$$\lim_{t\to+\infty} \|n_y(t)\|^2 = 0. \tag{4.3.29}$$

By the Poincaré inequality, we derive

$$\|n(t) - \bar{n}\|_{H^1} \le C_1\|n_y(t)\|$$

which, together with (4.3.28), leads to

$$\lim_{t\to+\infty}\|n(t)-\bar{n}\|_{H^1}=0. \tag{4.3.30}$$

By lemma 4.2.7, we can obtain

$$\frac{d}{dt}\|n_t(t)\|^2+\left\|n_{ty}(t)\right\|^2\le\left\|n_y(t)\right\|_{L^\infty}^2\left\|u_y(t)\right\|^2$$

which, combined with (4.2.12), (4.2.31) and lemma 1.1.2, gives

$$\lim_{t\to+\infty}\|n_t(t)\|=0. \tag{4.3.31}$$

From (4.1.3) and (4.2.31), and using the Poincaré inequality, we can infer

$$\begin{aligned}\left\|n_{yy}(t)\right\|^2&\le C_1\left(\|n_t(t)\|^2+\left\|n_y(t)v_y(t)\right\|^2+\left\|n_y^2(t)\right\|^2\right)\\&\le C_1\left(\|n_t(t)\|^2+\left\|n_y(t)\right\|_{L^\infty}^2\left\|v_y(t)\right\|^2+\left\|n_y(t)\right\|_{L^\infty}^2\left\|n_y(t)\right\|^2\right)\\&\le C_1\left(\|n_t(t)\|^2+\left\|n_{yy}(t)\right\|^2\left\|v_y(t)\right\|^2+\left\|n_{yy}(t)\right\|^2\left\|n_y(t)\right\|^2\right)\\&\le C_1\left(\|n_t(t)\|^2+\left\|v_y(t)\right\|^2+\left\|n_y(t)\right\|^2\right),\end{aligned}$$

which, along with (4.3.2), (4.3.30) and (4.3.31), gives

$$\lim_{t\to+\infty}\left\|n_{yy}(t)\right\|^2=0. \tag{4.3.32}$$

Thus (4.3.25) follows from (4.3.32). This completes the proof. □

Proof of Theorem 4.1.1. Combining lemmas 4.2.3–4.2.7 and lemmas 4.3.1–4.3.3, we complete the proof of theorem 4.1.1. □

Lemma 4.3.4. *If assumptions in theorem 4.1.2 are valid, then there holds*

$$\lim_{t\to+\infty}\|v(t)-\bar{v}\|_{H^2}=0 \tag{4.3.33}$$

where $\bar{v}=\int_0^1 v(y,t)\,dy=\int_0^1 v_0(y)\,dy$.

Proof. Differentiating (4.1.1) with respect to twice y, multiplying the resulting equation by v_{yy}, then integrating it over $[0,1]$, by the Young inequality, we can deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|v_{yy}(t)\right\|^2&=\int_0^1 u_{yyy}v_{yy}\,dy\\&\le\frac{1}{2}\left\|u_{yyy}(t)\right\|^2+\frac{1}{2}\left\|v_{yy}(t)\right\|^2\le\frac{1}{4}+\frac{1}{4}\left\|v_{yy}(t)\right\|^4+\frac{1}{2}\left\|u_{yyy}(t)\right\|^2\end{aligned}$$

which, together with lemmas 4.2.8 and 1.1.2, implies

$$\lim_{t\to+\infty}\left\|v_{yy}(t)\right\|^2=0. \tag{4.3.34}$$

By the embedding theorem, we have

$$\lim_{t\to+\infty}\|v(t)-\bar{v}\|_{H^2}=0.$$

This proves the proof. □

Lemma 4.3.5. *If assumptions in theorem 4.1.2 are valid, then the following estimate holds*

$$\lim_{t\to+\infty}\|u(t)\|_{H^2}=0. \tag{4.3.35}$$

Proof. By (4.3.5), we have

$$\begin{aligned}\widehat{p}_y &= \left(\frac{1}{v}+\frac{|n_y|^2}{v^2}\right)_y\\ &= -\frac{v_y}{v^2}+\frac{2n_y\cdot n_{yy}}{v^2}-\frac{2|n_y|^2v_y}{v^3} \qquad (4.3.36)\\ &= -u_t+\left(\frac{u_y}{v}\right)_y. \qquad (4.3.37)\end{aligned}$$

Hence, by lemmas 4.2.3–4.2.9, and using the interpolation inequality, we obtain

$$\begin{aligned}\|\widehat{p}_y(t)\| &\le C_1\Big(\|v_y(t)\|+\|n_y(t)\cdot n_{yy}(t)\|+\||n_y(t)|^2v_y(t)\|\Big)\\ &\le C_1\Big(\|v_y(t)\|+\|n_{yy}(t)\|_{L^\infty}\|n_y(t)\|+\|n_y(t)\|^2_{L^\infty}\|v_y(t)\|\Big)\\ &\le C_1\Big[\|v_y(t)\|+\Big(\|n_{yy}(t)\|^{\frac12}\|n_{yyy}(t)\|^{\frac12}+\|n_{yy}(t)\|\Big)\|n_y(t)\|+\|n_{yy}(t)\|^2\|v_y(t)\|\Big]\\ &\le C_1\big(\|v_y(t)\|+\|n_y(t)\|\big)\end{aligned}$$

which gives

$$\lim_{t\to+\infty}\|\widehat{p}_y(t)\|=0. \tag{4.3.38}$$

Differentiating (4.3.37) with respect to t, multiplying the result by u_t, then integrating it over [0, 1], employing an integration by parts, and using the Young inequality, we can obtain for any $\varepsilon>0$,

$$\begin{aligned}\frac12\frac{d}{dt}\|u_t(t)\|^2+\int_0^1\frac{u_{ty}^2}{v}dy &= \int_0^1\widehat{p}_t u_{ty}\,dy+\int_0^1\frac{u_y^2u_{ty}}{v^2}dy\\ &\le \varepsilon\|u_{ty}(t)\|^2+C_1\Big(\|\widehat{p}_t(t)\|^2+\|u_y(t)\|^4_{L^4}\Big). \qquad (4.3.39)\end{aligned}$$

Choosing $\varepsilon>0$ small enough, using (4.2.21), (4.2.31), (4.2.32) and the interpolation inequality, we can conclude

$$\begin{aligned}
&\frac{d}{dt}\|u_t(t)\|^2+\|u_{ty}(t)\|^2\\
&\le C_1(\|u_y(t)\|^2+\|n_y(t)\|_{L^\infty}^2\|n_{ty}(t)\|^2\\
&\quad+\|n_y(t)\|_{L^\infty}^2\|v_y(t)\|^2+\|u_y(t)\|^3\|u_{yy}(t)\|+\|u_y(t)\|^4)\\
&\le C_1\Big(\|u_y(t)\|^2+\|n_{ty}(t)\|^2+\|n_y(t)\|_{L^\infty}^2\Big).
\end{aligned} \tag{4.3.40}$$

Hence we infer from (4.3.40), (4.2.6), (4.2.12), (4.2.31) and lemma 1.1.2

$$\lim_{t\to+\infty}\|u_t(t)\|^2=0. \tag{4.3.41}$$

By (4.3.37), (4.1.8) and the interpolation inequality, we derive

$$\begin{aligned}
\|u_{yy}(t)\|&\le C_1\big(\|\widehat{p}_y(t)\|+\|u_t(t)\|+\|u_y(t)v_y(t)\|\big)\\
&\le C_1(\|\widehat{p}_y(t)\|+\|u_t(t)\|+\Big(\|v_y(t)\|^{\frac12}\|v_{yy}(t)\|^{\frac12}+\|v_y(t)\|\Big)\|u_y(t)\|)\\
&\le C_1\big(\|\widehat{p}_y(t)\|+\|u_t(t)\|+\|u_y(t)\|\big)
\end{aligned}$$

which, along with (4.3.25), (4.3.37), (4.3.40), gives

$$\lim_{t\to+\infty}\|u_{yy}(t)\|^2=0. \tag{4.3.42}$$

Thus (4.3.35) follows from (4.3.42) and (4.1.9). This completes the proof. □

Lemma 4.3.6. *If assumptions in theorem 4.1.2 are valid, then there holds that*

$$\lim_{t\to+\infty}\|n(t)-\bar{n}\|_{H^3}=0 \tag{4.3.43}$$

where $\bar{n}=\int_0^1 n(y,t)\,dy$.

Proof. By lemma 3.3.2, we can derive

$$\frac{d}{dt}\|n_{ty}(t)\|^2\le C_1\Big(\|n_{ty}(t)\|^2+\|n_{yy}(t)\|^2\Big)$$

which, along with (4.2.21), (4.2.31) and lemma 1.1.2, leads to

$$\lim_{t\to+\infty}\|n_{ty}(t)\|^2=0. \tag{4.3.44}$$

Hence it follows from (4.2.41), (4.3.32), (4.3.34) and (4.3.44) that

$$\lim_{t\to+\infty}\|n_{yyy}(t)\|^2=0, \tag{4.3.45}$$

which, using the embedding theorem and (4.3.26), gives (4.3.43). This proves the lemma. □

Proof of Theorem 4.1.2. Combining lemmas 4.2.8–4.2.9 and lemmas 4.3.4–4.3.6, we can complete the proof of theorem 4.1.2. □

Lemma 4.3.7. *If assumptions in theorem 4.1.3 are valid, then there holds that*

$$\lim_{t\to+\infty}\|v(t)-\bar{v}\|_{H^4}=0 \tag{4.3.46}$$

where $\bar{v}=\int_0^1 v(y,t)\,dy=\int_0^1 v_0(y)\,dy$.

Proof. Differentiating (4.1.1) with respect to y three times, multiplying the resulting equation by v_{yyy}, integrating it over [0, 1], and then using the Young inequality, we can derive

$$\begin{aligned}\frac{d}{dt}\left\|v_{yyy}(t)\right\|^2 &\le \left\|u_{yyyy}(t)\right\|^2+\left\|v_{yyy}(t)\right\|^2\\ &\le \frac{1}{2}+\frac{1}{2}\left\|v_{yyy}(t)\right\|^4+\left\|u_{yyyy}(t)\right\|^2\end{aligned}$$

which, together with lemma 4.2.10 and lemma 1.1.2, yields

$$\lim_{t\to+\infty}\left\|v_{yyy}(t)\right\|^2=0. \tag{4.3.47}$$

Differentiating (4.1.1) with respect to y four times, multiplying the resulting equation by v_{yyyy}, integrating it over [0, 1], and then using the Young inequality, we can derive

$$\begin{aligned}\frac{d}{dt}\left\|v_{yyyy}(t)\right\|^2 &\le \left\|u_{yyyyy}(t)\right\|^2+\left\|v_{yyyy}(t)\right\|^2\\ &\le \frac{1}{2}+\frac{1}{2}\left\|v_{yyyy}(t)\right\|^4+\left\|u_{yyyyy}(t)\right\|^2\end{aligned}$$

which, along with lemma 4.2.10 and lemma 1.1.2, leads to

$$\lim_{t\to+\infty}\left\|v_{yyyy}(t)\right\|^2=0. \tag{4.3.48}$$

Therefore, using the embedding theorem, we conclude from (4.3.33), (4.3.47) and (4.3.48),

$$\lim_{t\to+\infty}\|v(t)-\bar{v}\|_{H^4}=0.$$

This completes the proof. □

Lemma 4.3.8. *If assumptions in theorem 4.1.3 are valid, then there holds that*

$$\lim_{t\to+\infty}\|u(t)\|_{H^4}=0. \tag{4.3.49}$$

Proof. Clearly, using lemma 4.2.10 and the Young inequality, we can derive

$$\begin{aligned}\frac{d}{dt}\|u_{ty}(t)\|^2 &\le C_2(\|u_{ty}(t)\|^2+\|u_{ty}(t)\|_{H^2}^2+\|u_y(t)\|_{H^2}^2+\|n_y(t)\|_{H^1}^2\\&\quad+\|n_{ty}(t)\|_{H^1}^2+\|v_y(t)\|_{H^1}^2)\\&\le C_2+C_2\|u_{ty}(t)\|^4+C_2(\|u_{ty}(t)\|_{H^2}^2+\|u_y(t)\|_{H^2}^2+\|n_y(t)\|_{H^1}^2\\&\quad+\|n_{ty}(t)\|_{H^1}^2+\|v_y(t)\|_{H^1}^2)\end{aligned}$$

which, along with lemmas 4.2.3–4.2.10 and lemma 1.1.2, leads to

$$\lim_{t\to+\infty}\|u_{ty}(t)\|^2=0. \tag{4.3.50}$$

Differentiating (4.3.36) with respect to y, using the interpolation inequality and lemmas 4.2.3–4.2.10, we infer

$$\begin{aligned}\|\widehat{p}_{yy}(t)\| &\le C_1(\|v_{yy}(t)\|+\|v_y(t)\|_{L^\infty}\|v_y(t)\|+\|n_{yy}(t)\|_{L^\infty}\|n_{yy}(t)\|\\&\quad+\|n_y(t)\|_{L^\infty}\|n_{yyy}(t)\|+\|n_y(t)\|_{L^\infty}\|v_y(t)\|_{L^\infty}\|n_{yy}(t)\|\\&\quad+\|n_y(t)\|_{L^\infty}^2\|v_{yy}(t)\|+\|v_y(t)\|_{L^\infty}^2\|n_y(t)\|_{L^\infty}\|n_y(t)\|)\\&\le C_1\big(\|v_y(t)\|+\|v_{yy}(t)\|+\|n_y(t)\|+\|n_{yy}(t)\|+\|n_{yyy}(t)\|\big)\end{aligned}$$

which, combined with (4.3.2), (4.3.29), (4.3.34), (4.3.32) and (4.3.45), implies

$$\lim_{t\to+\infty}\|\widehat{p}_{yy}(t)\|=0. \tag{4.3.51}$$

Differentiating (4.3.37) with respect to y, using the interpolation inequality and lemmas 4.2.3–4.2.9, we derive

$$\begin{aligned}\|u_{yyy}(t)\| &\le C_1\Big(\|u_{ty}(t)\|+\|\widehat{p}_{yy}(t)\|+\|u_{yy}(t)v_y(t)\|+\|u_y(t)v_{yy}(t)\|+\Big\|u_y(t)v_y^2(t)\Big\|\Big)\\&\le C_1\Big(\|u_{ty}(t)\|+\|\widehat{p}_{yy}(t)\|+\|u_y(t)\|_{H^1}+\|v_y(t)\|_{H^1}\Big)\end{aligned}$$

which, together with (4.3.50), (4.3.51), (4.3.25), (4.3.34), (4.3.2) and (4.3.42), leads to

$$\lim_{t\to+\infty}\|u_{yyy}(t)\|=0. \tag{4.3.52}$$

Differentiating (4.3.36) with respect to y twice, using the interpolation inequality and lemmas 4.2.3–4.2.10, leads to

$$\|\widehat{p}_{yyy}(t)\|\le C_4\big(\|v_{yyy}(t)\|+\|v_{yy}(t)\|+\|n_{yyy}(t)\|+\|v_y(t)\|+\|n_y(t)\|\big)$$

which, along with (4.3.2), (4.3.29), (4.3.34), (4.3.45) and (4.3.47), results in

$$\lim_{t\to+\infty}\|\widehat{p}_{yyy}(t)\|=0. \tag{4.3.53}$$

Differentiating (4.3.36) with respect to t and y, using the Poincaré and the Gagliardo–Nirenberg interpolation inequalities, and using lemmas 4.2.3–4.2.10, we can obtain

$$\left\|\widehat{p}_{yyt}(t)\right\|^2 \leq C_2\Big(\left\|u_y(t)\right\|_{H^2}^2 + \left\|n_{ty}(t)\right\|_{H^2}^2 + \left\|n_y(t)\right\|_{H^2}^2 + \left\|v_y(t)\right\|_{H^2}^2\Big). \tag{4.3.54}$$

Differentiating (4.3.37) with respect to t once and y twice, multiplying the resulting equation by u_{tyy}, employing an integration by parts, and using the Young inequality, we can conclude that for any $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u_{tyy}(t)\right\|^2 + \int_0^1 \frac{u_{tyyy}^2}{v}dy \leq{}& \varepsilon\left\|u_{tyyy}(t)\right\|^2 + C_1(\varepsilon)\left\|\widehat{p}_{yyt}(t)\right\|^2 + C_1\Big(\left\|u_{tyy}v_y\right\|^2\\ &+ \left\|u_{ty}v_{yy}\right\|^2 + \left\|u_{ty}v_y^2\right\|^2 + \left\|u_{yy}\right\|_{L^4}^4 + \left\|u_y u_{yyy}\right\|^2\\ &+ \left\|u_y u_{yy} v_y\right\|^2 + \left\|u_y^2 v_{yy}\right\|^2 + \left\|u_y v_y\right\|_{L^4}^4\Big).\end{aligned}$$

Choosing $\varepsilon \in (0, 1)$ small enough, using the interpolation inequality and lemmas 4.2.3–4.2.9, we deduce

$$\begin{aligned}\frac{d}{dt}\left\|u_{tyy}(t)\right\|^2 + \left\|u_{tyyy}(t)\right\|^2 \leq{}& C_1\left\|\widehat{p}_{yyt}(t)\right\|^2 + C_2(\left\|u_{ty}(t)\right\|_{H^2}^2\\ &+ \left\|u_y(t)\right\|_{H^2}^2 + \left\|v_y(t)\right\|_{H^2}^2).\end{aligned} \tag{4.3.55}$$

Inserting (4.3.54) into (4.3.55), using lemmas 4.2.3–4.2.10 and lemma 1.1.2, we can derive

$$\lim_{t\to+\infty}\left\|u_{tyy}(t)\right\|^2 = 0. \tag{4.3.56}$$

Differentiating (4.3.37) with respect to y twice, using the Gagliardo–Nirenberg interpolation inequality and lemmas 4.2.3–4.2.10, we obtain

$$\begin{aligned}\left\|u_{yyyy}(t)\right\| \leq{}& C_1\Big(\left\|\widehat{p}_{yyy}(t)\right\| + \left\|u_{tyy}(t)\right\| + \left\|u_{yyy}v_y\right\| + \left\|u_{yy}v_y\right\| + \left\|u_{yy}v_y^2\right\|\\ &+ \left\|u_{yy}v_{yy}\right\| + \left\|u_y v_{yyy}\right\| + \left\|u_y v_{yy} v_y\right\| + \left\|u_y v_y^3\right\|\Big)\\ \leq{}& C_1\Big(\left\|\widehat{p}_{yyy}(t)\right\| + \left\|u_{tyy}(t)\right\| + \left\|u_y(t)\right\| + \left\|u_{yy}\right\| + \left\|v_y(t)\right\| + \left\|v_{yyy}(t)\right\|\Big)\\ &+ C_4\left\|v_{yy}(t)\right\|\end{aligned}$$

which, together with (4.3.2), (4.3.25), (4.3.34), (4.3.42), (4.3.47), (4.3.53) and (4.3.56), gives

$$\lim_{t\to+\infty}\left\|u_{yyyy}(t)\right\| = 0. \tag{4.3.57}$$

Therefore, it follows from (4.3.33), (4.3.52) and (4.3.57) that

$$\lim_{t\to+\infty}\left\|u(t)\right\|_{H^4} = 0.$$

Thus the proof follows immediately. □

Lemma 4.3.9. *If assumptions in theorem 4.1.3 are valid, then the following estimate holds*

$$\lim_{t\to+\infty}\|n(t)-\bar{n}\|_{H^4}=0 \tag{4.3.58}$$

where $\bar{n}=\int_0^1 n(y,t)\,dy$.

Proof. Exploiting lemma 4.2.10 and the Young inequality, we easily infer that

$$\begin{aligned}\frac{d}{dt}\|n_{tt}(t)\|^2+\left\|n_{tty}(t)\right\|^2 &\le C_2\Big(\|n_{tt}(t)\|^2+\left\|n_{ty}(t)\right\|^2_{H^1}+\left\|u_y(t)\right\|^2_{H^1}+\left\|u_{ty}(t)\right\|^2\Big)\\ &\le C_2+C_2\|n_{tt}(t)\|^4+C_2(\left\|n_{ty}(t)\right\|^2_{H^1}\\ &\quad+\left\|u_y(t)\right\|^2_{H^1}+\left\|u_{ty}(t)\right\|^2)\end{aligned}$$

which, along with (4.2.33), (4.2.31), (4.2.41) and lemma 1.1.2, implies

$$\lim_{t\to+\infty}\|n_{tt}(t)\|^2=0. \tag{4.3.59}$$

Then (4.3.55) follows from lemma 4.2.10, (4.3.31), (4.3.43), (4.3.47), (4.3.59) and lemmas 4.3.1–4.3.6 immediately. Thus the proof follows readily. □

Proof of Theorem 4.1.3. Using lemma 4.2.10 and lemmas 4.3.7–4.3.9, we can finally complete the proof of theorem 4.1.3. □

4.4 Bibliographic Comments

In addition to the comments in section 3.5, we would like to mention here that the global existence and regularity of solutions to (3.1.1)–(3.1.5) or (3.1.6)–(3.1.10) have been established for the pressure $P=\rho^\gamma=\frac{1}{v^\gamma}(\gamma\ge 1)$, while the large-time behavior of global solutions to (3.1.6)–(3.1.10) (*i.e.*, (4.1.1)–(4.1.5)) has only been proved for $\gamma=1$ in P and it is still open for $\gamma>1$. For the large-time behavior of solutions for incompressible liquid crystal system, we would like to refer to Wu [144] and the references therein.

Bibliography

[1] Adams R. A. (1975) *Sobolev space*. Academic Press, New York (Vol. 65 in the series *Pure and applied mathematics*).

[2] Adams R. A., Fournier, J. J. F. (2003) *Sobolev spaces*, section edn. Academic Press, New York, Oxford, Singapore, Sydney, Tokyo.

[3] Antontsev S. N., Kazhikhov A. V., Monakhov V. N. (1990) *Boundary value problem in mechanics of nonhomogeneous fluids*. North-Holland, Amsterdam, New York, Oxford, Tokyo.

[4] Batchelor G. K. (1967) *An introduction to fluids dynamics*. Cambridge University Press, London.

[5] Bellman R. (1948) On the existence and boundedness of solutions of nonlinear partial differential equations of parabolic type, *Trans. Amer. Math. Soc.* **64**(1), 21.

[6] Bellman R. (1953) On an inequality of Weinberger, *Am. Math. Monthly* **60**(6), 402.

[7] Bellman R. (1954) *Stability theory of differential equations*. McGraw-Hill Book Co., Inc., New York.

[8] Bellman R. (1967) Upper and lower bounds for solutions of the matrix Riccati equation, *J. Math. Anal. Appl.* **17**(2), 373.

[9] Blanca E., Francisco G., Marko R. (2006) Reproductivity for a nematic liquid crystal model, *Z. Angew. Math. Phys.* **57**(6), 984.

[10] Buet C., Després B. (2004) Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics, *J. Quant. Spectrosc. Radiat. Transfer* **85**(3–4), 385.

[11] Calderer M. C., Golovaty D., Lin F., Liu C. (2002) Time evolution of nematic liquid crystals with variable degree of orientation, *SIAM J. Math. Anal.* **33**(5), 1033.

[12] Chandrasekhar S. (1960) *Radiative transfer*. Dover Publications, Inc., New York.

[13] Choe H., Kim H. (2003) Strong solutions of the Navier–Stokes equations for isentropic compressible fluids, *J. Differ. Equ.* **190**(2), 504.

[14] Choe H., Kim H. (2004) Unique solvability for the density-dependent Navier–Stokes equations, *Nonlinear Anal., TMA* **59**(4), 465.

[15] Constantin P., Foias C., Temam R. (1983) Connexion entre la théorie mathematique des équations de Navier–Stokes et la théorie conventionnelle de la turbulence, *C. R. Acad. Sci. Paris, Série I* **297**(11), 599.

[16] Ding S. (April 1, 2009) Global solutions of one-dimensional liquid crystals system. A presentation in a conference at Wuhan University.

[17] Dubroca B., Feugeas J. L. (1999) Etude théorique et numérique d'une hiérarchie de modèles aux moments pour le transfert radiatif, *C. R. Acad. Sci. Paris, Série I* **329**(10), 915.

[18] Ducomet B., Nečasová Š. (2010) Global existence of solutions for the one-dimensional motions of a compressible gas with radiation: An “infrarelativistic model”, *Nonlinear Anal., TMA*, **72**(7–8), 3258.

[19] Ducomet B., Nečasová Š. (2010) Global weak solutions to the 1D compressible Navier–Stokes equations with radiation, *Commun. Math. Anal.* **8**(3), 23.

[20] Ducomet B., Nečasová Š. (2012) Large-time behavior of the motion of a viscous heat-conducting one-dimensional gas coupled to radiation, *Ann. Mat. Pura Appl.* **191**(2), 219.

[21] Ducomet B., Nečasová Š. (2013) Asymptotic behavior of the motion of a viscous heat-conducting one-dimensional gas with radiation: The pure scattering case, *Anal. Appl.* **11**(1), 1350003, 1.

[22] Ducomet B., Zlotnik A. (2003) Stabilization for equations of one-dimensional viscous compressible heat-conducting media with nonmonotone equation of state, *J. Differ. Equ.* **194** (1), 51–81.

[23] Ericksen J. (1987) Continuum theory of nematic liquid crystals, *Res. Mech.* **21**, 381.

[24] Ericksen J. (1961) Conservation laws for liquid crystals, *Trans. Soc. Rheol.* **5**, 22.

[25] Fan J., Ozawa T. (2009) Regularity criteria for a simplified Ericksen–Leslie system modeling the flow of liquid crystals, *Discrete Conti. Dyna. Syst.* **25**(3), 859.

[26] Fang D., Zhang T. (2004) A note on compressible Navier–Stokes equations with vacuum state in one dimension, *Nonlinear Anal., TMA* **58**(5–6), 719.

[27] Fang D., Zhang T. (2006) Global solutions of the Navier–Stokes equations for compressible flow with density-dependent viscosity and discontinuous initial data, *J. Differ. Equ.* **222**(1), 63.

[28] Feireisl E., *Dynamics of viscous compressible fluids.* Oxford University Press, Oxford (2003).

[29] Feireisl E., Novotný A., *Singular limits in thermodynamics of viscous fluids.* Birkhäuser Verlag, Basel-Boston-Berlin (2009).

[30] Feireisl E., Novotný A., Petzeltova H. (2001) On the existence of globally defined weak solutions to the Navier–Stokes equations, *J. Math. Fluid. Mech.* **3**(4), 358.

[31] Feireisl E., Petzeltova H. (2001) Bounded absorbing sets for the Navier–Stokes equations of compressible fluid, *Commun. Partial Differ. Equ.* **26**(7-8), 1133.

[32] Feireisl E., Petzeltova H. (2001) Asymptotic compactness of global trajectories generalized by the Navier–Stokes equations of a compressible fluid, *J. Differ. Equ.* **173**(2), 390.

[33] Feireisl E., Petzeltova H. (2002) The zero-velocity limit solutions of the Navier–Stokes equations of compressible fluid revisited. Navier–Stokes equations and related nonlinear problems, *Ann. Univ. Ferrara-Sez. VII-Sc. Mat.* **46**(1), 209.

[34] Foias C., Temam R. (1977) Structure of the set of stationary solutions of the Navier–Stokes equations, *Commun. Pure Appl. Math.* **30**(2), 149.

[35] Foias C., Temam R. (1987) The connection between the Navier–Stokes equations, dynamical systems and turbulence, *New directions in partial differential equations.* Academic Press, New York.

[36] Frid H., Shelukhin V. (2000) Vanishing shear viscosity in the equations of compressible fluids for the flows with the cylinder symmetry, *SIAM J. Math. Anal.* **31**(5), 1144.

[37] Friedman A. (1964) *Partial differential equations of parabolic type.* Prentice-Hall, Englewood Cliffs, NJ.

[38] Friedman A. (1976) *Partial differential equations.* Krieger, Huntington, New York.

[39] Fujita H., Kato T. (1964) On the Navier–Stokes initial value problem (I), *Arch. Rational Mech. Anal.* **16**(4), 269.

[40] Gagliardo E. (1958) Proprietá di alcune classi di funzioni in piú variabili, *Ricerche di Mat.* **7**, 102.

[41] Gagliardo E. (1959) Ulteriori proprietá di alcune classi di funzioni in piú variabili, *Ricerche di Mat.* **8**, 24.

[42] Galdi G. P. (1994) *An introduction to the mathematical theory of the Navier–Stokes equations*, Vol. I, *Linearized steady problems, Springer tracts in numerical philosophy,* Vol. 38. Springer-Verlag.

[43] Gallavotti G. (2002) *Foundations of fluid mechanics.* Springer-Verlag, Berlin, Heidelberg, New York.
[44] Golse F., Perthame B. (1986) Generalized solutions of the radiative transfer equations in a singular case, *Commun. Math. Phys.* **106**(2), 211.
[45] Gronwall T. H. (1919) Note on the derivatives with respect to a parameter of the solutions of a system of differential equation, *Ann. Math.* **20**(4), 292.
[46] Hoff D. (1992) Global well-posedness of the Cauchy problem for the Navier–Stokes equations of nonisentropic flow with discontinuous initial data, *J. Differ. Equ.* **95**(1), 33.
[47] Hoff D. (1995) Global solutions of the Navier–Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differ. Equ.* **120**(1), 215.
[48] Hoff D. (1996) Continuous dependence on initial data for discontinuous solutions of the Navier–Stokes equations for one-dimensional compressible flow, *SIAM J. Math. Anal.* **27**(5), 1193.
[49] Hoff D. (1997) Discontinuous solutions of the Navier–Stokes equations for multidimensional flow of heat-conducting fluids, *Arch. Rat. Mech. Anal.* **139**(4), 303.
[50] Hoff D. (2006) Uniqueness of weak solutions of the Navier–Stokes equations of multidimensional, compressible flow, *SIAM J. Math. Anal.* **37**(6), 1742.
[51] Hoff D., Serre D. (1991) The failure of continuous dependence on initial data for the Navier–Stokes equations of compressible flow, *SIAM J. Appl. Math.* **51**(4), 887.
[52] Hoff D., Smoller J. (2001) Non-formation of vacuum states for compressible Navier–Stokes equations, *Commun. Math. Phys.* **216**(2), 255.
[53] Hoff D., Ziane M. (1999) Compact attractors for the Navier–Stokes equations of onedimensional compressible flow, *C. R. Acad. Sci. Paris, Ser. I.* **328**(3), 239.
[54] Hoff D., Ziane M. (2000) The global attractor and finite determining nodes for the Navier–Stokes equations of compressible flow with singular initial data, *Indiana Univ. Math. J.* **49**, 843.
[55] Hölder O. (1889) Über einen Mittelworthssatz, *Nachr. Ges. Wiss. Göttingen*, 38.
[56] Hong M. (2011) Global existence of solutions of the simplified Ericksen–Leslie system in dimension two, *Calc. Var.* **40**(1–2), 15.
[57] Hu X., Wang D. (2010) Global solution to the three-dimensional incompressible flow of liquid crystals, *Commun. Math. Phys.* **296**(3), 861.
[58] Huang F., Matsumura A., Xin Z. (2006) Stability of contact discontinuities for the 1-D compressible Navier–Stokes equations, *Arch. Rat. Mech. Anal.* **179**(1), 55.
[59] Jiang P., Wang D. (2010) Formation of singularities of solutions to the three-dimensional Euler–Boltzmann equations in radiation hydrodynamics, *Nonlinearity* **23**, 809.
[60] Jiang P., Wang D. (2012) Global weak solutions to the three-dimensional Euler–Boltzmann equations in radiation hydrodynamics, *Quart. Appl. Math.* **70**, 25.
[61] Jiang S. (1994) On initial boundary value problems for a viscous heat-conducting one-dimensional gas, *J. Differ. Equ.* **110**(2), 157.
[62] Jiang S. (1994) On the asymptotic behavior of the motion of a viscous, heat-conducting, one-dimensional real gas, *Math. Z.* **216**(1), 317.
[63] Jiang S. (1996) Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain, *Commun. Math. Phys.* **178**(2), 339.
[64] Jiang S., Global solutions of the Cauchy problem for a viscous polytropic ideal gas, *Ann. Scuola Norm Sup, Pisa Cl. Sci.* **XXVI**(4), 47.
[65] S. Jiang (1998) Global smooth solutions to the equations of a viscous heat conducting one-dimensional gas with density-dependent viscosity, *Math. Nachr.* **190**(1), 163.
[66] Jiang S. (1998) Large-time behavior of solutions to the equations of a viscous polytropic ideal gas, *Ann. Mate. Pura Appl.* **175**(1), 253.
[67] Jiang S. (1999) Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains, *Commun. Math. Phys.* **200**(1), 181.
[68] Kawashima S. (1987) Large-time behavior of solutions to hyperbolic-parabolic systems of conservation laws and applications, *Proc. Roy. Soc. Edinburgh* **106**(1–2), 169.

[69] Kawashima S., Nishida T. (1981) Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases, *J. Math. Kyoto. Univ.* **21** (4), 825.

[70] Kawashima S., Nishibata S., Zhu P. (2003) Asymptotic stability of the stationary solution to the compressible Navier–Stokes equations in the half space, *Commun. Math. Phys.* **240**(3), 483.

[71] Kazhikhov A. V. (1977) Sur la solubilité globale des problèmes monodimensionnels aux valeurs initiales-limités pour les équations d̆un gaz visqueux et calorifére, *C. R. Acad.Sci. Paris, Ser. A* **284**, 317.

[72] Kazhikhov A. V. (1981) To a theory of boundary value problems for equations of one dimensional nonstationary motion of viscous heat-conduction gases, *Boundary Value Prob. Hydrodynam. Equ.* (in Russian), No. 50 Inst. Hydrodynamics, Siberian Branch Akad., USSR, pp. 37–62.

[73] Kazhikov A. V., Shelukhin V. V. (1977) Unique global solution with respect to time of initial boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* **41**(2), 273.

[74] Kippenhahn R., Weigert A. (1994) *Stellar structure and evolution.* Springer-Verlag, Berlin, Heidelberg.

[75] Kuang J. (2004) *Applied inequalities*, 3nd edn. Shangdong Science and Technology Press (in Chinese).

[76] Lech M. (1998) Why Hölder's inequality should be called Rogers' inequality, *Math. Inequal. Appl.* **1**(1), 69.

[77] Lesile F. (1968) Some constitutive equations for liquid crystals, *Arch. Rational Mech. Anal.* **28**(4), 265.

[78] Lesile F. (1979) Theory of flow phenomena in liquid crystals advances in liquid crystals, *Adv. Liq. Cryst.* **4**, 1.

[79] Lin C. (2007) Mathematical analysis of radiative transfer models, Ph.D. Thesis.

[80] Lin C., Coulombel J. F., Goudon T. (2006) Shock profiles for non-equilibrium radiating gases, *Physica D* **218**(1), 83.

[81] Lin F., Liu C. (1995) Nonparabolic dissipative systems modeling the flow of liquid crystals, *Commun. Pure Appl. Math.* **48**(5), 501.

[82] Lin F., Liu C. (1996) Partial regularity of the dynamical system modeling the flow of liquid crystals, *Discrete Conti. Dyna. Syst.* **2**(1), 1.

[83] Lin F., Liu C. (2000) Existence of solutions for the Ericksen–Leslie system, *Arch. Rational Mech. Anal.* **154**(2), 135.

[84] Lions P. L. (1998) *Mathematical topics in fluid dynamics*, Vol. 1, *Incompressible models*, Vol. 2 *Compressible models*. Oxford Science Publication, Oxford.

[85] Liu C. (2000) Dynamic theory for incompressible smectic-A liquid crystals: Existence and regularity, *Discrete Conti. Dyna. Syst.* **6**(3), 591.

[86] Liu C., Shen J. (2001) On liquid crystal flows with free-slip boundary conditions, *Discrete Conti. Dyna. Syst.* **7**(2), 307.

[87] Liu C., Walkington N. (2000) Approximation of liquid crystal flows, *SIAM J. Appl. Math.* **37** (3), 725.

[88] Liu C., Walkington N. (2002) Mixed methods for the approximation of liquid crystal flows, *ESAIM: Math. Model. Numer. Anal.* **36**(2), 205.

[89] Lowrie R. B., Morel J. E., Hittinger J. A. (1999) The coupling of radiation and hydrodynamics, *Astrophys. J.* **521**(1), 432.

[90] Matsumura A., Nishida T. (1979) The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, *Proc. Jap. Acad. Ser. A* **55**, 337.

[91] Matsumura A., Nishida T. (1980) The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto. Univ.* **20**(1), 67.

[92] Matsumura A., Nishida T. (1982) Initial boundary value problems for the equations of motion of general fluids, *Computing Meth. in Appl. Sci. and Engin. V.* (R. Glowinski, J. L. Lions, Eds.). North-Holland, Amsterdam, pp. 389–406.

[93] Matsumura A., Nishida T. (1983) Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, *Commun. Math. Phys.* **89**(4), 445.

[94] Mihalas D., Weibel-Mihalas B. (1984) *Foundations of radiative hydrodynamics*. Oxford University Press, New York.

[95] Nirenberg L. (1959) On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa* **13**(3), 113.

[96] Nirenberg L. (1974) *Topics in nonlinear functional analysis*. Courant Institute of Mathematical Sciences.

[97] Okada M., Kawashima S. (1983) On the equations of one-dimensional motion of compressible viscous fluids, *J. Math. Kyoto Univ.* **23**(1), 55.

[98] Paicu M., Zhang P. (2012) Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system, *J. Funct. Anal.* **262**(8), 3556.

[99] Pomraning G. C. (1973) *The equations of radiation hydrodynamics, international series' of monographs in natural philosophy*, Vol. 54. Pergamon Press, Oxford-New York-Toronto-Springer-BRAUNSCHWEIG.

[100] Pomraning G. C. (2005) *Radiation hydrodynamics*. Dover Publications, Inc., Mineola, New York.

[101] Qin Y. (2001) Global existence and asymptotic behavior for a viscous, heat-conductive, one-dimensional real gas with fixed and thermally insulated endpoints, *Nonlinear Anal., TMA* **44**(4), 413.

[102] Qin Y. (2002) Exponential stability for a nonlinear one-dimensional heat-conductive viscous real gas, *J. Math. Anal. Appl.* **272**(2), 507.

[103] Qin Y. (2004) Universal attractor in H^4 for the nonlinear one-dimensional compressible Navier–Stokes equations, *J. Differ. Equ.* **207**(1), 21.

[104] Qin Y., *Nonlinear parabolic-hyperbolic coupled systems and their attractors*, Operator *theory, advances in PDEs*, Vol. 184. Basel-Boston-Berlin, Birkhäuser (2008).

[105] Qin Y. (2010) Exponential stability for the compressible Navier–Stokes equations with the cylinder symmetry in $\mathbb{R}^3$, *Nonlinear Anal., RWA* **11**(5), 3590.

[106] Qin Y. (2017) *Analytic inequalities and their applications in PDEs*. Birkhäuser Verlag AG.

[107] Qin Y. (2016) *Integral and discrete inequalities and their applications*, Vol. I, *Linear inequalities*. Springer International Publishing AG.

[108] Qin Y. (2016) *Integral and discrete inequalities and their applications*, Vol. II, *Nonlinear inequalities*. Springer International Publishing AG.

[109] Qin Y., Dong X., Liu X. (2021) Global existence and asymptotic behavior of solutions for the one-dimensional infrarelativistic model of a compressible viscous gas with radiation, *Math. Meth. Appl. Sci.* doi: 10.1002/mma.7760.

[110] Qin Y., Feng B. (2017) Large-time behavior of solutions to a 1D liquid crystal system, *Math. Meth. Appl. Sci.* **50**(1), 7077.

[111] Qin Y., Feng B., Zhang M. (2012) Large-time behavior of solutions for the one-dimensional infrarelativistic model of a compressible viscous gas with radiation, *J. Differ. Equ.* **252**(12), 6175.

[112] Qin Y., Feng B., Zhang M. (2014) Large-time behavior of solutions for the 1D viscous heat-conducting gas with radiation: The pure scattering case, *J. Differ. Equ.* **256**(3), 989.

[113] Qin Y., Hu F. (2009) Global existence and exponential stability for a real viscous heat-conducting flow with shear viscosity, *Nonlinear Anal., RWA* **10**(1), 298.

[114] Qin Y., Hu G. (2011) Global smooth solutions for 1D thermally radiative magnetohydrodynamics, *J. Math. Phys.* **52**(2), 023102.

[115] Qin Y., Hu G., Wang T. (2011) Global smooth solutions for the compressible viscous and heat-conductive gas, *Quart. Appl. Math.* **69**(3), 509.

[116] Qin Y., Hu G., Wang T., Huang L., Ma Z. (2013) Remarks on global smooth solutions to a 1D self-gravitating viscous radiative and reactive gas, *J. Math. A7nal. Appl.* **408**(1), 19.

[117] Qin Y., Huang L. (2014) Global existence and regularity of a 1D liquid crystal system, *Nonlinear Anal., RWA* **15**, 172.

[118] Qin Y., Huang L., Ma Z. (2009) Global existence and exponential stability in H^4 for the nonlinear compressible Navier–Stokes equations, *Commun. Pure Appl. Anal.* **8**(6), 1991.

[119] Qin Y., Huang L., Yao Z. (2008) Regularity of one-dimensional compressible isentropic Navier–Stokes equations with density-dependent viscosity, *J. Differ. Equ.* **245**(12), 3956.

[120] Qin Y., Huang L., Yao Z. (2009) A remark on regularity of one-dimensional compressible isentropic Navier–Stokes equations with density-dependent viscosity, *J. Math. Anal. Appl.* **351**(2), 497.

[121] Qin Y., Huang L. (2010) Regularity and exponential stability of the p-th Newtonian fluid in one space dimension, *Math. Mode Meth. Appl. Sci.* **20**(4), 589.

[122] Qin Y., Huang L. (2010) Global existence and exponential stability for the p-th power viscous reactive gas, *Nonlinear Anal., TMA* **73**(9), 2800.

[123] Qin Y., Huang L. (2012) *Global well-posedness of nonlinear parabolic-hyperbolic coupled systems*, *Frontiers in mathematics*. Basel-Boston-Berlin, Birkhäuser.

[124] Qin Y., Huang L. (2014) Global existence and regularity of a 1D liquid crystal system, *Nonlinear Anal., RWA*, **15**, 172.

[125] Qin Y., Jiang L. (2010) Global existence and exponential stability of solutions in H^4 for the compressible Navier–Stokes equations with the cylinder symmetry, *J. Differ. Equ.* **249**(6), 1353.

[126] Qin Y., Liu X., Yang X. (2012) Global existence and exponential stability of solutions to the one-dimensional full non-Newtonian fluids, *Nonlinear Anal., RWA* **13**(2), 607.

[127] Qin Y., Liu X., Yang X. (2012) Global existence and exponential stability for a 1D compressible and radiative MHD flow, *J. Differ. Equ.* **253**(5), 1439.

[128] Qin Y., Liu X., Wang T. (2015) *Global wellposedness of nonlinear evolutionary fluid equations*. Springer Basel, 978-3-0348-0593-3.

[129] Qin Y., Rivera J. E. M. (2003) Exponential stability and universal attractors for the Navier–Stokes equations of compressible fluids between two horizontal parallel plates in $\mathbb{R}^3$, *Appl. Numer. Math.* **47**(2), 209.

[130] Qin Y., Song J. (2010) Maximal attractors for the compressible Navier–Stokes equations of viscous and heat conductive fluid, *Acta Math. Sci.* **30B**(1), 289.

[131] Qin Y., Su X., Deng S. (2008) Remarks on self-similar solutions to the compressible Navier–Stokes equations of a one-dimensional viscous polytropic ideal gas, *Appl. Math. Sci.* **2**(30), 1493.

[132] Qin Y., Wen S. (2008) Global existence of spherically symmetric solutions for nonlinear compressible Navier–Stokes equations, *J. Math. Phys.* **49**(2), 023101.

[133] Qin Y., Yu X. (2009) Global existence and asymptotic behavior for the compressible Navier–Stokes equations with a non-autonomous external force and heat source, *Math. Meth. Appl. Sci.* **32**(8), 1011.

[134] Qin Y., Zhao Y. (2008) Global existence and asymptotic behavior of compressible Navier–Stokes equations for a one-dimensional isothermal viscous gas, *Math. Mode. Meth. Appl. Sci.* **18**(8), 1183.

[135] R. Racke (1992) *Lectures on nonlinear evolution equations*, Aspects *of mathematics*, Vol. 19. VIEWEG, Bonn.

[136] J. Serrin (1962) On the interior regularity of weak solutions of the Navier–Stokes equations, *Arch. Rational Mech. Anal.* **9**(1), 187.

[137] Shen W., Zheng S. (1993) On the coupled Cahn–Hilliard equations, *Commun. Partial Differ. Equ.* **18**(3–4), 701.

[138] Sobolev S. L. (1938) On a theorem of functional analysis, *Mat. Sbornik* **4**, 471; English transl: *Amer. Math. Soc. Trans.* **34**, 39 [II.2, II.9, Notes for II].

[139] Temam R. (1979) *Navier–Stokes equations, theory and numerical analysis.* North Holland, Amsterdam.

[140] Temam R. (1983) *Navier–Stokes equations and nonlinear functional analysis.* Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

[141] Temam R. (1997) *Infinite-dimensional dynamical system in mechanics and physics*, 2nd edn. Springer, New York.

[142] Valli A., Zajaczkowski W. M. (1986) Navier–Stokes equation for compressible fluids: Global existence and qualitative properties of the solutions in the general case, *Commun. Math. Phys.* **103**(2), 259.

[143] Wen H., Ding S. (2011) Solutions of incompressible hydrodynamic flow of liquid crystals, *Nonlinear Anal., RWA* **12**(3), 1510.

[144] Wu H. (2010) Long-time behavior for nonlinear hydrodynamic system modeling the nematic liquid crystal flows, *Discrete Conti. Dyna. Syst.* **26**(1), 379.

[145] Wu S. T. (2011) General decay of solutions for a viscoelastic equation with nonlinear damping and source terms, *Acta Math. Sci.* **31**(4), 1436.

[146] Xin Z. (1998) Blowup of smooth solutions to the compressible Navier–Stokes equation with compact density, *Commun. Pure Appl. Math.* **51**(3), 229.

[147] Xin Z., Yan W. (2013) On blowup of classical solutions to the compressible Navier–Stokes equations, *Commun. Math. Phys.* **321**(2), 529.

[148] Yosida K. (1980) *Functional analysis*, 6th edn. Springer-Verlag.

[149] Young W. H. (1912) On classes of summable functions and their Fourier series, *Proc. Roy. Soc. London A* **87**(594), 225.

[150] Zhang T., Fang D. (2006) Global behavior of compressible Navier–Stokes equations with a degenerate viscosity coefficient, *Arch. Rational Mech. Anal.* **182**(2), 223.

[151] Zhang T., Fang D. (2007) Global behavior of spherically symmetric Navier–Stokes equations with density-dependent viscosity, *J. Differ. Equ.* **236**(1), 293.

[152] Zhang T., Fang D. (2008) Global wellposed problem for the 3-D incompressible anisotropic Navier–Stokes equations, *J. Math. Pures Appl.* **90**(5), 413.

[153] Zhang T., Fang D. (2009) A note on spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients, *Nonlinear Anal., RWA* **10**(4), 2272.

[154] Zhang T., Fang D. (2012) Random data Cauchy theory for the generalized incompressible Navier–Stokes equations, *J. Math. Fluid Mech.* **14**(2), 311.

[155] Zheng S. (1995) *Nonlinear parabolic equations and hyperbolic-parabolic coupled systems*, Pitman *series monographs and surveys in pure and applied mathematics*, Vol. 76. Longman Group Limitted, London.

[156] Zheng S., Qin Y. (2001) Universal attractors for the Navier–Stokes equations of compressible and heat conductive fluids in bounded annular domains in $\mathbb{R}^n$, *Arch. Rational Mech. Anal.* **160**(2), 153.

[157] Zhong X., Jiang S. (2007) Local existence and finite-time blow-up in multidimensional radiation hydrodynamics, *J. Math. Fluid Mech.* **9**(4), 543.

Index