Sabrine Arfaoui, Imen Rezgui, Anouar Ben Mabrouk

Wavelet Analysis on the Sphere

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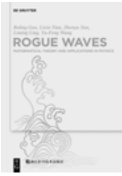

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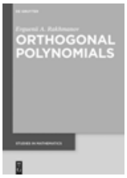

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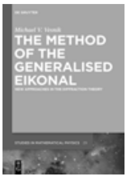

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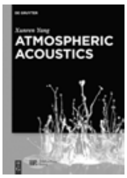

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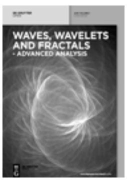

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Sabrine Arfaoui, Imen Rezgui,
Anouar Ben Mabrouk

Wavelet Analysis on the Sphere

Spheroidal Wavelets

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Authors

Dr. Anouar Ben Mabrouk
Computational Mathematics Laboratory
Department of Mathematics
Faculty of Sciences
University of Monastir
Avenue of Kairouan
5000 Monastir
Tunisia
&
Department of Mathematics
Higher Institute of Applied Mathematics and Informatics
Kairouan University
Avenue Assad Ibn Al-Fourat
3100 Kairouan
Tunisia
Email: anouar.benmabrouk@fsm.rnu.tn

Dr. Sabrine Arfaoui
Computational Mathematics Laboratory
Department of Mathematics
Faculty of Sciences
University of Monastir
Avenue of Kairouan
5000 Monastir
Tunisia
Email: sabrine.arfaoui@issatm.rnu.tn

Dr. Imen Rezgui
Computational Mathematics Laboratory
Department of Mathematics
Faculty of Sciences
University of Monastir
Avenue of Kairouan
5000 Monastir
Tunisia
Email: amounaa.rezgui@gmail.com

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Preface

The present volume comprises topics about the theory of harmonic analysis and its applications. We hope that it is a worthy addition to the references on orthogonal polynomials, homogenous polynomials, and wavelet spheroidal analysis.

All scientific work involves an armada of people. First, we wish to thank our professor, Samir Ben Ammou, professor of mathematics in the Department of Mathematics in the Faculty of Sciences at Monastir University, Tunisia, who supported this work by accepting us as members of his laboratory, The Computational Mathematics Laboratory He also provided funding for participation in scientific conferences where discussions with specialists occurred and thus improved the work.

We would also like to thank Professor Slaiem Ben Farah, the dean of the Faculty of Sciences at Monastir University, for his unforgotten help with providing funding for us to participate in scientific conferences, especially the *International Colloquium on Harmonic Analysis, Probability Theory and Their Interactions* held in Hammamet, Tunisia, December 15–19, 2014, the *International Colloquium on Random Matrices and Orthogonal Polynomials* held in Hammamet, Tunisia, March 22–27, 2015, and the *Fourth Tunisian–Japanese Conference on Geometric and Harmonic Analysis on Homogenous Spaces and Applications* held at Monastir, Tunisia, 18–23 December, 2015.

We also wish to thank the staff at De Gruyter, especially the executive board members, Dr. Anke Beck, the editorial director for mathematics, physics, and engineering, Dr. Konrad Kieling, and also the project editor, Dr. Astrid Seifert. We thank all of them for their hospitality, co-operation, collaboration, and for the time they spent on our project.

The present book stems from lectures and papers on the topics developed and are gathered, re-developed, improved, and sometimes completed when there are missing developments. However, naturally, it is not exhaustive and may be critiqued, corrected, and improved by readers. So, we welcome comments and suggestions.

We also want to stress the fact that originally we planned to present more applications, especially applications of spheroidal wavelets. We regret their absence, which was due to the time constraints of the editor. We hope that what is presented in this volume will allow readers to become acquainted with the topics presented.

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1 Introduction

The present work provides the scientific community with a unified collection of developments in applied mathematical problems and as mathematical theory. First, we wish to acknowledge that this book is the result of the work of two doctoral students, Sabrine Arfaoui and Imen Rezgui, who were supervised by the third author until October 2014. These studies took place at the Computational Mathematics Laboratory UR11ES51 headed by Professor Samir Ben Ammou, a recent student of École Nationale des Ponts et Chaussées de Paris. The work focused on a special class of wavelets and their applications. It is proposed to develop special wavelet bases that are related to special functions in one part and adapted to spherical geometry in another.

Since their appearance, and especially over the last decades, wavelets have proved to be powerful bases for many domains, such as numerical analysis, signal/image processing, physics, biomaths, medicine, and data analysis. Their power stems from the fact that they do not require a large number of coefficients to accurately represent general functions and large data sets. This allows compression and efficient computations. Wavelets also offer both theoretical characterization of smoothness, insights into the structure of functions and operators, and practical numerical tools that lead to faster computational algorithms. Classical constructions have been limited to simple domains, such as intervals, cubes, Cartesian representations, tensor products, etc. So, one main challenge may be the construction of wavelets on general domains as they appear in graphics applications. In the present context, we aim to present wavelet constructions for functions defined on the sphere. We aim to show that using special functions, such as orthogonal polynomials, homogenous polynomials, and Bessel functions and their relatives, can be sources for well-adapted wavelets. Readers will notice that the constructed schemes lead to extremely easily implemented bases and allow fully adaptive algorithms.

In [14], a polynomial wavelet-type system adapted to the sphere is presented in order to expand continuous functions into wavelet series on the sphere. The method is characterized by an optimum order of growth of the degrees of polynomials. However, and as the authors themselves have already noticed and declared, the wavelet-type system presented is not suitable for implementations as no explicit formulas for coefficient functionals have been provided and the fact remains that the growth of the degrees of polynomials is too rapid.

In [142], a simple technique for constructing biorthogonal wavelets on the sphere with customized properties is developed. The construction is an incidence of a fairly general scheme compared to [152] and [153]. The authors mentioned an important task about wavelets on the sphere showing that efficient wavelet algorithms have practical applications since many computational problems are naturally stated on the sphere.

The first notion developed in this book is orthogonal polynomials. These are well-known because of their link to many mathematical, physical, engineering and com-

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puter sciences topics, such as scattering theory, automatic control, signal analysis, potential theory, approximation theory, and numerical analysis. Orthogonal polynomials are special as they are orthogonal with respect to some special weights allowing them to satisfy some properties that are not fulfilled with other polynomials. Such properties have made them useful candidates to resolve enormous problems in physics, probability, statistics, and other fields. In the present work, we aim to review orthogonal polynomials by recalling the original definitions, reproduce their properties, and develop some cases related to the most well-known method to reproduce some classes of them.

Next, as a natural extension of orthogonal polynomials, we present a review of homogenous polynomials and their interactions with harmonic analysis on the sphere. Specifically, we study the constructions of the spherical harmonics and develop the main results of the theory of harmonic analysis on the sphere, such as the addition theorem and the Fourier transformation. The link with some special features, such as ultra-spherical polynomials and Bessel functions are also reviewed.

As in all research studies where the track is unpredictable, this work uses many notions. As mentioned, we are exploring special wavelets. These are naturally related to special functions. That is why we immediately plunged into the context of special functions, that is, some particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications. A detailed study of the most well-known types of these functions has been conducted. We detail the definitions, properties, and characterizations of Bessel, Hankel, and zonal functions. Proofs have been developed, sometimes in detail, relative to the base references and sometimes originally developed in the case of a lack of references. Graphic illustrations and some examples of applications are sometimes mentioned, such as differential equations, integro-differential equations, and time series.

Special functions are indispensable in many topics ranging from pure mathematics to applied fields. Thus, it is important to study their properties. Although many properties and characteristics of such functions appear in many mathematical documents, there is no unified treatment of the topic. With this book, we are filling this hole in the literature.

The last topic is spherical wavelets, which may be considered as a class of special functions. We made use of zonal, spherical harmonics, homogenous, as well as orthogonal polynomials. Recall that the spherical harmonics form the basis of the Hilbert space $L^2(S^n)$, where S^n is the unit sphere of R^n, $n \in \mathbb{N}$. Harmonic analysis on the sphere is the natural extension of Fourier series, which studies the expressibility of functions and generalized functions as sums of the fundamental exponential functions. The exponential functions are simpler functions, and are both eigenfunctions of the translation-invariant differential operator and group homomorphisms. Here also, spherical harmonics are simple and eigenfunctions of some differential operators.

2 Review of orthogonal polynomials

2.1 Introduction

Developments and interests in orthogonal polynomials have seen continuous and great progress since their appearance. Orthogonal polynomials are connected with many mathematical, physical, engineering, and computer sciences topics, such as trigonometry, hypergeometric series, special and elliptic functions, continued fractions, interpolation, quantum mechanics, partial differential equations. They are also be found in scattering theory, automatic control, signal analysis, potential theory, approximation theory, and numerical analysis.

Orthogonal polynomials are special polynomials that are orthogonal with respect to some special weights allowing them to satisfy some properties that are not generally fulfilled with other polynomials or functions. Such properties have made them well-known candidates to resolve enormous problems in physics, probability, statistics and other fields.

Since their origin in the early 19th century, orthogonal polynomials have formed a somehow classical topic related to Legendre polynomials, Stieltjes' continued fractions, and the work of Gauss, Jacobi, and Christoffel, which has been generalized by Chebyshev, Heine, Szegö, Markov, and others. The most popular orthogonal polynomials are Jacobi, Laguerre, Hermite polynomials, and their special relatives, such as Gegenbauer, Chebyshev, and Legendre polynomials. An extending family has been developed from the work of Wilson, inducing a special set of orthogonal polynomials known by his name, which generalizes the Jacobi class. This new family has given rise to other previously unknown sets of orthogonal polynomials, including Meixner Pollaczek, Hahn, and Askey polynomials.

Orthogonal polynomials may also be classified according to the measure applied to define the orthogonality. In this context, we cite the class of discrete orthogonal polynomials that form a special case based on some discrete measure. The most common are Racah polynomials, Hahn polynomials, and their dual class, which in turn include Meixner, Krawtchouk, and Charlier polynomials.

Already with the classification of orthogonal polynomials, one can distinguish circular and generally spherical orthogonal polynomials, which consists of some special sets related to measures supported by the circle or the sphere. One well-known class is composed of Rogers–Szegö polynomials on the unit circle and Zernike polynomials, which are related to the unit disk.

Orthogonal polynomials, and especially classical ones, can generally be introduced by three principal methods. A first method is based on the Rodrigues formula which consists of introducing orthogonal polynomials as outputs of a derivation. The second method consists of introducing orthogonal polynomials as eigenvectors

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of Sturm–Liouville operators, or equivalently, solutions of second-order differential equations. The last method is based on a three-level recurrence formula.

In this chapter, we aim to review orthogonal polynomials by recalling the original definitions, reproduce their properties, and develop some cases related to the most known method to reproduce some classes of them.

2.2 Generalities

This section reviews basic definitions as well as properties of orthogonal polynomials. To do this, we first restrict ourselves to the field $\mathbb{R}$, and when it is necessary we recall that the development remains valid on the complex field $\mathbb{C}$.

Definition 1. A Hilbert space is a vector space equipped with a scalar product, which makes it a complete space relative to the scalar product induced norm.

Definition 2. A polynomial P of degree n on $\mathbb{R}$ is formally defined by the expression

$$P(X) = \sum_{k=0}^{n} a_k X^k ,$$

where X is the variable and a_ks, $0 \le k \le n$, are elements of $\mathbb{R}$ called scalars and known as the polynomial coefficient such that $a_n \neq 0$.

Remark 3. The polynomial function associated with the polynomial P, which will also be denoted by P, is the function defined on the whole space $\mathbb{R}$ by $P(x) = \sum_{k=0}^{n} a_k x^k$. We denote by $\mathbb{R}[X]$ the set of all polynomials on $\mathbb{R}$. Of course, it is well known that $\mathbb{R}[X]$ is a vector space on $\mathbb{R}$ with infinite dimension and that for any $n \in \mathbb{N}$, the set $\mathbb{R}_n[X]$ of polynomials on $\mathbb{R}$ with degree at most n is a vector space with dimension $n + 1$ on $\mathbb{R}$.

Definition 4. A set of polynomials $\mathcal{B} = (P_0, P_1, \dots, P_n, \dots)$ in $\mathbb{R}[X]$ is said to be staggered with the degrees iff $\deg(P_i) = i$, $\forall i$.

The following result shows one important property of staggered degrees polynomials confirming the ability of such polynomials to be good candidates for polynomial spaces bases.

Proposition 5. *Any finite set $\mathcal{B} = (P_0, P_1, \ldots, P_n)$ of staggered degrees polynomials in $\mathbb{R}_n[X]$ is linearly independent.*

Proof. Let $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ be scalars in $\mathbb{R}$ such that $\sum_{i=0}^{n} \alpha_i P_i = 0$. This means that for all $x \in \mathbb{R}$, $\sum_{i=0}^{n} \alpha_i P_i(x) = 0$. By considering the nth-order derivative on x, we obtain $\alpha_n \frac{d^n P_n}{dx^n} = 0$. Consequently, $\alpha_n = 0$. Next, proceeding by induction on n, we prove that all the coefficients α_i are null. Hence, $\mathcal{B}$ is a free set in E. Observe next that the dimension of E ($\dim E = n + 1$) coincides with the cardinality of $\mathcal{B}$. Therefore, $\mathcal{B}$ is a basis of E. □

Theorem 6 (GRAM–SCHMIDT). *Let $\{f_n\}_{n\geq 0}$ be a countable system of linearly independent elements in a prehilbertian space. Then, there exists an orthonormal system $\{g_n\}_{n\geq 0}$ such that for any n, Vect$\{g_0, g_1, \ldots, g_n\}$ = Vect$\{f_0, f_1, \ldots, f_n\}$.*

Proof. We proceed by induction to construct the system $\{g_n\}_{n\geq 0}$. Let $g_0 = f_0$. Then element g_1 will be defined by

$$g_1 = f_1 - \alpha g_0 .$$

As we want g_0 and g_1 to be orthogonal, we obtain

$$\langle g_0, g_1 \rangle = \langle g_0, f_1 \rangle - \alpha \langle g_0, g_0 \rangle = 0 .$$

So that, $\alpha = \frac{\langle g_0, f_1 \rangle}{\langle g_0, g_0 \rangle}$. Otherwise, we subtract from f_1 its orthogonal projection on g_0, i.e.,

$$g_1 = f_1 - \frac{\langle f_1, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0 .$$

Hence, clearly we have Vect$\{_0, g_1\}$ = Vect$\{f_0, f_1\}$.
Next, g_2 is defined analogously by subtracting from f_2 its orthogonal projections on (g_0, g_1). In other words,

$$g_2 = f_2 - \frac{\langle f_2, g_1 \rangle}{\langle g_1, g_1 \rangle} g_1 - \frac{\langle f_2, g_0 \rangle}{\langle g_0, g_0 \rangle} g_0 .$$

It is straightforward that g_2 is orthogonal to g_0 and g_1. Assume next that g_n is well known. g_{n+1} will be obtained as follows:

$$g_{n+1} = f_{n+1} - \sum_{i=0}^{n} \frac{\langle f_{n+1}, g_i \rangle}{\langle g_i, g_i \rangle} g_i .$$

We check easily that for all $k \leq n$,

$$\begin{aligned}\langle g_{n+1}, g_k\rangle &= \langle f_{n+1}, g_k\rangle - \sum_{i=1}^{n} \frac{\langle f_{n+1}, g_i\rangle}{\langle g_i, g_i\rangle}\langle g_i, g_k\rangle \\ &= \langle f_{n+1}, g_k\rangle - \frac{\langle f_{n+1}, g_k\rangle}{\langle g_k, g_k\rangle}\langle g_k, g_k\rangle = 0 .\end{aligned}$$

Obviously, the elements g_n are not normalized. To do this, we divide each one by its norm. The equality $\mathrm{Vect}\{g_0, g_1, \dots, g_n\} = \mathrm{Vect}\{f_0, f_1, \dots, f_n\}$ is straightforward. □

Definition 7. Let I be an interval in $\mathbb{R}$ nonreduced to a point and let ω be a positive continuous function on I. ω is said to be a *weight function* iff

$$\int_I |x|^d \omega(x)\, \mathrm{d}x < \infty, \qquad \forall d \in \mathbb{N} .$$

We denote by the next $\mathcal{C}_\omega(I)$ the vector space of continuous functions on the interval I, satisfying

$$\int_I |f(x)|^2 \omega(x)\, \mathrm{d}x < \infty . \tag{2.1}$$

It results from hypothesis 7 that the polynomials are elements of $\mathcal{C}_\omega(I)$. On this space of functions, a *scalar product* can be defined by

$$\langle f, g\rangle = \int_I f(x)g(x)\omega(x)\, \mathrm{d}x . \tag{2.2}$$

The integration interval I will be called the *orthogonality interval*.

Definition 8. A set of polynomials $(P_i)_{i\geq 0}$ is said to be orthogonal iff it satisfies
(1) $\mathrm{Degree}(P_i) = i$; $\forall i \in \mathbb{N}$.
(2) $\langle P_i, P_j\rangle = 0$; $\forall (i, j) \in \mathbb{N}^2$; $i \neq j$.

The following result shows some generic properties of orthogonal polynomials, as they are special cases of staggered degree polynomials and consequently they also form good candidates for polynomial spaces orthogonal bases.

Proposition 9. *Let $(P_i)_{i\geq 0}$ be a set of orthogonal polynomials. Then*
(1) $\forall n \in \mathbb{N}$; *$(P_0, P_1, \dots, P_n)$ is an orthogonal basis of* $\mathbb{R}_n[X]$.
(2) $\forall (n, p) \in \mathbb{N}^2$; $n \geq p + 1 \Longrightarrow P_n \in (\mathbb{R}_p[X])^\perp$.

Proof. The first assertion is a consequence of Proposition 5 and the orthogonality of the set $(P_0, P_1, \dots, P_n)$. (We can also use the second point in Definition 8 to prove the independence of the P_js, $j = 0, \dots, n$). Next, as $\mathbb{R}_p[X]$ is generated by the set $(P_0, P_1, \dots, P_p)$ and $n \geq p+1$, which means that $P_n \perp P_j$, for all $j = 0, \dots, p$, so it is orthogonal to $\mathbb{R}_p[X]$. □

Remark 10. Sometimes we need to use unitary orthogonal polynomials P_n. Thus, we need to multiply them by constants so that $\lambda_n P_n$ becomes unitary or not. So, in the following, we will not differentiate between the two notions and will use the notation $(P_n)_n$ and $\lambda_n P_n$ depending on the context.

Properties 11. The unitary orthogonal polynomials satisfy the following assertions:
(1) $P_0(x) = 1$.
(2) $\text{Degree}(P_n) = n, \forall n \in \mathbb{N}$.
(3) $\int_I P_n(x)Q(x)w(x)\,\mathrm{d}x = 0, \forall Q \in \mathbb{R}[X]$ such that $\text{Degree}(Q) < n$.
(4) $\mathbb{R}_n(X) = \text{Vect}(P_0, \dots, P_n), \forall n \in \mathbb{N}$.

Proof. (1) P_0 is a unitary constant polynomial. So, it is equal to 1.
(2) It follows from the first assertion in Definition 8.
(3) As $\text{Degree}(Q) > n$ so $Q \in \mathbb{R}_n[X]^{\perp}$. Thus, assertion (3).
(4) Holds from Proposition 9. □

Lemma 12. *Let $(P_0, \dots, P_n)$ be a unitary orthogonal polynomial set. Hence,*
(1) *$(P_0, \dots, P_n)$ is a basis of $\mathbb{R}_n[X]$.*
(2) *P_n is orthogonal to $\mathbb{R}_{n-1}[X]$.*

Indeed, Firstly, we know that $\dim \mathbb{R}_n[X] = n+1 = \text{card}(P_0, \dots, P_n)$. On the other hand, $(P_0, \dots, P_n)$ is orthogonal; hence, it is linearly independent. Thus, it consists of a basis in $\mathbb{R}_n[X]$.
The second point follows from the fact that P_n is orthogonal to $(P_0, \dots, P_{n-1})$, which means that it is orthogonal to $\mathbb{R}_{n-1}(X) = \text{Vect}(P_0, \dots, P_{n-1})$.

2.3 Orthogonal polynomials via a three-level recurrence

Theorem 13 (Recurrence rule). *Let $(P_i)_{i\geq 0}$ be a set of orthogonal polynomials. There exist scalars $(a_n)_n$, $(b_n)_n$, and $(c_n)_n$ such that*

$$P_{n+1} = (a_n X + b_n)P_n + c_n P_{n-1}; \quad \forall n \in \mathbb{N}^* .$$

More precisely,

$$a_n = \frac{k_{n+1}}{k_n}, \quad b_n = -a_n \frac{\langle XP_n, P_n\rangle}{\|P_n\|^2} \quad \text{and} \quad c_n = -\frac{a_n}{a_{n-1}} \frac{\langle P_n, P_n\rangle}{\langle P_{n-1}, P_{n-1}\rangle} ,$$

where k_n is the coefficient of X^n in $P_n(X)$.

Proof. Without loss of generality, we can assume that $(P_i)_{i\geq 0}$ is orthonormal. Let $\mathcal{B} = (XP_n, P_n, P_{n-1}, \ldots, P_0)$ be a set of staggered degree polynomials in $\mathbb{R}_{n+1}[X]$. So, it is linearly independent in $\mathbb{R}_{n+1}[X]$. Consequently, it forms a basis of $\mathbb{R}_{n+1}[X]$. Consequently, there exist then scalars a_n, b_n, c_n and α_i, $0 \leq i \leq n-2$ such that

$$P_{n+1} = a_n XP_n + b_n P_n + c_n P_{n-1} + \sum_{i=0}^{n-2} \alpha_i P_i .$$

Next, using the orthogonality property of $(P_i)_{i\geq 0}$, we obtain

$$\langle P_{n+1}, P_i\rangle = a_n \langle XP_n, P_i\rangle + \alpha_i \|P_i\|^2 = 0, \quad \forall 0 \leq i \leq n-2 .$$

On the other hand,

$$\langle XP_n, P_i\rangle = \langle P_n, XP_i\rangle .$$

Since $XP_i \in \mathbb{R}_{n-1}[X]$, we obtain

$$\langle XP_n, P_i\rangle = 0 .$$

Consequently,

$$\alpha_i = 0, \quad \forall 0 \leq i \leq n-2 .$$

Hence,

$$P_{n+1} = (a_n X + b_n)P_n + c_n P_{n-1} . \tag{2.3}$$

We now evaluate the coefficients a_n, b_n, and c_n. Recall that P_n can be written as

$$P_n(X) = k_n X^n + k_{n-1} X^{n-1} + \cdots + k_0 .$$

By identification of the higher degree monomials in (2.3), we obtain

$$a_n = \frac{k_{n+1}}{k_n} .$$

Next, the inner product of (2.3) with P_n gives

$$\langle P_{n+1}, P_n \rangle = a_n \langle XP_n, P_n \rangle + b_n \langle P_n, P_n \rangle + c_n \langle P_{n-1}, P_n \rangle \,.$$

Using the orthogonality of the set, we get

$$a_n \langle XP_n, P_n \rangle + b_n \langle P_n, P_n \rangle = 0 \,.$$

Hence,

$$b_n = -a_n \frac{\langle XP_n, P_n \rangle}{\langle P_n, P_n \rangle} \,.$$

Next, using the inner product with P_{n-1} and using again the orthogonality of the set, we obtain

$$a_n \langle XP_n, P_{n-1} \rangle + c_n \langle P_{n-1}, P_{n-1} \rangle = 0 \,.$$

Hence,

$$c_n = -a_n \frac{\langle XP_n, P_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle} = -a_n \frac{\langle P_n, XP_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle} \,.$$

Next, denote $XP_{n-1} = \sum_{i=0}^{n} \alpha_i P_i$ as the decomposition of XP_{n-1} in the basis of polynomials $(P_i)_{0 \le i \le n}$. By observing the higher degree monomials in the decomposition, we get

$$X k_{n-1} X^{n-1} = \alpha_n k_n X^n \qquad \Longleftrightarrow \qquad \alpha_n = \frac{k_{n-1}}{k_n} = \frac{1}{a_{n-1}} \,.$$

On the other hand,

$$\begin{aligned} \langle XP_n, P_{n-1} \rangle &= \langle P_n, XP_{n-1} \rangle \\ &= \alpha_n \langle P_n, P_n \rangle + \sum_{i=0}^{n-1} \alpha_i \langle P_n, P_i \rangle \\ &= \alpha_n \langle P_n, P_n \rangle + \sum_{i=0}^{n-1} \alpha_i 0 \\ &= \alpha_n \langle P_n, P_n \rangle . \end{aligned}$$

Consequently,

$$c_n = -a_n \frac{\langle P_n, XP_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle} = -a_n \frac{\alpha_n \langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle} = -\frac{a_n}{a_{n-1}} \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle} \,.$$

Hence,

$$P_{n+1} = a_n X P_n + b_n P_n + c_n P_{n-1} \,,$$

where

$$a_n = \frac{k_{n+1}}{k_n}, \quad b_n = -a_n \frac{\langle XP_n, P_n \rangle}{\langle P_n, P_n \rangle} \quad \text{and} \quad c_n = -\frac{a_n}{a_{n-1}} \frac{\langle P_n, P_n \rangle}{\langle P_{n-1}, P_{n-1} \rangle} \,.$$

In the case where $(P_0, \ldots, P_n)$ is orthonormal, we obtain

$$a_n = \frac{k_{n+1}}{k_n}, \quad b_n = -a_n \langle XP_n, P_n \rangle \quad \text{and} \quad c_n = -\frac{a_n}{a_{n-1}} \,.$$ □

Favard presented the converse of Theorem 13, which states that under suitable conditions, a sequence of polynomials satisfying the three-level equation stated there can be orthogonal relative to a suitable weight function.

Theorem 14 (Favard's theorem). *Let* $\{c_n\}_{n=0}^{\infty}$ *and* $\{\lambda_n\}_{n=0}^{\infty}$ *be sequences in* $\mathbb{R}$, *and* $\{P_n\}_{n=0}^{\infty}$, *a set of polynomials satisfying*

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad \forall n \in \mathbb{N}^* ,$$

where $P_0(x) = 1$ *and* $P_1(x) = x - c_1$. *Then, there exists a unique linear form* φ *on* $\mathbb{R}_n(X)$ *for which* $\varphi(P_k P_m) = 0$ *whenever* $k \neq m$.

Proof. We proceed by steps.
Step 1. We claim that Degree$(P_n) = n$, $\forall n \in \mathbb{N}$. Indeed, for $n = 0$, $P_0(x) = 1$. Hence, Degree$(P_0) = 0$. For $n = 1$, $P_1(x) = x - c$, so it is of degree 1. Assume next that Degree$(P_n) = n$ and prove the same for P_{n+1}. The three-level relation above yields that

$$\text{Degree}(P_{n+1}) = \text{Degree}\big((x - c_{n+1})P_n(x)\big) = 1 + n .$$

Hence, we proved by recurrence on n that Degree$(P_n) = n$, $\forall n \in \mathbb{N}$.
Step 2. Consider the space $\mathbb{R}_n[X]$ of polynomials on $\mathbb{R}$ with degrees at most n. It results from Step 1 that the set $\mathcal{B}_n = (P_0, \dots, P_n)$, satisfying that the three-level relation is a degree-straggled set of polynomials. Henceforth, it is a basis of $\mathbb{R}_n[X]$. Let $\varphi\colon \mathbb{R}_n[X] \longrightarrow \mathbb{R}$ be the continuous linear form defined on such a basis by

$$\varphi(P_0) = 1,\ \varphi(P_1) = \cdots = \varphi(P_n) = 0 .$$

It holds from the Riez–Fréchet theorem that there exists a function ω such that

$$\varphi(P) = \langle P, \omega \rangle = \int_{\mathbb{R}} P(x)\omega(x)dx .$$

We now prove that

$$\varphi(P_k P_m) = 0, \quad \forall 0 \le k \neq m \le n .$$

For $k < m$, denote $P_k(x) = \sum_{s=0}^{k} \alpha_s(k)x^s$. We get

$$\varphi(P_k P_m) = \sum_{s=0}^{k} \alpha_s(k)\varphi(x^s P_m) .$$

On the other hand, $x^s P_m$ can be written as

$$x^s P_m = \sum_{i=m-s}^{m+s} d_i P_i .$$

Hence,

$$\varphi(x^s P_m) = \sum_{i=m-s}^{m+s} d_i \varphi(P_i) = 0 .$$

□

2.4 Darboux–Christoffel rule

Recall that an element P in a vector space equipped with an orthonormal basis $(\tilde{P_0}\dots,\tilde{P_n})$ can be written as

$$P = \sum_{k=0}^{n} \langle \tilde{P_k}, P\rangle \tilde{P_k} \ .$$

In the case of $\mathcal{C}_\omega(I)$, this means that

$$\begin{aligned} P(x) &= \sum_{k=0}^{n} \left(\int_I \tilde{P_k}(y)P(y)\omega(y)dy \right) \tilde{P_k}(x) \\ &= \int_I \left(\sum_{k=0}^{n} \tilde{P_k}(y)\tilde{P_k}(x) \right) P(y)\omega(y)dy \\ &= \int_I K_n(x,y)P(y)\omega(y)dy \\ &= \langle K_n(x,y), P\rangle \ , \end{aligned}$$

where we denoted

$$K_n(x,y) = \sum_{k=0}^{n} \tilde{P_k}(y)\tilde{P_k}(x) = \sum_{k=0}^{n} \frac{P_k(y)P_k(x)}{\|P_k\|^2} \ .$$

Definition 15. $K_n(x,y)$ is called the *Darboux–Christoffel kernel.*

We now state the famous Darboux–Christoffel theorem, which characterizes orthogonal polynomial sequences [98].

Theorem 16. *Let $\{P_n\}_{n\geq 0}$ be a set of orthogonal polynomials. Then the following assertions hold:*
(1) $K_n(x,y) = \frac{k_n}{k_{n+1}h_n}\frac{P_{n+1}(x)P_n(y)-P_n(x)P_{n+1}(y)}{x-y}$*;* $x \neq y$.
(2) $K_n(x,x) = \frac{k_n}{k_{n+1}h_n}(P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x))$.
where $h_n = \|P_n\|^2$.

Proof. Without loss of generality, we can assume that the system $\{P_n\}_{n\geq 0}$ is orthonormal. So that $h_n = 1$ for all n. For the first point, we proceed by induction on n. Let $x \neq y$. When $n = 0$, the left-hand side term becomes

$$K_0(x,y) = P_0(x)P_0(y) = 1 \ .$$

The right-hand side term becomes

$$\frac{k_0}{k_1}\frac{P_1(x)P_0(y)-P_0(x)P_1(y)}{x-y}=\frac{k_0}{k_1}\frac{P_1(x)-P_1(y)}{x-y}=\frac{k_0}{k_1}\frac{k_1x-k_1y}{x-y}=k_0=1\,.$$

Hence for $n=0$, assume next that the property is true for $n-1$. This means that

$$K_{n-1}(x,y)=\frac{k_{n-1}}{k_n}\frac{P_n(x)P_{n-1}(y)-P_{n-1}(x)P_n(y)}{x-y}\,, \tag{2.4}$$

and we check the validity for n. To do this, recall that the three-level induction rule in Theorem 13 implies that

$$P_{n+1}=(a_nX+b_n)P_n+c_nP_{n-1}\,.$$

Thus,

$$\begin{aligned}P_{n+1}(x)P_n(y)-P_n(x)P_{n+1}(y)&=[(a_nx+b_n)P_n(x)+c_nP_{n-1}(x)]\,P_n(y)\\&\quad-P_n(x)\,[(a_ny+b_n)P_n(y)+c_nP_{n-1}(y)]\\&=a_n(x-y)P_n(x)P_n(y)\\&\quad+c_n\,[P_{n-1}(x)P_n(y)-P_n(x)P_{n-1}(y)]\,.\end{aligned}$$

Using the induction hypothesis (2.4), we obtain

$$\begin{aligned}c_n\,[P_{n-1}(x)P_n(y)-P_n(x)P_{n-1}(y)]&=-c_n\frac{k_n}{k_{n-1}}(x-y)K_{n-1}(x,y)\\&=\frac{a_n}{a_{n-1}}\frac{k_n}{k_{n-1}}(x-y)K_{n-1}(x,y)\\&=a_n(x-y)K_{n-1}(x,y)\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{k_n}{k_{n+1}}\frac{P_{n+1}(x)P_n(y)-P_n(x)P_{n+1}(y)}{(x-y)}&=\frac{k_n}{k_{n+1}}\frac{a_n(x-y)P_n(x)P_n(y)}{(x-y)}\\&\quad+\frac{k_n}{k_{n+1}}\frac{a_n(x-y)K_{n-1}(x,y)}{(x-y)}\\&=P_n(x)P_n(y)+K_{n-1}(x,y)\\&=K_n(x,y)\,.\end{aligned}$$

The next assertion is obtained from 1 by letting $y\to x$. □

Example 2.1. We set here some examples of induction relations for the most known orthogonal polynomials.

(1) *Legendre polynomials*

$$P_{n+1} = \frac{2n+1}{n+1} X P_n - \frac{n}{n+1} P_{n-1}, \quad \forall n \in \mathbb{N}^* .$$

(2) *Chebyshev polynomials*

$$P_{n+1} = 2XP_n - P_{n-1}, \quad \forall n \in \mathbb{N}^* .$$

(3) *Hermite polynomials*

$$P_{n+1} = 2XP_n - 2nP_{n-1}, \quad \forall n \in \mathbb{N}^* .$$

Proposition 17 (Existence of real zeros). *Let $(P_n)_{n\geq 0}$ be a set of orthogonal polynomials. Then, for all $n \geq 0$, P_n has n distinct real zeros in the integration interval.*

Proof. Let $a_1, \ldots, a_m$ be the real zeros of P_n in the orthogonality interval, each one used just one time. It is straightforward that P_n is sign changing on the orthogonality interval. $P_n(X)$ can be written in the form

$$P_n(X) = \prod_{1\leq i\leq n} (X - a_i) Q(X) ,$$

where Q is a nonsign changing polynomial on the orthogonality interval. We shall prove that $m = n$. For this, let $S(X) = \prod_{1\leq i\leq m}(X - a_i)$. It consists of an m'' degree polynomial that is sign changing at each point a_i, $1 \leq i \leq m$. $S(X)P_n(X)$ is then not sign changing on the orthogonality interval and hence for $S(X)P_n(X)\omega(X)$, where ω is a weight. Thus, $\langle P_n, S\rangle \neq 0$. On the other hand, Lemma 12 yields that P_n is orthogonal to all polynomials with lower degrees. Hence $\langle P_n, S\rangle = 0$ which is a contradiction. So, it results that $m = n$, and thus, P_n has n zeros in the orthogonality interval which are simple. □

2.5 Continued fractions

In this section, we emphasize the relation between continued fractions and orthogonal polynomials.

Definition 18. A continued fraction is a formal expression

$$F = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\cdots + \frac{a_n}{b_n + \cdots}}}$$

with either finite or infinite stages.

Remark 19.
(a) When $\forall x \in \mathbb{R}$, there exists a continued fraction representing it, with $a_0 \in \mathbb{Z}$ and $a_j \in \mathbb{N}$, $\forall j \geq 0$, and $b_j = 1$, $\forall j \geq 0$.
(b) Some functions can also be presented with continued fractions.

Notations. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be in $\mathbb{R}$. We will apply the following notation for the continued fraction F:

$$F = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} + \cdots . \tag{2.5}$$

We also denote $F_n = \frac{R_n}{S_n}$, $n \in \mathbb{N}$, to designate the fraction obtained by the truncation of F of the order n:

$$F_n = \frac{R_n}{S_n} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n} .$$

The following results have been proved in [98].

Proposition 20. *Let $(R_n)_n$ and $(S_n)_n$ be defined by*

$$R_0 = b_0, \quad S_0 = 1 ,$$

$$R_1 = b_0 b_1 + a_1, \quad S_1 = b_1 ,$$

$$R_n = b_n R_{n-1} + a_n R_{n-2}, \quad S_n = b_n S_{n-1} + a_n S_{n-2} \quad \textit{for} n \geq 2 .$$

Then,

$$F_n = \frac{R_n}{S_n} .$$

Proof. Define the function

$$f_n(x) = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_n|}{|b_n + x} ,$$

and remark firstly that

$$f_n(x) = f_{n-1}\left(\frac{a_n}{b_n + x}\right) .$$

We claim that

$$f_n(x) = \frac{R_{n-1}x + R_n}{S_{n-1}x + S_n} .$$

The proposition is obtained by setting $x = 0$. We now proceed by recurrence. For $n = 1$, we have

$$f_1(x) = b_0 + \frac{a_1}{b_1 + x} = \frac{b_0 x + (b_0 b_1 + a_1)}{1.x + b_1} = \frac{R_0 x + R_1}{S_0 x + S_1} .$$

So, the proposition is valid for $n = 1$. Assume next that it remains valid for $n \leq k$. We check it for $n = k + 1$.

$$\begin{aligned} f_{k+1}(x) &= f_k\left(\frac{a_{k+1}}{b_{k+1} + x}\right) \\ &= \frac{R_{k-1}(\frac{a_{k+1}}{b_{k+1}+x}) + R_k}{S_{k-1}(\frac{a_{k+1}}{b_{k+1}+x}) + S_k} \\ &= \frac{a_{k+1}R_{k-1} + b_{k+1}R_k + R_k x}{a_{k+1}S_{k-1} + b_{k+1}S_k + S_k x} \\ &= \frac{a_{k+1}R_{k-1} + R_k(b_{k+1} + x)}{a_{k+1}S_{k-1} + S_k(b_{k+1} + x)} . \end{aligned}$$

Setting $x = 0$, we get

$$f_{k+1}(0) = \frac{R_{k+1}}{S_{k+1}} = \frac{a_{k+1}R_{k-1} + R_k b_{k+1}}{a_{k+1}S_{k-1} + S_k b_{k+1}} . \qquad \square$$

Definition 21. R_n and S_n are called, respectively, the partial nth numerator and the partial nth denominator of F_n.

Corollary 22. *It holds that*

$$R_n S_{n-1} - R_{n-1} S_n = (-1)^{n+1} a_1, \ldots, a_n, \quad \forall n \geq 1 .$$

Proof. By induction on n. For $n = 1$, we have

$$R_1 S_0 - S_1 R_0 = (b_0 b_1 + a_1)1 - b_1(b_0) = a_1 .$$

So, assume that the result is valid for n, i.e.,

$$R_k S_{k-1} - R_{k-1} S_k = (-1)^{k+1} a_1, \ldots, a_k, \quad \forall 1 \leq k \leq n .$$

Then,

$$\begin{aligned} R_{n+1}S_n - S_{n+1}R_n &= S_n(b_{n+1}R_n + a_{n+1}R_{n-1}) - R_n(b_{n+1}S_n + a_{n+1}S_{n-1}) \\ &= -a_{n+1}(-R_{n-1}S_n + R_nS_{n-1}). \end{aligned}$$

The induction hypothesis yields that

$$R_{n+1}S_n - S_{n+1}R_n = -a_{n+1}\left[(-1)^{n+1}a_1, \dots, a_n\right] = (-1)^{n+2}a_1, \dots, a_{n+1}\,. \qquad \square$$

Recall that in Proposition 20, we have

$$R_n = b_nR_{n-1} + a_nR_{n-2} \quad \text{and} \quad S_n = b_nS_{n-1} + a_nS_{n-2} \quad .$$

By setting

$$b_0 = 0, \quad b_n = x - c_n, \quad a_1 = \lambda_1 \neq 0, \quad a_{n+1} = -\lambda_{n+1} \neq 0 \quad \forall n \geq 2\,,$$

we obtain $R_n = (x-c_n)R_{n-1} - \lambda_nR_{n-2}$. And thus, we obtain the orthogonal polynomials recurrence formula

$$P_{n+1}(x) = (x - c_{n+1})P_n(x) - \lambda_{n+1}P_{n-1}(x)\,.$$

2.6 Orthogonal polynomials via Rodrigues rule

A literature review of orthogonal polynomials reveals that there are many methods to obtain such polynomials. One is explicit and based on the Rodrigues rule, which applies derivation. Let

$$P_n(x) = \frac{1}{k_n\omega(x)}\frac{d^n}{dx^n}\left[\omega(x)S^n\right]\,,$$

where S is a polynomial in x, ω is a weight function, and k_n is a constant. We have precisely the following result.

Theorem 23 ([98]). *Let $I = [a, b[\subset \mathbb{R}$ and ω is a weight function on I and $(\phi_n)_{n\in\mathbb{N}}$ be a set of real functions on I satisfying*
(1) *ϕ_n is C^n on $]a, b[$ for all n.*
(2) *$\phi_n^{(k)}(a^+) = \phi_n^{(k)}(b^-) = 0$ for all k, $0 \leq k \leq n-1$.*
(3) *$T_n = \frac{1}{k_n\omega}(\omega\phi_n)^{(n)}$ is a polynomial of degree n, (k_n is a normalization constant).*
Then, $(T_n)_{n\in\mathbb{N}}$ is orthogonal. The converse is true iff ω is C^∞.

Proof. It suffices to prove the orthogonality. For $n < m$, we have

$$\begin{aligned}
\langle T_n, T_m\rangle &= \int_a^b T_n(x)T_m(x)\omega(x)dx \\
&= \int_a^b T_n(x)\frac{1}{k_m\omega(x)}(\omega\phi_m)^{(m)}\omega(x)dx \\
&= \int_a^b T_n(x)\frac{1}{k_m}(\omega\phi_m)^{(m)}dx \\
&= (-1)^m\int_a^b \Big(T_n(x)\Big)^{(m)}\frac{\omega\phi_m}{k_m}dx \\
&= 0\,.
\end{aligned}$$

The fourth equality is a consequence of Hypothesis (2) and the integration by the parts rule. The last equality is a consequence of Hypothesis (3). □

2.7 Orthogonal polynomials via differential equations

A large class of orthogonal polynomials is obtained from first-order linear differential equations of the type

$$a(x)y'' + b(x)y' - \lambda_n y = 0\,, \tag{2.6}$$

where a is a polynomial of degree 2, and b is a polynomial of degree 1, where both are independent of the integer parameter n, and finally, λ_n are scalars. y is the unknown function. By introducing the operator $T\colon \mathbb{R}[X] \longrightarrow \mathbb{R}[X]$ such that $T(y) = ay'' + by'$, the solution y appears as an eigenvector of T associated with the eigenvalue λ_n. We introduce next a resolvent function $\omega > 0$, which permits us to express the operator T on the form $T(y) = \frac{1}{w}(awy')'$. The equality $T(y) = ay'' + a'y' + \frac{aw'}{w}y'$ shows that ω is a solution of the differential equation $a\omega' + (a' - b)\omega = 0$. So, it is of the form $\omega = e^A$, where A is a primitive of $\frac{b-a'}{a}$. Recall now that

$$\langle T(f), g\rangle = \int_I (a\omega f')'(x)g(x)\omega(x)dx = \big[a\omega f'g\big]_I - \int_I a(x)f'(x)g'(x)\omega(x)dx\,.$$

Iff the weight ω vanishes on the frontier of the integration interval I, we obtain

$$\langle T(f), g\rangle = -\int_I a(x)f'(x)g'(x)\omega(x)dx = \langle f, T(g)\rangle\,.$$

This means that the operator T is symmetric.

Denote for the next $T_n\colon \mathbb{R}_n[X] \longrightarrow \mathbb{R}_n[X]$ the restriction of T on $\mathbb{R}_n[X]$. It is straightforward that $\mathbb{R}_n[X]$ is invariant under the action of T_n since the degrees of a

and b are less than 2 and 1, respectively. So, we can arrange the pairs (λ, y) into a sequence (λ_k, y_k), where we re-obtain the eigenpairs of the operator T_n for $k = 0, \dots, n$. Next, observing that $a(x) = a_2X^2 + a_1X + a_0$ and $b(x) = b_1X + b_0$, it results that T_n is an endomorphism on $\mathbb{R}_n[X]$. Thus, there exists a T-eigenvector's orthonormal basis of such a space. In particular, there exists at least an eigenvector P_n of degree n, which may be assumed to be unitary and satisfying

$$aP_n'' + bP_n' = \lambda_n P_n \,.$$

This means that for $n \neq m$, we obtain $\lambda_n \neq \lambda_m$ and thus the polynomials P_n are orthogonal.

2.8 Some classical orthogonal polynomials

In the previous sections, we reviewed the three most well-known schemes to obtain orthogonal polynomials. The first one is based on the explicit Rodrigues derivation rule, which states that the nth element of the set of orthogonal polynomials, which is also of degree n, is obtained by

$$P_n(x) = \frac{1}{k_n\omega(x)}\frac{d^n}{dx^n}\left[\omega(x)S^n\right] \,,$$

where S is a suitable polynomial in x.

The next method is based on an induction rule as in (2.3) and eventually necessitates that the first and the second elements of the desired set of orthogonal polynomials be known. It states that

$$P_{n+1} = (a_nX + B_n)P_n + c_nP_{n-1} \,, \tag{2.7}$$

where a_n, b_n, and c_n are known scalars.

Finally, the last scheme consists of introducing orthogonal polynomials as the solutions of ordinary differential equations (ODEs) of the form

$$a(x)y'' + b(x)y - \lambda_n y = 0 \,,$$

where a is a 2-degree polynomial and b is a polynomial with degree 1 and λ_n are scalars. The idea consists of developing polynomial solutions of the ODEs. According to the coefficients of each equation, we obtain the desired class of polynomials, such as Legendre and Laguerre.

In this section, we propose to revisit some classical classes of orthogonal polynomials and show their construction with the three schemes.

2.8.1 Legendre polynomials

From Rodrigues rule

Legendre polynomials consist of polynomials defined on the orthogonality interval $I =]-1,1[$ relative to the weight function $\omega \equiv 1$, the polynomial $S(x) = (x^2-1)$, and the constant $k_n = 2^n n!$. The nth Legendre polynomial, usually denoted in the literature by L_n, is obtained by

$$L_n(x) = \frac{d^n}{dx^n}\left[\frac{(x^2-1)^n}{2^n n!}\right].$$

Using the Leibniz rule of derivation, $L_n(x)$ can be explicitly computed. We have

$$L_n(x) = \frac{1}{2^n n!}\sum_{k=0}^{n} C_n^k \left((x-1)^n\right)^{(k)}\left((x+1)^n\right)^{(n-k)} = \frac{1}{2^n}\sum_{k=0}^{n}\left(C_n^k\right)^2 (x-1)^{n-k}(x+1)^k .$$

For example,

$$L_0(x) = 1, \qquad L_1(x) = x\,,$$
$$L_2(x) = \frac{1}{2}(3x^2-1), \qquad L_3(x) = \frac{1}{2}(5x^3-3x)\,,$$
$$L_4(x) = \frac{1}{8}(35x^4-30x^2+3), \qquad L_5(x) = \frac{1}{8}(63x^5-70x^3+15x)\,.$$

From the induction rule

Legendre polynomials can also be introduced via the induction rule

$$L_{n+1} = \frac{2n+1}{n+1}XL_n - \frac{n}{n+1}L_{n-1}, \qquad \forall n \in \mathbb{N}^*$$

with initial data $L_0(x) = 1$ and $L_1(x) = x$. It yields, for $n = 1$, that

$$L_2(x) = \frac{3}{2}xL_1(x) - \frac{1}{2}L_0(x) = \frac{3}{2}x^2 - \frac{1}{2}\,.$$

For $n = 2$, it yields that

$$L_3(x) = \frac{5}{3}xL_2(x) - \frac{2}{3}L_1(x) = \frac{5}{3}x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{2}{3}x = \frac{5}{2}x^3 - \frac{3}{2}x\,.$$

Applying the same procedure, we obtain

$$L_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \quad \text{and} \quad L_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)\,.$$

From ODEs

Legendre polynomials are obtained as the polynomial solutions of the following ODE:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,\ x \in I =]-1,1[\,. \tag{2.8}$$

Using the notations of Section 2.7, this means that $a(x) = 1 - x^2$, $b(x) = -2x$ and $\lambda_n = -n(n+1)$. In the sense of the linear operator T, the polynomials L_n can be introduced via the operator

$$T(y) = (1 - x^2)y'' - 2xy' = ((1 - x^2)y')' ,$$

which corresponds to the weight function $\omega(x) = 1$ and $a(x)\omega(x) = 1 - x^2$. Note that $a\omega$ vanishes at the frontiers ± 1 of the orthogonality interval I. Furthermore, in terms of eigenvalues as in equation (2.7), if we suppose that the same eigenvalue λ_n is associated with at least two eigenvectors P_n and P_m, we obtain $(n - m)(n + m - 1) = 0$, which has no integer solutions except $n = m$. This confirms that the eigenvalues and eigenvectors are one to one, which means that the eigenvectors (polynomials) are orthogonal. Figure 2.1 illustrates the graphs of the first Legendre polynomials.

For clarity and convenience, we will develop the polynomial solutions. So, denote $P(x) = a_p x^p + a_{p-1}x^{p-1} + \cdots + a_1 x + a_0$ as a polynomial solution of degree p of equation (2.8). We obtain the following system:

$$\begin{cases} 2a_2 + n(n+1)a_0 = 0, \\ 6a_3 + (n^2 + n - 2)a_1 = 0, \\ [(n(n+1) - (p-1)(p+1)]a_{p-1} = 0, \\ [(n(n+1) - p(p+1)]a_p = 0, \\ (k+1)(k+2)a_{k+2}[(n(n+1) - k(k+1)]a_k = 0, \quad 2 \le k \le p-2. \end{cases}$$

Hence, $p = n$ and

$$\begin{cases} 2a_2 + n(n+1)a_0 = 0, \\ 6a_3 + (n^2 + n - 2)a_1 = 0, \\ [(n(n+1) - n(n-1)]a_{p-1} = 0, \\ (k+1)(k+2)a_{k+2}[(n(n+1) - k(k+1)]a_k = 0, \quad 2 \le k \le p-2. \end{cases}$$

For example, for $n = 0$, we obtain

$$P(x) = a_0 .$$

For $n = 1$, we get

$$P(x) = a_1 x .$$

For $n = 2$, we obtain

$$P(x) = -a_0(3x^2 - 1) .$$

For $n = 3$,

$$P(x) = -a_1\left(\frac{5}{3}x^3 - x\right) .$$

For $n = 4$, we have

$$P(x) = -a_0\left(\frac{35}{3}x^4 - 10x^2 + 1\right) .$$

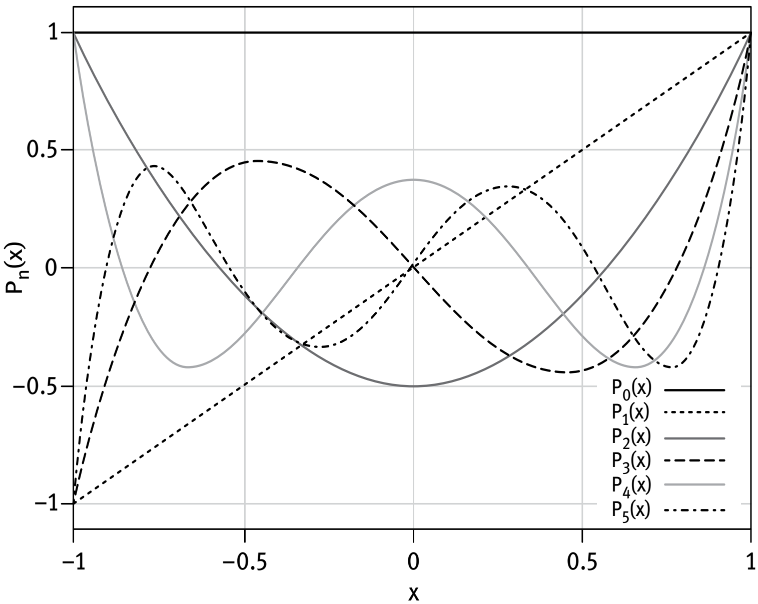

Fig. 2.1: Legendre polynomials.

Next, for $n = 5$, we obtain

$$P(x) = a_1 \left(\frac{21}{5} x^5 - \frac{14}{3} x^3 + x \right) .$$

Now, using the orthogonality of these polynomials on $[-1, 1]$, we obtain the same polynomials.

2.8.1.1 Commentaries

One important question is how to choose the polynomial S in Rodrigues rule to be equivalent with the same outputs of the recurrence rule and the ODE scheme.

Firstly, the degree of S is fixed in an obvious way as $\deg P_n = n, \forall n$.

Hence, for example, in the Legendre case, S should be of degree 2, that is,

$$S(x) = a + bx + cx^2, \quad c \neq 0 .$$

Thus,

$$L_n(x) = e_n \frac{d^n}{dx^n} (S^n(x)), \quad e_n = \frac{1}{2^n n!} .$$

Consequently, from the induction rule of Legendre polynomials we obtain, for $n = 2$,

$$L_2 = \frac{3}{2} x L_1 - \frac{1}{2} L_0$$

$$e_2 (S^2(x))'' = \frac{3}{2} e_1 x (S'(x)) - \frac{1}{2} e_0 .$$

As a result,

$$\begin{cases} b^2 + 2ac + 2 = 0 \\ 6bc - 3b = 0 \\ 6c^2 - 6c = 0 \,. \end{cases}$$

Hence, we obtain

$$c = 1, \quad b = 0, \quad a = -1 \,.$$

Or equivalently,

$$S(x) = x^2 - 1 \,.$$

2.8.2 Laguerre polynomials

From the Rodrigues rule

These polynomials are obtained via the Rodrigues rule with $\omega(x) = e^{-x}$, $S(x) = x$ and the constant $k_n = n!$ by

$$\mathcal{L}_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n) = \sum_{k=0}^{n} \frac{C_n^k}{k!}(-x)^k \,.$$

The first polynomials are then

$$\mathcal{L}_0(x) = 1, \qquad \mathcal{L}_1(x) = 1 - x \,,$$

$$\mathcal{L}_2(x) = \frac{1}{2}(x^2 - 4x + 2), \qquad \mathcal{L}_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6) \,,$$

$$\mathcal{L}_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24) \,,$$

$$\mathcal{L}_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120) \,.$$

It holds clearly from simple calculus that these polynomials are orthogonal in the interval $[0, \infty[$ relative to the weight function $\omega(x) = e^{-x}$.

From the induction rule

Laguerre polynomials are solutions of the following recurrent relation:

$$(n+1)\mathcal{L}_{n+1}(x) + (x - 2n - 1)\mathcal{L}_n(x) + n\mathcal{L}_{n-1}(x) = 0$$

with the first and second elements $\mathcal{L}_0(x) = 1$ and $\mathcal{L}_1(x) = 1 - x$. For $n = 1$, we get

$$2\mathcal{L}_2(x) + (x - 3)\mathcal{L}_1(x) + \mathcal{L}_0(x) = 0,$$

which implies that

$$\mathcal{L}_2(x) = \frac{1}{2}(3 - x)(1 - x) + \frac{1}{2} = \frac{1}{2}(x^2 - 4x + 2) \,.$$

Next, for $n = 2$, we obtain

$$3\mathcal{L}_3(x) + (x-5)\mathcal{L}_2(x) + 2\mathcal{L}_1(x) = 0,$$

which means that

$$\mathcal{L}_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)\,.$$

Similarly, we can obtain

$$\mathcal{L}_4(x) = \frac{1}{24}(-x^4 - 16x^3 + 72x^2 - 96x + 24)$$

and

$$\mathcal{L}_5(x) = \frac{1}{120}(-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)\,.$$

From ODEs

To apply the ODE procedure, we set $I =]0, \infty[$ as the orthogonality interval, $a(x) = x$, $b(x) = 1 - x$, $\omega(x) = e^{-x}$ and consequently, the operator T will be $T(y) = xy'' - (1 - x)y'$. We observe immediately that $a(x)\omega(x) = xe^{-x}$ is null at 0 and has the limit 0 at $+\infty$. Furthermore, $P(x)\omega(x)$ is integrable on I for all polynomial P. In the present case, equation (2.7) becomes $(n+m)(n-m) = 0$ and the eigenvalues are $\lambda_n = -n$. The associated ODE is

$$xy'' - (1-x)y' + ny = 0\,.$$

It is straightforward that for all n, the polynomial $\mathcal{L}_n$ is a solution of this differential equation. Figure 2.2 illustrates the graphs of some examples of Laguerre polynomials.

2.8.3 Hermite polynomials

From Rodrigues rule

Hermite polynomials are related to the orthogonality interval $I = \mathbb{R}$ with the weight function $\omega(x) = e^{-x^2}$. Denote H_n as the nth element, i.e., a Hermite polynomial of degree n. H_n is explicitly expressed via Rodrigues rule as follows:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)\,.$$

As examples, we get

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2\,,$$
$$H_3(x) = 8x^3 - 12x, \quad \text{and} \quad H_4(x) = 16x^4 - 48x^2 + 12\,.$$

From the induction rule

Hermite polynomials H_n can be obtained by means of the induction rule

$$H_{n+1} = 2XH_n - 2nH_{n-1}, \quad \forall n \in \mathbb{N}^*\,,$$

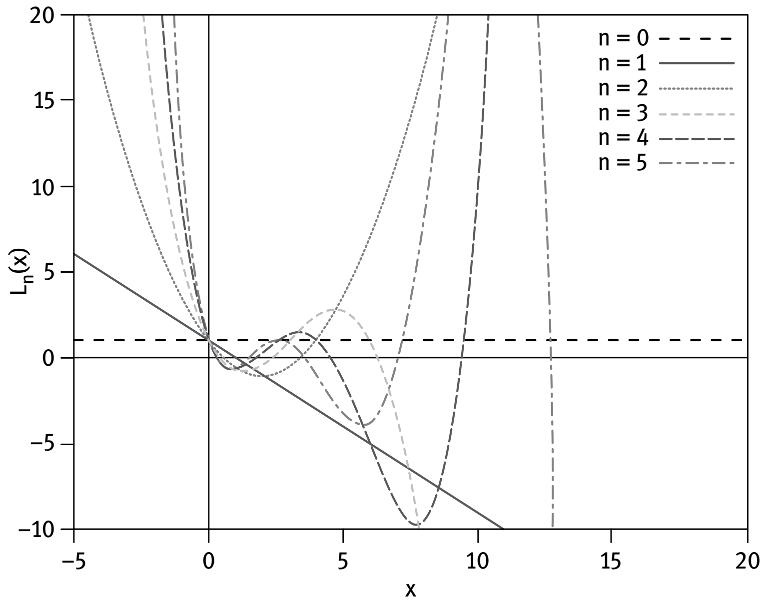

Fig. 2.2: Laguerre polynomials.

with the initial data $H_0(X) = 1$ and $H_1(X) = 2X$. So, for $n = 1, 2, 3, 4, 5$ we obtain as examples

$$H_2(X) = 4X^2 - 2, \quad H_3(X) = 8X^3 - 12X, \quad H_4(X) = 16X^4 - 48X^2 + 12\,,$$
$$H_5(X) = X^5 - 10X^3 + 15X, \quad H_6(X) = X^6 - 15X^4 + 45X^2 - 15\,.$$

From ODEs

Hermite polynomials are also solutions of a second-order ODE in the interval $I = \mathbb{R}$. Using the notations of Section 2.7 this means that $a(x) = 1$, $b(x) = -2x$, and $\omega(x) = e^{-x^2}$ as a weight function. It is immediate that $a(x)\omega(x) = e^{-x^2}$, which has 0 limits at the boundaries of the interval I. Furthermore, $P(x)\omega(x)$ is integrable on I for all polynomials P. By means of the eigenvalues of the linear operator T, Hermite polynomials are eigenvectors of $T(y) = y'' - 2xy'$ associated with eigenvalues $\lambda_n = -2n$. The corresponding ODE is

$$y'' - 2xy' + 2ny = 0\,.$$

Some examples of Hermite polynomials are illustrated in figure 2.3.

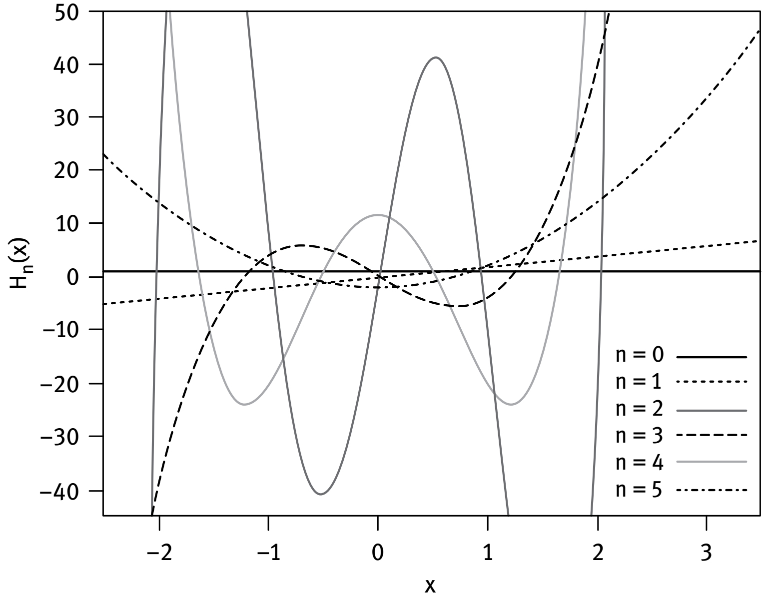

Fig. 2.3: Hermite polynomials.

2.8.4 Chebyshev polynomials

From Rodrigues rule

Chebyshev polynomials are related to the orthogonality interval $I =]-1, 1[$ and the weight function $\omega(x) = (1-x^2)^{-1/2}$. Denoted usually by T_n for the Chebyshev polynomial of degree n, these are explicitly expressed via the Rodrigues rule as

$$T_n(x) = \frac{(-1)^n (1-x^2)^{\frac{1}{2}} \sqrt{\pi}}{2^n \Gamma(n+\frac{1}{2})} \frac{d^n}{dx^n} \left((1-x^2)^{n-\frac{1}{2}} \right) ,$$

where $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ is Euler's well-known function. It is immediately seen (by recurrence for example) that

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!}, \quad \forall n \in \mathbb{N} ,$$

and hence, the first Chebyshev polynomials can be obtained as

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1, \quad T_5(x) = 16x^5 - 20x^3 + 5x .$$

From the induction rule

Chebyshev polynomials are solutions of the induction formula

$$T_{n+1} = 2xT_n - T_{n-1}, \quad \forall n \in \mathbb{N}^* ,$$

with initial data, $T_0(x) = 1$ and $T_1(x) = x$. Let, $T_n(x) = \sum_{k=0}^{n} a_k^n x^k$, then

$$\begin{aligned}
\sum_{k=0}^{n+1} a_k^{n+1} x^k &= 2x \sum_{k=0}^{n} a_k^n x^k - \sum_{k=0}^{n-1} a_k^{n-1} x^k \\
\sum_{k=0}^{n+1} a_k^{n+1} x^k + a_{n+1}^{n+1} x^{n+1} + a_0^{n+1} &= \sum_{k=0}^{n} 2a_k^n x^{k+1} - \sum_{k=0}^{n-1} a_k^{n-1} x^k \\
&= \sum_{k=1}^{n+1} 2a_{k-1}^n x^k - \sum_{k=0}^{n-1} a_k^{n-1} x^k
\end{aligned}$$

$$\sum_{k=1}^{n-1} \left(a_k^{n+1} - 2a_{k-1}^n + a_k^{n-1}\right) x^k + a_0^{n+1} + a_0^{n-1} + \left(a_n^{n+1} - 2a_{n-1}^n\right) x^n + \left(a_{n+1}^{n+1} - 2a_n^n\right) x^{n+1} = 0 .$$

We obtain the following system:

$$\begin{cases} a_0^{n+1} + a_0^{n-1} = 0 \\ a_n^{n+1} = 2a_{n-1}^n . \\ a_{n+1}^{n+1} = 2a_n^n \\ a_k^{n+1} = 2a_{k-1}^n - a_k^{n-1}, \quad 1 \le k \le n-1 . \end{cases}$$

We have

$$T_0(x) = 1 \Longleftrightarrow a_0^0 = 1$$

and

$$T_1(x) = x \Longleftrightarrow a_0^1 = 0, a_1^1 = 1 .$$

Hence,

$$T_2(x) = a_2^2 x^2 + a_1^2 x + a_0^2$$

From the above system, we obtain

$$\begin{cases} a_0^2 = -a_0^0 = -1 . \\ a_1^2 = 2a_0^1 = 0 . \\ a_2^2 = 2a_1^1 - a_2^0 = 2 , \end{cases}$$

which means that

$$T_2(x) = 2x^2 - 1 .$$

Now, replacing n by 2 in the system we obtain

$$\begin{cases} a_0^3 = -a_0^1 = 0 . \\ a_2^3 = 2a_1^2 = 0 . \\ a_1^3 = 2a_0^2 - a_1^1 = -3 . \\ a_3^3 = 2a_2^2 - a_3^1 = 4 . \end{cases}$$

Therefore,

$$T_3(x) = 4x^3 - 3x\ .$$

Next, for $n = 3$, the system becomes

$$\begin{cases} a_0^4 = -a_0^2 = 1\ . \\ a_3^4 = 2a_2^3 = 0\ . \\ a_1^4 = 2a_0^3 - a_1^2 = 0\ . \\ a_2^4 = 2a_1^3 - a_2^2 = -8\ . \\ a_4^4 = 2a_3^3 - a_4^2 = 8\ . \end{cases}$$

Thus,

$$T_4(x) = 8x^4 - 8x^2 + 1\ .$$

Now, by replacing n with 4, the system yields

$$\begin{cases} a_0^5 = -a_0^3 = 0\ . \\ a_4^5 = 2a_3^4 = 0\ . \\ a_1^5 = 2a_0^4 - a_1^3 = 5\ . \\ a_2^5 = 2a_1^4 - a_2^3 = 0\ . \\ a_3^5 = 2a_2^4 - a_3^3 = -20\ . \\ a_5^5 = 2a_4^4 - a_5^3 = 16\ . \end{cases}$$

Hence,

$$T_5(x) = 16x^5 - 20x^3 + 5x\ .$$

So, we obtain the same Techebythev polynomials as for the Rodrigues and ODE rules.

From ODEs

We set $I =]-1, 1[$, $a(x) = 1 - x^2$, $b(x) = -x$ and $\omega(x) = (1 - x^2)^{-1/2}$. The linear operator T is then given by

$$T(y) = (1 - x^2)y'' - xy'\ .$$

It is straightforward that $a(x)\omega(x) = \sqrt{1 - x^2}$ vanishes at ± 1 and the eigenvalues $\lambda_n = -n^2$ give rise to eigenvectors (polynomials), T_ns. This yields that the T_ns are the corresponding solutions of the ODE

$$(1 - x^2)y'' - xy' + n^2 y = 0\ .$$

Remark 24. Chebyshev polynomials T_n can be explicitly defined on $[-1,1]$ by

$$T_n(x) = \cos(n\mathrm{Arc}\cos(x))\ .$$

Indeed, by considering Moivre's rule $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, and by setting for $\theta \in [0,\pi]$, $x = \cos\theta$, we obtain $\sin\theta\sqrt{1-x^2}$. This implies that

$$\cos(n\theta) = \cos(n\arccos(x)) = \sum_{m=0}^{[\frac{n}{2}]} C_n^{2m}(-1)^m x^{n-2m}(1-x^2)^m, \quad n \in \mathbb{N}.$$

Next, we observe that

$$\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta).$$

Henceforth, we obtain explicit T_ns as above. Figure 2.4 illustrates the graphs of the first Chebyshev polynomials.

Remark 25. It holds that a second kind of Chebyshev polynomial already exists. It is defined by means of the Rodrigues rule as

$$U_n(x) = \frac{(-1)^n(n+1)\sqrt{\pi}}{2^{n+1}\Gamma(n+\frac{3}{2})(1-x^2)^{\frac{1}{2}}}\frac{d^n}{dx^n}\left((1-x^2)^{n+\frac{1}{2}}\right), \quad x \in [-1,1]$$

or by means of trigonometric functions as

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \forall n \in \mathbb{N}^*.$$

These polynomials satisfy the same induction rule as the previous but with different initial data $U_0(x) = 1$ and $U_1(x) = 2x$. Finally, similar to other classes of orthogonal polynomials, they satisfy the ODE

$$\forall x \in \mathbb{R}, \quad (1-x^2)U_n''(x) - 3xU_n'(x) + n(n+2)U_n(x) = 0.$$

2.8.5 Gegenbauer polynomials

From Rodrigues rule

Gegenbauer polynomials, also called ultraspherical polynomials, are defined relative to the weight function $\omega(x) = (1-x^2)^{p-1/2}$, where p is a real parameter, and to the orthogonality interval $I =]-1,1[$. From the Rodrigues rule, these are defined as

$$G_m^p(x) = \frac{(-1)^m\Gamma(p+\frac{1}{2})\Gamma(n+2p)}{2^m m!\Gamma(2p)\Gamma(p+m+\frac{1}{2})}(1-x^2)^{\frac{1}{2}-p}\frac{d^m}{dx^m}\left((1-x^2)^{p+m-\frac{1}{2}}\right). \tag{2.9}$$

Hence, by applying the Leibniz derivation rule, we obtain

$$G_m^p(x) = C_m^p\left[x^m - a_{m-2}x^{m-2} + a_{m-4}x^{m-4} + \cdots\right]$$

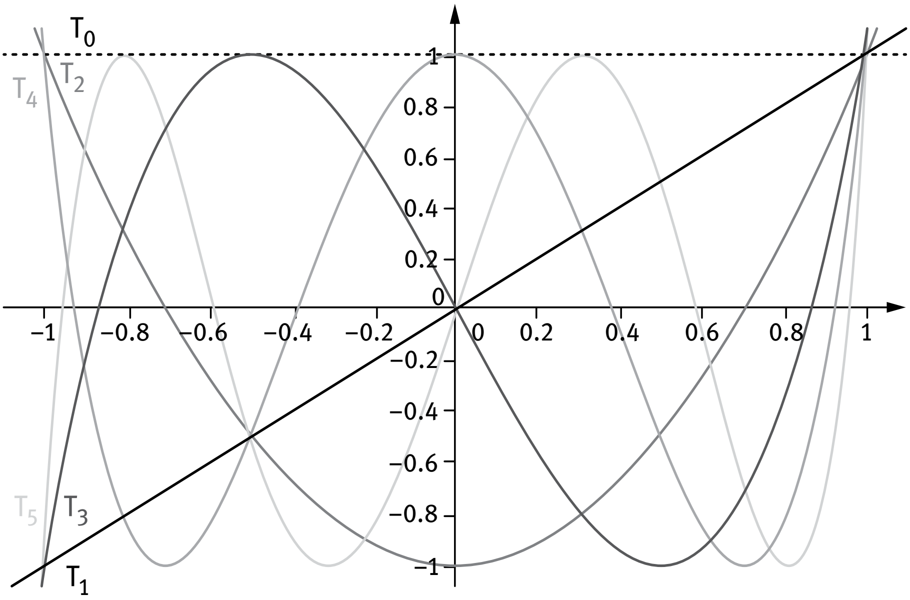

Fig. 2.4: Chebyshev polynomials.

where

$$C_m^p = \frac{2^m \Gamma(p+m)}{m!\Gamma(p)} ,$$

$$a_{m-2} = \frac{m(m-1)}{2^2(p+m-1)}, \quad a_{m-4} = \frac{m(m-1)(m-2)(m-3)}{2^4(p+m-1)(p+m-2)}, \dots .$$

From the induction rule

Gegenbauer polynomials G_m^p can also be introduced via the induction rule stated for $p \geq \frac{-1}{2}$ by

$$mG_m^p(x) = 2x(m+p-1)G_{m-1}^p(x) - (m+2p-2)G_{m-2}^p(x) , \tag{2.10}$$

already with

$$G_0^p(x) = 1 \qquad \text{and} \qquad G_1^p(x) = 2p(1-x) .$$

This gives, for example,

$$G_2^p(x) = 2p(p+1)\left[x^2 - \frac{1}{2p+2}\right]$$

and

$$G_3^p(x) = \frac{4}{3}p(p+1)(p+2)\left[x^3 - \frac{3}{2p+4}x\right] .$$

Furthermore, we notice that G_m^p is composed of monomials having the same parity of the index m.

Remark 26. The following assertions hold:
- $G_m^p(-x) = (-1)^m G_m^p(x)$.
- $G_{2m+1}^p(0) = 0$.
- $G_{2m}^p(0) = \frac{(-1)^m \Gamma(p+m)}{\Gamma(p)\Gamma(m+1)}$.

From ODEs

Gegenbauer polynomials G_m^p are solutions of the ODE

$$(1-x^2)y'' - (2p+1)xy' + m(m+2p)y = 0\,,$$

in the interval $I =]-1, 1[$ with coefficients $a(x) = 1 - x^2$, $b(x) = -(2p+1)x$ and $c(x) = m(m+2p)$. These polynomials can be introduced as the eigenvectors of the linear Sturm–Liouville-type operator T defined by

$$T(y) = (1-x^2)y'' - (2p+1)xy'\,.$$

By choosing $\omega(x) = (1-x^2)^{p-\frac{1}{2}}$, we observe that $a(x)\omega(x) = (1-x^2)^{p+\frac{1}{2}}$ vanishes at the boundary points ± 1. The eigenvalues are $\lambda_m = -m(m+2p)$.

2.9 Conclusion

In this chapter, we outlined the concepts and the main properties and characteristics of orthogonal polynomials. Some basic notions concerning orthogonal polynomials are recalled that are related to weight functions, integration theory, linear algebra theory of vector spaces, their basis, and orthogonal systems and their relation to orthogonal polynomials. Next, the three main methods for introducing orthogonal polynomials were reviewed. The first method uses Rodrigues formula and it yields orthogonal polynomials as outputs of a higher order derivatives of some special functions. The second is based on recurrence relations, which yield orthogonal polynomials as sequences of functions defined by a three-level induction rule. We recalled and redeveloped Favard's results on orthogonal polynomials as well as its reciprocals. The last method consists of orthogonal polynomials as solutions to ordinary differential equations or equivalently as eigenfunctions of Sturm–Liouville operators. Some concluding and illustrating examples are provided to enlighten theoretical developments.

3 Homogenous polynomials and spherical harmonics

3.1 Introduction

In this chapter, we present a review of homogenous polynomials and their interactions with harmonic analysis on the sphere. Specifically, we will study constructions of spherical harmonics and develop the main results of the theory of harmonic analysis on the sphere, such as the addition theorem and the Fourier transformation. We will prove the link with some special features, such as ultraspherical polynomials and Bessel functions.

Spherical harmonics are initially derived from the Laplace equation on the sphere. They are found in many scientific fields, starting with pure mathematics, where they appear as an extension of Fourier analysis of spherical domains. In physics, spherical harmonics have been used as basic solutions or modes of well-known equations, such as the Laplace, Poisson, Schrödinger, diffusion, and wave equations. Spherical harmonics also appear in acoustics, geophysics, computer graphics, crystallography, and recently in 3D image processing, where they are used to model complex phenomena and therefore provide models or approximations of the solutions to the equations governing them.

Because of their relationships and interactions with all these areas, spherical harmonics have been the subject of numerous classical and modern mathematical works. A first theory related to the basic constructions of spherical harmonics is the concept of homogenous polynomials. These polynomials are the combinations of monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $|\alpha| = \sum \alpha_i$ is a fixed integer that acts as a degree as for single-variable polynomials. This will partly be relevant to the development of homogenous polynomial theory.

As a first step, we will recall the exact solution of the Laplace equation in spherical coordinates in the three-dimensional Euclidean space as a concrete example of construction and proof of the existence of spherical harmonics. Next, homogenous polynomial theory will be developed. We will show especially that the homogenous and harmonic polynomials may serve as generator systems and bases in the Hilbert space of square integrable functions on the sphere. This marks the starting point for the basic construction of spherical harmonics. Such bases, when projected orthogonally with respect to some elements of the sphere, reproduces some kernels known as zonal harmonics, which will be revisited and explained. Recall that zonal functions are widely applied in harmonic analysis and approximation theory, where they have a central role.

The first section of this chapter will be devoted to the development of some differential operators on the sphere, mainly the Laplace operator. A spherical coordinates

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solution will be provided in Section 3 in order to show examples of spherical harmonics. Section 4 will cover homogenous polynomials. Basic properties will be revisited and applied next for harmonic polynomials in Section 5. Later in this section, harmonic homogenous polynomials will be proved to have a central role for providing orthonormal bases in $L^2(S^{n-1})$. Fourier transforms, and thus convolution operators, will be presented in Section 6. Hecke and Bochner–Hecke theorems will be proved to show the strong relation with special functions, especially Bessel functions. Finally, the theory of zonal functions will be developed in Section 7, where we revisit the famous addition theorem.

3.2 Spherical Laplace operator

In this section, we review some basic concepts of spherical analysis, such as differential operators and especially the Laplacian. Recall that the n-sphere is

$$S^{n-1} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n;\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}\,.$$

Recall also that the polar representation on $\mathbb{R}^n$ is defined by

$$x(r, \varphi, \theta_1, \dots, \theta_{n-2}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} r \sin\theta_{n-2} \sin\theta_{n-3} \dots \sin\theta_2 \sin\theta_1 \cos\varphi \\ r \sin\theta_{n-2} \sin\theta_{n-3} \dots \sin\theta_2 \sin\theta_1 \sin\varphi \\ r \sin\theta_{n-2} \sin\theta_{n-3} \dots \sin\theta_2 \cos\theta_1 \\ r \sin\theta_{n-2} \sin\theta_{n-3} \dots \cos\theta_2 \\ \vdots \\ r \sin\theta_{n-2} \cos\theta_{n-3} \\ r \cos\theta_{n-2} \end{pmatrix}$$

The parameter $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the Euclidian distance to the origin O, $\theta_i \in [0, \pi[$ and $\varphi \in [0, 2\pi[$.
For example, on the real space $\mathbb{R}^3$, we obtain the spherical coordinates' system

$$x(r, \varphi, \theta) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}$$

where $r > 0$ is always the Euclidian distance to O, $\theta \in [0, \pi]$ is the polar distance, and $\varphi \in [0, 2\pi[$ is the longitude. The orthonormal vector system is locally composed of

$$e^r = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}, \quad e^\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

and

$$e^\theta = e^r \wedge e^\varphi = \begin{pmatrix} -\cos\theta \cos\varphi \\ -\cos\theta \sin\varphi \\ \sin\theta \end{pmatrix}.$$

By setting $t = \cos\theta$, we obtain the explicit gradient on $\mathbb{R}^3$ in spherical coordinates

$$\nabla = e^r \frac{\partial}{\partial r} + \frac{1}{r}\left(e^\varphi \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} + e^t \sin\theta \frac{\partial}{\partial t}\right) = e^r \frac{\partial}{\partial r} + \frac{1}{r}\nabla^* \,. \tag{3.1}$$

∇^* is said to be the *surface gradient*. We can also define the rotational operator, also called the curl, for all $\mathcal{C}^1$-functions F as

$$L^*_\xi F(\xi) = \xi \wedge \nabla^*_\xi F(\xi) \,. \tag{3.2}$$

Already on $\mathbb{R}^3$, the curl is expressed by means of spherical coordinates as

$$L^* = -e^\varphi \sin\theta \frac{\partial}{\partial t} + e^t \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} \,. \tag{3.3}$$

Definition 27 (Laplace operator [162]). The Laplace operator, also called the Laplacian and denoted by ∇^2 or Δ, is the second-order differential operator defined explicitly on second-order differentiable functions by means of Cartesian coordinates as follows:

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \,.$$

It is related to many problems in both mathematics and physics and it appears in quasi-differential equations found in natural phenomena. We recall as an example the famous Dirichlet problem.

Definition 28 (Spherical Laplacian [162]). Let u be a C^2 function on $\mathbb{R}^n$ and denote $\tilde{f}(x) = f(\frac{x}{\|x\|})$. The spherical Laplacian known also as Laplace–Beltrami operator of u is defined by

$$\Delta_{S^{n-1}} f = (\Delta \tilde{f})_{/S^{n-1}} \,,$$

where Δ is the Laplace operator on $\mathbb{R}^n$.

Remark 29. $\Delta_{S^{n-1}}$ can be defined equivalently by means of the following relation:

$$\Delta u = \frac{1}{r^{n-1}}\left[\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) + \Delta_{S^{n-1}} u\right] \,. \tag{3.4}$$

Lemma 30. *For all f, that is C^2 on the sphere and all $\rho \in SO(n)$, it holds that*

$$\Delta_{S^{n-1}}(f \circ \rho) = (\Delta_{S^{n-1}} f) \circ \rho \,.$$

Let $M \in \mathcal{M}_n(\mathbb{R})$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and denote f_M the function defined on $\mathbb{R}^n$ by $f_M(x) = f(Mx)$. Then, for all $x \in \mathbb{R}^n$, we have

$$\Delta f_M(x) = \sum_{i,k=1}^{n} \langle L_i, L_k \rangle \frac{\partial^2 f}{\partial x_i \partial x_k}(Mx) \,,$$

where L_i are the rows of M and $\langle L_i, L_k \rangle$ is their natural inner product $\mathbb{R}^n$. Consequently, for $M = \rho$, this leads to $\langle L_i, L_k \rangle = \delta_{ik}$ the Kronecker product. Hence,

$$\Delta(f \circ \rho)(x) = (\Delta f) \circ \rho(x) \,.$$

For $x \in S^{n-1}$, this yields that

$$\Delta_{S^{n-1}}(f \circ \rho) = (\Delta_{S^{n-1}} f) \circ \rho \,.$$

The following lemma shows that $\Delta_{S^{n-1}}$ is also symmetric.

Lemma 31. *Let f and g be C^2 on the sphere S^{n-1}. Then,*

$$\langle \Delta_{S^{n-1}} f, g \rangle = \langle f, \Delta_{S^{n-1}} g \rangle \,.$$

The *proof* is a simple application of the Green–Ostrogradsky formula.

3.3 Some direct computations on S^2

The purpose of this section is to provide some examples of spherical harmonics by means of the resolution of the Laplace equation $\Delta P = 0$ on the sphere S^2 and an introduction to spherical harmonics. The general definition and general properties will be introduced later. In the spherical coordinate system, the Laplace equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial P}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 P}{\partial \varphi^2} = 0 \,. \tag{3.5}$$

For a solution P with separated variables, $P(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$, this yields that

$$\frac{1}{r^2}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = 0 \,.$$

Multiplying with r^2, this becomes

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}=0\,.$$

Or equivalently,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=-\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}=0\,.$$

The left-hand side part is independent of (θ,φ); however, the right-hand side part is independent of the variable r. Hence, these are constant, which means that

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=K \tag{3.6}$$

and

$$\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}=-K \tag{3.7}$$

for some constant K. The elementary solutions of (3.6) are of the form

$$R(r)=K(r^l+r^{-(l+1)})\,.$$

Equation (3.7) yields that

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+K\sin^2\theta=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}\,. \tag{3.8}$$

Using analogous arguments and by seeking Fourier modes solutions, we obtain

$$\Phi(\varphi)=Ke^{im\varphi}\,,$$

where m is an appropriate constant. Now, the first part of (3.8) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0\,. \tag{3.9}$$

Denoting $x=\cos\theta$, we obtain

$$\frac{\partial\Theta}{\partial\theta}=\frac{\partial\Theta}{\partial x}\frac{\partial x}{\partial\theta}=-\frac{\partial\Theta}{\partial x}\sin\theta\,.$$

Consequently, (3.9) becomes

$$(1-x^2)\frac{\partial^2\Theta}{\partial x^2}-2x\frac{\partial\Theta}{\partial x}+\left(l(l+1)-\frac{m^2}{1-x^2}\right)\Theta=0\,.$$

The polynomial solutions of such equation are the well-known Legendre polynomials. Such polynomials are defined for $l\in\mathbb{N}$ and $m\in[-l,l]\cap\mathbb{Z}$. Explicitly, these are expressed as

$$L_{l,m}(\cos\theta)=\frac{(-1)^m}{2^l l!}\sqrt{\frac{2(l-m)!}{(l+m)!}}\left(1-\cos^2\theta\right)^{\frac{m}{2}}\frac{\partial^{l+m}(\cos^2\theta-1)^l}{\partial(\cos\theta)^{l+m}}\,.$$

Hence, the Laplace equation solutions are finally the so-called *spherical harmonics* defined by

$$Y_{l,m}(r,\theta,\varphi) = \left(C_l^m r^l + D_l^m r^{-(l+1)}\right) L_l^m(\cos\theta) e^{im\varphi} \,.$$

On the sphere (r=constant), these functions become

$$Y_{l,m}(\theta,\varphi) = C_l^m L_l^m(\cos\theta) e^{im\varphi} \,,$$

where C_l^m is a normalization constant given by

$$C_l^m = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \,.$$

The system $(Y_{l,m})_{-l\leq m\leq l}$ is orthonormal in the vector space $L^2(S*n-1)$. Here, we list some explicit expressions of these spherical harmonics.

– $\underline{l=0}$:

$$Y_{0,0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

– $l=1$:

$$Y_{1,-1} = (\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\varphi}$$
$$Y_{1,0} = (\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$
$$Y_{1,1} = (\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\varphi}$$

– $l=2$:

$$Y_{2,-2} = (\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{-2i\varphi}$$
$$Y_{2,-1} = (\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{-i\varphi}$$
$$Y_{2,0} = (\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$$
$$Y_{2,1} = (\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\varphi}$$
$$Y_{2,2} = (\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\varphi}$$

– $l=3$:

$$Y_{3,0} = (\theta,\varphi) = \frac{1}{4}\sqrt{\frac{7}{\pi}}(5\cos^3\theta - 3\cos\theta) \,.$$

3.4 Homogenous polynomials

Definition 32. Let E and F be the $\mathbb{R}$-vector spaces and $f: E \to F$ be a function. f is said to be *homogenous with degree k* iff

$$\forall \lambda \in \mathbb{R}, \quad \forall x \in E, \qquad f(\lambda x) = \lambda^k f(x) .$$

We say that f is *positively homogenous with degree k* iff

$$\forall t \geq 0, \quad \forall x \in E, \qquad f(tx) = t^k f(x) .$$

Definition 33. A polynomial P on $\mathbb{R}^n$, $n \in \mathbb{N}$, is said to be *homogenous with degree* $k \in \mathbb{N}^*$, if it is of the form

$$P(x) = \sum_{|\alpha|=k} C_\alpha x^\alpha, \quad \forall x \in \mathbb{R}^n ,$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_*^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$, $x \in \mathbb{R}^n$ and C_α are real numbers called the coefficients of P.

Corollary 34. *The polynomial function associated with a homogenous polynomial P is homogenous with the same degree.*

The following result shows the first characterizations of the space of homogenous polynomials [162].

Theorem 35. *For all k, denote $\mathcal{P}_k(\mathbb{R}^n)$ the vector space of all homogenous polynomials with degree k on $\mathbb{R}^n$ and d_k^n its dimension. It holds that*
(1) $d_k^n = \dim \mathcal{P}_k(\mathbb{R}^n) = C_{n+k-1}^k$.
(2) $\mathcal{P}_k(\mathbb{R}^n)$ *is invariant by means of* $O(n)$.

To prove this theorem, we need the following preliminary result.

Lemma 36. *Let Γ_n^k be the number of possible choices of n elements (not necessarily different) from $\{0, 1, \dots, k\}$ with their sum equal to k. Then,*
(1) $\Gamma_n^1 = n$.
(2) $k\Gamma_n^k = (n + k - 1)\Gamma_n^{k-1}$.
(3) $\Gamma_n^k = C_{n+k-1}^k$.

Proof. (1) The set of choices is composed of an n-tuple where all the components are zero except one of them, which should be equal to 1. Thus, a total number of n combinations.
(2) Consider the alphabet $\mathcal{A}_n = \{x_1, x_2, \ldots, x_n\}$. It consists of combining words with k letters from $\mathcal{A}_n$. A word is a series of characters $a = x_{i_1} x_{i_2} \ldots x_{i_k}$. Two situations are possible. The word a is of the form $a = x_1 x_{i_2} \ldots x_{i_k}$, or it did not start with the letter x_1. In the first case, the total number is the same as the words composed of $k-1$ letters from the alphabet $\mathcal{A}_n$, and since the choice of the x_is is the same for all of them, we obtain a total number $n\Gamma_n^{k-1}$. In the second case, where the word a did not start with x_i, we obtain $k-1$ possibilities to fix its first letter. Next, complete the word a with $k-1$ letters from $\mathcal{A}_n$. Thus, a total number of $(k-1)\Gamma_n^{k-1}$. Finally, since the order of the letters in a is the same for all the alphabets A_n, we obtain

$$k\Gamma_n^k = n\Gamma_n^{k-1} + (k-1)\Gamma_n^{k-1} .$$

Hence,

$$\Gamma_n^k = \frac{(n+k-1)}{k}\Gamma_n^{k-1} .$$

(3) It reposes an iteration procedure of the previous relation. We get

$$\begin{aligned}\Gamma_n^k &= \frac{(n+k-1)}{k}\Gamma_n^{k-1}\\ &= \frac{(n+k-1)}{k}\frac{(n+k-2)}{k-1}\Gamma_n^{k-2}\\ &= \cdots\\ &= C_{n+k-1}^k .\end{aligned}$$

□

Proof of Theorem 35. The first assertion is a consequence of Lemma 36. We proceed to proving the second. Let $\rho \in O(n)$ and $P \in \mathcal{P}_k(\mathbb{R}^n)$. It is straightforward that $P \circ \rho(\lambda x) = P(\lambda\rho(x)) = \lambda^k P(\rho(x))$, hence, a homogenous polynomial with degree k. □

Example 3.1.

$$\dim \mathcal{P}_k(\mathbb{R}^2) = k+1 \quad \text{and} \quad \dim \mathcal{P}_k(\mathbb{R}^3) = \frac{(k+1)(k+2)}{2} .$$

This may be checked directly. Indeed, a basis of $\mathcal{P}_k(\mathbb{R}^2)$ is formed with all products $X^i Y^{k-i}$, $i = 0, \ldots, k+1$. Similarly, a natural basis of $\mathcal{P}_k(\mathbb{R}^3)$ is formed with all products $X^i Y^j Z^{k-i-j}$, $i, j = 0, \ldots, k+1$. For example, for $i = 0$, we get $k+1$ couples $(j, k-j)$, $j = 0, \ldots, k$. For $i = 1$, we get k couples $(j, k-1-j)$, $j = 0, \ldots, k-1$. And so on. We get a sum $k+1+k+k-1+\cdots 1 = \frac{(k+1)(k+2)}{2}$.

Definition 37. A polynomial P is said to be harmonic iff its Laplacian is zero, i.e., $\Delta P = 0$. The space of all harmonic homogenous polynomials of degree k on $\mathbb{R}^n$ will be denoted $\mathcal{PH}_k(\mathbb{R}^n)$ and s_k^n its dimension.

Theorem 38. *For all k, it holds that*

$$\mathcal{P}_k(\mathbb{R}^n) = \mathcal{PH}_k(\mathbb{R}^n) \oplus |x|^2 \mathcal{P}_{k-2}(\mathbb{R}^n) .$$

For $k = 2p$ even,

$$\mathcal{P}_{2p}(\mathbb{R}^n) = \bigoplus_{j=0}^{p} |x|^{2p-2j} \mathcal{PH}_{2j}(\mathbb{R}^n) .$$

For $k = 2p + 1$ odd,

$$\mathcal{P}_{2p+1}(\mathbb{R}^n) = \bigoplus_{j=0}^{p} |x|^{2p-2j} \mathcal{PH}_{2j+1}(\mathbb{R}^n) .$$

Furthermore, for all k, $s_k^n = \dim \mathcal{PH}_k(\mathbb{R}^n) = d_k^n - d_{k-2}^n$.

Proof. Consider the inner product $\langle ., . \rangle$, which corresponds to (x^α, x^β) the quantity

$$\langle x^\alpha, x^\beta \rangle = \alpha!\beta! \text{ if } \alpha = \beta \quad \text{and} \quad 0 \text{ else},$$

and also consider the mapping

$$\begin{aligned} \Phi \colon \mathcal{P}_{k-2}(\mathbb{R}^n) &\longrightarrow \mathcal{P}_k(\mathbb{R}^n) \\ P &\longmapsto \Phi(P) = |x|^2 P. \end{aligned}$$

It is straightforward that Φ is injective. Consequently, $\mathcal{P}_{k-2}(\mathbb{R}^n)$ is isomorphic to $\Phi(\mathcal{P}_{k-2}(\mathbb{R}^n)) = |x|^2 \mathcal{P}_{k-2}(\mathbb{R}^n)$. Now, consider similarly the mapping

$$\begin{aligned} \varphi \colon \mathcal{P}_k(\mathbb{R}^n) &\longrightarrow \mathcal{P}_{k-2}(\mathbb{R}^n) \\ P &\longmapsto \varphi(P) = \Delta P , \end{aligned}$$

which is surjective with kernel

$$\ker(\varphi) = \mathcal{PH}_k(\mathbb{R}^n) .$$

Hence,

$$\dim \mathcal{P}_k(\mathbb{R}^n) = \dim \mathcal{P}_{k-2}(\mathbb{R}^n) + \dim \mathcal{PH}_k(\mathbb{R}^n) .$$

Furthermore,

$$\mathcal{P}_k(\mathbb{R}^n) = |x|^2 \mathcal{P}_{k-2}(\mathbb{R}^n) \oplus \mathcal{PH}_k(\mathbb{R}^n) . \qquad \square$$

Proposition 39. *Let $P \in \mathcal{P}_k(\mathbb{R}^n)$. There exists $Q \in \mathcal{PH}(\mathbb{R}^n)$ such that*

$$P_{/_{S^{n-1}}} = Q_{/_{S^{n-1}}} .$$

Proof. We divide the proof into two cases.
– $k = 2p$. Theorem 38 implies that

$$P(x) = \sum_{j=0}^{p} C_j |x|^{2p-2j} P_{2j+1}(x), \quad \text{or} \quad P_{2j+1} \in \mathcal{PH}_{2j+1} .$$

So, consider the polynomial $Q(x) = \sum_{j=0}^{p} C_j P_{2j+1}(x)$. It is obvious that Q is harmonic, not necessarily homogenous, and it satisfies

$$P(u) = \sum_{j=0}^{p} C_j P_{2j}(u) = Q(u); \quad \forall u \in S^{n-1} .$$

– $k = 2p + 1$. Again, Theorem 38 implies that

$$P(x) = \sum_{j=0}^{p} C_j |x|^{2p-2j} P_{2j}(x), \quad \text{or} \quad P_{2j} \in \mathcal{PH}_{2j} .$$

Consider analogously $Q(x) = \sum_{j=0}^{p} C_j P_{2j}(x)$, which is also harmonic, not necessarily homogenous, and also satisfies

$$P(u) = \sum_{j=0}^{p} C_j P_{2j+1}(u) = Q(u); \quad \forall u \in S^{n-1} . \qquad \square$$

3.5 Spherical harmonics

Definition 40. A function f defined on the sphere S^{n-1} is said to be a *spherical harmonic* iff it is the restriction of a harmonic homogenous polynomial P on S^{n-1}, that is, $\exists P \in \mathcal{PH}(\mathbb{R}^n)$ such that

$$f(u) = P(u), \quad \forall u \in S^{n-1} .$$

The degree of f is that of P. The space of all spherical harmonics on $\mathbb{R}^n$ with degree k will be denoted by $\mathcal{HS}_k(S^{n-1})$.

The following proposition is proved in [162].

Proposition 41.
(1) $\dim \mathcal{HS}_k(S^{n-1}) = \dim \mathcal{PH}_k(\mathbb{R}^n)$.
(2) $\mathcal{HS}_m(S^{n-1}) \perp \mathcal{HS}_n(S^{n-1}), \quad \forall m \neq n$.
(3) $L^2(S^{n-1}) = \overline{\bigoplus_{k=0}^{+\infty} \mathcal{HS}_k(S^{n-1})}$.
(4) $\forall f \in \mathcal{HS}_k, \Delta_S f = -k(k+n-2)f$.

Proof. (1) Consider the linear mapping

$$\begin{aligned} L: \mathcal{PH}_k(\mathbb{R}^n) &\longrightarrow \mathcal{HS}_k(S^{n-1}) \\ P &\longmapsto L(P) = P_{/S^{n-1}} . \end{aligned}$$

Observing the definition of $\mathcal{HS}_k(S^{n-1})$, we deduce that L is surjective. Next, let P be such that $L(P) = 0$, then, for all $x \in \mathbb{R}^n \setminus \{0\}$, we obtain

$$P(x) = P\left(|x|\frac{x}{|x|}\right) = |x|^k P\left(\frac{x}{|x|}\right) = |x|^k L(P)(x) = 0 .$$

So, L is also injective. Consequently, $\mathcal{HS}_k(S^{n-1})$ and $\mathcal{PH}_k(\mathbb{R}^n)$ are isomorphic, thus, with the same dimension.
(2) Let $P \in \mathcal{HS}_k(S^{n-1})$ and $Q \in \mathcal{HS}_l(S^{n-1})$, $n \neq m$ in $\mathbb{N}$. We claim that

$$\langle P, Q\rangle_{L^2(S^{n-1})} = 0 .$$

Indeed, P and Q are restrictions of harmonic homogenous polynomials $\widetilde{P} \in \mathcal{HS}_n(\mathbb{R}^n)$ and $\widetilde{Q} \in \mathcal{HS}_m(\mathbb{R}^n)$ on S^{n-1}. So, it results from Green's formula that

$$\int_{B(0,1)} \widetilde{P}(x) \underbrace{\Delta\widetilde{P}(x)}_{=0} - \widetilde{Q}(x) \underbrace{\Delta\widetilde{Q}(x)}_{=0} = \int_{S^{n-1}} P(\xi)\frac{\partial P}{\partial \nu}(\xi) - Q(\xi)\frac{\partial Q}{\partial \nu}(\xi) d\sigma(\xi) = 0 ,$$

where ν is the outward normal vector of S^{n-1}. Observing next that for a homogenous polynomial P of degree s that

$$\frac{\partial P}{\partial \nu} = sP ,$$

we obtain

$$\int_{S^{n-1}} P(\xi)Q(\xi)d\sigma(\xi)(m-n) = 0 ,$$

which yields the orthogonality of $\mathcal{HS}_m(S^{n-1})$ and $\mathcal{HS}_n(S^{n-1})$.
(3) Observe firstly that on the S^{n-1}, the quadratic elementary polynomial

$$Q(x) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1 .$$

Otherwise, from Theorem 38, the sum of the spaces $\mathcal{HS}_k(S^{n-1})$ gives the space of restrictions on S^{n-1} of all homogenous polynomials. So, it results from the compactness of S^{n-1} and the well-known Stone–Weierstrass theorem that such a sum is dense in the space of continuous functions on S^{n-1} relative to the uniform topology and hence relative to that of L^2. Thus,

$$L^2\left(S^{n-1}, d\sigma\right) = \overline{\bigoplus_{k=0}^{+\infty} \mathcal{HS}_k(S^{n-1})} .$$

(4) Let $f \in \mathcal{HS}_k(S^{n-1})$, restriction of $P \in \mathcal{PH}_k(\mathbb{R}^n)$, and denote

$$\widetilde{f}(x) = \frac{1}{\|x\|^k} P(x) .$$

Obviously, for $x \in S^{n-1}$, we have $f(x) = \widetilde{f}(x) = P(x)$. By applying the Leibniz law, we obtain

$$\begin{aligned}
\Delta\widetilde{f} &= \Delta\left(\frac{1}{\|x\|^k}\right)P + 2\left\langle \nabla\frac{1}{\|x\|^k}, \nabla P\right\rangle + \frac{1}{\|x\|^k}\Delta P \\
&= k(k-n+2)\frac{1}{\|x\|^{k+2}}P - 2k^2\frac{1}{\|x\|^{k+2}}\Delta P .
\end{aligned}$$

Observing that on the sphere $\|x\| = 1$, we obtain

$$\Delta_S f = -k(k+n-2)f. \qquad \square$$

Lemma 42. *There exists an orthonormal basis $\{Y_{n,j}\}_{1\le j\le s_k^n}$ of $\mathcal{HS}_k(S^{n-1})$ composed of spherical harmonics of degree k so that any spherical harmonics Y in $\mathcal{HS}_k(S^{n-1})$ is written in a unique way as*

$$Y = \sum_{j=1}^{s_k^n} C_{n,j}Y_{n,j} .$$

Proof. $\mathcal{HS}_k(S^{n-1})$ is a finite-dimensional vector space with the canonic basis formed by the restriction of $(x^\alpha)_{|\alpha|=k}$ on the sphere S^{n-1}. So, the Gram–Schmidt procedure yields an orthonormal basis.

The following result is known as the addition theorem of spherical harmonics. It is a basic result in harmonic analysis of homogenous polynomials that relates the theory of spherical harmonics to the theory of orthogonal polynomials. For more details on such relationships and a proof in a special case, refer to [162]. $\square$

Theorem 43. *Consider an orthonormal basis $\{Y_{n,j}\}_{1\le j\le s_k^n}$ of $\mathcal{P}_k(S^{n-1})$. The following assertion holds:*

$$\sum_{j=1}^{s_k^n} Y_{k,j}(\xi)Y_{k,j}(\eta) = \frac{s_k^n}{\omega_{n-1}} P_{k,n}(\xi.\eta), \quad \forall \xi, \eta \in S^{n-1} ,$$

where $P_{k,n}$ is the Legendre polynomial defined by

$$P_{k,n}(t) = k!\Gamma\left(\frac{n-1}{2}\right)\sum_{j=0}^{[k/2]}(-1)^j \frac{(1-t^2)^j t^{k-2j}}{4^j j!(k-2j)!\Gamma(j+\frac{n-1}{2})} .$$

Proof. Denote $F(\xi,\eta)$ the left-hand term in the addition rule above, i.e.,

$$F(\xi,\eta) = \sum_{j=1}^{s_k^n} Y_{k,j}(\xi)Y_{k,j}(\eta), \quad \xi,\eta \in S^{n-1} . \tag{3.10}$$

We claim that F is invariant under the action of $O(n)$. Indeed, let $A \in O(n)$ and $H_{k,j} \in \mathcal{P}_k(\mathbb{R}^n)$ be such that

$$H_{k,j}|_{S^{n-1}} = Y_{k,j}, \quad \forall j .$$

It is straightforward that the function $x \longmapsto H_{n,j}(Ax)$ is also a harmonic homogenous polynomial on $\mathbb{R}^n$ of degree k. Consequently, $\xi \longmapsto Y_{k,j}(A\xi)$, $j = 1,\dots,s_k^n$ is a spherical harmonic of degree k. Hence, it can be written as

$$Y_{n,j}(A\xi) = \sum_{m=1}^{s_k^n} \alpha_m^{(j)} Y_{k,m}(\xi) \quad \forall \xi \in S^{n-1} .$$

Next, observing that A is orthogonal, we get

$$\int_{S^{n-1}} Y_{k,m}(A\xi)Y_{k,l}(A\xi)d\omega(\xi) = \int_{S^{n-1}} Y_{k,m}(\xi)Y_{k,l}(\xi)d\omega(\xi) = \delta_{ml} .$$

Hence,

$$\int_{S^{n-1}} Y_{k,m}(A\xi)Y_{k,l}(A\xi)d\omega(\xi) = \sum_{i,m=1}^{s_k^n} b_i^{(j)} b_m^{(l)} \delta_{im} .$$

Or equivalently,

$$\delta_{ml} = \int_{S^{n-1}} Y_{k,m}(A\xi)Y_{k,l}(A\xi)d\omega(\xi) = \sum_{m=1}^{s_k^n} b_m^{(j)} b_m^{(l)}, \quad \forall m,l \in \{1,\dots,s_k^n\} .$$

This means that the matrix $B = (b_m^{(j)})_{m,j=1,\dots,s_k^n}$ is orthogonal. Consequently, $\forall \xi$, $\eta \in S^{n-1}$

$$\begin{aligned}
F(A\xi, A\eta) &= \sum_{j=1}^{s_k^n} Y_{k,j}(A\xi) Y_{k,j}(A\eta) \\
&= \sum_{j=1}^{s_k^n} \sum_{m,l=1}^{s_k^n} b_m^{(j)} b_l^{(j)} Y_{k,m}(\xi) Y_{k,l}(\eta) \\
&= \sum_{m=1}^{s_k^n} Y_{k,m}(\xi) Y_{k,m}(\eta) \\
&= F(\xi, \eta) .
\end{aligned}$$

Hence, F is invariant under the action of orthogonal transformations. It follows (see [16, 54, 59, 143, 162]) that

$$F(\xi, \eta) = F(\xi, \xi) P_{k,n}(\xi\eta) \quad \text{and} \quad F(\xi, \eta) = F(\eta, \eta) P_{k,n}(\xi\eta), \qquad \forall \xi, \eta \in S^{n-1}$$

and that $P_{k,n}$ is the Legendre polynomial of degree k defined precisely in Theorem 43. This means in particular that $F(\xi, \xi) = F(\eta, \eta)$ and thus constant on S^{n-1}. Otherwise, observing that

$$F(\xi, \xi) = \sum_{j=1}^{s_k^n} |Y_{k,j}(\xi)|^2 ,$$

we obtain by means of the integration on the whole sphere S^{n-1} that

$$F(\xi, \xi)\omega_{n-1} = s_k^n . \qquad \square$$

Remark 44. Let $f : S^{n-1} \longrightarrow \mathbb{R}$. Then, it can be written as a linear combination of spherical harmonics

$$f(\eta) = \sum_{k=0}^{\infty} \sum_{m=1}^{s_k^n} K_{m,n} Y_{k,n}(\eta),$$

where $K_{m,n}$ are called the spherical harmonic coefficients or spherical harmonic transforms of f at the order (m, n), and are obtained from the inner products $\langle f, Y_{k,n} \rangle$.

3.6 Fourier transform of spherical harmonics

Let P be a homogenous polynomial on $\mathbb{R}^n$. Denote

$$L_P^2(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n);\ \exists f_0 : \mathbb{R}_+ \to \mathbb{R} \text{ such that } f(x) = f_0(|x|) P(x) \right\}$$

and for $\lambda \in \mathbb{R}$, we denote

$$\mathcal{H}_\lambda(\mathbb{R}^n) = \left\{ f : \mathbb{R}_+ \to \mathbb{R} \text{ such that } \int_{\mathbb{R}_+} |f(x)|^2 x^\lambda dx < +\infty \right\} .$$

It holds that for $\lambda = n + 2k - 1$, $\mathcal{H}_\lambda(\mathbb{R}^n)$ is the adherence of $\Sigma = (E^{-\pi\alpha r^r};\ \alpha > 0)$. We can consider $\Sigma^\perp$ in $\mathcal{H}_\lambda(\mathbb{R}^n)$ and prove that $\Sigma^\perp = \{0\}$.

Definition 45. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. The *profile* of f is defined by $\widetilde{f} : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $f(x) = \widetilde{f}(|x|)$.

Definition 46. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. The radial transform of f is $\overline{f} : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

$$\overline{f}(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(|x|u)d\sigma(u) ,$$

where ω_{n-1} is the volume of the sphere S^{n-1} and $d\sigma$ is the Lebesgue measure on S^{n-1}.

Lemma 47. *Let $d\sigma$ be the Lebesgue measure on S^{n-1}. We have*

$$\widehat{\sigma}(t) = 2\pi|t|^{1-n/2} J_{\frac{n}{2}-1}(2\pi|t|) ,$$

where for $\lambda \in \mathbb{R}$, J_λ is the Bessel function

$$J_\lambda(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(\lambda t - x \sin t)} dt .$$

Proof. Recall that the Fourier transform of σ is defined by

$$\widehat{\sigma}(t) = \int_{S^{n-1}} e^{-2i\pi t\xi} d\sigma(\xi) .$$

As the measure $d\sigma$ is invariant under rotations, we can assume without loss of the generality that $t = (|t|, 0, \dots, 0)$. We obtain

$$\widehat{\sigma}(t) = \int_{S^{n-1}} e^{-2i\pi|t|\cos\theta} d\sigma(\xi) ,$$

where θ is the angle (t, e_1), with $e_n = (1, 0, \ldots, 0)$. In spherical coordinates, this means that

$$\widehat{\sigma}(t) = \omega_{n-2} \int_0^{\pi} e^{-2i\pi|t|\cos\theta} \sin^{n-2}\theta d\theta \,.$$

Denote next $r = \cos\theta$. We obtain

$$\widehat{\sigma}(t) = \omega_{n-2} \int_{-1}^{1} e^{-2i\pi|t|r}(1-r^2)^{(n-3)/2} dr \,.$$

Observe next that

$$\omega_{n-2} = \frac{\pi^{\frac{n-3}{2}}}{\Gamma(\frac{n-1}{2})}$$

and on the other hand,

$$J_\lambda(2\pi|t|) = \frac{|\pi t|^\lambda}{\pi^{1/2}\Gamma(\lambda+1/2)} \int_{-1}^{1} e^{-2i\pi|t|r}(1-r^2)^{\lambda-1/2} dr \,,$$

so we obtain the desired result. □

Lemma 48. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a radial function with profile $\widetilde{f}$. Then,*

$$\widetilde{\widehat{f}}(r) = 2\pi r^{1-n/2} \int_0^{+\infty} \widetilde{f}(s) J_{\frac{n}{2}-1}(2\pi rs) s^{n/2} ds = \int_0^{+\infty} \widetilde{f}(s)\widehat{\sigma}(rs) s^{n-1} ds \,.$$

Indeed, for $x = r\xi \in \mathbb{R}^n$, $(\xi \in S^{n-1})$, we have

$$\begin{aligned} \widehat{f}(x) &= \int_{\mathbb{R}^n} f(\zeta) e^{-2i\pi x\zeta} d\zeta \\ &= \int_0^{+\infty} {}_{S^{n-1}} f(s) e^{-2i\pi rs\xi\eta} s^{n-1} ds d\sigma(\eta) \\ &= \int_0^{+\infty} \widetilde{f}(s)\widehat{\sigma}(rs) s^{n-1} ds \,. \end{aligned}$$

Hence, Lemma 48.

Theorem 49. *Let f be a function of the form $f(x) = e^{-\pi|x|^2}P(x)$, where $P \in \mathcal{P}_k(\mathbb{R}^n)$. Then,*

$$\widehat{f}(x) = i^{-k} f(x) \,.$$

Proof. It suffices to show that $f(x) = e^{-\pi|x|^2}x^\alpha$, with $|\alpha| = k$. For this choice, we have

$$
\begin{aligned}
\widehat{f}(x) &= \int_{\mathbb{R}^n} e^{-\pi|y|^2} y^\alpha e^{-2i\pi xy}\, dy \\
&= \prod_{m=1}^{n} \int_{\mathbb{R}} e^{-\pi y_m^2} y_m^{\alpha_m} e^{-2i\pi x_m y_m}\, dy_m \\
&= \prod_{m=1}^{n} e^{-\pi x_m^2} (-i x_m)^{\alpha_m} \\
&= i^{-k} e^{-\pi|x|^2} x^\alpha \, .
\end{aligned}
$$

The following theorem is known as the Bochner–Hecke theorem. □

Theorem 50. *Let $n \geq 2$, $k \geq 0$, $P \in \mathcal{P}_k(\mathbb{R}^n)$ and f_0 be measurable on $[0, +\infty[$ such that*

$$\int_0^{+\infty} |f_0(r)|^2 r^{n+2k-1}\, dr < +\infty \, .$$

Then,

$$F(x) = f_0(|x|)P(x) \in L^2(\mathbb{R}^n) \quad \text{and} \quad \widehat{F}(x) = g_0(|x|)P(x) \, ,$$

where g_0 is measurable on $[0, +\infty[$ such that $\int_0^{+\infty} |g_0(r)|^2 r^{n+2k-1}\, dr < +\infty$. More precisely,

$$g_0(r) = \frac{2\pi}{i^k r^\lambda} \int_0^{+\infty} f_0(t) J_\lambda(2\pi r t) t^{\lambda+1}\, dt, \quad \text{with } \lambda = \frac{n + 2k - 2}{2} \, .$$

Proof. The first part is trivial as we have

$$\int_{\mathbb{R}^n} |F(x)|^2\, dx = \int_0^{+\infty} |f_0(r)|^2 r^{n+2k-1}\, dr \int_{S^{n-1}} |P(\eta)|^2\, d\sigma(\eta) \, .$$

The second part follows from Lemma 48 and Theorem 49. □

3.7 Zonal functions

In this section, we review the basic concepts of zonal functions. We revisit their Fourier transforms and reproduce the proof of the well-known Bochner–Hecke theorem. We first recall a result on the existence of zonal function sites (see [48, 65, 162]).

Theorem 51. *For all $u \in \mathcal{S}^{n-1}$, there exists $Z_u^k \in \mathcal{HS}_k(S^{n-1})$ such that*

$$Y(u) = \int_{\mathcal{S}^{n-1}} Y(v) Z_u^k(v) d\sigma(v) \,.$$

Furthermore, Z_u^k satisfies the following assertions:
(1) $Z_u^k(v) = \sum_l \overline{Y_l(u)} Y_l(v)$*; for all orthonormal basis $(Y_l)_l$ of $\mathcal{HS}_k$ and $\forall u, v \in \mathcal{S}^{n-1}$.*
(2) $Z_{\rho u}^k(\rho v) = Z_u^k(v), \forall \rho \in SO(n)$ *and* $\forall u, v \in \mathcal{S}^{n-1}$.
(3) $Z_u^k(u) = \frac{d_k}{w_{r-1}}, \forall u \in \mathcal{S}^{n-1}$.

Proof. Let $Y \in \mathcal{HS}_k(S^{n-1})$ and $\forall (Y_l)_l$ be an orthonormal basis of $\mathcal{HS}_k$. Then, Y is of the form

$$Y = \sum_m \alpha_m Y_m, \quad \text{where} \quad \alpha_m = \int_{S^{n-1}} Y(v) \overline{Y_m(v)} d\sigma(v) \,.$$

Consequently, for $u \in S^{n-1}$, we obtain

$$\begin{aligned} Y(u) &= \sum_m \int_{S^{n-1}} Y(v) \overline{Y_m(v)} d\sigma(v) Y_m(u) \\ &= \int_{S^{n-1}} Y(v) \underbrace{\sum_m \overline{Y_m(v)} Y_m(u)}_{Z_u^k(v)} \, d\sigma(v) \\ &= \int_{S^{n-1}} Y(v) Z_u^k(v) d\sigma(v). \end{aligned}$$

Hence, $Y(u)$ and (1) hold.
(2) For $\rho \in SO(n)$ and $u, v \in \mathcal{S}^{n-1}$, we have

$$Z_{\rho u}^k(\rho v) = \sum_m \overline{Y_m(\rho u)} Y_m(\rho v).$$

Recall that the right-hand side part is invariant under the action of $O(n)$. Thus,

$$Z_{\rho u}^k(\rho v) = \sum_m \overline{Y_m(u)} Y_m(v) = Z_u^k(v).$$

(3) For $u \in S^{n-1}$, we have

$$Z_u^k(u) = \sum_m \overline{Y_m(u)} Y_m(u) = \sum_m |Y_m(u)|^2 = \frac{s_k^n}{\omega_{n-1}}. \qquad \square$$

Definition 52. Z_u^k defined in Theorem 51 is called the zonal function of degree k and the pole u.

Proposition 53. *Let $v \in S^{n-1}$. Then there exists φ_u^k such that*

$$\varphi_u^k(uv) = Z_u^k(v) \quad \forall u \in \mathcal{S}^{n-1}.$$

Indeed, it follows from Theorem 51 that Z_u^k is invariant under orthogonal transformations. Thus, it depends only on the angle (u, v). Consequently, there exists φ_u^k defined on $[-1, 1]$ such that

$$Z_u^k(v) = \varphi_u^k(uv) \quad \forall u \in \mathcal{S}^{n-1}.$$

This result allows us to link with the definition of the zonal function based on $L^2[-1, 1]$.

Definition 54. Let $\xi \in S^{n-1}$ and $G : [-1, 1] \longrightarrow \mathbb{R}$ be a function. The function

$$\begin{aligned} G_\xi : \Omega &\longrightarrow \mathbb{R} \\ \eta &\longmapsto G_\xi(\eta) = G(\xi.\eta) \end{aligned}$$

is called the *ξ-zonal function* on Ω.

These definitions confirm that it suffices to define zonal functions relative to one of the vectors in a coordinate system to cover the entire sphere.

Proposition 55. *Let $e_n = (0, \dots, 0, 1)$ be one pole of the sphere S^{n-1}. The following assertions hold.*

(1) *For all $u \in S^{n-1}$, there exists $\rho \in SO(n)$ such that $\rho e_n = u$.*
(2) $Z_u^k(v) = Z_{e_n}^k(l^{-1}v); \quad \forall u, v \in \mathcal{S}^{n-1}$.
(3) $\varphi_u^k(uv) = \varphi_{e_n}^k(uv); \quad \forall u, v \in \mathcal{S}^{n-1}$.

Proof. If $u = e_n$, it suffices to take $\rho - Id$. If not, we consider the rotation ρ centered at O and transforming e_n to u. Hence, (1).
(2) We have

$$Z_u^k(v) = Z_{\rho e_n}^k(\rho\rho^{-1}v) = Z_{e_n}^k(\rho^{-1}v).$$

(3) We have

$$\varphi_u^k(uv) = Z_u^k(v) = Z_{e_n}^k(\rho^{-1}v) = \varphi_{e_n}^k(e_n\rho^{-1}v) = \varphi_{e_n}^k(\rho e_n \rho\rho^{-1}v) = \varphi_{e_n}^k(uv). \qquad \square$$

This means that to compute the zonal function Z_u^k, it suffices to evaluate $\varphi_{e_n}^k(t)$ for $t \in [-1, 1]$. So, let $v \in \mathbb{R}^n$, $v = r\eta$ with $r > 0$ and $\eta \in S^{n-1}$. Let $r\cos\theta = e_n v$. Hence,

$$\varphi_{e_n}^k(e_n v) = r^k P_k(\cos\theta)\,,$$

where P_k is a polynomial of degree k. Since Z_u^k is harmonic, it holds that

$$(1 - t^2)P_k''(t) - (n-1)tP_k'(t) + k(n+k-2)P_k(t) = 0.$$

This leads to the following characterization of zonal functions.

Theorem 56. *Let $n \geq 3$, $k \geq 0$, and $u \in S^{n-1}$. Then,*

$$Z_u^k(v) = \frac{s_n^k}{w_{n-1}} P_k^{\frac{n-2}{2}}(uv)\,,$$

where P_k^λ is the unique polynomial of degree k associated with λ, i.e., a polynomial solution of

$$\begin{cases} (1-t^2)P'' - (2\lambda+1)tP' + k(2\lambda+k)P = 0, \\ P(1) = 1. \end{cases}$$

3.8 Conclusion

In this chapter, the basic concepts and properties of the spherical Laplace operator and homogenous harmonic polynomials were reviewed. In addition, spherical harmonics as well as their Fourier transforms were studied. Finally, zonal functions as special cases of spherical harmonics have been revisited. Basic theorems, such as the addition theorem, Hecke theorem, and Bochner–Hecke theorem have been presented with more detail. As we have seen in this chapter, spherical harmonics are strongly related to orthogonal polynomials, such as Gegenbauer polynomials, which are linked to the well-known Bessel function, which is a particular case of a wider class of function known as special functions. These will be the subject of a forthcoming work.

4 Review of special functions

4.1 Introduction

The main motivation behind this chapter about special functions is that these functions are applied in the quasi-field of mathematical physics and that there is no literature dedicated on them and their basic properties with efficient proofs and original references.

Special functions are, as their name indicates, special in their definitions, applications, proofs as well as their interactions with other fields. It is thus important to understand their basic properties.

They appear in the treatment of differential equations, such as heat and Schrödinger equations, quantum mechanics, approximation theory, communication systems, wave propagation, probability theory, and number theory.

Special functions are also related to orthogonal polynomials, as both of them are generated by second-order ordinary differential equations. We cite mainly Legendre, Gegenbauer, and Jacobi polynomials. They are also associated with infinite series, improper integrals, and Fourier transforms, yielding special transforms, such as Bessel, Jakobi, Hankel, and Dunkl transforms of functions.

Historically, special functions differ from elementary ones, such as powers, roots, trigonometric, and their inverses, mainly with the limitations that these latter classes have known. Many fundamental problems such as orbital motion, simultaneous oscillatory chains, and spherical body gravitational potential were not best described using elementary functions. This makes it necessary to extend elementary function classes to more general ones that may describe unresolved problems.

In the present chapter, we aim to recall special functions most frequently applied in scientific fields, such as Bessel functions, Mathieu functions, the Gamma function, the Beta function, and Jacobi functions.

4.2 Classical special functions

4.2.1 Euler's Γ function

Euler's Gamma function was introduced by Bernoulli and Christian Goldbach in the 17th century by extending the factorial to nonintegers. But the problem remained unsolved until the work of Leonhard Euler, who was the first to point out a rigorous formulation based on infinite products. Next, Euler's Gamma function has been applied in numerous contexts in both mathematics and physics, such as integration theory, number theory, probability, group theory, and partial differential equations (PDEs), and has also been extended to the meromorphic function on the whole complex plane.

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Definition 57. Euler's Γ function is defined by the following integral expression known sometimes as the second-kind Euler integral, defined for $x \in \mathbb{R}_+^*$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \,.$$

Proposition 58.
(1) *Euler's Γ integral converges for all $x > 0$.*
(2) *The function Γ is C^∞ on $]0, +\infty[$ and we have*

$$\Gamma^{(k)}(x) = \int_0^{+\infty} e^{-t} (\ln t)^k t^{x-1} dt, \qquad \forall x > 0, \forall k \in \mathbb{N} \,.$$

(3) *Euler's Γ function can be extended on the half-plane* $\mathrm{Re}(z) > 0$.

Proof. (1) For $x > 0$, denote $f(t,x) = t^{x-1}e^{-t}$. First note that $f(t,x) > 0$ for all $t \in (0, +\infty)$. When $t \to 0$, $f(t,x) \sim t^{x-1}$ and

$$\int_0^1 t^{x-1} dt = \frac{1}{x}$$

is convergent. So

$$\int_0^1 f(t,x) dt$$

is also convergent. Now, note that there exist $A, M > 0$ constants such that $t^2 f(t,x) < M$ whenever $t > A$ and thus

$$\int_A^{+\infty} f(t,x) dt \le \int_A^{+\infty} \frac{1}{t^2} dt \,.$$

The last integral is convergent. So

$$\int_A^{+\infty} f(t,x) dt$$

is also convergent. Finally, the integral $\Gamma(x)$ is convergent for all $x > 0$.

(2) Let $a, b \in \mathbb{R}$ with $0 < a < b$ and $\phi\colon [a,b]\times]0,+\infty[\longrightarrow \mathbb{R}$ such that $\phi(x,t) = t^{x-1}e^{-t}$. It consists of a C^∞ function that satisfies

$$\frac{\partial^{(k)}\phi}{\partial x^k}(x,t) = (\ln t)^k t^{x-1}e^{-t}\,,$$

which is also continuous on $[a,b]\times]0,+\infty[$. In addition, for $k \in \mathbb{N}^*$, we have

- $\forall x \in [a,b]$, the function $t \mapsto \frac{\partial^k\phi}{\partial x^k}(x,t)$ is continuous on $]0,+\infty[$.
- $\forall t \in]0,+\infty[$, the function $x \mapsto \frac{\partial^k\phi}{\partial x^k}(x,t)$ is continuous on $[a,b]$.
- $\forall (x,t) \in [a,b]\times]0,+\infty[$, we have

$$\left|\frac{\partial^k\phi}{\partial x^k}(x,t)\right| \le (\ln t)^k \max(t^{a-1}, t^{b-1})e^{-t}\,.$$

Hence, the function Γ is $\mathcal{C}^\infty$ on $]0,+\infty[$ and $\forall k \in \mathbb{N}^*$, $\forall x > 0$,

$$\Gamma^{(k)}(x) = \int_0^{+\infty} (\ln t)^k t^{x-1}e^{-t}dt\,.$$

Hence (3.3).
(3) We will prove by recurrence the proposal

$$P_m\colon \Gamma \text{ can be extended on } -m+1 > \mathrm{Re}(z) > -m, \forall m \in \mathbb{N}.$$

Indeed, P_0 holds because Γ is analytic on $\{\mathrm{Re}(z) > 0\}$. Therefore, it is analytic on $\{1 > \mathrm{Re}(z) > 0\}$. Next, for $0 > \mathrm{Re}(z) > -1$, we have $1 > \mathrm{Re}(z+1) > 0$. Hence, $\Gamma(z+1)$ is analytic. In addition, $\Gamma(z) = \frac{\Gamma(z+1)}{z}$. Thus, Γ is holomorphic on $\{0 > \mathrm{Re}(z) > -1\}$ with 0 being a simple pole corresponding to the residues 1. So, Γ can be extended to a meromorphic function $\{\mathrm{Re}(z) > -1\}$ with a simple pole at 0. Hence, the property P_1. Next, applying the recurrence rule, we obtain

$$\Gamma(z) = \frac{\Gamma(z+n)}{\prod_{k=0}^{n-1}(z+k)}\,.$$

□

Properties 59. The following assertions are satisfied.
(1) $\Gamma(x+1) = x\Gamma(x); \quad \forall x > 0.$
(2) $\Gamma(n+1) = n!, \forall n \in \mathbb{N}.$
(3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}.$
(4) $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}, \forall n \in \mathbb{N}.$

Proof. (1) An integration by parts gives

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t}dt = x\int_0^{+\infty} t^{x-1}e^{-t}dt = x\Gamma(x)\,.$$

(2) Putting $x = n \in \mathbb{N}^*$ in assertion (1), we get

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n!\Gamma(1) = n!\,.$$

Hence the appointment of *generalized factorial function* for Γ.
(3) We have

$$\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt\,.$$

Putting $x = \sqrt{t}$, we get

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx\,.$$

Hence,

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy\,.$$

Now, using the polar coordinates system, $x = r\cos\theta$ and $y = r\sin\theta$, with $r \in (0,\infty)$ and $\theta \in (0, \frac{\pi}{2})$ we get

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2} r dr d\theta = \pi\,.$$

Therefore, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
(4) By recurrence on n. For $n = 0$, we have $\Gamma(0 + \frac{1}{2}) = \Gamma(\frac{1}{2})$ on the left and $\sqrt{\pi}$ on the right. So, the assertion is true for $n = 0$. Assume next that it is true for n. We shall then check it for $n + 1$.

$$\begin{aligned}\Gamma\left(n+1+\frac{1}{2}\right) &= \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\\ &= \left(n+\frac{1}{2}\right)\frac{(2n)!\sqrt{\pi}}{2^{2n}n!}\\ &= \frac{(2n+2)!\sqrt{\pi}}{2^{2n+2}(n+1)!}\\ &= \frac{(2(n+1))!\sqrt{\pi}}{2^{2(n+1)}(n+1)!}\,.\end{aligned}$$

□

The next result shows some asymptotic behaviors of Euler's Γ function.

Theorem 60. *Euler's Γ function satisfies the so-called Stirling formula,*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \quad as \quad x \longrightarrow +\infty\,.$$

Proof. Recall that

$$\Gamma(x+1) = \int_0^{+\infty} t^x e^{-t} dt\,.$$

By setting $t = x + \sqrt{x}u$, we obtain

$$\begin{aligned}\Gamma(x+1) &= \int_{-\sqrt{x}}^{+\infty} e^{-x-\sqrt{x}u} e^{x\ln(x+\sqrt{x}u)} \sqrt{x} du \\ &= \left(\frac{x}{e}\right)^x \sqrt{x} \int_{-\sqrt{x}}^{+\infty} e^{-\sqrt{x}u + x\ln(1+\frac{u}{\sqrt{x}})} du\,.\end{aligned}$$

Now, it suffices to prove that the last integral tends to $\sqrt{2\pi}$ as $x \to +\infty$. Denote $\Gamma_1(x)$ as this integral and let

$$f(x,u) = \begin{cases} e^{-\sqrt{x}u + x\ln(1+\frac{u}{\sqrt{x}})} & \text{if } u \geq -\sqrt{x} \\ 0 & \text{if not}\,. \end{cases}$$

We get

$$\Gamma_1(x) = \int_{-\infty}^{+\infty} f(x,u) du\,.$$

For fixed $u \in \mathbb{R}$, we have

$$\begin{aligned}\lim_{x\longrightarrow+\infty} f(x,u) &= \lim_{x\longrightarrow+\infty} \exp\left(-\sqrt{x}u + x\ln\left(1+\frac{u}{\sqrt{x}}\right)\right) \\ &= \lim_{x\longrightarrow+\infty} \exp\left(-\sqrt{x}u + x\left(\frac{u}{\sqrt{x}} - \frac{1}{2}\frac{u^2}{x} + \theta\left(\frac{1}{x}\right)\right)\right) \\ &= \exp\left(\frac{-u^2}{2}\right)\,.\end{aligned}$$

On the other hand, if $u \in]-\sqrt{x}, 0]$, as $\frac{|u|}{\sqrt{x}} < 1$, we obtain

$$f(x,u) \leq \exp\left(-\frac{u^2}{2}\right)\,.$$

Finally, for $u \in]0,+\infty[$, $f(x,u)$ is a decreasing function of x on $]0,+\infty[$. We deduce for $u > 0$ and $x \in [1,+\infty[$ that

$$f(x,u) \leq f(1,u) = (1+u)e^{-u}\,.$$

So, for all $u \in \mathbb{R}$ and all $x \in [1,+\infty[$, we have $0 \leq f(x,u) \leq g(u)$, where g is the integrable function defined by

$$g(u) = \begin{cases} e^{-\frac{u^2}{2}}, & \text{if } u \leq 0 \\ (1+u)e^{-u}, & \text{if not } u \geq 0\,. \end{cases}$$

By the dominated convergence theorem, we obtain

$$\Gamma_1(x) \to \int_{-\infty}^{+\infty} e^{\frac{-u^2}{2}}\,du = \sqrt{2\pi}, \quad \text{as } x \to +\infty\,.$$

Now, by setting $t = \frac{\sqrt{x}}{u}$, we get $f(x,u) = e^{h(t,u)}$, where

$$h(t,u) = u^2t^2\left(-\frac{1}{t} + \ln\left(1 + \frac{1}{t}\right)\right),$$

which is decreasing in t. □

Proposition 61. *Euler's Γ function satisfies the so-called Gauss formula for all $x > 0$,*

$$\frac{1}{\Gamma(x)} = \lim_{n\longrightarrow+\infty} \frac{x(x+1)\cdots(x+n)}{n!n^x}\,.$$

Proof. Applying the recurrence relation n times, we obtain

$$\Gamma(x)x(x+1)\cdots(x+n) = \Gamma(x+n+1)\,.$$

Therefore,

$$\begin{aligned}
\frac{x(x+1)\cdots(x+n)}{n!n^x} &= \frac{\Gamma(x+n+1)}{\Gamma(x)n!n^x}\\
&\sim \frac{\sqrt{2\pi}(x+n+1)^{x+n+\frac{1}{2}}e^{-(x+n+1)}}{\Gamma(x)\sqrt{2\pi}(n+1)^{n+\frac{1}{2}}e^{-n-1}n^x}\\
&\sim \frac{1}{\Gamma(x)}\left(\frac{x+n+1}{n}\right)^x\left(\frac{x+n+1}{n+1}\right)^{n+1} e^{-x}\left(1+\frac{x}{n}\right)^n\\
&= \frac{1}{\Gamma(x)}\,.
\end{aligned}$$

□

Proposition 62.
(1) *The infinite product $\prod_{k=1}^{+\infty}(1+\frac{z}{k})e^{\frac{-z}{k}}$ is normally convergent on every compact of $\mathbb{C}$ and therefore defines an analytic function of z.*
(2) *Euler's Γ function satisfies the so-called Gauss–Weierstrass formula for $z \notin -\mathbb{N}$,*

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{n\to+\infty} \frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z}\\
&= ze^{\gamma z}\lim_{n\longrightarrow+\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{\frac{-z}{k}}\,,
\end{aligned}$$

where γ is the Euler–Mascheroni constant given by

$$\gamma = \lim_{n\longrightarrow+\infty}\left(\sum_{k=1}^{n-1}\frac{1}{k} - \ln n\right).$$

Proof. (1) Put $u_k(z) = (1+\frac{z}{k})e^{\frac{-z}{k}} - 1$. A simple Taylor development gives

$$|u_k(z)| \le \frac{|z|^2}{n^2}$$

whenever $\frac{|z|}{n}$ is bounded independent of n and z. Hence, the infinite product converges uniformly on every compact set in $\mathbb{C}$ to an analytic function.
(2) Observe that

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt = \lim_{n\to+\infty}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n dt\,.$$

This is a consequence of the application of the dominated convergence theorem. So, next, integrating by parts, we obtain

$$\Gamma(z) = \lim_{n\to+\infty}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}$$

whenever $z \notin -\mathbb{N}$. So the first part is proved. Next, observe that

$$\frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} = zn^{-z}\prod_{k=1}^n\frac{z+k}{k} = ze^{z(\gamma_n-\log n)}\prod_{k=1}^n e^{-\frac{z}{k}}\prod_{k=1}^n\left(1+\frac{z}{k}\right)$$

where $\gamma_n = \sum_{k=1}^{n-1}\frac{1}{k}$. Next, we obtain

$$\frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} = ze^{z(\gamma_n-\log n)}\prod_{k=1}^n e^{-\frac{z}{k}}\left(1+\frac{z}{k}\right),$$

which implies by the limit on n that

$$\frac{1}{\Gamma(z)} = \lim_{n\to+\infty}\frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} = ze^{\gamma z}\lim_{n\longrightarrow+\infty}\prod_{k=1}^n\left(1+\frac{z}{k}\right)e^{\frac{-z}{k}}.$$ □

Proposition 63. *For $z \in \mathbb{C}\setminus\mathbb{Z}$,*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

The proof is based on the following lemma, which canbe obtained by simple application of Fourier series theory on the function $f(t) = \cos(st)$, where $s \in \mathbb{C}$. The result may also be established by direct methods based on the simple relation $e^z = \lim_{k\to+\infty}(1+\frac{z}{k})^k$. Thus, the proof of this lemma is left to the reader.

Lemma 64. *$\forall z \in \mathbb{C}$, we have*

$$\sin(\pi z) = \pi z \prod_{n\geq 1}\left(1 - \frac{z^2}{n^2}\right) . \tag{4.1}$$

Proof of Proposition 63. Let

$$a_n(z) = \frac{z(z+1)(z+2)\cdots(z+n)}{n!n^z} .$$

Proceeding as in the proof of Proposition 62, we obtain

$$a_n(z)a_n(1-z) = z\left(1 + \frac{1-z}{n}\right)\prod_{k=1}^{n}\left(1 - \frac{z^2}{k^2}\right) .$$

The limit on n gives

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \lim_{n\to+\infty} a_n(z)a_n(1-z) = z\prod_{k\geq 1}\left(1 - \frac{z^2}{k^2}\right) = \frac{\sin(\pi z)}{\pi z} .$$

Consequently,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} .$$

□

Remark 65. The meromorphic function Γ has no roots on $\mathbb{C}$.

Proposition 66.
- *Convexity of Γ: Γ is strictly convex on* $]0, +\infty[$.
- *Asymptotic behavior of Γ at ∞:* $\lim_{x\longrightarrow+\infty}\Gamma(x) = +\infty$.
- *Asymptotic behavior of Γ at ∞:* $\lim_{x\longrightarrow+\infty}\frac{\Gamma(x)}{x} = +\infty$.
- *Asymptotic behavior of Γ at 0^+:* $\lim_{x\longrightarrow 0^+}\Gamma(x) = +\infty$.

Proof. (1) Recall that the function Γ is twice differentiable on $]0, +\infty[$ and $\forall x > 0$, so we have

$$\Gamma''(x) = \int_0^{+\infty}(\ln t)^2 t^{x-1}e^{-t}dt > 0 .$$

Hence, it is convex.

(2) Since the function Γ is increasing on $)0,+\infty[$, for x big enough, we have

$$\Gamma(x)=(x-1)\Gamma(x-1)\geq(x-1)\Gamma(1)=x-1\,.$$

We deduce that $\lim_{x\longrightarrow+\infty}\Gamma(x)=+\infty$.
(3) For $x>1$, we have

$$\frac{\Gamma(x)}{x}=\frac{(x-1)}{x}\Gamma(x-1)\longrightarrow+\infty\quad\text{as }x\longrightarrow+\infty\,.$$

We deduce that the graph of the function Γ has at $+\infty$ a vertical asymptotic direction.
(4) For $x>0$, $\Gamma(x)=\frac{\Gamma(x+1)}{x}\to\frac{\Gamma(1)}{0^+}=+\infty$ when $x\longrightarrow0^+$.
So $\lim_{x\longrightarrow0^+}\Gamma(x)=+\infty$. In addition, we have precisely, $\Gamma(x)\sim\frac{1}{x}$ as $x\to0^+$. □

4.2.2 Euler's beta function

The origin of Euler's beta function goes back to differential calculus and integrals. It was introduced in the Arithmetica Infinitorum published by Wallis. Newton next discovered the binomial formula and introduced Euler's beta function, which was then developed for other versions, such as the incomplete and the corrected versions. The beta function is given by Euler in the following form:

$$\beta(p,q)=\int_0^1 t^p(1-t)^q\,dt$$

and is known as the first-kind Euler integral. But since Legendre's work, it appears in a slightly modified form

$$\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx,\quad p>0,q>0\,.$$

It is apparent that such a function is symmetrical in (p,q), i.e.,

$$\beta(p,q)=\beta(q,p)\,.$$

The beta function also has another integral representation. Indeed, by setting $t=\frac{y}{a}$, $a>0$, it becomes

$$\beta(p,q)=\frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}\,dy\,.$$

Again, setting $t=\sin^2\theta$, we get a trigonometric form

$$\beta(p,q)=2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta\,.$$

Finally, with the variable change $t = \frac{y}{(1+y)}$, we get

$$\beta(p, q) = \int_0^{+\infty} \frac{y^{p-1}}{(1+y)^{p+q}} dy \, .$$

In the following, we will apply one of these representations without mentioning it each time. The form applied will be understood from the development.

Proposition 67.
(1) *The beta integral converges whenever* $x, y > 0$.
(2) *The beta integral is continuous on* $]0, +\infty[\times]0, +\infty[$.
(3) *The beta integral remains valid on the quarter complex plane* $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

Proof. (1) Whenever $p, q > 0$ we have

$$t^p(1-t)^q \sim t^p, \ t \to 0^+ \quad \text{and} \quad t^p(1-t)^q \sim (1-t)^q, \ t \to 1^- \, .$$

Hence, the integral is convergent.
(2) On $]0, +\infty[\times]0, +\infty[$, the function $(p, q) \mapsto f_t(p, q) = t^p(1-t)^q$ is continuous for all $t \in (0, 1)$. Furthermore, $t^p(1-t)^q \le 1, \forall t, p, q$. Thus, the integral is uniformly convergent to a continuous function on $]0, +\infty[\times]0, +\infty[$. Next, by recurrence on $k \in \mathbb{N}$, we can prove that beta is k-times differentiable according to p and q. We can also prove that

$$\frac{\partial^k \beta}{\partial p^k}(p, q) = \int_0^1 (\log t)^k t^p (1-t)^q dt \, ,$$

$$\frac{\partial^k \beta}{\partial q^k}(p, q) = \int_0^1 (\log(1-t))^k t^p (1-t)^q dt$$

and for $n + m = k$,

$$\frac{\partial^k \beta}{\partial p^n q^m}(p, q) = \int_0^1 (\log t)^n (\log(1-t))^m t^p (1-t)^q dt \, .$$

(3) For $p, q \in \mathbb{C}$, we have

$$|t^p(1-t)^q| = t^{\mathrm{Re}(p)}(1-t)^{\mathrm{Re}(q)}, \quad \forall t \in (0, 1) \, . \qquad \square$$

Properties 68.

(1) $p\beta(p, q+1) = q\beta(p+1, q)$, $\forall p, q \geq 0$.
(2) $\beta(p, 1) = \frac{1}{p}$.
(3) $\beta(\frac{1}{2}, \frac{1}{2}) = \pi$.
(4) $\forall n \in \mathbb{N}$ and $\forall p > 0$, $\beta(p, n) = \frac{n-1}{p}\beta(p+1, n-1)$.
(5) $\forall n \in \mathbb{N}$ and $\forall p > 0$, $\beta(p, n) = \frac{(n-1)(n-2)\cdots 2\cdot 1}{p(p+1)\cdots(p+n-1)}$.
(6) $\forall m, n \in \mathbb{N}$, $\beta(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$.
(7) $\forall p, q > 0$, $\beta(p, q) = \int_0^1 \frac{y^{p-1}+y^{q-1}}{(1+y)^{p+q}}dy$.
(8) $\forall p$, $0 < p < 1$, $\beta(p, 1-p) = \int_0^1 \frac{y^{p-1}+y^{-p}}{(1+y)}dy$.
(9) $\forall p$, $0 < p < 1$, $\beta(p, 1-p) = \frac{\pi}{\sin \pi p}$.
(10) $\forall p, q > 0$, $\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Proof. (1) Integrating by parts, we get

$$\beta(p, q+1) = \int_0^1 \frac{q}{p} x^p (1-x)^{q-1} dx = \frac{q}{p}\beta(p+1, q) .$$

(2) We have

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx .$$

So,

$$\beta(p, 1) = \int_0^1 x^{p-1} dx = \frac{1}{p} .$$

(3) Taking $q + 1 = n \in \mathbb{N}$, we get

$$\beta(p, n) = \frac{n-1}{p}\beta(p+1, n-1) .$$

(4) Observing that $\beta(p, 1) = \frac{1}{p}$, we get by iteration

$$\beta(p, n) = \frac{1 \cdot 2 \cdots (n-1)}{p(p+1)\cdots(p+n-1)} .$$

(5) If we take $p = m \in \mathbb{N}$ in the previous equation, we obtain

$$\beta(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} .$$

(6)

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx \quad (x = u^2)$$
$$= 2\int_0^1 \frac{du}{\sqrt{1-u^2}}$$
$$= \pi\,.$$

(7) We have

$$\beta(p,q) = \int_0^{+\infty} \frac{y^{p-1}}{(1+y)^{p+q}}dy$$
$$= \int_0^1 \frac{y^{p-1}}{(1+y)^{p+q}}dy + \underbrace{\int_1^{+\infty} \frac{y^{p-1}}{(1+y)^{p+q}}dy}_{I} \quad \left(y \text{ with } \frac{1}{y} \text{ in I}\right)$$
$$= \int_0^1 \frac{y^{p-1}+y^{q-1}}{(1+y)^{p+q}}dy\,.$$

(8) For $0 < p < 1$, we get

$$\beta(p,1-p) = \int_0^{\infty} \frac{y^{p-1}}{(1+y)}dy$$
$$= \int_0^1 \frac{y^{p-1}}{(1+y)}dy + \int_1^{\infty} \frac{y^{p-1}}{(1+y)}dy$$
$$= \int_0^1 \frac{y^{p-1}+y^{-p}}{(1+y)}\,.$$

(9) Recall that $\frac{1}{1+y} = \sum_{n=0}^{\infty}(-1)^n y^n$ whenever $0 < y < 1$. Hence,

$$\int_0^1 \frac{y^{p-1}}{1+y}dy = \sum_{n=0}^{\infty} \frac{(-1)^n}{p+n}\,.$$

Similarly, we have

$$\int_0^1 \frac{y^{-p}}{1+y}dy = \sum_{n=1}^{\infty} \frac{(-1)^n}{p-n}\,.$$

Therefore,

$$\beta(p,1-p) = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{p-n} = \frac{\pi}{\sin \pi p}\,.$$

(10) By setting $t = y^2$ in the Γ integral, we obtain

$$\Gamma(p) = 2 \int_0^{+\infty} y^{2p-1} e^{-y^2} dy .$$

Thus,

$$\Gamma(p)\Gamma(q) = 4 \int_0^{+\infty} \int_0^{+\infty} x^{2q-1} y^{2p-1} e^{-(x^2+y^2)} dxdy .$$

Next, applying polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, this yields that

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= 4 \int_0^{+\infty} \int_0^{\frac{\pi}{2}} (r\cos\theta)^{2q-1} (r\sin\theta)^{2p-1} e^{-r^2} drd\theta \\
&= 4 \int_0^{+\infty} r^{2(p+q-1)} e^{-r^2} dr \int_0^{+\frac{\pi}{2}} (\cos\theta)^{2q-1} (\sin\theta)^{2p-1} d\theta \\
&= 4 \frac{1}{2} \Gamma(p+q) \frac{1}{2} \beta(p,q) \\
&= \Gamma(p+q)\beta(p,q) .
\end{aligned}$$

□

The following result relates the differentiability of beta to Euler's Γ function. The proof is an immediate consequence of the last property above.

Proposition 69. *The function beta is differentiable and we have*

$$\frac{\partial}{\partial p}\beta(p,q) = \beta(p,q)\left(\frac{\Gamma'(p)}{\Gamma(p)} - \frac{\Gamma'(p+q)}{\Gamma(p+q)}\right) = \beta(p,q)\left(\psi(p) - \psi(p+q)\right) ,$$

where ψ is the so-called di-Gamma function defined by $\psi(p) = \frac{\Gamma'(p)}{\Gamma(p)}$.

In the following, we introduce the complete and incomplete beta functions.

Definition 70. The *complete Beta function* is defined for $a, b > 0$ by

$$\beta(p; a, b) = \int_0^p t^{a-1}(1-t)^{b-1} dt . \tag{4.2}$$

The incomplete (regularized) beta function is

$$I_p(a,b) = \frac{\beta(p;a,b)}{\beta(a,b)}; \quad a, b > 0 . \tag{4.3}$$

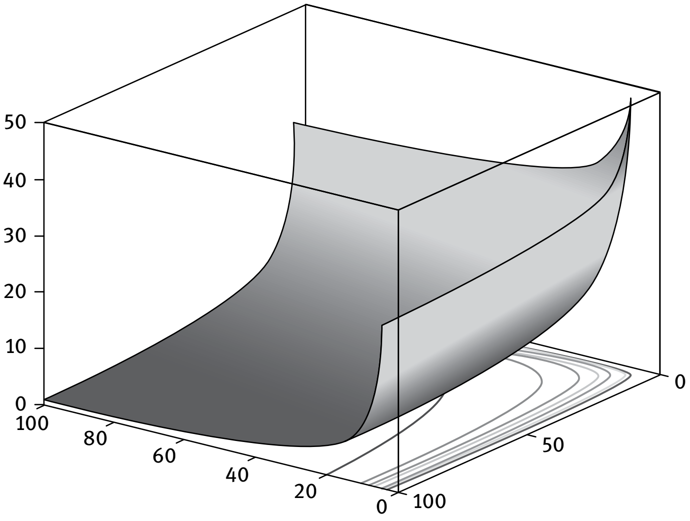

Fig. 4.1: Representations of the beta function.

Figure 4.1 illustrates the graph of the beta function.

4.2.3 Theta function

The theta function appears in many areas, such as manifolds, quadratic forms, soliton theory, and quantum theory.

Definition 71. The function θ is defined for $(z, \tau) \in \mathbb{C}^2$ such that $\mathrm{Im}(\tau) > 0$, by

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2in\pi z} . \tag{4.4}$$

Proposition 72. *We have*

(1) $\forall \tau$ *such that* $\mathrm{Im}(\tau) > 0$, $\theta(., \tau)$ *is a holomorphic function on* $\mathbb{C}$.

(2) $\theta(z + 1, \tau) = \theta(z, \tau)$, $\forall \tau$ *such that* $\mathrm{Im}(\tau) > 0$.

(3) $\theta(z + \tau, \tau) = e^{-i\pi\tau} e^{-2i\pi z} \theta(z, r)$.

Proof. (1) For all τ, the function $z \mapsto e^{i\pi n^2 \tau} e^{2i\pi n z}$ is holomorphic on $\mathbb{C}$. Moreover, for all compact $K \subset \mathbb{C}$, we have

$$\sup_{z \in K} |e^{i\pi n^2 \tau} e^{2i\pi n z}| \le e^{-\pi n^2 \,\mathrm{Im}(\tau)} e^{2\pi R n} ,$$

with R such that $K \subset D(0, R)$. Hence, the series $\sum_n e^{-\pi n^2 \operatorname{Im}(\tau)} e^{Rn}$ is convergent, which yields that $\sum_n e^{i\pi n^2 \tau} e^{2i\pi nz}$ is holomorphic.
(2) $\forall z, \tau$, we have

$$\begin{aligned}
\theta(z+1, \tau) &= \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2i\pi n(z+1)} \\
&= \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2i\pi nz} (e^{2i\pi})^n \\
&= \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2i\pi nz} \\
&= \theta(z, \tau) \, .
\end{aligned}$$

(3) $\forall z, \tau$, we have

$$\begin{aligned}
\theta(z+\tau, \tau) &= \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2i\pi n(z+\tau)} \\
&= \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2i\pi nz} (e^{2i\pi})^n \\
&= \sum_{n \in \mathbb{Z}} e^{i\pi (n^2+2n) \tau} e^{2i\pi nz} \\
&= \sum_{n \in \mathbb{Z}} e^{i\pi ((n+1)^2-1) \tau} e^{2i\pi nz} \\
&= e^{-i\pi\tau} \sum_{n \in \mathbb{Z}} e^{i\pi ((n+1)^2) \tau} e^{2i\pi nz} \\
&= e^{-i\pi\tau} \sum_{n \in \mathbb{Z}} e^{i\pi n^2 \tau} e^{2i\pi (n-1) z} \\
&= e^{-i\pi\tau} e^{-2i\pi z} \theta(z, r) \, .
\end{aligned}$$

□

Proposition 73.
(1) *For all τ such that* $\operatorname{Im}(\tau) > 0$, *we have*

$$\sqrt{\frac{\tau}{i}} \theta(z, \tau) = e^{\frac{-i\pi z^2}{\tau}} \theta\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) \, . \tag{4.5}$$

(2) *For $t > 0$, let $\Theta(t) = \theta(0, it)$. Then,*

$$\sqrt{t} \Theta(t) = \Theta\left(\frac{1}{t}\right) \, .$$

Proof. Denote for $x \in \mathbb{R}$, $f(x) = e^{i\pi x^2 \tau} e^{2i\pi xz}$. From the well-known Poisson summation formula, we obtain

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) \, .$$

On the other hand, note that

$$f(x)=G_\alpha\left(x+\frac{z}{\tau}\right)e^{\frac{-i\pi z^2}{\tau}}\ \text{with } \alpha=-2i\pi\tau \quad\text{and}\quad G_\alpha(t)=e^{-\frac{\alpha t^2}{2}}\,.$$

Therefore,

$$\begin{aligned}\widehat{f}(\omega)&=\widehat{G_\alpha}\left(.+\frac{z}{\tau}\right)(\omega)e^{\frac{-i\pi z^2}{\tau}}\\ &=e^{2i\pi\frac{z}{\tau}\omega}e^{\frac{-2\pi^2\omega^2}{\alpha}}\sqrt{\frac{2\pi}{\alpha}}e^{\frac{-i\pi z^2}{\tau}}\\ &=e^{2i\pi\frac{z}{\tau}\omega}e^{\frac{\pi\omega^2}{i\tau}}\sqrt{\frac{1}{-i\tau}}e^{\frac{-i\pi z^2}{\tau}}\\ &=e^{2i\pi\frac{z}{\tau}\omega}e^{\frac{-i\pi\omega^2}{\tau}}\sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}}\,.\end{aligned}$$

Hence,

$$\sum_{n\in\mathbb{Z}}\widehat{f}(n)=\sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}}\sum_{n\in\mathbb{Z}}e^{2i\pi\frac{z}{\tau}n}e^{\frac{-i\pi n^2}{\tau}}=\sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}}\theta\left(\frac{z}{\tau},\frac{-1}{\tau}\right)\,.$$

Consequently,

$$\theta(z,\tau)=\sqrt{\frac{i}{\tau}}e^{\frac{-i\pi z^2}{\tau}}\theta\left(\frac{z}{\tau},\frac{-1}{\tau}\right)\,,$$

or equivalently,

$$\sqrt{\frac{\tau}{i}}\theta(z,\tau)=e^{\frac{-i\pi z^2}{\tau}}\theta\left(\frac{z}{\tau},\frac{-1}{\tau}\right)\,.$$

□

4.2.4 Riemann zeta function

The Riemann zeta function is often known in number theory and in particular in the study of the distribution of prime numbers.

Definition 74. The Riemann zeta function is defined for $x>1$ by

$$\zeta(x)=\sum_{n=1}^{+\infty}\frac{1}{n^x}\,. \tag{4.6}$$

Remark 75. The definition may be extended to complex numbers $x=a+ib$ with $a>1$.

Proposition 76. *The ζ Riemann's function satisfies the so-called Euler's multiplication*

$$\zeta(x) = \prod_{p\in\mathcal{P}} \frac{1}{(1-p^{-x})}, \qquad \forall x > 1\,,$$

where $\mathcal{P}$ is the set of prime numbers.

Proof. For $x > 1$, we have

$$\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \frac{1}{5^x} + \cdots$$

Thus,

$$\frac{1}{2^x}\zeta(x) = \frac{1}{2^x} + \frac{1}{4^x} + \frac{1}{6^x} + \frac{1}{8^x} + \frac{1}{10^x} + \cdots$$

Or equivalently,

$$\left(1 - \frac{1}{2^x}\right)\zeta(x) = 1 + \frac{1}{3^x} + \frac{1}{5^x} + \frac{1}{7^x} + \frac{1}{9^x} + \cdots$$

Multiplying again by $\frac{1}{3^x}$, we get

$$\frac{1}{3^x}\left(1 - \frac{1}{2^x}\right)\zeta(x) = \frac{1}{3^x} + \frac{1}{9^x} + \frac{1}{15^x} + \frac{1}{21^x} + \frac{1}{27^x} + \cdots$$

Hence,

$$\left(1 - \frac{1}{3^x}\right)\left(1 - \frac{1}{2^x}\right)\zeta(x) = 1 + \frac{1}{5^x} + \frac{1}{7^x} + \cdots$$

Next, by following the same process we get for $p \in \mathcal{P}$,

$$\left(1 - \frac{1}{p^x}\right)\cdots\left(1 - \frac{1}{11^x}\right)\left(1 - \frac{1}{7^x}\right)\left(1 - \frac{1}{5^x}\right)\left(1 - \frac{1}{3^x}\right)\left(1 - \frac{1}{2^x}\right)\zeta(x) = 1 + \sum_{n>p} \frac{1}{n^x}\,.$$

Next, note that the last summation goes to 0 as $p \to \infty$. Therefore,

$$\zeta(x) \prod_{p\in\mathcal{P}} (1 - p^x) = 1\,.$$

Hence,

$$\zeta(x) = \prod_{p\in\mathcal{P}} \frac{1}{(1-p^{-x})}\,. \qquad \square$$

Proposition 77.
(1) *ζ is continuous, nonincreasing, and convex on $]1, +\infty[$.*
(2) *ζ is $\mathcal{C}^\infty$ on $]1, +\infty[$ and*

$$\zeta^{(k)}(x) = (-1)^k \sum_{n=2}^{\infty} \frac{(\ln n)^k}{n^x}; \qquad \forall k \in \mathbb{N} \text{ and } x > 1\,.$$

Proof. (1) Let $a > 1$. For $n \in \mathbb{N}^*$, the function $x \mapsto \frac{1}{n^x}$ is continuous on $[a, +\infty[$. Moreover, $\forall x \in [a, +\infty[$,

$$|\frac{1}{n^x}| = \frac{1}{n^x} \leq \frac{1}{n^a} .$$

Thus, the series $\sum_n \frac{1}{n^a}$ is normally convergent. Hence, the sum ζ is continuous on $[a, +\infty[$. This being true for all real $a \in]1, +\infty[$. Henceforth, ζ is continuous on $]1, +\infty[$.

Next, the *monotony of* ζ follows from the fact that for all $n \in \mathbb{N}$, the functions $x \mapsto \frac{1}{n^x}$ is nonincreasing on $]1, +\infty[$.

Finally, to prove the *convexity of the function* ζ, recall that for all $n \in \mathbb{N}$, the functions $x \mapsto \frac{1}{n^x}$ is convex on $]1, +\infty[$. So, ζ is convex on $]1, +\infty[$ as a sum of convex functions $]1, +\infty[$.

(2) Let $a > 1$. For all $n \in \mathbb{N}$, the function $f_n : x \mapsto \frac{1}{n^x}$, is $\mathcal{C}^\infty$ on $[a, +\infty[$ and for $x \geq a$ and $k \geq 1$, we have

$$\left|f_n^k(x)\right| = \left|(-1)^k \frac{(\ln n)^k}{n^a}\right| \leq \frac{(\ln n)^k}{n^a} .$$

Note that $\sum_n \frac{(\ln n)^k}{n^a}$ converges by the Bertrand rule of numerical series. So, we deduce that for $k \geq 1$, the series $\sum_n f_n^{(k)}$ is normally convergent on $[a, +\infty[$. As a result, ζ is $\mathcal{C}^k$ on $[a, +\infty[$ for all k. Hence, it is $\mathcal{C}^\infty$ on $[a, +\infty[$ for all $a > 1$. So, it is $\mathcal{C}^\infty$ on $]1, +\infty[$ and the derivatives are obtained as stated above. □

Proposition 78. *The ζ function satisfies*
- $\lim_{x \longrightarrow +\infty} \zeta(x) = 1$.
- $\lim_{x \longrightarrow 1^+} \zeta(x) = +\infty$.

Proof. (1) Note first that the series $\sum_{n \geq 1} \frac{1}{n^2}$ converges. Henceforth, the series $\zeta(x)$ is uniformly convergent on the interval $[2, +\infty[$. Furthermore,

$$\lim_{x \to +\infty} \frac{1}{n^x} = \begin{cases} 1, & \text{for } n = 1 , \\ 0, & \text{for } n > 1 . \end{cases}$$

So, by applying the limit on $\zeta(x)$ at infinity we get

$$\lim_{x \longrightarrow +\infty} \zeta(x) = 1 + \sum_{n \geq 2} 0 = 1 .$$

(2) holds from the fact that ζ is nonincreasing on $]1, +\infty[$ and that $\sum_{n \geq 1} \frac{1}{n} = +\infty$. □

Proposition 79. *The ζ function can be extended on the band $\Omega = \{z \in \mathbb{C}; \operatorname{Re}(s) > 1\}$ in a holomorphic function. With higher derivative $\zeta^{(k)}$, $k \in \mathbb{N}$ is given by*

$$\zeta^{(k)}(z) = \sum_{n=1}^{+\infty} \frac{(-1)^k \ln^k n}{n^z} . \tag{4.7}$$

Proof. (1) The function $f_n(z) = \frac{1}{n^z}$, $n \geq 1$ is holomorphic, and the series $\sum_n f_n$ is uniformly convergent on all sets of the form $\Omega_a = \{z \in \mathbb{C}; \operatorname{Re}(z) > a\}$ for all $a > 1$. So the sum ζ is holomorphic on $\operatorname{Re}(z) > 1$.
(2) For $k \in \mathbb{N}$, we have $f_n^{(k)}(z) = \frac{(-1)^k \ln^k n}{n^z}$. On any set Ω_a, the series $\sum_n f_n^{(k)}$ is uniformly convergent. Hence, ζ is $\mathcal{C}^k$ and its derivative of order k is given by (4.7). □

Proposition 80. *The function ζ has a meromorphic extension on $\mathbb{C}$, with a single pole in 1 which is simple.*

To prove this result, we need to recall that the well-known Bernoulli numbers, denoted by B_n, form a sequence of rational numbers. These numbers were first studied by Jacques Bernoulli in the context of computing summations of the form $S_m(n) = \sum_{k=0}^{n-1} k^m$ for different integer values m. It holds that these quantities are polynomials of the variable n with degree $m+1$. Hence, we can write them in the form

$$S_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} C_{m+1}^k B_k n^{m+1-k} . \tag{4.8}$$

The numbers B_k are called the Bernoulli numbers. These numbers may also be defined by means of a generator function as

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k . \tag{4.9}$$

Generally, these numbers may be extended to polynomials. The well-known Bernoulli polynomials are obtained from the following relation:

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} x^k . \tag{4.10}$$

It yields a sequence of polynomials of degree k in x. For more details, refer to [67]. These are applied in numerous fields. We recall here one application that will be used later. It consists of the well-known Euler–Maclaurin summation rule for functions.

Proposition 81. *Let f be $\mathcal{C}^{2k}$ function on $[p,q]$, $p,q \in \mathbb{Z}$ and $k \in \mathbb{N}$. It holds that*

$$\frac{f(p)+f(q)}{2} + \sum_{i=p+1}^{q-1} f(i) = \sum_{j=1}^{k} \frac{B_{2j}}{(2j!)} \left(f^{(2j-1)}(q) - f^{(2j-1)}(p)\right) + \int_p^q f(x)dx + R_{p,q}^k \,,$$

where $R_{p,q}^k$ is the rest

$$R_{p,q}^k = -\frac{1}{(2k)!} \int_p^q f^{(2k)}(x) B_{2k}(x-[x])dx$$

where $B_{2k}(.)$ is the Bernoulli polynomial of degree $2k$.

Proof of Proposition 80. By applying Euler–Maclaurin summation to the function $f(x) = \frac{1}{(1+x)^z}$ on the interval $[0,n]$, we get

$$\frac{1+(1+n)^{-z}}{2} + \sum_{i=1}^{n-1} f(i) = \int_0^n f(x)dx + \sum_{j=1}^{k} \frac{b_{2j}}{(2j)!} \left(f^{(2j-1)}(n) - f^{(2j-1)}(0)\right) + R_k \,.$$

Letting n tend to $+\infty$, we will have

$$\frac{1}{2} + \underbrace{\int_0^{+\infty} (1+t)^{-z} dt}_{(z-1)^{-1}} - \sum_{l=1}^{p} \frac{b_{2l}}{2l!} f^{(2l-1)}(0) - \int_0^{\infty} \frac{B_{2p}(t)}{(2p)!} f^{(2p)}(t)dt \,,$$

where

$$f^{(k)}(x) = \frac{-z(-z-1)\cdots(-z-k+1)}{(1+x)^{z+k}} = (-1)^k \frac{z(z+1)\cdots(z+k-1)}{(1+x)^{z+k}} \,.$$

So for $\mathrm{Re}(z) > 1$,

$$\zeta(z) = \underbrace{\frac{1}{2} + \frac{1}{z-1}}_{\text{meromorphic}} + \underbrace{\sum_{l=1}^{p} \frac{b_{2l}}{(2l)!} z\cdots(z+2l-2) + I_p}_{\text{holomorphic function}}$$

with

$$I_p(z) = -\int_0^{+\infty} \frac{z\cdots(z+2p-1)}{(1+t)^{z+2p}} B_{2p}(t)dt \,.$$

Now, the function $z \mapsto \frac{z\cdots(z+2p-1)}{(1+t)^{z+2p}}B_{2p}(t)$ is holomorphic and we have for all $\delta > 0$ and all z; $\mathrm{Re}(z) \geq 1-2p+\delta$,

$$\left|\frac{z\cdots(z+2p-1)}{(1+t)^{z+2p}}B_{2p}(t)\right| \leq \frac{|b_{2p}|z\cdots(z+2p-1)}{(1+t)^{1+\delta}} .$$

So, I_p is holomorphic on $\mathrm{Re}(z) > 1-2p$. □

Proposition 82. *The function ζ can be expressed in the integral form as follows.*

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^1 \frac{(-\ln u)^{z-1}}{1-u}\mathrm{d}u = \frac{1}{\Gamma(z)}\int_0^{+\infty}\frac{t^{z-1}}{\mathrm{e}^t-1}\mathrm{d}t, \quad \mathrm{Re}(z) > 1 ,$$

where Γ is the Euler function.

Proof. We have

$$\zeta(z)\Gamma(z) = \sum_{n\geq 1}\frac{\Gamma(z)}{n^z} = \sum_{n\geq 1}\int_0^{+\infty}\mathrm{e}^{-u}\left(\frac{u}{n}\right)^{z-1}\frac{\mathrm{d}u}{n} .$$

By setting $u = nt$, we obtain

$$\int_0^{+\infty}\mathrm{e}^{-u}\left(\frac{u}{n}\right)^{z-1}\frac{\mathrm{d}u}{n} = \int_0^{+\infty}\mathrm{e}^{-nt}t^{z-1}\mathrm{d}t .$$

Hence, using the monotone convergence theorem, we obtain

$$\zeta(z)\Gamma(z) = \sum_{n\geq 1}\int_0^{+\infty}\mathrm{e}^{-nt}t^{z-1}\mathrm{d}t = \int_0^{+\infty}\mathrm{e}^{-t}\frac{1}{1-\mathrm{e}^{-t}}t^{z-1}\mathrm{d}t = \int_0^{+\infty}\frac{t^{z-1}}{\mathrm{e}^t-1}\mathrm{d}t .$$ □

Proposition 83. *The function ζ satisfies the following quasi-induction rule:*

$$\zeta(x) = 2^x\pi^{x-1}\sin\left(\frac{\pi x}{2}\right)\Gamma(1-x)\zeta(1-x); \quad \forall x \in \mathbb{C}\setminus\{0,1\} .$$

Proof. Let ε be such that $0 < \varepsilon < \pi$ and $n \in \mathbb{N}$. Consider the path C_ε^n represented in Figure 4.2 and the function

$$f_s(z) = \frac{(-z)^{s-1}}{e^z-1}$$

with s being fixed. So, applying the residue theorem and letting $R \to +\infty$, $\varepsilon \to 0$, and next $n \to +\infty$, we get

$$2i\pi(2\pi)^{s-1}\zeta(1-s)2\sin\left(\frac{\pi x}{2}\right) = \Gamma(s)\zeta(s)2i\sin(s\pi) .$$

Next, using Proposition 63, we get

$$(2\pi)^{s-1}\zeta(1-s)2\sin\left(\frac{\pi x}{2}\right) = \zeta(s)\frac{1}{\Gamma(1-s)}\,.$$

Or equivalently

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi x}{2}\right)\zeta(1-s)\Gamma(1-s)\,.$$ □

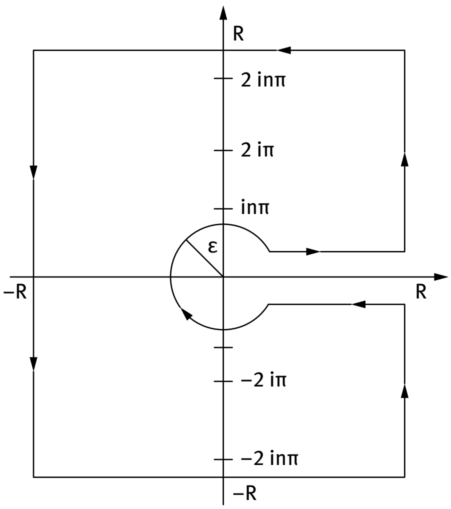

Fig. 4.2: The path C_ε^n.

Finally, Figure 4.3 graphically illustrates the ζ function.

4.2.5 Hypergeometric function

The origin of hypergeometric functions goes back to the early 19th century, when Gauss studied the second-order ordinary differential equation

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0 \tag{4.11}$$

with some constants a, b, and c in $\mathbb{R}$. Next, by developing a solution of (4.11) on a series of form $\sum_n \alpha_n x^x$, we obtain for c, $a-b$, and $c-a-b$ not integers, a general solution given by

$$y = F(a,b,c,x) + Bx^{1-c}F(a-c+1, b-c+1, 2-c, x) \tag{4.12}$$

where F is the series

$$F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{x^n}{n!}\,,$$

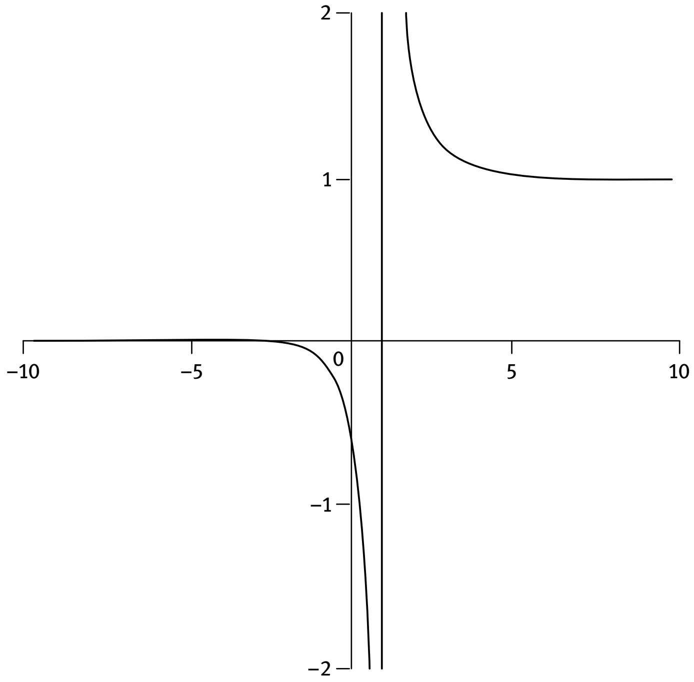

Fig. 4.3: General shape of the zeta function for (−10) to +10.

which is often denoted by ${}_2F_1(a, b, c, x)$, converges uniformly inside the unit disk and is known as the hypergeometric function.

When a, b, and c are integers, the hypergeometric function can be reduced to a transcendental function such as

$$ {}_2F_1(1, 1; 2; x) = -x^{-1} \ln(1 - x) \,. $$

Theorem 84. *F is differentiable with respect to x and*

$$ \frac{\partial F}{\partial x}(a, b, c, x) = \frac{ab}{c} F(a + 1, b + 1, c + 1, x) \,. $$

Proof. Write

$$ F(a, b, c, x) = \sum_{n=0}^{\infty} \alpha_n(a, b, c) x^n \,, $$

where

$$ \alpha_n = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)\Gamma(n + 1)} \,. $$

Inside its convergence domain, we have

$$\frac{\partial F}{\partial x} = \sum_{n=0}^{\infty} (n+1)\alpha_{n+1} x^n \,.$$

Observe next that

$$\begin{aligned}(n+1)\alpha_{n+1}(a,b,c) &= \frac{a(a+1)\cdots(a+n)b(b+1)\cdots(b+n)}{n!c(c+1)\cdots(c+n)}\\ &= \frac{ab}{c}\alpha_n(a+1,b+1,c+1)\,.\end{aligned}$$

Hence,

$$\frac{\partial F}{\partial x} = \frac{ab}{c}\sum_{n=0}^{\infty} \alpha_n(a+1,b+1,c+1)x^n = \frac{ab}{c}F(a+1,b+1,c+1,x)\,. \qquad \square$$

Theorem 85. *For* $0 < \operatorname{Re} b < \operatorname{Re} c$, $\operatorname{Re} a < \operatorname{Re} c - \operatorname{Re} b$, *and* $|x| \le 1$, *it holds that*

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;x) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt\,.$$

Proof. Let, for $|x| < 1$,

$$I = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt\,.$$

It is straightforward that I is a convergent integral. Next, we have

$$\begin{aligned}(1-tx)^{-a} &= \sum_{n=0}^{\infty} \frac{(-a)(-a-1)\cdots(-a-n+1)}{n!}(-tx)^n\\ &= \sum_{n=0}^{\infty} \frac{(a)(a+1)\cdots(a+n-1)}{n!}(tx)^n\\ &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} t^n x^n\,.\end{aligned}$$

Hence,

$$
\begin{aligned}
I &= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} x^n \int_0^1 t^{b+n-1}(1-t)^{c-b-1}dt \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)\Gamma(n+1)} x^n \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)} \\
&= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)\Gamma(n+1)\Gamma(b)} x^n \\
&= \frac{\Gamma(c-b)\Gamma(b)}{\Gamma(a)} F(a,b;c;x) .
\end{aligned}
$$

□

Theorem 86.

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} .$$

Proof. Taking $x = 1$ in the integral expression of the hypergeometric function in Theorem 85, one obtains

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b;c;1) = \int_0^1 t^{b-1}(1-t)^{c-a-b-1}dt = \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} .$$

Therefore,

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} .$$

□

Proposition 87. *We have*
(1) $F(n,1,1,x) = (1-x)^{-n}$.
(2) $xF(1,1,2,x) = -\log(1-x)$.
(3) $\lim_{\beta\leftrightarrow\infty} F(1,\beta,1,\frac{x}{\beta}) = e^x$.
(4) $\lim_{\beta\to\infty} xF(\alpha,\beta,\frac{3}{2},\frac{-x^2}{4\alpha\beta}) = \sin x$.
(5) $\lim_{\beta\to\infty} xF(\alpha,\beta,\frac{1}{2},\frac{-x^2}{4\alpha\beta}) = \cos x$.
(6) $xF(\frac{1}{2},\frac{1}{2},\frac{3}{2},x^2) = \arcsin x$.
(7) $xF(\frac{1}{2},1,\frac{3}{2},-x^2) = \arccos x$.

Proof. (1) Denote $y(x) = (1-x)^{-n}$. It is straightforward that y is a solution of (4.11) for $a = n$, and $b = c = 1$, and with $y(0) = 1$. So, (4.12) says that

$$y(x) = F(n,1,1,x) + Bx^{1-1}F(n-1+1, 1-1+1, 2-1, x) = CF(n,1,1,x)$$

with B and thus C being constants. Observing next that $F(n, 1, 1, 0) = 1$, we get $C = 1$ or equivalently $B = 0$.
(2) Again the function $y(x) = -x^{-1}\log(1-x)$ is a solution of (4.11) for $a = b = 1$, and $c = 2$. Hence, it is of the form

$$y(x) = F(1, 1, 2, x) + Bx^{1-2}F(0, 0, 0, x) ,$$

or equivalently,

$$-\log(1-x) = xF(1, 1, 2, x) + Be^x ,$$

which by setting $x = 0$ gives $B = 0$.
(3) Recall firstly that

$$F\left(1, \beta, 1, \frac{x}{\beta}\right) = \frac{\Gamma(1)}{\Gamma(1)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\Gamma(\beta+n)}{\Gamma(1+n)} \frac{x^n}{\beta^n n!} ,$$

which means that

$$F\left(1, \beta, 1, \frac{x}{\beta}\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)}{\beta^n \Gamma(\beta)} \frac{x^n}{n!} .$$

So, let $K \in \mathbb{N}$ be fixed such that $2|x| \le K$ and denote

$$u_n(\beta) = \frac{\Gamma(\beta+n)}{\beta^n \Gamma(\beta)} \frac{x^n}{n!} .$$

It is straightforward that for $\beta \ge K$, we have

$$|u_n(\beta)| \le v_n = |u_n(K)| = \frac{\Gamma(K+n)}{K^n \Gamma(K)} \frac{|x|^n}{n!} .$$

Next, observe that

$$\lim_{n\to+\infty} \frac{v_{n+1}}{v_n} = \frac{|x|}{K} < 1 .$$

Hence, the D'Alembert rule affirms that the series $F(1, \beta, 1, \frac{x}{\beta})$ converges uniformly in β in the interval $[K, +\infty[$. Observing now that

$$\lim_{\beta\to+\infty} u_n(\beta) = 1 ,$$

we get

$$\lim_{\beta\to+\infty} F\left(1, \beta, 1, \frac{x}{\beta}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x .$$

(4) Recall that

$$F\left(\alpha, \beta, \frac{3}{2}, \frac{-x^2}{4\alpha\beta}\right) = \frac{\Gamma(\frac{3}{2})}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\frac{3}{2}+n)} \frac{(-1)^n x^{2n}}{4^n \alpha^n \beta^n n!} .$$

Denote

$$\sigma_n(\alpha) = \frac{\Gamma(\alpha+n)}{\alpha^n \Gamma(\alpha)} \frac{x^n}{n!} .$$

We get

$$F\left(\alpha, \beta, \frac{3}{2}, \frac{-x^2}{4\alpha\beta}\right) = \sum_{n=0}^{\infty} \sigma_n(\alpha)\sigma_n(\beta)\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+n)}\frac{(-1)^n x^{2n}}{4^n n!} .$$

Using similar arguments as for (3) above and the properties of Euler's Γ function, we get

$$\lim_{\alpha,\beta\to+\infty} F\left(\alpha, \beta, \frac{3}{2}, \frac{-x^2}{4\alpha\beta}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \frac{\sin x}{x} .$$

(5) Follows by quite the same techniques as the previous assertion.
(6) Observe that

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) = \frac{1}{2\Gamma(\frac{1}{2})}\sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n)}{\frac{1}{2}+n}\frac{x^{2n}}{n!} .$$

Next, using the well-known relation $\Gamma(x+1) = x\Gamma(x)$, for $x > 0$, we obtain

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{n+1}(2n+1)(n!)^2}x^{2n} = \frac{\arcsin x}{x} .$$

(7) Follows by the same arguments as assertion (6). □

Definition 88. The hypergeometric function may be generalized for $a = (a_1, \dots, a_p)$ and $b = (b_1, \dots, b_q)$, $p, q \in \mathbb{N}$ by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q, x) = \sum_{n=0}^{\infty} \alpha_n x^n ,$$

where

$$\alpha_0 = 1 \quad \text{and} \quad \frac{\alpha_{n+1}}{\alpha_n} = \frac{(n+a_1)(n+a_2)\cdots(n+a_p)}{(n+b_1)(n+b_2)\cdots(n+b_q)}\frac{1}{n+1} ,$$

or differently by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_k(b_2)_n\cdots(b_q)_n}\frac{x^n}{n!} ,$$

where $(a)_n$ is the *increasing factorial* or the *Pochhammer symbol* given by

$$(a)_n = \frac{(a+n-1)!}{(a-1)!} = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+n-1) .$$

4.2.6 Legendre function

Legendre functions are fundamental solutions of the Laplace equation on the sphere. There are two classes of solutions that are related to the parameters λ and μ, which will be explained later. In the following, we denote the first kind by P_λ and the second kind by Q_λ. The associated Legendre functions corresponding to P_λ and Q_λ are denoted by P_λ^μ and Q_λ^μ, respectively. These are respective generalizations of Legendre polynomials $P_\ell(x)$ and associated Legendre polynomials $P_\ell^m(x)$, to noninteger values of ℓ and m.

Definition 89. The Legendre functions are solutions of the general Legendre equation

$$(1-x^2)y'' - 2xy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-x^2}\right] y = 0\,,$$

where λ and μ are generally complex numbers called, respectively, the degree and the order of the associated Legendre function.

The case of Legendre functions corresponding to $\mu = 0$ and $\lambda \in \mathbb{N}$ reduces to *orthogonal Legendre polynomials.*

Proposition 90.
(1) *For $\mu = 0$, the following integral form is a Legendre function:*

$$F_\lambda(z) = \frac{1}{2\pi i} \int_C \frac{(t^2-1)^\lambda}{2^\lambda (t-z)^{\lambda+1}} dt\,,$$

for $|z-1| < 2$ where $\mathcal{C}$ is a circle surrounding the points 1 and z and not -1.
(2) *For $\lambda \in \mathbb{C}$ and $|x| > 1$, $x \in \mathbb{R}$, we get*

$$\begin{aligned} F_\lambda(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(x + \sqrt{x^2-1}\cos\theta\right)^\lambda d\theta \\ &= \frac{1}{\pi} \int_0^1 \left(x + \sqrt{x^2-1}(2t-1)\right)^\lambda \frac{dt}{\sqrt{t(1-t)}}\,. \end{aligned}$$

Proof. (1) Applying the derivatives of F_λ, we get

$$\begin{aligned} &(1-z^2)F_\lambda''(z) - 2zF_\lambda'(z) + \lambda(\lambda+1)F_\lambda(z) \\ &= \frac{\lambda+1}{2\pi i} \int_C \frac{(t^2-1)^\lambda}{2^\lambda (t-z)^{\lambda+3}} (\lambda t^2 - 2(\lambda+1)zt + \lambda + 2) dt \\ &= \frac{\lambda+1}{2\pi i} \int_C \frac{d}{dt} \frac{(t^2-1)^{\lambda+1}}{2^\lambda (t-z)^{\lambda+2}} dt = 0\,. \end{aligned}$$

So, F_λ satisfies the Legendre equation.
(2) Consider for the integral form the circle $\mathcal{C}$ centered at x with radius $r = \sqrt{x^2-1}$. We first obtain

$$t^2 - 1 = \sqrt{x^2-1}e^{i\theta}2(x + \sqrt{x^2-1}\cos\theta), \ \theta \in [-\pi, \pi] .$$

Hence,

$$\frac{(t^2-1)^\lambda}{2^\lambda (t-z)^{\lambda+1}}dt = \frac{\sqrt{x^2-1}^\lambda e^{i\lambda\theta}2^\lambda (x+\sqrt{x^2-1}\cos\theta)^\lambda}{2^\lambda \sqrt{x^2-1}^{\lambda+1} e^{i(\lambda+1)\theta}} i\sqrt{x^2-1}e^{i\theta}d\theta .$$

As a result,

$$F_\lambda(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(x + \sqrt{x^2-1}\cos\theta\right)^\lambda d\theta .$$

Next, setting $t = \frac{1+\cos\theta}{2}$, we obtain

$$F_\lambda(x) = \frac{1}{\pi}\int_0^1 \left(x + \sqrt{x^2-1}(2t-1)\right)^\lambda \frac{dt}{\sqrt{t(1-t)}} . \qquad \square$$

Proposition 91. *The following are Legendre functions:*
- *The first-kind function P_λ^μ defined for $|1-z| < 2$ by*

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right) ,$$

  where Γ is Euler's Gamma function.
- *The second-kind function $Q_\lambda^\mu(z)$ defined for $|z| > 1$ by*

$$Q_\lambda^\mu(z) = C_\mu^\lambda \frac{(z^2-1)^{\mu/2}}{z^{\lambda+\mu+1}} {}_2F_1\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}; \lambda+\frac{3}{2}; \frac{1}{z^2}\right) ,$$

where $C_\mu^\lambda = \frac{\sqrt{\pi}\Gamma(\lambda+\mu+1)}{2^{\lambda+1}\Gamma(\lambda+3/2)}e^{i\mu\pi}$, and ${}_2F_1$ is the hypergeometric function.

Proof. It suffices to show that the functions

$$F(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right), \quad |1-z| < 2$$

and for $|z| > 1$,

$$F(z) = C_\mu^\lambda \frac{(z^2-1)^{\mu/2}}{z^{\lambda+\mu+1}} {}_2F_1\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}; \lambda+\frac{3}{2}; \frac{1}{z^2}\right)$$

are solutions of the general Legendre Definition 89. We will develop the first part. The second follows by similar techniques. So, for simplicity denote

$$H(z) = \frac{1}{\Gamma(1-\mu)} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right) ,$$

$$A(z) = \frac{1}{1-z^2}, \quad B(z) = \left[\frac{1+z}{1-z}\right]^{\mu/2}$$

and $Z = \frac{1-z}{2}$. Standard calculus yields that

$$(1-z^2)H''(z) + 2(\mu_z)H'(z) + \lambda(\lambda+1)H(z) = 0 , \tag{4.13}$$

$$F'(z) = \mu A(z)B(z)H(z) + B(z)H'(z) ,$$

and

$$F''(z) = \mu(\mu+2z)A^2(z)B(z)H(z) + 2\mu A(z)B(z)H'(z) + B(z)H''(z) . \qquad \square$$

Now, recall the Legendre equation

$$(1-z^2)y'' - 2zy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-z^2}\right] y = 0 .$$

Replacing y by F and taking into account equation (4.13), we show that F is a Legendre function.

Proposition 92. *The Legendre function F_λ satisfies the following three-level induction rule:*

$$(\lambda+1)F_{\lambda+1}(z) - (2\lambda+1)zF_\lambda(z) + \lambda F_{\lambda-1}(z) = 0 .$$

Proof. Let C be the contour as above. We have

$$F_\lambda(z) = \frac{1}{2^{\lambda+1}\pi i} \int_C \frac{(t^2-1)^\lambda}{(t-z)^{\lambda+1}} dt .$$

Classical arguments show that F_λ is holomorphic and

$$F'_\lambda(z) = \frac{(\lambda+1)}{2^{\lambda+1}\pi i} \int_C \frac{(t^2-1)^\lambda}{(t-z)^{\lambda+2}} dt .$$

On the other hand,

$$\frac{d}{dt} \frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+1}} = \frac{2(\lambda+1)t(t^2-1)^\lambda}{(t-z)^{\lambda+1}} - \frac{(\lambda+1)(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}} .$$

Hence,

$$0 = \int_C \left(\frac{2t(t^2-1)^\lambda}{(t-z)^{\lambda+1}} - \frac{(t^2-1)^{\lambda+1}}{(t-z)^{\lambda+2}} \right) dt .$$

Consequently,

$$\begin{aligned}\frac{1}{2^{\lambda+1}\pi i}\int_C \frac{(t^2-1)^\lambda}{(t-z)^\lambda} &= \frac{1}{2^{\lambda+1}\pi i}\int_C \frac{t(t^2-1)^\lambda}{(t-z)^{\lambda+1}} \\ &\quad - \frac{1}{2^{\lambda+1}\pi i}\int_C \frac{z(t^2-1)^\lambda}{(t-z)^{\lambda+1}} \\ &= F_{\lambda+1} - zF_\lambda(z)\,. \end{aligned} \tag{4.14}$$

Differentiating with respect to z, we obtain

$$F'_{\lambda+1}(z) - zF'_\lambda(z) = (\lambda+1)F_\lambda(z)\,.$$

Thus,

$$\begin{aligned} 0 &= \int_C \frac{d}{dt}\left[\frac{t(t^2-1)^\lambda}{(t-z)^\lambda}\right] dt \\ &= \int_C \left[\frac{(t^2-1)^\lambda}{(t-z)^\lambda} + \frac{2\lambda t^2(t^2-1)^{\lambda-1}}{(t-z)^\lambda} - \frac{\lambda t(t^2-1)^\lambda}{(t-z)^{\lambda+1}}\right] dt \\ &= \int_C \frac{(t^2-1)^\lambda + 2\lambda[(t^2-1)+1](t^2-1)^{\lambda-1}}{(t-z)^\lambda} - \frac{\lambda[(t-z)+z](t^2-1)^\lambda}{(t-z)^{\lambda+1}} \\ &= \int_C \left[(\lambda+1)\frac{(t^2-1)^\lambda}{(t-z)^\lambda} + 2\lambda\frac{(t^2-1)^{\lambda-1}}{(t-z)^\lambda} - \lambda z\frac{(t^2-1)^\lambda}{(t-z)^{\lambda+1}}\right] dt\,. \end{aligned}$$

Finally using (4.14), we deduce that

$$\begin{aligned} 0 &= (\lambda+1)[F_{\lambda+1}(z) - zF_\lambda(z)] + 2\lambda F_{\lambda-1}(z) - \lambda z F_\lambda(z) \\ &= (\lambda+1)F_{\lambda+1}(z) - (2\lambda+1)zF_\lambda(z) + 2\lambda F_{\lambda-1}(z)\,. \end{aligned}$$ □

4.2.7 Bessel function

Bessel functions form an important class of special functions and are applied almost everywhere in mathematical physics. They are also known as cylindrical functions, or cylindrical harmonics, because they are part of the solutions of the Laplace equation in cylindrical coordinates met in heat propagation along a cylinder. In pure mathematics, Bessel functions can be introduced in three ways: as solutions of second-order differential equations, through a recurrent procedure as solutions of a three-level recurrent functional equation, and via the Rodrigues derivation formula.

Definition 93. The Bessel equation is a linear differential equation of second order written in the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{v^2}{x^2}\right)y = 0 \,,$$

where v is a positive constant.

Remark 94.

(1) Any solution of Bessel's equation is called the *Bessel function*.

(2) Given two linearly independent solutions y_1 and y_2 of Bessel's differential equation, the general solution is expressed as a linear combination

$$y = C_1 y_1 + C_2 y_2 \,,$$

where C_1 and C_2 are two constants.

Theorem and Definition 95. *Bessel's differential equation has a solution of the form*

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k} \,. \tag{4.15}$$

The function J_v is called the Bessel function of the first kind of the order v.

Proof of the Theorem. We will search a nontrivial solution of the form

$$y = x^p \sum_{i\geq 0} a_i x^i = \sum_{i\geq 0} a_i x^{i+p} \,,$$

where p is a real parameter. By replacing y and its derivatives in Definition 93, we get

$$\sum_{i\geq 0} a_i(i+p)(i+p-1)x^{i+p} + \sum_{i\geq 0} a_i(i+p)x^{i+p} + (x^2 - v^2)\sum_{i\geq 0} a_i x^{i+p} = 0 \,.$$

Or equivalently,

$$\sum_{i\geq 0} \left[(i+p)(i+p-1) + (i+p) - v^2\right] a_i x^{i+p} + \sum_{i\geq 0} a_i x^{i+p-2} = 0 \,,$$

which means that

$$\sum_{i\geq 0} \left[(i+p)^2 - v^2\right] a_i x^{i+p} + \sum_{j\geq 2} a_{j-2} x^{j+p} = 0 \,.$$

Therefore,

$$a_0(p^2 - v^2) = a_1((p+1)^2 - v^2) = 0$$

and

$$a_i((i+\nu)^2 - \nu^2) + a_{i-2} = 0, \quad \forall i \geq 2 .$$

For $p = \nu$, we get $a_1 = 0$ and

$$i(i+2\nu)a_i = -a_{i-2}, \quad \forall i \geq 2 .$$

Thus,

$$a_i = -\frac{a_{i-2}}{i(2\nu+i)}, \quad \forall i \geq 2 .$$

Hence, the coefficients a_{2k+1}, and

$$a_{2k} = (-1)^k \frac{a_0}{2^{2k} k!(\nu+k)(\nu+k-1)\cdots(\nu+1)}, \quad \forall k \geq 0 .$$

Taking $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$, and observing that

$$\Gamma(\nu+k+1) = (\nu+k)(\nu+k-1)\cdots(\nu+1)\Gamma(\nu+1) ,$$

we get

$$a_{2k} = (-1)^k \frac{1}{2^{2k+\nu} k! \Gamma(\nu+k+1)}, \quad k \geq 0.$$

As a result, the solution of the equation will be

$$y = \left(\frac{x}{2}\right)^\nu \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k} . \qquad \square$$

Remark 96.

(1) For $p = -\nu$, the solution of Bessel's equation in Definition 93 is called Bessel's function of the first kind with the order $-\nu$ and is denoted by $J_{-\nu}(x)$ with

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k} .$$

(2) For ν, noninteger J_ν and $J_{-\nu}$ are linearly independent and therefore the general solution of the Bessel equation is of the form

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x) .$$

(3) The same solution can be obtained by choosing $p + 1 = \nu$ in the proof of Theorem 95.

Proposition 97. *For $\nu = n \in \mathbb{N}$, we have*

$$J_n = (-1)^n J_{-n} .$$

Proof. We have

$$
\begin{aligned}
J_{-n}(x) &= \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k-n+1)} \left(\frac{x}{2}\right)^{2k-n} \\
&= \sum_{m\geq 0} \frac{(-1)^{m+n}}{(m+n)!\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+n} \\
&= (-1)^n \sum_{m\geq 0} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} \\
&= (-1)^n J_n(x)\,.
\end{aligned}
$$

□

Example 4.1. For $v = 0$,

$$
J_0(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} = \sum_{k\geq 0} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k},
$$

which is an even function. Else, $J_0(0) = 1$. For $v = 1$, we obtain

$$
J_1(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(k+2)} \left(\frac{x}{2}\right)^{2k+1} = \sum_{k\geq 0} \frac{(-1)^k}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1},
$$

which is an odd function and satisfies $J_1(0) = 0$.

Definition 98. The Bessel function of the second kind of the order α denoted usually by Y_α and is given by

$$
Y_\alpha(x) = \begin{cases} \frac{\cos(\pi\alpha)J_\alpha(x) - J_{-\alpha}(x)}{\sin(\pi\alpha)}, & \text{for } \alpha \notin \mathbb{Z} \\ \lim_{v\to\alpha} \frac{\cos(\pi v)J_v(x) - J_{-v}(x)}{\sin(\pi v)}, & \text{for } \alpha \in \mathbb{Z}\,. \end{cases}
$$

Proposition 99. *For $\alpha \in \mathbb{Z}$, Y_α is a solution of Bessel's differential equation, singular at* 0 *and satisfying precisely* $\lim_{x\to 0} Y_0(x) = \infty$.

Proof. For $\alpha \notin \mathbb{Z}$, Y_α is a linear combination of J_α and $J_{-\alpha}$. Hence it is a solution of the Bessel's differential equation. We now prove this for $\alpha \in \mathbb{Z}$. It holds for all $v \notin \mathbb{Z}$ and all x that

$$
x^2 Y_v''(x) + xY_v'(x) + (x^2 - v^2)Y_v(x) = 0\,.
$$

Letting $\nu \to \alpha \in \mathbb{Z}$, we obtain

$$x^2 Y_\alpha''(x) + xY_\alpha'(x) + (x^2 - \alpha^2)Y_\alpha(x) = 0\,.$$

Next, we show that $\lim_{x\longrightarrow 0} Y_\alpha(x) = +\infty$. Indeed, for $\alpha \notin \mathbb{Z}$, Y_α is a linear combination of J_α and $J_{-\alpha}$. So it is a solution of Bessel's differential equation. Next, substituting Y_ν for $\nu \notin \mathbb{Z}$ in the differential equation and letting ν tend to α we get a solution Y_α for $\alpha \in \mathbb{Z}$. Y_α is singular at 0 because of the powers $(\frac{x}{2})^\alpha$ and $(\frac{x}{2})^{-\alpha}$. We now prove the remaining part. Recall that

$$Y_\alpha(x) = \lim_{\nu\to\alpha} \frac{\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)}\,.$$

We have for $\nu = \alpha$, $\sin(\pi\nu) = 0$, $\cos(\pi\nu) = (-1)^\alpha$, $(-1)^\alpha J_\alpha(x) = J_{-\alpha}(x)$. By applying L'Hôpital's rule, we obtain

$$\begin{aligned}
Y_\alpha(x) &= \lim_{\nu\longrightarrow\alpha} \frac{\frac{\partial}{\partial\nu}\left[\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)\right]}{\frac{\partial}{\partial\nu}\sin(\pi\nu)} \\
&= \frac{2}{\pi}J_\alpha(x)\left[\ln\frac{x}{2} + C\right] - \frac{1}{\pi}\sum_{k=0}^{\alpha-1}\frac{\Gamma(\alpha-k)}{k!}\left(\frac{x}{2}\right)^{2k-\alpha} \\
&\quad - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k-\alpha}}{k!\Gamma(\alpha+k+1)}\left[\sum_{m=1}^{\alpha+k}\frac{1}{m} + \sum_{m=1}^{k}\frac{1}{m}\right],
\end{aligned}$$

where C is Euler's constant. For $\alpha = 0$, we obtain

$$Y_0(x) = \frac{2}{\pi}J_0(x)\left[\ln\frac{x}{2} + C\right] - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\sum_{m=1}^{k}\left(\frac{1}{m}\right)\left(\frac{x}{2}\right)^{2k-\alpha}\,.$$

Thus $\lim_{x\longrightarrow 0} Y_0(x) = +\infty$. □

Definition 100. The Bessel generating function of the first kind is given by

$$u(x,t) = \sum_{n=-\infty}^{+\infty} J_n(x)t^n\,.$$

Lemma 101. *For all $x \in \mathbb{R}$ and $t \in \mathbb{R}^*$, we have*

$$u(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\,.$$

Proof. We have

$$
\begin{aligned}
e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} &= \sum_{k\geq 0} \frac{\left(\frac{xt}{2}\right)^k}{k!} \sum_{m\geq 0} \frac{\left(-\frac{x}{2t}\right)^m}{m!} \\
&= \sum_{k\geq 0} \sum_{m\geq 0} (-1)^m \frac{t^{k-m}}{m!k!} \left(\frac{x}{2}\right)^{k+m} .
\end{aligned}
$$

Setting $k = m + n$, we get

$$
\begin{aligned}
e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} &= \sum_{m\geq 0} \sum_{n+m\geq 0} (-1)^m \frac{t^n}{m!(n+m)!} \left(\frac{x}{2}\right)^{2m+n} \\
&= \sum_{m\geq 0} \sum_{n\geq -m} (-1)^m \frac{t^n}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n} \\
&= \sum_{m\geq 0} \sum_{n=-\infty}^{+\infty} (-1)^m \frac{\left(\frac{x}{2}\right)^{2m+n}}{m!\Gamma(n+m+1)} t^n \\
&= \sum_{n=-\infty}^{+\infty} \sum_{m\geq 0} (-1)^m \frac{\left(\frac{x}{2}\right)^{2m+n}}{m!\Gamma(n+m+1)} t^n \\
&= \sum_{n=-\infty}^{+\infty} J_n(x) t^n \\
&= u(x,t) .
\end{aligned}
$$

□

Theorem 102. *The Bessel function J_n satisfies*

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \quad \forall n \in \mathbb{N} .$$

Proof. Differentiating the generating function u with respect to the variable t we obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\sum_{n=-\infty}^{+\infty} J_n(x) t^n\right) = \sum_{n=-\infty}^{+\infty} n J_n(x) t^{n-1} = \sum_{n=-\infty}^{+\infty} (n+1) J_{n+1}(x) t^n .$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\right) = \frac{x}{2}e^{\frac{x}{2}t} - \frac{x}{2t^2}e^{\frac{x}{2t}} \\
&= \frac{x}{2}e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\left(1-\frac{1}{t^2}\right) \\
&= \frac{x}{2}u(x,t)\left(1-\frac{1}{t^2}\right) \\
&= \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^n\left(1-\frac{1}{t^2}\right) \\
&= \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^n - \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^{n-2} \\
&= \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_n(x)t^n - \frac{x}{2}\sum_{n=-\infty}^{+\infty} J_{n+2}(x)t^n \,.
\end{aligned}$$

By identification, we obtain

$$nJ_n(x) = \frac{x}{2}J_{n-1}(x) + \frac{x}{2}J_{n+1}(x), \quad \forall n \geq 0\,.$$

Therefore

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x), \quad \forall n \geq 0\,.$$

□

Theorem 103. *The Bessel function J_n is differentiable and its derivative satisfies*

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\,.$$

Proof. Differentiating the generating function u with respect to x, we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \frac{1}{2}\left[\sum_{n=-\infty}^{+\infty} J_n(x)t^{n+1} - \sum_{n=-\infty}^{+\infty} J_n(x)t^{n-1}\right]\,.$$

On the other hand,

$$\frac{\partial u}{\partial x} = \sum_{n=-\infty}^{+\infty} J_n'(x)t^n, \quad \forall n \geq 1\,.$$

Consequently

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]\,.$$

□

Remark 104. In the particular case $n = 0$, we obtain

$$J_0'(x) = -J_1(x)\,.$$

Theorem 105. *The first-kind Bessel function can be expressed by the integral form*

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\varphi - n\varphi)d\varphi\ . \tag{4.16}$$

In particular, for $n = 0$*, we have*

$$J_0(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(x\sin\varphi)d\varphi\ .$$

Proof. Recall that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{k=-\infty}^{+\infty} J_k(x)t^k\ .$$

Setting $t = e^{i\varphi}$, we get

$$e^{ix\sin\varphi} = \sum_{k=-\infty}^{+\infty} J_k(x)e^{ik\varphi}\ , \tag{4.17}$$

which is the Fourier series of the 2π-periodic function $f(\varphi) = e^{ix\sin\varphi}$. Therefore,

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\varphi}e^{-in\varphi}d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(x\sin\varphi - n\varphi\right)d\varphi\ .$$

In particular, for $n = 0$, we have

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(x\sin\varphi\right)d\varphi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos\left(x\sin\varphi\right)d\varphi\ . \qquad \square$$

Proposition 106. *Let* λ *and* μ *be two different roots of the Bessel function* $J_\nu(x)$*. The Bessel functions* $J_\nu(x)$ *satisfy the following orthogonality property:*

$$\int_0^1 xJ_\nu(\lambda x)J_\nu(\mu x)dx = 0\ .$$

Proof. Denote

$$y_{\nu,\lambda}(x) = J_\nu(\lambda x) \qquad \text{and} \qquad y_{\nu,\mu}(x) = J_\nu(\mu x)\ .$$

Then, $y_{\nu,\lambda}$ and $y_{\nu,\mu}$ are solutions of the following Bessel-type differential equations:

$$(xy'_{\nu,\lambda})'(x) + \left(\lambda^2 x - \frac{\nu^2}{x}\right) y_{\nu,\lambda}(x) = 0 \tag{4.18}$$

$$(xy'_{\nu,\mu})'(x) + \left(\mu^2 x - \frac{\nu^2}{x}\right) y_{\nu,\mu}(x) = 0 \,. \tag{4.19}$$

Multiplying the first one by $y_{\nu,\mu}$ and the second by $y_{\nu,\lambda}$ and integrating on $(0, 1)$, we get

$$(\lambda^2 - \mu^2) \int_0^1 x J_\nu(\lambda x) J_\nu(\mu x) dx = 0 \,.$$

Therefore, since $\lambda \neq \mu$, we get

$$\int_0^1 x J_\nu(\lambda x) J_\nu(\mu x) dx = 0 \,. \qquad \square$$

Figures 4.4 and 4.5 illustrate the graphs of the first and second kind Bessel functions.

4.2.8 Hankel function

Hankel functions are applied as physical solutions for incoming or outgoing waves in cylindrical geometry. These are linearly independent solutions of the complex-parameter Bessel equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2) y = 0 \,, \tag{4.20}$$

where α is an arbitrary complex number.

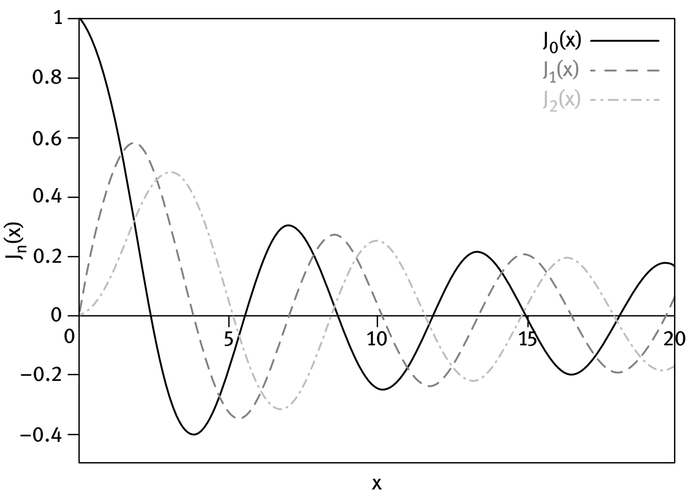

Fig. 4.4: Graphs of the first three first-kind Bessel functions.

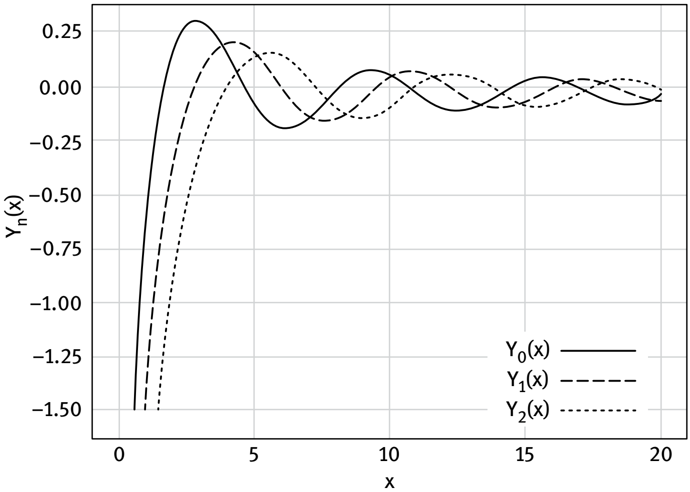

Fig. 4.5: Graphs of the first three second-kind Bessel functions.

Definition 107. Hankel functions of the first and second kind are defined, respectively, by

$$H_\alpha^1(x) = J_\alpha(x) + iY_\alpha(x) \quad \text{and} \quad H_\alpha^2(x) = J_\alpha(x) - iY_\alpha(x) \,,$$

where J_α and Y_α are the Bessel functions of the first and second kind, respectively.

Proposition 108. *The following assertions are true.*

(1) $H_\alpha^1(x) = \frac{J_{-\alpha}(x) - e^{-i\alpha\pi} J_\alpha(x)}{i \sin(\alpha\pi)}$.

(2) $H_\alpha^2(x) = \frac{J_{-\alpha}(x) - e^{i\alpha\pi} J_\alpha(x)}{-i \sin(\alpha\pi)}$.

(3) $H_{-\alpha}^1(x) = e^{i\alpha x} H_\alpha^1(x)$.

(4) $H_{-\alpha}^2(x) = e^{-i\alpha x} H_\alpha^2(x)$.

Proof. (1) Recall that the second-kind Bessel function is

$$Y_\alpha(x) = \frac{\cos(\pi v) J_v(x) - J_{-v}(x)}{\sin(\pi v)} \,.$$

Therefore,

$$
\begin{aligned}
H^1_\alpha(x) &= J_\alpha(x) + i\frac{\cos(\pi v)J_v(x) - J_{-v}(x)}{\sin(\pi v)} \\
&= \frac{J_\alpha(x)[\sin(\pi v) + i\cos(\pi v)] - iJ_{-v}(x)}{\sin(\pi v)} \\
&= \frac{J_\alpha(x)[i\sin(\pi v) - \cos(\pi v)] + J_{-v}(x)}{i\sin(\pi v)} \\
&= \frac{J_{-v}(x) - J_\alpha(x)[\cos(\pi v) - i\sin(\pi v)]}{i\sin(\pi v)} \\
&= \frac{J_{-\alpha}(x) - e^{-i\alpha\pi}J_\alpha(x)}{i\sin(\alpha\pi)} .
\end{aligned}
$$

(2) Similarly to (1), we have

$$
\begin{aligned}
H^2_\alpha(x) &= J_\alpha(x) - i\frac{\cos(\pi v)J_v(x) - J_{-v}(x)}{\sin(\pi v)} \\
&= \frac{J_\alpha(x)[\sin(\pi v) - i\cos(\pi v)] + iJ_{-v}(x)}{\sin(\pi v)} \\
&= \frac{J_\alpha(x)[-i\sin(\pi v) - \cos(\pi v)] + J_{-v}(x)}{i\sin(\pi v)} \\
&= \frac{J_{-v}(x) - J_\alpha(x)[\cos(\pi v) + i\sin(\pi v)]}{-i\sin(\pi v)} \\
&= \frac{J_{-\alpha}(x) - e^{i\alpha\pi}J_\alpha(x)}{-i\sin(\alpha\pi)} .
\end{aligned}
$$

(3) It follows from (1) that

$$
\begin{aligned}
H^1_{-\alpha}(x) &= \frac{J_\alpha(x) - e^{i\alpha\pi}J_{-\alpha}(x)}{-i\sin(\alpha\pi)} \\
&= -e^{i\alpha\pi}\frac{e^{-i\alpha\pi}J_\alpha(x) - J_{-\alpha}(x)}{i\sin(\alpha\pi)} \\
&= e^{i\alpha\pi}\frac{J_{-\alpha}(x) - e^{-i\alpha\pi}J_\alpha(x)}{i\sin(\alpha\pi)} \\
&= e^{i\alpha\pi}H^1_\alpha(x) .
\end{aligned}
$$

(4) Similarly to (3), we have

$$
\begin{aligned}
H^2_{-\alpha}(x) &= \frac{J_\alpha(x) - e^{-i\alpha\pi}J_{-\alpha}(x)}{i\sin(\alpha\pi)} \\
&= e^{-i\alpha\pi}\frac{e^{i\alpha\pi}J_\alpha(x) - J_{-\alpha}(x)}{i\sin(\alpha\pi)} \\
&= e^{-i\alpha\pi}\frac{J_{-\alpha}(x) - e^{i\alpha\pi}J_\alpha(x)}{-i\sin(\alpha\pi)} \\
&= e^{-i\alpha\pi}H^2_\alpha(x) .
\end{aligned}
$$

□

Proposition 109. *The first-kind Hankel function H_n^1, $n \in \mathbb{Z}$ can be expressed in the integral form as*

$$H_n^1(x) = \frac{1}{i\pi} \int_0^1 \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{n+1}} dt \,. \tag{4.21}$$

Proof. It follows from Proposition 97 and Definition 93 that

$$H_n^1(z) = \frac{1}{i\pi} J_n'(t) \,.$$

On the other hand, from Theorem 105, equation (4.16), we have that

$$J_n'(x) = -\frac{1}{2\pi} \int_0^{2\pi} \sin(x \sin\varphi - n\varphi) \sin\varphi d\varphi \,.$$

Now, standard computations as in Theorem 105 yield that

$$\int_0^1 \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{n+1}} dt = J_n'(x) \,.$$

Hence,

$$H_n^1(z) = \frac{1}{i\pi} \int_0^1 \frac{e^{\frac{x}{2}(t-\frac{1}{t})}}{t^{n+1}} dt \,.$$ □

Theorem 110. *Hankel functions H_α^i are differentiable and we have*

$$\frac{d}{dz} H_\alpha^i(z) = \frac{1}{2}\left(H_{\alpha-1}^i(z) - H_{\alpha+1}^i(z)\right), \quad i = 1, 2 \,,$$

and

$$\frac{2\alpha}{z} H_\alpha^i(z) = H_{\alpha-1}^i(z) + H_{\alpha+1}^i(z), \quad i = 1, 2 \,.$$

Proof. We have

$$H_n^1(z) = \frac{1}{i\pi} J_n'(t) = \frac{1}{2}\left[\frac{1}{i\pi} J_{n-1}(x) - \frac{1}{i\pi} J_{n+1}(x)\right] \,.$$

Hence,

$$\frac{d}{dz} H_n^1(z) = \frac{1}{2}\left[\frac{1}{i\pi} J_{n-1}'(z) - \frac{1}{i\pi} J_{n+1}'(z)\right] = \frac{1}{2}\left[H_{n-1}(z) - H_{n+1}(z)\right] \,.$$

We now prove the next part. To do this, we recall the explicit form of Bessel function J_v from (4.15), which states that

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k} .$$

Now, for $i = 1$, we get

$$H^1_{\alpha-1}(z) + H^1_{\alpha+1}(z) = \frac{(J_{-\alpha+1} - J_{-\alpha-1}) + e^{-i\alpha\pi}(J_{\alpha-1} + J_{\alpha+1})}{-i \sin \alpha\pi} .$$

Next, it suffices to evaluate the quantities in the numerator. We evaluate one quantity and leave The others for readers. Using the above expression, we get

$$\begin{aligned} J_{\alpha-1} + J_{\alpha+1} &= \left(\frac{x}{2}\right)^\alpha \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\alpha+k)} \left(\frac{x}{2}\right)^{2k-1} \\ &\quad + \left(\frac{x}{2}\right)^\alpha \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\alpha+k+2)} \left(\frac{x}{2}\right)^{2k-1} \\ &= \frac{\alpha+1}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha-1} - \frac{2\alpha}{x} J_\alpha(x) . \end{aligned}$$

Using the same techniques and next substituting into the equality above, we get the desired result. □

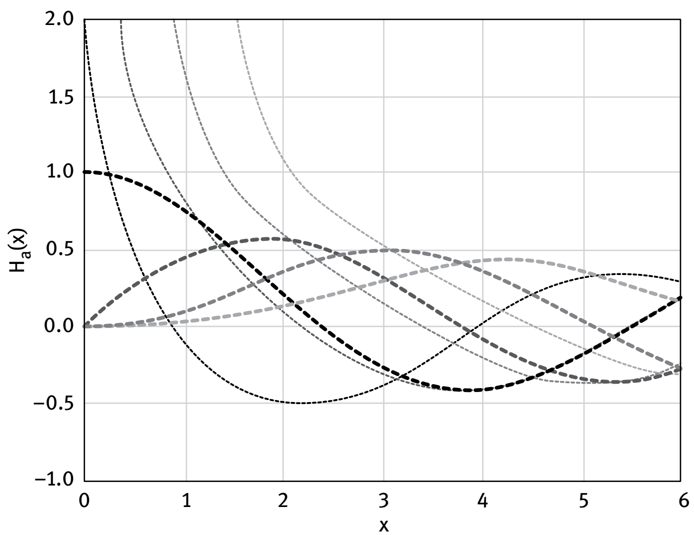

Fig. 4.6: Hankel function.

4.2.9 Mathieu function

Mathieu functions were originally introduced as solutions of the Mathieu differential equation

$$\frac{d^2x}{dt^2} + \omega^2(t)x = 0; \quad \text{or} \quad \omega^2(t) = \omega_0^2[1 - \xi_0 \cos(t)] \,. \tag{4.22}$$

It is a special case of the general Hill equation given by

$$\frac{d^2y}{dt^2} + f^2(t)y = 0 \,,$$

where f is a periodic function.

The Mathieu differential equation has in fact many variants. One variant may be obtained by a scaling modification by setting $y(t) = x(2t)$, which therefore satisfies the equation

$$\frac{d^2y}{dt^2} + [a - 2q\cos(2t)]\, y = 0 \,, \tag{4.23}$$

where a and q are constant coefficients. By setting $u = it$ in (4.23), we get the Mathieu modified differential equation

$$\frac{d^2y}{du^2} - [a - 2q\cosh(2u)]\, y = 0 \,. \tag{4.24}$$

By setting $x = \cos(t)$, we obtain a second Mathieu modified differential equation

$$(1-t)^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} + (a + 2q(1 - 2t^2))y = 0 \,.$$

As in the theory of the Schrödinger equation, we can guess stationary solutions of the form

$$F(a, q, x) = e^{i\mu x}P(a, q, x) \,, \tag{4.25}$$

where μ is a complex number called the Mathieu exponent and P is a periodic complex valued function. The following graph is illustrated with $a = 1$, $q = \frac{1}{5}$, and $\mu = 1 + 0.0995i$.

Definition 111. For fixed a, q we define

– The Mathieu cosine $C(a, q, x)$ by

$$C(a, q, x) = \frac{F(a, q, x) + F(a, q, -x)}{2F(a, q, 0)} \,.$$

– The Mathieu sine $S(a, q, x)$ by

$$S(a, q, x) = \frac{F(a, q, x) - F(a, q, -x)}{2F'(a, q, 0)} \,.$$

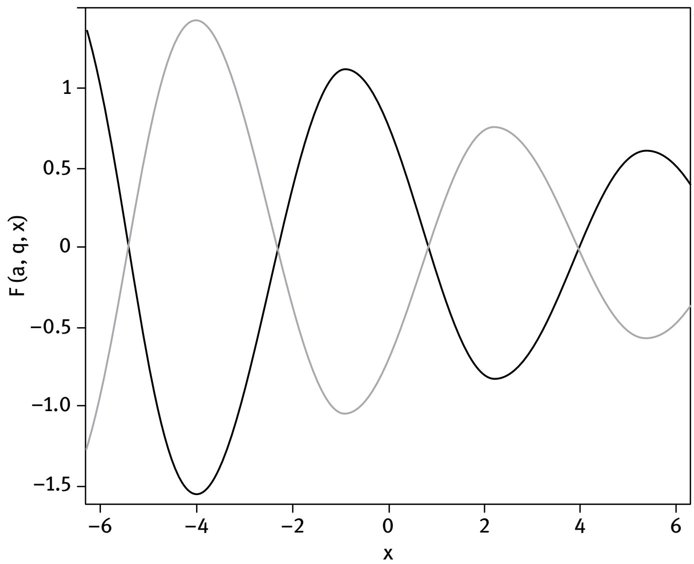

Fig. 4.7: Mathieu function: Real part and imaginary part, $a = \mu = 1$ and $q = 0.2$.

Properties 112. The following assertions hold:
(1) $C(a, q, 0) = 1$ and $S(a, q, 0) = 0$.
(2) $C'(a, q, 0) = 0$ and $S'(a, q, 0) = 1$.
(3) $C(a, q, -x) = C(a, q, x)$: The Mathieu cosine is an even function.
(4) $S(a, q, -x) = -S(a, q, x)$: The Mathieu sine is an odd function.
(5) $C(a, 0, x) = \cos(\sqrt{a}x)$ and $S(a, 0, x) = \frac{\sin(\sqrt{a}x)}{\sqrt{a}}$.

Proof. (1) We have

$$\begin{aligned} C(a, q, 0) &= \frac{F(a, q, 0) + F(a, q, 0)}{2F(a, q, 0)} \\ &= \frac{2F(a, q, 0)}{2F(a, q, 0)} \\ &= 1 \,. \end{aligned}$$

Similarly, for the sine function, we have

$$\begin{aligned} S(a, q, 0) &= \frac{F(a, q, 0) - F(a, q, 0)}{2F'(a, q, 0)} \\ &= \frac{0}{2F'(a, q, 0)} \\ &= 0 \,. \end{aligned}$$

(2) We have

$$
\begin{aligned}
C'(a,q,0) &= \frac{F'(a,q,0) - F'(a,q,0)}{2F(a,q,0)} \\
&= \frac{0}{2F(a,q,0)} \\
&= 0\,,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
S'(a,q,0) &= \frac{F'(a,q,0) + F'(a,q,0)}{2F'(a,q,0)} \\
&= \frac{2F'(a,q,0)}{2F'(a,q,0)} \\
&= 1\,.
\end{aligned}
$$

(3) We have

$$
\begin{aligned}
C(a,q,-x) &= \frac{F(a,q,-x) + F(a,q,x)}{2F(a,q,0)} \\
&= \frac{F(a,q,x) + F(a,q,-x)}{2F(a,q,0)} \\
&= C(a,q,x)\,.
\end{aligned}
$$

Then, the Mathieu cosine is an even function.
(4) Similarly,

$$
\begin{aligned}
S(a,q,-x) &= \frac{F(a,q,-x) - F(a,q,x)}{2F'(a,q,0)} \\
&= -\frac{F(a,q,x) - F(a,q,-x)}{2F'(a,q,0)} \\
&= -S(a,q,x)\,.
\end{aligned}
$$

Then, the Mathieu sine is an odd function.
(5) Follows from the fact that $S(a,0,.)$ and $C(a,0,.)$ are solutions of the Mathieu equation

$$
\frac{d^2y}{dx^2} + ay = 0
$$

and the assertions (1) and (2). ☐

Remark 113.

- The general solution of the Mathieu equation (for fixed a and q) is a linear combination of the Mathieu cosine and sine.
- In general, the Mathieu cosine and sine are not periodic. However, for small values of q we have

$$C(a,q,x) \sim \cos(\sqrt{a}x) \quad \text{and} \quad S(a,q,x) \sim \frac{\sin(\sqrt{a}x)}{\sqrt{a}}\,.$$

The Mathieu cosine is illustrated graphically in Figure 4.8.

4.2.10 Airy function

The Airy function was introduced by the astronomer George Biddell Airy in optical calculations. These are solutions of the second-order differential equation known as the Airy differential equation

$$y'' - xy = 0\,. \tag{4.26}$$

One idea to resolve such an equation is to use the well-known Fourier Transform, which leads formally to a set of solutions called Airy functions based on the following

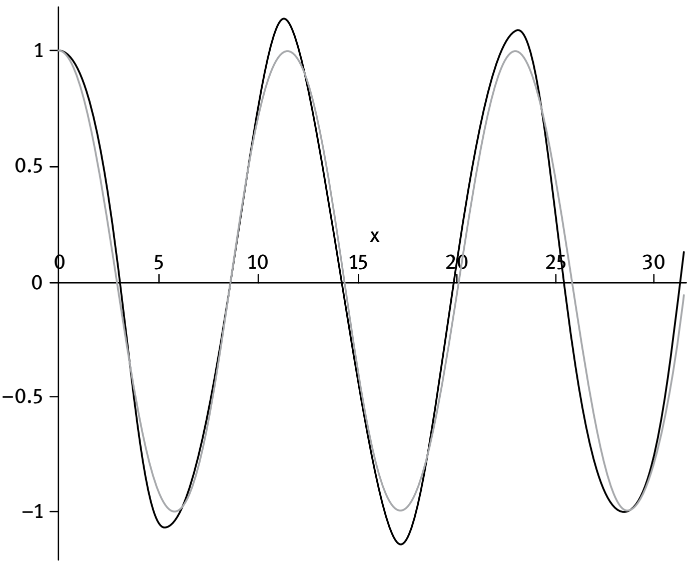

Fig. 4.8: Mathieu cosine: C(0.3;0.1;x) (Grey).

integral representation:

$$A(x) = \frac{1}{\pi} \int_0^{+\infty} \cos\left(\xi x + \frac{\xi^3}{3} \right) d\xi \,,$$

which is in fact a divergent integral. In fact, the integral is a semi-convergent integral. Indeed, for $0 < a < L < +\infty$, an integration by parts yields that

$$\int_a^L \cos\left(\xi x + \frac{\xi^3}{3} \right) d\xi = 2 \int_a^L \sin\left(\xi x + \frac{\xi^3}{3} \right) \frac{\xi}{(x+\xi^2)^2} d\xi + \left[\frac{\sin\left(\xi x + \frac{\xi^3}{3}\right)}{x + \xi^2} \right]_a^L \,.$$

As the integral $\int_a^\infty \sin(\xi x + \frac{\xi^3}{3}) \frac{\xi}{(x+\xi^2)^2} d\xi$ is absolutely convergent, the desired result follows.

Definition 114. For $\eta > 0$, we define the Airy function Ai by means of the following integral:

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{\mathbb{R}+i\eta} e^{i\xi x} e^{i\frac{\xi^3}{3}} d\xi \,.$$

Furthermore, applying classical techniques of parameter-depending integrals, we can prove that
(1) Ai is continuous on $\mathbb{R}$.
(2) $\lim_{x\to+\infty} \mathrm{Ai}(x) = 0$.

Indeed, note that $\mathrm{Ai}(x)$ may be written in the form

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix(\xi+i\eta)} e^{i\frac{(\xi+i\eta)^3}{3}} d\xi \,.$$

Next, as for $\eta > 0$, we get

$$\mathrm{Re}\left(ix(\xi + i\eta) + i\frac{(\xi + i\eta)^3}{3} \right) = -x\eta - \xi^2\eta + \frac{\eta^3}{3} \,,$$

the last integral is then absolutely convergent. Furthermore, it is uniformly convergent on any compact set in $\mathbb{R}$. So, since the function $x \mapsto e^{ix(\xi+i\eta)} e^{i\frac{(\xi+i\eta)^3}{3}}$ is continuous for all η and ξ, the function Ai is then continuous on $\mathbb{R}$. In fact, we may prove that Ai is $\mathcal{C}^\infty$ and that for all $k \in \mathbb{N}$,

$$\mathrm{Ai}^{(k)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} (i(\xi + i\eta))^k e^{ix(\xi+i\eta)} e^{i\frac{(\xi+i\eta)^3}{3}} d\xi \,.$$

We prove further that Ai is independent of the parameter η. Indeed,

$$\frac{d\mathrm{Ai}}{d\eta} = \frac{d}{d\eta}\frac{1}{2\pi}\int_{\mathbb{R}} \left(e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}\right)d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{d}{d\xi}\left\{e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}\right\}d\xi = 0\,,$$

as the function

$$\xi \mapsto e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}$$

is in the Schwartz class.

Properties 115. The following properties of the Airy function Ai hold:
(1) The function Ai satisfies the Airy differential equation (4.26).
(2) $\mathrm{Ai}(j.)$ is a solution of the Airy differential equation (4.26), whenever $j^3 = 1$.
(3) The function Ai is an entire function of x.
(4) For all $x \in \mathbb{R}$, $\mathrm{Ai}(x) \in \mathbb{R}$.
(5) $A_i(0) = \frac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}$ and $A_i'(0) = \frac{-1}{3^{\frac{1}{3}}\Gamma(\frac{1}{3})}$.

Proof. (1) As noted above, the Airy function Ai is twice differentiable and

$$\begin{aligned}\mathrm{Ai}''(x) &= \frac{1}{2\pi}\int_{\mathbb{R}+i\eta} (i\xi)^2 e^{ix\xi}e^{i\frac{\xi^3}{3}}\,d\xi \\ &= \frac{i}{2\pi}\int_{\mathbb{R}+i\eta} e^{ix\xi}\frac{d}{d\xi}\left(e^{i\frac{\xi^3}{3}}\right)d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}+i\eta} \xi e^{ix\xi}e^{i\frac{\xi^3}{3}}\,d\xi = x\mathrm{Ai}(x)\,.\end{aligned}$$

(2) Let $\widetilde{\mathrm{Ai}}(x) = \mathrm{Ai}(jx)$. We have

$$\widetilde{\mathrm{Ai}}''(x) = j^2\mathrm{Ai}''(jx) = j^2(jx\mathrm{Ai}(jx)) = j^3x\widetilde{\mathrm{Ai}}(x) = x\widetilde{\mathrm{Ai}}(x)\,.$$

(3) The function f_η defined by $f_\eta(x,\xi)) = e^{ix(\xi+i\eta)}e^{i\frac{(\xi+i\eta)^3}{3}}$ is analytic as a function of x for all ξ. Furthermore, for all $R > 0$ and $|x| \le R$, we have

$$|f_\eta(x,\xi)| \le e^{-R\eta}e^{-R\xi}e^{-\eta\xi^2}\,.$$

The last function is integrable according to ξ. So, Ai is analytic.

(4) For $x \in \mathbb{R}$ we have

$$\begin{aligned}\overline{\mathrm{Ai}(x)} &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix(\xi-i\eta)}e^{-i\frac{(\xi-i\eta)^3}{3}}\,d\xi\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix(-\xi+i\eta)}e^{i\frac{(-\xi+i\eta)^3}{3}}\,d\xi\,.\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix(\omega+i\eta)}e^{i\frac{(\omega+i\eta)^3}{3}}\,d\omega\,.\\ &= \mathrm{Ai}(x)\,.\end{aligned}$$

(5) As Ai is independent of $\eta > 0$, and Ai(0) is real, we can write

$$\mathrm{Ai}(0) = \frac{1}{2\pi}\mathrm{Re}\left(\int_{\mathbb{R}} e^{i\frac{(\xi+i)^3}{3}}\,d\xi\right)\,.$$

Denote I as the last integral and $J = \frac{1}{2}I$. Simple computations yield that

$$I = 2\int_0^{+\infty} e^{i\frac{(\xi+i)^3}{3}}\,d\xi\,,$$

which means that

$$\mathrm{Ai}(0) = \frac{1}{\pi}\mathrm{Re}\left(\int_0^{+\infty} e^{i\frac{(\xi+i)^3}{3}}\,d\xi\right) = \frac{1}{\pi}\mathrm{Re}(J)\,.$$

Next, for $R > 0$ large enough consider the points $O(z_0 = 0)$, $A(z_A = R)$, $B = (z_B = \mathrm{Re}^{i\frac{\pi}{6}})$, $C(z_C = i + z_B)$, and $D = (z_D = i)$ and the contours γ_R composed of the juxtaposition of the segment $[o, A]$, the arc (AB) and the segment BO in the positive sense, and δ_R the parallelogram contour $OBCDO$ countered also in the positive sense. So, applying the residues theory on the function $f(z) = e^{i\frac{(z+i)^3}{3}}$ and the contour γ_R, we get

$$\pi\mathrm{Ai}(0) = \mathrm{Re}\left(\lim_{R\to+\infty} K_R\right)\,,$$

where K_R is the integral given by

$$K_R = \int_{[B,O]} f(z)dz = \int_{[C,D]} e^{i\frac{z^3}{3}}\,dz\,.$$

Now, applying again the residues theory with the function $g(z) = e^{i\frac{z^3}{3}}$ on the parallelogram contour $OBCDO$, we obtain

$$\lim_{R\to+\infty} K_R = \int_0^{+\infty} g\left(te^{i\frac{\pi}{6}}\right)e^{i\frac{\pi}{6}}\,dt = e^{i\frac{\pi}{6}}\int_0^{+\infty} e^{-\frac{t^3}{3}}\,dt\,.$$

Hence,

$$\pi\mathrm{Ai}(0) = \frac{\sqrt{3}}{2}\int_0^{+\infty} e^{-\frac{t^3}{3}}\,dt = \frac{\sqrt{3}}{2}3^{-2/3}\int_0^{+\infty} x^{-2/3}e^{-x}\,dx = \frac{\Gamma(\frac{1}{3})}{2.3^{1/6}}\,,$$

which means that

$$\mathrm{Ai}(0) = \frac{\Gamma(\frac{1}{3})}{2\pi 3^{1/6}}\,.$$

Analogous techniques may be applied to obtain $\mathrm{Ai}'(0)$. □

Now, we introduce the second-kind Airy function ([18]).

Definition 116. The second-kind Airy function is defined by

$$\mathrm{Bi}(x) = e^{i\pi/6}\mathrm{Ai}(jx) + e^{-i\pi/6}\mathrm{Ai}(j^2x)\,, \tag{4.27}$$

where $j = e^{i2\pi/3}$.

Proposition 117. *The second-kind Airy function Bi is a solution of the Airy differential equation* (4.26) *and satisfies*

$$Bi(0) = \frac{1}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)} \quad \textit{and} \quad B_i'(0) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}\,.$$

Furthermore, Bi is real on the real axis $\mathbb{R}$.

Proof. We have

$$\begin{aligned}\mathrm{Bi}''(x) &= j^2e^{i\pi/6}\mathrm{Ai}''(jx) + j^4e^{-i\pi/6}\mathrm{Ai}''(j^2x)\\ &= j^2e^{i\pi/6}jx\mathrm{Ai}(jx) + j^4e^{-i\pi/6}j^2x\mathrm{Ai}(j^2x)\\ &= x(e^{i\pi/6}\mathrm{Ai}(jx) + e^{-i\pi/6}\mathrm{Ai}(j^2x))\\ &= x\mathrm{Bi}(x)\,.\end{aligned}$$

Hence, Bi satisfies (4.26). Next,

$$\mathrm{Bi}(0) = e^{i\pi/6}\mathrm{Ai}(0) + e^{-i\pi/6}\mathrm{Ai}(0) = \sqrt{3}\mathrm{Ai}(0) = \sqrt{3}\frac{\Gamma(\frac{1}{3})}{2\pi 3^{1/6}}\,.$$

Now, observing that

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}\,,$$

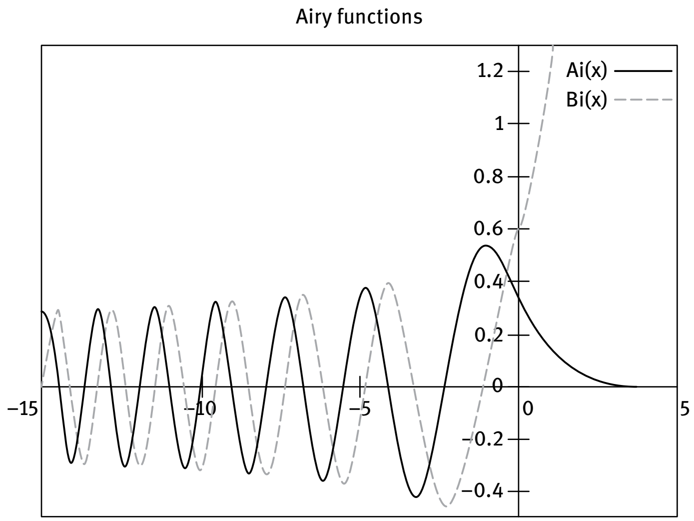

Fig. 4.9: Airy function Ai and Bi.

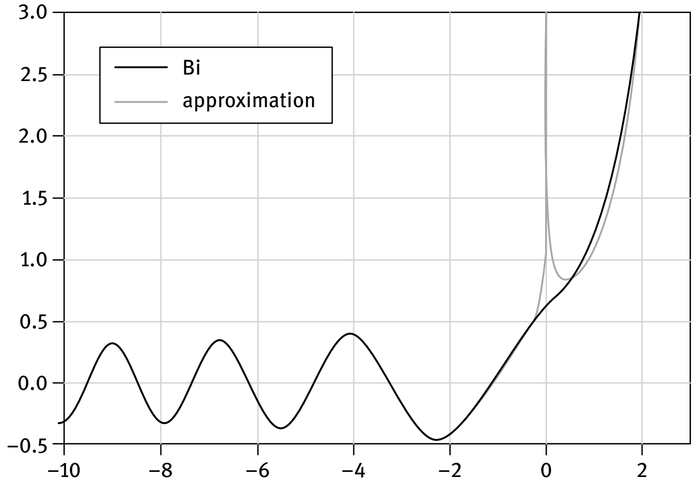

Fig. 4.10: The Airy function Bi and its approximation.

we get

$$\mathrm{Bi}(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)}\,.$$

The same techniques yield $\mathrm{Bi}'(0)$. Finally, for $x \in \mathbb{R}$, we have

$$\overline{\mathrm{Bi}(x)} = e^{-i\pi/6}\mathrm{Ai}(\bar{j}x) + e^{i\pi/6}\mathrm{Ai}(\bar{j}^2x) = e^{-i\pi/6}\mathrm{Ai}(j^2x) + e^{i\pi/6}\mathrm{Ai}(jx) = \mathrm{Bi}(x)\,. \qquad \square$$

Airy functions Ai and Bi are illustrated in Figure 4.9. Furthermore, Figure 4.10 illustrates the Airy function Bi and its approximation.

4.3 Hankel–Bessel transform

In this section, we focus on the most known transform associated with the special functions developed previously. We will review the Hankel–Bessel transform of functions. Readers are referred to [53] for more details. We denote the inner product in $L^2(\mathbb{R}^+, dx)$ by

$$\langle f, g\rangle = \int_0^{\infty} f(x)\overline{g}(x)dx$$

and the associated norm by $\|.\|_2$. Similarly, we denote the inner product in $L^2(\mathbb{R}^+, \xi d\xi)$ by

$$\langle f, g\rangle_{\xi} = \int_0^{\infty} f(\xi)\overline{g}(\xi)\xi d\xi$$

and the associated norm by $\|.\|_{\xi,2}$.

Definition 118. Let $f \in L^2(\mathbb{R}^+, dx)$. The Bessel transform of f is defined by

$$\mathcal{B}(f)(\xi) = \int_0^{+\infty} f(x)\sqrt{x}J_v(x\xi)dx, \quad \forall \xi > 0\,,$$

where J_v is the Bessel function of first kind and index v.

We immediately have the following characteristics:

Proposition 119.

(1) *For all* $f \in L^2(\mathbb{R}^+, dx)$, $\mathcal{B}(f) \in L^2(\mathbb{R}^+, \xi d\xi)$.

(2) *The Bessel transform* $\mathcal{B}$ *is invertible and its inverse is*

$$\mathcal{B}^{-1}(g)(x) = \int_0^{+\infty} g(\xi)\sqrt{x}J_v(x\xi)\xi d\xi, \quad \forall g \in L^2(\mathbb{R}^+, \xi d\xi)\,.$$

Proof. (1) Let f and g be in $L^2(\mathbb{R}^+, dx)$. We have

$$\begin{aligned}
\langle \mathcal{B}(f), \mathcal{B}(g)\rangle_\xi &= \int_0^{+\infty} \mathcal{B}(f)(\xi)\mathcal{B}(g)(\xi)\xi d\xi \\
&= \int_{\mathbb{R}_+^3} \sqrt{x}\sqrt{y} f(x)g(y)J_\nu(x\xi)J_\nu(y\xi)\xi dx dy d\xi \\
&= \int_{\mathbb{R}_+^2} \sqrt{x}\sqrt{y} f(x)g(y)\frac{\delta(x-y)}{x} dx dy \\
&= \int_{\mathbb{R}_+} \sqrt{x}\sqrt{x} f(x)g(x)\frac{1}{x} dx \\
&= \langle f, g\rangle \ .
\end{aligned}$$

So, taking $g = f$, we get

$$\|\mathcal{B}(f)\|_{\xi,2} = \|f\|_2$$

which means that $\mathcal{B}$ is an isometry.
(2) Denote $\widetilde{\mathcal{B}}(f)$ the right-hand quantity. We will prove that $\mathcal{B}(\widetilde{\mathcal{B}}(f)) = f$. Indeed,

$$\begin{aligned}
\mathcal{B}(\widetilde{\mathcal{B}}(f))(\xi) &= \int_0^{+\infty} \widetilde{\mathcal{B}}(f)(x)\sqrt{x}J_\nu(x\xi)dx \\
&= \int_0^{+\infty}\int_0^{+\infty} f(\eta)\sqrt{x}J_\nu(x\eta)\eta\sqrt{x}J_\nu(x\xi)d\eta dx \\
&= \int_0^{+\infty} f(\eta)\eta\frac{\delta(\eta-\xi)}{\eta} d\eta \\
&= f(\xi) \ .
\end{aligned}$$

□

Definition 120. The Hankel transform, also called Fourier–Bessel transform of the order ν, is defined by

$$\mathcal{H}(f)(\xi) = \int_0^{\infty} f(x)J_\nu(x\xi)x dx; \quad \forall f \ . \tag{4.28}$$

Remark 121. Hankel transform $\mathcal{H}$ and Bessel one $\mathcal{B}$ are related via the equality

$$\mathcal{H}(f)(\xi) = \mathcal{B}(\sqrt{.}f)(\xi) \ .$$

5 Spheroidal-type wavelets

5.1 Introduction

Wavelet analysis was introduced in the early 1980s in the context of signal analysis and exploration for petroleum to give a representation of signals and detect their characteristics. Several methods have been applied for the task; the most known is the Fourier transform. A major drawback of this method is its limitation to stationary and periodic signals. Furthermore, the description of signals is limited to the global behavior and cannot provide any detailed information. Also, in its numerical computer processing, Fourier analysis often yields nonfast algorithms.

Progress has been made by introducing the windowed Fourier transform (WFT) to address the problems of time-frequency localization. The WFT acts on signals by computing the classical Fourier transform of the signal multiplied by a time-localized function known as the window. However, the situation was not resolved, especially with the emergence of new problems, such as irregular signals or high-frequency variations.

The major drawback with the WFT is the fact that the shape of the window is fixed and may not be adapted to the fluctuations of nonstationary signals. Thus, the need for an analysis taking into account nonlinear algorithms, nonstationary signals, as well as nonperiodical and volatile ones has become a necessity for both theory and application. Wavelet analysis was introduced, developed, and has proved its power despite these obstacles. In this chapter, we review a special case of wavelet analysis adapted especially to spheroidal wavelets. We recall the strong relationship with orthogonal polynomials, homogenous polynomials, spherical harmonics, as well as special functions, and develop some details and examples.

5.2 Wavelets on the real line

Wavelet analysis is primarily based on an effective representation for standard functions on the real line and a robustness to the specification models. It also permits a reduction in time computation algorithms compared to other methods. This is essentially due to the simplicity of the analysis and the ease of generalization and efficiency according to the dimension. It permits one to analyze functions from different horizons starting from one horizon, which is not possible with Fourier analysis, for example. There, the number of coefficients to be computed is the standard point behind any approximation. Finally, wavelet analysis permits one to relate time localization to frequency.

Mathematically speaking, a wavelet or an analyzing wavelet on the real line is a function $\psi \in L^2(\mathbb{R})$, which satisfies some conditions, such as the admissibility condi-

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tion, which somehow describes Fourier–Plancherel identity and which says that

$$\int_{\mathbb{R}^+} |\hat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} = C_\psi < \infty \,. \tag{5.1}$$

The function ψ has to also satisfy a number of vanishing moments, which is related in wavelet theory to its regularity order. It states that

$$p = 0, \ldots, m-1, \quad \int_{\mathbb{R}} \psi(t) t^p \, dt = 0 \,. \tag{5.2}$$

Sometimes, we say that ψ is $\mathcal{C}^m$ on $\mathbb{R}$. The time-localization chart is a normalization form that is resumed in the identity

$$\int_{-\infty}^{+\infty} |\psi(u)|^2 du = 1 \,. \tag{5.3}$$

To analyze a signal by wavelets, one passes via the so-called wavelet transforms. A wavelet transform is a representation of the signal by means of an integral form similar to Fourier in which the Fourier sine and/or cosine is replaced by the analyzing wavelet ψ. In Fourier transform, the complex exponential source function yields the copies $e^{is\cdot}$ index by the indices $s \in \mathbb{R}$, which somehow represent frequencies. This transform is continuous in the sense that it is indexed by the whole line of indices $s \in \mathbb{R}$.

In wavelet theory, the situation is more unified. A continuous wavelet transform (CWT) is also well known. First, a frequency, scale, or a dilation/compression parameter $s > 0$ and a second one related to time or position $u \in \mathbb{R}$ have to be fixed. The source function ψ, known as the analyzing wavelet, is next transformed to yield some copies (replacing the $e^{is\cdot}$)

$$\psi_{s,u}(x) = \frac{1}{\sqrt{s}} \psi\left(\frac{x-u}{s}\right) \,. \tag{5.4}$$

The CWT of a real valued function f defined on the real line at the position u and the scale s is defined by

$$d_{s,u}(f) = \int_{-\infty}^{\infty} f(t) \psi_{s,u}(t) dt, \quad \forall u, s \,. \tag{5.5}$$

By varying the parameters s and u, we can completely cover the time-frequency plane. This gives a full and redundant representation of the whole signal to be analyzed (see [99]). This transform is called continuous because of the nature of the parameters s and u that can operate at all levels and positions.

So, wavelets operate according to two parameters: the parameter u which permits one to translate the graph of the source wavelet mother ψ and the parameter s which permits one to compress or to dilate the graph of ψ. Computing or evaluating the coefficients $d_{u,s}$ means analyzing the function f with wavelets.

Properties 122. The wavelet transform $d_{s,u}(f)$ possesses some properties, such as
(1) the linearity, in the sense that

$$d_{s,u}(\alpha f + \beta g) = \alpha d_{s,u}(f) + \beta d_{s,u}(g), \quad \forall f, g \,,$$

(2) the translation-invariance, in the sense that

$$d_{s,u}(\tau_t f) = d_{s,u-t}(f), \quad \forall f; \text{ and } \forall u, s, t \,,$$

and where

$$(\tau_t f)(x) = f(x - t) \,,$$

(3) the dilation-invariance, in the sense that

$$d_{s,u}(f_a) = \frac{1}{\sqrt{a}} d_{as,au}(f), \quad \forall f; \text{ and } \forall u, s, a \,,$$

and where for $a > 0$,

$$(f_a)(x) = f(ax) \,.$$

The proof of these properties is easy and readers can refer to [8] for a review.

It holds in wavelet theory, as in Fourier analysis theory, that the original function f can be reproduced via its CWT by an L^2-identity.

Theorem 123. *For all $f \in L^2(\mathbb{R})$, we have the L^2-equality*

$$f(x) = \frac{1}{C_\psi} \int\int d_{s,u}(f) \psi\left(\frac{x-u}{s}\right) \frac{dsdu}{s^2} \,.$$

The proof of this result is based on the following lemma.

Lemma 124. *Under the hypothesis of Theorem 123, we have*

$$\int\int d_{s,u}(f)\overline{d_{s,u}(g)} \frac{dsdu}{s} = C_\psi \int f(x)\overline{g(x)} dx, \; \forall f, g \in L^2(\mathbb{R}) \,.$$

Proof. We have

$$d_{s,u}(f) = \frac{1}{s} f * \psi_s(u) = \frac{1}{s} \int f(x) \psi\left(\frac{x-u}{s}\right) dx = \frac{1}{2\pi} \mathcal{F}\left(\hat{f}(y)\overline{\hat{\psi}(sy)} e^{-iuy}\right) \,.$$

Consequently,

$$\int_u d_{s,u}(f)\overline{d_{s,u}(g)}du = \frac{1}{2\pi}\int_y \widehat{f}(y)\overline{\widehat{g}(y)}|\hat{\psi}(sy)|^2 dy\,.$$

By application of Fubini's rule, we get

$$\begin{aligned}\int_{s>0}\int_u d_{s,u}(f)\overline{d_{s,u}(g)}\frac{dsdu}{s} &= \frac{1}{2\pi}\int_{s>0}\int_y \hat{f}(y)\overline{\hat{g}(y)}|\hat{\psi}(sy)|^2\frac{dsdy}{s}\\ &= \frac{1}{2\pi}d_\psi\int_y \hat{f}(y)\overline{\hat{g}(y)}dy\\ &= C_\psi\int_y f(y)\overline{g(y)}dy\,. \end{aligned}$$

□

Proof of Theorem 123. By applying the Riesz rule, we get

$$\begin{aligned}&\left\|F(x)-\frac{1}{C_\psi}\int_{1/A\le a\le A}\int_{|b|\le B} C_{a,b}(F)\psi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2}\right\|_{L^2}\\ &= \sup_{\|G\|=1}\left(\int F(x)-\frac{1}{C_\psi}\int_{1/A\le a\le A}\int_{|b|\le B} C_{a,b}(F)\psi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2}\right)\overline{G(x)}dx\,.\end{aligned}$$

Next, using Fubini's rule, we observe that the last line is equal to

$$\begin{aligned}&= \sup_{\|G\|=1}\left(\int F(x)\overline{G(x)}dx-\frac{1}{C_\psi}\int_{1/A\le a\le A}\int_{|b|\le B} C_{a,b}(F)\overline{C_{a,b}(G)}\frac{dadb}{a}\right)\\ &= \sup_{\|G\|=1}\frac{1}{C_\psi}\int_{(a,b)\notin[1/A,A]\times[-B,B]} C_{a,b}(F)\overline{C_{a,b}(G)}\frac{dadb}{a}\,,\end{aligned}$$

which by Cauchy–Schwartz inequality is bounded by

$$\begin{aligned}&\le \frac{1}{C_\psi}\left[\int_{(a,b)\notin[1/A,A]\times[-B,B]}|C_{a,b}(F)|^2\frac{dadb}{a}\right]^{1/2}\\ &\quad\left[\sup_{\|G\|=1}\int_{(a,b)\notin[1/A,A]\times[-B,B]}|C_{a,b}(G)|^2\frac{dadb}{a}\right]^{1/2}\,.\end{aligned}$$

Now, Lemma 124 shows that the last quantity goes to 0 as R tends to $+\infty$. □

On the real line, the most well-known examples are Haar and Schauder wavelet, where explicit computations are always possible. The Haar example is the simplest example in the theory of wavelets. It is based on the wavelet mother expressed by

$$\psi(x) = \chi_{[0,1/2[}(x) - \chi_{[1/2,1[}(x)\,.$$

The Schauder wavelet is based on the explicit wavelet mother

$$\begin{aligned}\psi(x) = \frac{1}{2}&(1 - |2x|)\chi_{[-1/2,1/2]}(x)\\ &- (1 - |2x - 1|)\chi_{[0,1[}(x)\\ &+ \frac{1}{2}(1 - |2x - 2|)\chi_{[1/2,3/2]}(x)\,.\end{aligned}$$

Readers can refer to [8, 75, 95, 99] for more details and examples of original wavelet analysis on the real line and Euclidian spaces in general.

5.3 Chebyshev wavelets

Chebyshev wavelets stem from one mother wavelet ψ^m depending on a parameter m, which represents the degree of Chebyshev polynomial of first kind associated with the wavelet. The source *Chebyshev wavelet* mother ψ^m is defined by

$$\psi^m(t) = \widetilde{T}_m(t),\ \ 0 \le t < 1 \quad \text{and} \quad 0,\ \text{else}$$

where

$$\widetilde{T}_m(t) = \sqrt{\frac{2}{\pi}}\,T_m(t), \qquad m = 0, 1, 2, \dots, M - 1\,. \tag{5.6}$$

Here $T_m(t)$ are the Chebyshev polynomials of the first kind of degree m, given by

$$T_m(t) = \cos(m \arccos t)\,.$$

Next, we perform the usual translation–dilation actions using parameters $j \in \mathbb{N}$ for the level and a parameter $n = 1, 2, \dots, 2^{j-1}$ for the position. Thus, we obtain the dilation–translation copies of ψ^m explicitly expressed by

$$\psi^m_{j,n}(t) = \begin{cases} 2^{\frac{j}{2}}\,\widetilde{T}_m(2^j t - 2n + 1), & \frac{n-1}{2^{j-1}} \le t < \frac{n}{2^{j-1}} \\ 0, & \text{else}\,. \end{cases} \tag{5.7}$$

The Chebyshev wavelets are orthonormal with respect to the weight function

$$\omega_j(t) = \omega_{n,k}(t) = \omega(2^{j-1}t - n + 1), \quad n = 1, 2, \dots, 2^{k-1} \text{ and } \frac{n-1}{2^{j-1}} \le t < \frac{n}{2^{j-1}}\,.$$

Denote next

$$L^2_\omega([0,1]) = \left\{ f,\ \int_0^1 |f(x)|^2\ \omega(x)dx < \infty \right\}\,,$$

where $\omega(x) = \frac{1}{2\sqrt{x(1-x)}}$. A function $f \in L^2_\omega([0,1])$ can be approximated in a series form as

$$f = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} C_{nm}\psi^m_{j,n}\,,$$

where

$$C^m_{j,n} = \langle f, \psi^m_{j,n}\rangle_{\omega_j} ,$$

in which $\langle ., .\rangle_{\omega_j}$ is the inner product in $L^2_{\omega_j}([0, 1])$.

5.4 Gegenbauer wavelets

Gegenbauer wavelets (GW) depend on four parameters: j, n, m, p. The parameter $j \in \mathbb{N}$ represents the level of resolution, $n \in \{1, 2, 3, \ldots, 2^{j-1}\}$, is related to the translation parameter, $m = 0, 1, 2, \ldots, M-1$, $M > 0$ is the degree of the Gegenbauer polynomial, and finally a real parameter $p > -\frac{1}{2}$. The mother Gegenbauer wavelet is defined on $[0, 1)$ by $\psi^{m,p}(x) = G^p_m(x)$, where G^p_m is the well-known Gegenbauer polynomial defined in Chapter 1. Next, the translation–dilation copies of $\psi^{m,p}$ are defined by

$$\psi^{m,p}_{j,n}(x) = \begin{cases} \frac{1}{\sqrt{L^p_m}} 2^{\frac{j}{2}} G^p_m(2^j x - 2n + 1), & \frac{2n-2}{2^j} \le t < \frac{2n}{2^j} , \\ 0, & \text{elsewhere} . \end{cases}$$

Note here that the translation parameter takes only odd values.

Remark 125. For $p = \frac{1}{2}$, we get Legendre wavelets. For $p = 0$ and $p = 1$, we obtain the Chebyshev wavelet of first and second kind, respectively.

To obtain the mutual orthogonality of Gegenbauer wavelets $\psi^{m,p}_{j,n}$, the weight function associated with the Gegenbauer polynomials has to be dilated and translated as for the Gegenbauer wavelets. Thus, we obtain a translation–dilation copy of the weight ω as

$$\omega_{j,n}(x) = \omega(2^j x - 2n + 1) = (1 - (2^j x - 2n + 1)^2)^{p-\frac{1}{2}} .$$

At a fixed level of resolution, we get

$$\omega_{j,n}(x) = \begin{cases} \omega_{j,1}(x), & 0 \le x < \frac{1}{2^{j-1}} , \\ \omega_{j,2}(x), & \frac{1}{2^{j-1}} \le x < \frac{2}{2^{j-1}} , \\ \omega_{j,3}(x), & \frac{2}{2^{j-1}} \le x < \frac{3}{2^{j-1}} , \\ \vdots & \\ \omega_{j,2^{j-1}}(x), & \frac{2^{j-1}-1}{2^{j-1}} \le x < 1 . \end{cases}$$

According to such wavelets, a function $f \in L^2[0,1)$ can be expressed in terms of the GW as

$$f = \sum_{j=1}^{\infty} \sum_{n \in \mathbb{Z}} d_{j,n}^{m,p} \psi_{j,n}^{m,p} , \tag{5.8}$$

where the coefficient $d_{j,}^{m,p}$ are the so-called wavelet coefficients given by

$$d_{j,n}^{m,p} = \langle f, \psi_{j,n}^{m,p} \rangle = \int_0^1 \omega_{j,n}(x) \psi_{j,n}^{m,p}(x) f(x) dx .$$

For more details, refer to [124, 126, 135, 139].

5.5 Hermite wavelets

Hermite wavelets are based on the well-known Hermite polynomials. Recall that such polynomials consist of a sequence of orthogonal polynomials with respect to the special weight function $\omega(x) = e^{-x^2}$ and are explicitly given by

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} (e^{-x^2}) .$$

The Hermite mother wavelet is given by

$$\psi^m(t) = \begin{cases} \widetilde{H}_m(2t), & 0 \le t < 1 , \\ 0, & \text{else} , \end{cases} \tag{5.9}$$

where

$$\widetilde{H}_m = \frac{1}{2^m l! \sqrt{\pi}} H_m .$$

The translation–dilation copies of ψ^m are next defined by

$$\psi_{k,n}^m(t) = \begin{cases} 2^{\frac{k}{2}} \widetilde{H}_m(2^{k+1} t - 2l + 1), & \frac{l-1}{2^k} \le t < \frac{n}{2^k} \\ 0, & \text{else} . \end{cases} \tag{5.10}$$

Note that such wavelets depend essentially on the parameter m, which is the degree of the m-Hermite polynomial H_m. Hermite wavelets are orthonormal with respect to the weight function

$$\omega_{l,k}(t) = \omega(2^{k-1} t - l + 1), \quad l = 0, 1, \ldots, x, 2k, \ \frac{l-1}{2^k} \le t < \frac{n}{2^k} .$$

Some propeties of Hermite wavelets are given in [1].

5.6 Laguerre wavelets

Laguerre wavelets are orthogonal wavelets defined in the interval (0, 1) and stem from one source mother function

$$\psi^m(t) = \widetilde{L}_m(t)\chi_{[0,1[}(t) = \frac{1}{m!}L_m(t)\chi_{[0,1[}(t)\,, \tag{5.11}$$

where L_m is the Laguerre polynomial of degree m. A wavelet basis is next expressed by

$$\psi^m_{k,n}(t) = \begin{cases} 2^{\frac{k}{2}}\widetilde{L}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}\,, \\ 0, & \text{else}\,. \end{cases} \tag{5.12}$$

For more details on these wavelets, see [79].

5.7 Bessel wavelets

There are several approaches to introduce Bessel wavelets [126, 127].

In the present section, we will present the most known approach. For $1 \le p < \infty$ and $\mu > 0$, denote

$$L^p_\sigma(\mathbb{R}_+) := \left\{ f \text{ such that } \|f\|_{p,\sigma} = \left(\int_0^\infty |f(x)|^p \, d\sigma(x) \right)^{\frac{1}{p}} < \infty \right\}\,,$$

where $d\sigma(x) = \frac{x^{2\mu}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}dx$. Denote also

$$j_\mu(x) = 2^{\mu-\frac{1}{2}}\Gamma\left(\mu + \frac{1}{2}\right) x^{\frac{1}{2}-\mu} J_{\mu-\frac{1}{2}}(x)\,,$$

where $J_{\mu-\frac{1}{2}}$ is the Bessel function of first kind and of order $\mu - \frac{1}{2}$. Denote next,

$$D(x, y, z) = \int_0^\infty j_\mu(xt)j_\mu(yt)j_\mu(zt)d\sigma(t)$$

and the translation

$$\tau_x f(y) = \widetilde{f}(x, y) = \int_0^\infty D(x, y, z)f(z)d\sigma(z), \quad \forall 0 < x, y < \infty\,.$$

Next, for a two-variable function f, we define the dilation operator

$$D_a f(x, y) = a^{-2\mu-1} f\left(\frac{x}{a}, \frac{y}{a}\right)\,.$$

Definition 126. Let $\Psi \in L^p_\sigma(\mathbb{R}_+)$. The Bessel wavelet copy $\Psi_{a,b}$ is defined by

$$\Psi_{a,b}(x) = D_a \tau_b \Psi(x) = a^{-2\mu-1} \int_0^\infty D\left(\frac{b}{a}, \frac{x}{a}, z\right) \Psi(z) d\sigma(z); \quad \forall a, b \geq 0 \,.$$

The Bessel wavelet transform (BWT) of a function $f \in L^q_\sigma(\mathbb{R}_+)$, at the scale a and the position b is defined by

$$(B_\Psi f)(a, b) = a^{-2\sigma-1} \int_0^\infty \int_0^\infty f(t)\overline{\Psi}(z) D\left(\frac{b}{a}, \frac{t}{a}, z\right) d\sigma(z) d\sigma(t) \,.$$

The following result shows one of the BWT of functions.

Theorem 127. *Let $f \in L^p_\sigma(\mathbb{R}_+)$, $\Psi \in L^q_\sigma(\mathbb{R}_+)$ with $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(B_\psi f)$ is continuous on $\mathbb{R}^2_+$.*

Proof. Let (a_0, b_0) be an arbitrary fixed point of $\mathbb{R}^2_+$. We have

$$\begin{aligned}
&|(B_\Psi f)(a, b) - (B_\Psi f)(a_0, b_0)| \\
&\leq a^{-2\mu-1} \left| \int_0^\infty \int_0^\infty f(t)\overline{\Psi(z)} \left[D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a}, z\right)\right] d\sigma(z) d\sigma(t) \right| \\
&\leq a^{-2\mu-1} \left[\int_0^\infty \int_0^\infty |f(t)|^p \left| D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a_0, z}\right)\right|^{\frac{1}{p}} d\sigma(t) d\sigma(z) \right] \\
&\quad \times \left[\int_0^\infty \int_0^\infty |\overline{\psi}(z)|^q \left| D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a_0, z}\right)\right|^{\frac{1}{q}} d\sigma(t) d\sigma(z) \right] .
\end{aligned}$$

Now, observe that

$$\left| D\left(\frac{b}{a}, \frac{t}{a}, z\right) - D\left(\frac{b_0}{a_0}, \frac{t}{a_0, z}\right)\right| \leq 2 \,.$$

Moreover, using the dominated convergence theorem and the continuity of $D(\frac{b}{a}, \frac{t}{a}, z)$ with respect to (a, b), we get

$$\lim_{(a,b)\to(a_0,b_0)} |(B_\Psi f)(a, b) - (B_\Psi f)(a_0, b_0)| = 0 \,,$$

which proves the continuity of the BWT on $\mathbb{R}^2_+$. ☐

Definition 128. Let $f, g \in L^p_\sigma(\mathbb{R}_+)$. We define the convolution product (usually known as the Hankel convolution) by

$$(f\sharp g)(x) = \int_0^\infty \tau_x f(y) g(y) d\sigma(y) \, .$$

The following result is a variant of Parseval/Plancherel rules for the case of BWT.

Theorem 129. *Let $\Psi \in L^2_\sigma(\mathbb{R}_+)$ and $f, g \in L^2_\sigma(\mathbb{R}_+)$. Then*

$$\int_0^\infty \int_0^\infty (B_\Psi f)(a,b)\overline{(B_\Psi g)}(a,b) \frac{d\sigma(a)}{a^{2\mu+1}} d\sigma(b) = C_\Psi \, \langle f, g \rangle \, ,$$

where

$$C_\Psi = \int_0^\infty t^{-2\mu-1} |\widehat{\Psi}(t)|^2 dt > 0 \, .$$

The proof follows similar techniques as for the case of real-line wavelets. Because of its importance, we reproduce it in detail.

Proof. Recall that

$$\begin{aligned}(B_\Psi f)(a,b) &= \int_{\mathbb{R}_+} f(t) \Psi_{a,b}(t) d\sigma(t) \\ &= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+^2} f(t) \Psi(z) D\left(\frac{b}{a}, \frac{t}{a}, z\right) d\sigma(z) d\sigma(t) \, .\end{aligned}$$

Now observe that

$$D\left(\frac{b}{a}, \frac{t}{a}, z\right) = \int_{\mathbb{R}_+} j\left(\frac{b}{a} u\right) j\left(\frac{t}{a} u\right) j(zu) d\sigma(u) \, .$$

Hence,

$$
\begin{aligned}
(B_\Psi f)(a,b) &= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+^3} f(t)\Psi(z) j\left(\frac{b}{a}u\right) j\left(\frac{t}{a}u\right) j(zu)\,d\sigma(u)\,d\sigma(z)\,d\sigma(t) \\
&= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+^2} \widehat{f}\left(\frac{u}{a}\right) \Psi(z) j\left(\frac{b}{a}u\right) j(zu)\,d\sigma(u)\,d\sigma(z) \\
&= \frac{1}{a^{2\sigma+1}} \int_{\mathbb{R}_+} \widehat{f}\left(\frac{u}{a}\right) \widehat{\Psi}(u) j\left(\frac{b}{a}u\right) d\sigma(u) \\
&= \int_{\mathbb{R}_+} \widehat{f}(\eta)\widehat{\Psi}(a\eta) j(b\eta)\,d\sigma(\eta) \\
&= \left(\widehat{f}(\eta)\widehat{\Psi}(a\eta)\right)^{\wedge}(b)\,.
\end{aligned}
$$

As a result

$$
\begin{aligned}
&\int_{\mathbb{R}_+^2} (B_\Psi f)(a,b)\overline{(B_\Psi g)}(a,b)\frac{d\sigma(a)}{a^{2\mu+1}}\,d\sigma(b) \\
&= \int_{\mathbb{R}_+^2} \widehat{f}(\eta)\widehat{\Psi}(a\eta)\widehat{g}(\eta)\widehat{\Psi}(a\eta)\,d\sigma(\eta)\frac{d\sigma(a)}{a^{2\mu+1}} \\
&= \int_{\mathbb{R}_+} \widehat{f}(\eta)\widehat{g}(\eta)\left(\int_{\mathbb{R}_+} |\widehat{\Psi}(a\eta)|^2 \frac{d\sigma(a)}{a^{2\mu+1}}\right) d\sigma(\eta) \\
&= C_\Psi \int_{\mathbb{R}_+} \widehat{f}(\eta)\widehat{g}(\eta)\,d\sigma(\eta) \\
&= C_\Psi\,\langle \widehat{f}, \widehat{g}\rangle \\
&= C_\Psi\,\langle f, g\rangle\,. \qquad \square
\end{aligned}
$$

5.8 Cauchy wavelets

Cauchy wavelets are one step in the direction of introducing spherical wavelets as they aim to take into account the angular behavior of the analyzed signals. In the one-dimensional case, Cauchy wavelets are defined via their Fourier transform

$$
\widehat{\psi}_m(\omega) = \begin{cases} 0, & \text{for} \quad \omega < 0 \\ \omega^m e^{-\omega}, & \text{for} \quad \omega \geq 0\,, \end{cases}
$$

with $m > 0$. In 1D, the positive half-line is a convex cone. Thus a natural generalization to 2D will be a wavelet whose support in spatial frequency space is contained in a convex cone with an apex at the origin. Let $C \equiv C(\alpha, \beta)$ be the convex cone determined by the unit vectors e_α, e_β, where $\alpha < \beta$, $\beta - \alpha < \pi$ and for all θ, $e_\theta \equiv (\cos\theta, \sin\theta)$. The axis of the cone is $\xi_{\alpha\beta} = e_{\frac{\alpha+\beta}{2}}$. In other words,

$$\begin{aligned} C(\alpha,\beta) &= \{k \in \mathbb{R}^2, \quad \alpha \le \arg(k \le \beta)\} \\ &= \{k \in \mathbb{R}^2, \quad k.\xi_{\alpha\beta} \ge e_\alpha.\xi_{\alpha\beta} = e_\beta.> \mathbf{0}\}\,. \end{aligned}$$

The dual cone to $C(\alpha,\beta)$ is

$$\widetilde{C}(\alpha,\beta) = \{k \in \mathbb{R}^2, \quad k.k' > 0, \quad \forall k' \in C(\alpha,\beta)\}\,.$$

Note that $\widetilde{C}(\alpha,\beta)$ may also be seen as

$$\widetilde{C}(\alpha,\beta) = C(\widehat{\alpha},\widehat{\beta})\,,$$

where $\widehat{\alpha} = \beta - \frac{\pi}{2}$, $\widehat{\beta} = \alpha + \frac{\pi}{2}$ and $e_\alpha.e_{\widehat{\alpha}} = e_\beta.e_{\widehat{\beta}} = 0$. Thus the axis of $\widetilde{C}$ is $\xi_{\alpha\beta}$.

The two-dimensional Cauchy wavelet is defined via its Fourier transform

$$\widehat{\psi}^{C,\eta}_{lm} = \begin{cases} (k.e_{\widetilde{\alpha}})^l (k.e_{\widetilde{\beta}})^m e^{-k.\eta}, & k \in C(\alpha,\beta)\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{5.13}$$

where $\eta \in \widetilde{C}$ and $l, m \in \mathbb{N}^*$. Note that such a wavelet is also supported by C. It satisfies the admissibility condition

$$c_{\psi^{C,\eta}_{lm}} \equiv (2\pi)^2 \int \frac{d^2k}{|k|^2} |\widehat{\psi}^{C,\eta}_{lm}(k)|^2 < \infty\,. \tag{5.14}$$

The following result obtained by Antoine et al. is proved in [12] and yields an explicit form for the two-dimensional Cauchy wavelet.

Proposition 130. *For even $\eta \in \widetilde{C}$ and $l, m \in \mathbb{N}^*$. The 2D Cauchy wavelet $\psi^{C,\eta}_{lm}(x)$ with support in C belongs to $L^2(\mathbb{R}^2, dx)$ and is given by*

$$\psi^{C,\eta}_{lm}(x) = \frac{i^{l+m+2}}{2\pi} l!m! \frac{[\sin(\beta-\alpha)]^{l+m+1}}{[(x+i\eta).e_\alpha]^{l+1}[(x+i\eta).e_\beta]^{m+l}}\,. \tag{5.15}$$

We can, with analogous techniques, define multidimensional Cauchy wavelets. See [12] and the references therein for more details.

5.9 Spherical wavelets

Spherical wavelets are adopted for understanding complicated functions defined or supported by the sphere. The classical spherical wavelets are essentially done by convolving the function against rotated and dilated versions of one fixed function ψ. To introduce a special wavelet analysis on the sphere related to zonals we first recall some useful topics. Let $F \in L^2[-1,1]$ and L_n be the Legendre polynomial of degree n. The coefficients

$$\widehat{F}(n) = 2\pi\langle F, L_n\rangle = 2\pi \int_{-1}^{1} F(x)L_n(x)dx, \quad n \in \mathbb{N}$$

are called *the Legendre coefficients* or *the Legendre transforms* of F. It is proved in harmonic Fourier analysis that F can be expressed in a series form

$$F = \sum_{n=0}^{\infty} \widehat{F}(n)\frac{2n+1}{4\pi}L_n \tag{5.16}$$

called *the Legendre series* of F.

Definition 131. A family $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$ is called a *spherical scaling function system* if the following assertions hold.

(1) For all $n, j \in \mathbb{N}$, we have $\widehat{\phi_j}(n) \leq \widehat{\phi_{j+1}}(n)$. In other words, for all $n \in \mathbb{N}$ the sequence $(\widehat{\phi_j}(n))_{j\in\mathbb{N}}$ is increasing
(2) $\lim_{j\longrightarrow\infty} \widehat{\phi_j}(n) = 1$ for all $n \in \mathbb{N}$
(3) $\widehat{\phi_j}(n) \geq 0$ for all $n, j \in \mathbb{N}$,

where $\widehat{\phi_j}(n)$ is the Legendre transform of $\widehat{\phi_j}$.

We will now investigate a way of constructing a scaling function [54].

Definition 132. A continuous function $\gamma : \mathbb{R}^+ \longmapsto \mathbb{R}$ is said to be admissible if it satisfies the *admissibility condition*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\left(\sup_{x\in[n,n+1]} |\gamma(x)|\right)^2 < +\infty\,. \tag{5.17}$$

In this case, γ is called *an admissible generator* of the function $\psi : [-1,1] \to \mathbb{R}$ given by

$$\psi = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi}\gamma(n)L_n\,. \tag{5.18}$$

We immediately obtain the following characteristics [162].

Proposition 133. *The following assertions are true:*
(1) *If γ is an admissible generator, then the generated function $\psi \in L^2[-1,1]$.*
(2) *For all $n \in \mathbb{N}$, $\widehat{\psi}(n) = \gamma(n)$.*

Proof. (1) Since the Legendre polynomials form an orthogonal basis for $L^2[-1,1]$ with $\langle L_n, L_n \rangle_{L^2[-1,1]} = \frac{4\pi}{2n+1}$, the admissibility condition imposed on γ yields that

$$\|\psi\|^2_{L^2[-1,1]} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (\gamma_0(n))^2 \leq \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\sup_{x\in[n,n+1]} |\gamma_0(x)| \right)^2 < +\infty$$

(2) is an immediate result from (5.16). □

We now investigate the idea to construct a whole family of admissible functions starting from one source admissible function.

Definition 134. The *dilation operator* is defined for $\gamma\colon [0,\infty) \to \mathbb{R}$ and $a > 0$ by

$$D_a\gamma(x) = \gamma(ax) \quad \forall x \in [0,\infty) .$$

For $a = 2^{-j}$, $j \in \mathbb{Z}$ we denote $\gamma_j = D_j\gamma = D_{2^{-j}}\gamma$.

Definition 135. An admissible function $\varphi\colon [0,\infty) \to \mathbb{R}$ is said to be a *generator of a scaling function* if it is monotonously decreasing, continuous at 0 and satisfies $\varphi(0) = 1$.
The system $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$, defined by

$$\phi_j = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \varphi_j(n) L_n$$

is said to be *the corresponding spherical scaling function* associated with φ.

It holds sometimes that for all j, the sequence $(\widehat{\phi_j}(n))_n$ is stationary with zero stationary value. In this case, the system $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$ is called *bandlimited*. It holds that for bandlimited scaling functions, each ϕ_j is a 1D polynomial, and for all $F \in L^2(S^2)$, $\phi_j * F$ is a polynomial on S^2. The following theorem affirms that scaling functions permit one to approximate L^2 functions with polynomial approximates (see [162]).

Now, we show that such scaling functions are suitable candidates to approximate functions in L^2 as it is needed in wavelet theory in general. Thus, they are suitable sources to define multiresolution analysis and/or a wavelet analysis on the sphere.

Theorem 136. *Let $\{\phi_j\}_{j\in\mathbb{N}}$ be a scaling function and $F \in L^2(S^2)$. Then*

$$\lim_{j\to\infty} \|F - \phi_j^{(k)} * F\|_{L^2(S^2)} = 0$$

for all levels of iterations $k \in \mathbb{N}$.

Here, for a function $\Phi \in L^2$, we designate by $\Phi^{(k)}$ the k-times self-convolution of Φ with itself. The last approximation is called *spherical approximate identity*. The next theorem shows the role of spherical scaling functions in the construction of multiresolution analysis on the sphere.

Proof. First observe that

$$\phi_j^{(k)} * F = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\Phi}_J(n)\widehat{F}(n,j)Y_{n,j} .$$

Thus,

$$F - \phi_j^{(k)} * F = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (1 - \widehat{\Phi}_J(n))\widehat{F}(n,j)Y_{n,j} ,$$

which by applying the Parseval identity yields that

$$\|F\phi_j^{(k)} * F\|_2^2 = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (1 - \widehat{\Phi}_J(n))^2(\widehat{F}(n,j))^2 .$$

Now, observing that the last series is J-uniformly convergent and the fact that

$$\lim_{J\to+\infty} (1 - \widehat{\Phi}_J(n)) = 0$$

for all n, it results that

$$\lim_{j\to\infty} \|F - \phi_j^{(k)} * F\|_{L^2(S^2)} = 0 .$$

□

Theorem 137. *Let for $j \in \mathbb{Z}$,*

$$V_j = \{\phi_j^{(2)} * F | F \in L^2(S^2)\} ,$$

where $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$ is a scaling function. Then, the sequence $(V_j)_j$ defines a multiresolution analysis on the sphere. That is,

(1) $V_j \subset V_{j+1} \subset L^2(S^2), \quad \forall j \in \mathbb{N}.$

(2) $\overline{\bigcup_{j=0}^{\infty} V_j} = L^2(S^2).$

For $j \in \mathbb{Z}$, the spaces V_j represents the so-called *scale or approximation space* at the level j.

Proof. (1) As $\Phi \in L^2$ and also F, the convolution $\Phi * F$ is also L^2.
Consider next, for $J \in \mathbb{Z}$, the function

$$\gamma_J(n) = \left(\frac{\widehat{\Phi}_J(n)}{\widehat{\Phi}_{J+1}(n)} \right)^2 \widehat{F}(n,j) \quad \text{if} \quad \Phi_{J+1}(n) \neq 0$$

and 0 else, and define the function G by

$$G = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \gamma_j(n) Y_{n,j} \, .$$

It is straightforward that $G \in L^2$ and that $\widehat{G}(n,j) = \gamma_j(n)$. Furthermore,

$$\begin{aligned} \phi_{J+1}^{(2)} * G &= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\Phi}_{J+1}(n) \widehat{G}(n,j) Y_{n,j} \\ &= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\Phi}_J(n) \widehat{F}(n,j) Y_{n,j} \\ &= \phi_J^{(2)} * F \, . \end{aligned}$$

Hence, $\phi_J^{(2)} * F = \phi_{J+1}^{(2)} * G \in V_{J+1}$. Consequently, $V_J \subset V_{J+1}$.
(2) The density property is an immediate consequence of the spherical approximate identity proved in Theorem 136. □

Based on this multiresolution analysis of $L^2(S^2)$, we can introduce spherical wavelets.

Definition 138. Let $\Phi = \{\phi_j\}_{j \in \mathbb{N}} \subset L^2[-1,1]$ be a scaling function and let $\Psi = \{\psi_j\}_{j \in \mathbb{N} \cup \{-1\}}$ and $\widetilde{\Psi} = \{\tilde{\psi}_j\}_{j \in \mathbb{N} \cup \{-1\}}$ be in $L^2[-1,1]$ satisfying the so-called *refinement equation*

$$\widehat{\psi_j}(n) \widehat{\tilde{\psi}_j}(n) = (\widehat{\phi_{j+1}}(n))^2 - (\widehat{\phi_j}(n))^2 \quad \forall n, j \in [0, +\infty) \, .$$

Then,
(a) Ψ and $\widetilde{\Psi}$ are called, respectively, (*spherical*) *primal wavelet* and (*spherical*) *dual wavelet* relative to Φ.
(b) The functions ψ_0 and $\tilde{\psi}_0$ are called the *primal mother wavelet* and the *dual mother wavelets*, respectively.
Here, we set $\psi_{-1} = \tilde{\psi}_{-1} = \phi_0$.

The following result obtained by Volker in [162] shows the existence of primal and dual wavelets.

Theorem 139. *Let φ_0 be a generator of a scaling function and $\psi_0, \tilde{\psi}_0$ be admissible function such that*

$$\psi_0\tilde{\psi}_0(x) = \left(\varphi_0\left(\frac{x}{2}\right)\right)^2 - (\varphi_0(x))^2 \quad \forall x \in \mathbb{R}^+ .$$

Then, ψ_0 and $\tilde{\psi}_0$ are generators of primal and dual mother wavelets, respectively.

Proof. We will prove precisely that the dilated copies $\{\psi_j\}_{j\in\mathbb{N}\cup\{-1\}}, \{\tilde{\psi}_j\}_{j\in\mathbb{N}\cup\{-1\}} \subset L^2[-1,1]$ defined via their Legendre coefficients by dilating ψ_0 and $\tilde{\psi}_0(x)$ as

$$\widehat{\psi_j}(n) = \psi_j(n) = \psi_0(2^{-j}n), \quad \widehat{\tilde{\psi}_j}(n) = \tilde{\psi}_j(n) = \tilde{\psi}_0(2^{-j}n); \quad \forall n, j \in \mathbb{N} .$$

and

$$\widehat{\psi_{-1}}(n) = \widehat{\tilde{\psi}_{-1}}(n) = \varphi_0(n); \quad \forall n \in \mathbb{N}$$

are a primal and dual wavelets, respectively. Indeed, considering these dilated copies we obtain for all $n, j \in \mathbb{N}$,

$$\begin{aligned}\widehat{\psi_j}(n)\widehat{\tilde{\psi}_j}(n) &= \psi_0(2^{-j}n)\tilde{\psi}_0(2^{-j}n)\\ &= (\varphi_0(2^{-j-1}n))^2 - (\varphi_0(2^{-j}n))^2\\ &= (\widehat{\phi_{j+1}}(n))^2 - (\widehat{\phi_j}(n))^2 .\end{aligned}$$ □

A fundamental property of spherical wavelets is the scale-step property proved below, which prepares us to introduce detail spaces.

Theorem 140. *Let $\Psi = \{\psi_j\}_{j\in\mathbb{N}\cup\{-1\}}$ and $\widetilde{\Psi} = \{\tilde{\psi}_j\}_{j\in\mathbb{N}\cup\{-1\}}$ be a primal and a dual wavelet corresponding to the scaling function $\{\phi_j\}_{j\in\mathbb{N}} \subset L^2[-1,1]$. The following assertions hold for all $F \in L^2(S^2)$.*
(i) $\phi_{J_2}^{(2)} * F = \phi_{J_1}^{(2)} * F + \sum_{j=J_1}^{J_2-1} \tilde{\psi}_j * \psi_j * F, \forall J_1 < J_2 \in \mathbb{N}$.
(ii) $F = \phi_J^{(2)} * F + \sum_{j=J}^{\infty} \tilde{\psi}_j * \psi_j * F, \forall J \in \mathbb{N}$.

Proof. (i) We will evaluate the last right-hand series term in the assertion. Using the definition of primal and dual wavelets, we obtain

$$\begin{aligned}\tilde{\psi}_j * \psi_j * F &= \sum_{n=0}^{+\infty}\sum_{s=1}^{2n+1} \widehat{\psi}_j(n)\widehat{\tilde{\psi}}_j(n)\widehat{F}(n,s)Y_{n,s}\\ &= \sum_{n=0}^{+\infty}\sum_{j=1}^{2n+1} \left[(\widehat{\phi}_{j+1}(n))^2 - (\widehat{\phi}_j(n))^2\right]\widehat{F}(n,s)Y_{n,s}\\ &= \phi_{j+1}^{(2)} * F - \phi_j^{(2)} * F .\end{aligned}$$

As a result,

$$\sum_{j=J_1}^{J_2-1} \tilde{\psi}_j * \psi_j * F = \phi_{J_2}^{(2)} * F - \phi_{J_1}^{(2)} * F .$$

(ii) is an immediate consequence of assertion (i). □

Theorem 141. *Denote for $j \in \mathbb{Z}$,*

$$W_j = \{\tilde{\psi}_j * \psi_j * F / F \in L^2(S^2)\} .$$

Then, for all $J \in \mathbb{Z}$,

$$V_{J+1} = V_J + W_J .$$

Proof. The inclusion $V_J \subset V_{J-1} + W_{J-1}$ is somehow easy and it is a consequence of Theorem 140. We will prove the opposite inclusion. So, let $F_1 \in V_J$ and $F_2 \in W_J$. We seek a function $F \in L^2$ for which we have

$$\Phi_{J+1}^{(2)} * F = F_1 + F_2 .$$

Since $F_1 \in V_J$ and $F_2 \in W_J$, there exist G_1 and G_2 in L^2 such that

$$F_1 = \Phi_J^{(2)} * G_1 \quad \text{and} \quad F_2 = \widetilde{\Psi}_J * \Psi_J * G_2 .$$

Now, consider the function γ defined by

$$\gamma(n,j) = \left(\frac{(\widehat{\Phi}_J(n))^2 \widehat{G}_1(n,j) + ((\widehat{\Phi}_{J+1}(n))^2 - (\widehat{\Phi}_J(n))^2)\widehat{G}_2(n,j)}{\widehat{\Phi}_{J+1}(n)} \right)^2 ,$$

whenever $\Phi_{J+1}(n) \neq 0$ and 0 else, and define the function F by

$$F = \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \gamma(n,j) Y_{n,j} .$$

It is straightforward that $F \in L^2$ and that $\widehat{F}(n,j) = \gamma(n,j)$. Furthermore,

$$\begin{aligned}
\Phi_{J+1}^{(2)} * F &= \sum_{n=0}^{+\infty,*} \sum_{j=1}^{2n+1} (\widehat{\Phi}_{J+1}(n))^2 \widehat{F}(n,j) Y_{n,j} \\
&= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (\widehat{\Phi}_J(n))^2 \widehat{G}_1(n,j) Y_{n,j} \\
&\quad + \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} ((\widehat{\Phi}_{J+1}(n))^2 - (\widehat{\Phi}_J(n))^2) \widehat{G}_2(n,j) Y_{n,j}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} (\widehat{\Phi}_J(n))^2 \widehat{G}_1(n,j) Y_{n,j} \\
&\quad + \sum_{n=0}^{+\infty} \sum_{j=1}^{2n+1} \widehat{\widetilde{\Psi}}_J(n) \widehat{\Psi}_J(n) \widehat{G}_2(n,j) Y_{n,j} \\
&= \phi_J^{(2)} * G_1 + \widetilde{\Psi}_J * \Psi_J * G_2 \\
&= F_1 + F_2 \,.
\end{aligned}
$$

Consequently, $F_1 + F_2 \in V_{J+1}$. □

Definition 142. For $j \in \mathbb{Z}$, the space W_j is called the detail space at the level j and the mapping

$$
\begin{aligned}
(\mathrm{SWT})_j : L^2(S^2) &\to L^2(S^2) \\
F &\longmapsto \psi_j * F
\end{aligned}
$$

is called the *spherical wavelet transform* at the scale j.

Based on this definition and the results above, any function $F \in L^2(S^2)$ will be represented by means of an L^2-convergent series

$$
F = \sum_{j=-1}^{\infty} \tilde{\psi}_j * (SWT)_j(F) \,. \tag{5.19}
$$

6 Some applications

6.1 Introduction

This chapter presents some applications related to the previous theories developed. The results presented are not new but consist of some developments in the direct application of some types of wavelets related to orthogonal polynomials and spherical calculus. We aimed to present especially some applications on differential equations and their numerical treatment with wavelets, some integrodifferential equations, and image processing and time-series processing related to spherical domains.

The developments stem from lectures and papers that are listed here and can be consulted by readers for more details and for complete study of the problems presented [3, 4, 9, 24, 50, 87, 90–92, 94, 110, 137, 139, 158, 160, 167, 169].

Of course, these applications may not be the best ones for the topic but we tried to reproduce the simplest ones so that they can be easily understood and redeveloped by the interested reader and make him/her familiar with wavelet theory and its interaction with other fields. More complicated applications can be found in the literature on wavelet theory, which is growing every day.

6.2 Wavelets for numerical solutions of PDEs

In this section, we propose to redevelop numerical solutions of the well-known Chebyshev and Hermite differential equations by applying the simplest Haar wavelets. The aim is to show explicit calculus based on wavelets and their interaction with the first chapter dealing with orthogonal polynomials. The Chebyshev differential equation is

$$(1 - x^2)y'' - xy' + \lambda^2 y = 0 \,. \tag{6.1}$$

The Hermite differential equation is

$$y'' - 2xy' + 2\lambda y = 0 \,. \tag{6.2}$$

$\lambda \in \mathbb{R}$ is a fixed parameter.

Recall that these equations already have exact polynomial solutions composed of the known classes of Chebyshev and Hermite polynomials. We propose in this section to reproduce some numerical studies based on Haar wavelets to compare wavelet results with the exact ones.

In this section, we reproduce the method developed in [136]. The crucial idea to develop here differs from classical ones in that instead of considering the development of the unknown solution y into a wavelet series

$$y = \sum_i y_i \psi_i ,$$

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where ψ_i is a suitable wavelet basis, we consider the development of the higher order derivative of y in the equation. So consider the Haar wavelet mother

$$\psi(x) = \chi_{[0,1/2[}(x) - \chi_{[1/2,1[}(x)$$

and its copies

$$\psi_{j,k}(x) = 2^{j/2}(\chi_{[k2^{-j},(k+1/2)2^{-j}[}(x) - \chi_{[(k+1/2)2^{-j},(k+1)2^{-j}[}(x)) .$$

Consider a level J of resolution and denote $M = 2^J$. For $j = 0, 1, \dots, J$ denote $m_j = 2^j$ and for $k = 0, 1, \dots, m_j - 1$, we consider the lexicographical index $i = m_j + k + 1$. Thus the index i lies in the grid $2, 3, \dots, 2M = 2^{J+1}$. With this notation, the orthogonal set $(\psi_{j,k})_{j,k}$ will be denoted by $(\psi_i)_i$, $i = 2, 3, \dots, 2M$. For $i = 1$, we set as a convention $\psi_1 \equiv \chi_{[0,1[}$, which means the Haar scaling function. So, the above series development is the approximation of y (or its derivatives) at the level J.

Now, denote

$$P_{i,1}(t) = \int_0^t \psi_i(s)ds \quad \text{and} \quad P_{i,l}(t) = \int_0^t P_{i,l-1}(s)ds; \qquad l = 2, 3, \dots$$

We get, for example,

$$P_{i,1}(t) = \sqrt{m_j} \begin{cases} t - \frac{k}{m_j}, & t \in [\frac{k}{m_j}, \frac{k+0.5}{m_j}[\\ \frac{k+1}{m_j} - t, & t \in [\frac{k+0.5}{m_j}, \frac{k+1}{m_j}[\\ 0, & \text{elsewhere} . \end{cases}$$

and

$$P_{i,2}(t) = \sqrt{m_j} \begin{cases} 0, & t \in [0, \frac{k}{m_j}] \\ \frac{1}{2}(t - \frac{k}{m_j})^2, & t \in [\frac{k}{m_j}, \frac{k+0.5}{m_j}] \\ \frac{1}{4m_j^2} - \frac{1}{2}(t - \frac{k+1}{m_j})^2, & t \in [\frac{k+0.5}{m_j}, \frac{k+1}{m_j}] \\ \frac{1}{4m_j^2}, & t \in [\frac{k+1}{m_j}, 1] \end{cases}$$

Now, write

$$y''(t) = \sum_{i=1}^{M} a_i \psi_i(t) .$$

Hence,

$$y'(t) = \sum_{i=1}^{M} a_i P_{i,1}(t) + C_1$$

and

$$y(t) = \sum_{i=1}^{M} a_i P_{i,2}(t) + C_1 t + C_2 ,$$

where C_1 and C_2 are constants related to the boundary (and/or initial) conditions. Next, it remains to compute on the grid $t_l = \frac{l-0.5}{m_j}$, $l = 1, 2, \ldots, 2m_j$, $j = 0, 1, \ldots, J$ to obtain a matrix system with unknown a_is.

In [136], the numerical solution of (6.1) with $\Lambda = 5$ and $J = 3$ has been provided and compared with the exact solution

$$y(t) = t - 4t^3 + \frac{16}{5}t^5$$

and the initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 1\,.$$

Table 6.1 is obtained.

Table 6.1: Exact and Haar wavelet numerical solutions of (6.1).

$t/32$	Numerical solution	Exact solution
1	0.0309	0.0311
3	0.0898	0.0905
5	0.1375	0.1413
7	0.1750	0.1758
9	0.1749	0.1979
11	0.1782	0.1966
13	0.1628	0.1735
15	0.1306	0.1292
17	0.0381	0.0669
19	− 0.0110	− 0.0074
21	− 0.0518	− 0.0848
23	− 0.0837	− 0.1527
25	− 0.0577	− 0.1948
27	− 0.0471	− 0.1905
29	− 0.0083	− 0.1148
31	− 0.0472	0.0624

In terms of error estimates, this gives an L^2-error

$$\|y_{\text{exact}} - y_{\text{approx}}\| = 6.63 \times 10^{-2}\,.$$

In terms of graphic illustration, we obtained Figure 6.1.

Now, the same techniques are applied to the Hermite equation (6.2) with $\lambda = 3$, $J = 3$, to obtain the exact solution

$$y(x) = x - \frac{2}{3}x^3$$

and the initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 1$$

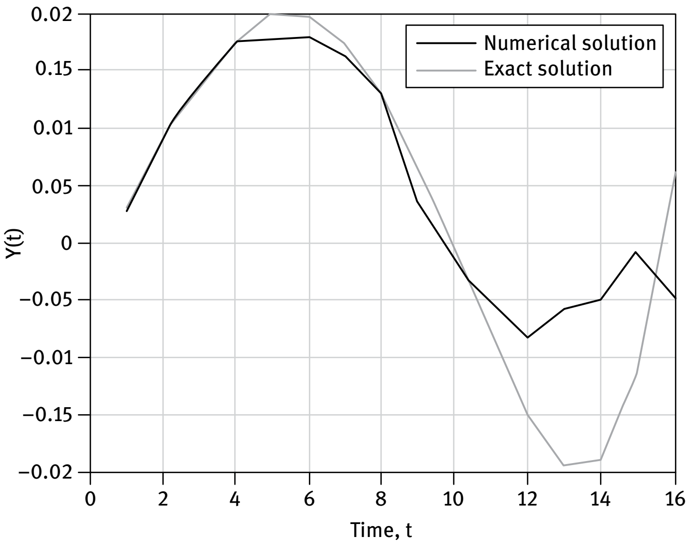

Fig. 6.1: Exact and Haar wavelet numerical solutions of (6.1).

Table 6.2: Exact and Haar wavelet numerical solutions of (6.2).

$t/32$	Numerical solution	Exact solution
1	0.0312	0.0312
3	0.0931	0.0932
5	0.1530	0.1537
7	0.2111	0.2118
9	0.2618	0.2664
11	0.3127	0.3167
13	0.3585	0.3616
15	0.3993	0.4001
17	0.4068	0.4313
19	0.4362	0.4542
21	0.4594	0.4678
23	0.4768	0.4712
25	0.4875	0.4634
27	0.4928	0.4433
29	0.4925	0.4101
31	0.4867	0.3627

yielded Table 6.2.

In terms of error estimates, this gives an L^2-error

$$\|y_{\text{exact}} - y_{\text{approx}}\| = 4.05 \times 10^{-2} .$$

In terms of graphic illustration, we obtained Figure 6.2.

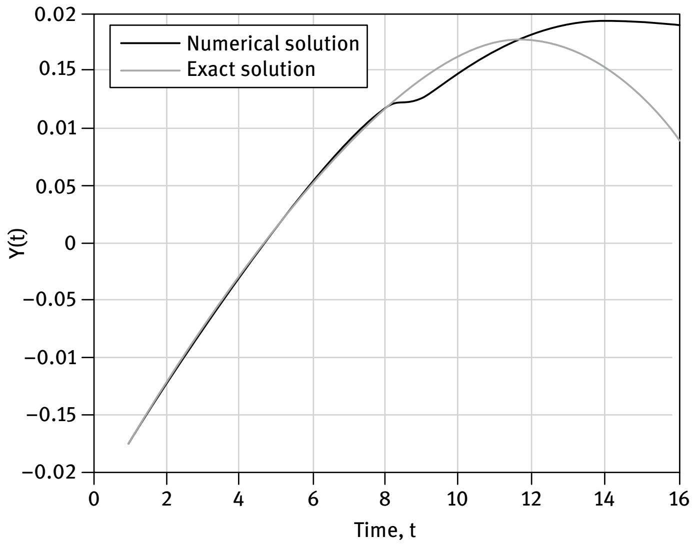

Fig. 6.2: Exact and Haar wavelet numerical solutions of (6.2).

6.3 Wavelets for integrodifferential equations

In this section, we present some methods based on wavelets to solve integrodifferential equations. We review one famous work developed in [87] where the authors applied some spherical types of wavelets to develop numerical solutions of the n-order integrodifferential problem

$$\begin{cases} \sum_{i=0}^{n} \lambda_i y^{(i)}(x) - \int_0^{2\pi} k(x,t)y(t)dt = g(x)\,, \\ y^{(i)}(0) = y_i, \quad i = 1, 2, \ldots, n \end{cases} \tag{6.3}$$

where k is an L^2 smooth function and 2π-periodic according to the first variable, and y and g are also L^2 2π-periodic. y is of course the unknown function to be approximated.

The basic idea is to apply trigonometric Hermite interpolation to obtain a periodic wavelet analysis. The authors in [87] noticed that one main difficulty in applying wavelets for the representation of integral operators is that the quadrature leads to potentially high cost with a sparse matrix. This was the starting point behind the application of a special type of wavelet bases to simplify the computation expense.

Next, trigonometric or circular wavelets were introduced based on Dirichlet kernels. For $m \in \mathbb{N}$, let

$$D_m(x) = \frac{1}{2} + \sum_{k=1}^{m} \cos(kx) \quad \text{and} \quad \widetilde{D}_m(x) = \sum_{k=1}^{m} \sin(kx)\,.$$

Denote

$$x_{j,m} = \frac{m\pi}{2^j}, \quad j \geq 0, \;\; m = 0, 1, \ldots, 2^{j+1} - 1$$

for the subdivision of the interval $[0, 2\pi)$ with a dyadic grid. The scaling functions are defined as follows:

$$\Phi^0_{j,0}(x) = \frac{1}{2^{2j+1}} \sum_{k=0}^{2^{j+1}-1} D_k(x) \,,$$

$$\Phi^1_{j,0}(x) = \frac{1}{2^{2j+1}} \left(\widetilde{D}_{2^{j+1}-1}(x) + \frac{1}{2} \sin(2^{j+1}x) \right) \,,$$

and for $s = 0; 1$ and $m = 0, 1, \ldots; 2^{j+1} - 1$,

$$\Phi^s_{j,m}(x) = \Phi^s_{j,0}(x - x_{j,m}) \,.$$

The approximation spaces V_j are defined as

$$V_j = \operatorname{span}(\Phi^0_{j,m}, \Phi^1_{j,m}; \; m = 0, 1, \ldots, 2^{j+1} - 1) \,.$$

The associated wavelet functions are defined as follows:

$$\Psi^0_{j,0}(x) = \frac{1}{2^{j+1}} \cos(2^{j+1}x) + \frac{1}{32^{2j+1}} \sum_{k=2^{j+1}+1}^{2^{j+2}-1} (32^{j+1} - k) \cos(kx) \,,$$

$$\Psi^1_{j,0}(x) = \frac{1}{2^{2j+3}} \sin(2^{j+2}x) + \frac{1}{32^{2j+1}} \sum_{k=2^{j+1}+1}^{2^{j+2}-1} \sin(kx)$$

and for $s = 0; 1$ and $m = 0, 1, \ldots; 2^{j+1} - 1$,

$$\Psi^s_{j,m}(x) = \Psi^s_{j,0}(x - x_{j,m}) \,.$$

The approximation spaces V_j are defined as

$$W_j = \operatorname{span}(\Psi^0_{j,m}, \Psi^1_{j,m}; \; m = 0, 1, \ldots, 2^{j+1} - 1) \,.$$

Next, the idea proceeds as usual to project the differential equation on an approximation space V_j in a given level j. So, the functions $y(x)$, $k(x, t)$, and $g(x)$ are approximated using trigonometric scaling functions as

$$\begin{cases} y(x) \approx \Phi(x)\alpha \,, \\ k(x, t) \approx \Phi(x)K\Phi(t) \,, \\ \text{and} \\ g(x) \approx \Phi(x)\beta \,, \end{cases} \tag{6.4}$$

where $\Phi(x)$ and $\Phi(t)$ are matrices depending on the scaling functions and the parameters of the integrodifferential equation (6.3), K and β are vectors obtained from the functions $k(x, t)$ and $g(x)$. Precisely, β is $2^{J+2} \times 1$ and K is $2^{J+2} \times 2^{J+2}$. Finally, α is composed of the coordinates of the projection vector of $y(x)$ on the space V_j. Substituting these approximations into the system (6.3) leads to a linear system of equations with 2^{J+2} unknowns and equations, which can be solved to find α and thus the unknown function $y(x)$.

The following result is proved in [87] and shows the convergence and its rate of the approximation solution to the exact one. Under the hypothesis of problem (6.3), we have

$$\|y - y_J\|_2 \leq C2^{-2(J+1)} \left(\sum_{i=0}^{n} \lambda_i J^i + 2 \right) .$$

6.4 Wavelets in image and signal processing

In this section, we explain some methods to construct wavelets adapted to image processing on the sphere. We aim to review the methods developed in [3] of constructing wavelets on the sphere. A first example is related to Haar wavelets. We introduce the so-called spherical Haar wavelets (SHW). For a resolution j and a pixel k on the sphere, we have one scaling function $\phi_{j,k}$ and three wavelet functions $\psi_{m,j,k}$; $m = 1, 2, 3$. Next, the sphere is subdivided according to the number N_{side} of pixels, $N_{\text{side}} = 2^{j-1}$, which yields a number $n_j = 12 \times 4^{j-1}$ of pixels for each surface μ_j. The scaling function and the three wavelets are defined by

$$\phi_{j,k}(x) = \begin{cases} 1, & \text{if} \quad x \in S_{j,k} \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_{1,j,k} = \frac{\phi_{j+1,k_0} + \phi_{j+1,k_2} - \phi_{j+1,k_1} - \phi_{j+1,k_3}}{4\mu_{j+1}}$$

$$\psi_{2,j,k} = \frac{\phi_{j+1,k_0} + \phi_{j+1,k_1} - \phi_{j+1,k_2} - \phi_{j+1,k_3}}{4\mu_{j+1}}$$

$$\psi_{3,j,k} = \frac{\phi_{j+1,k_0} + \phi_{j+1,k_3} - \phi_{j+1,k_1} - \phi_{j+1,k_2}}{4\mu_{j+1}}$$

The k_js $j = 0, 1, 2, 3$ represent the four pixels at the resolution or the level $j+1$ obtained from the pixel k of the resolution level j. The scaling (approximation) coefficients at the level j and the position k are then evaluated using those of the $j + 1$ level by means of a filter relation as

$$a_{j,k} = \frac{1}{4} \sum_{m=0}^{3} a_{j+1,k_m} . \tag{6.5}$$

The wavelet (detail) coefficients are also related by means of a similar filter,

$$d_{1,j,k} = \mu_{j+1}(a_{j+1,k_0} + a_{j+1,k_2} - a_{j+1,k_1} - a_{j+1,k_3})$$
$$d_{2,j,k} = \mu_{j+1}(a_{j+1,k_0} + a_{j+1,k_1} - a_{j+1,k_2} - a_{j+1,k_3})$$
$$d_{3,j,k} = \mu_{j+1}(a_{j+1,k_0} + a_{j+1,k_3} - a_{j+1,k_1} - a_{j+1,k_2})$$

It holds that such a transformation is orthogonal and leads to an exact reconstruction of the function f analyzed. Let

$$f(x_i) = \sum_{l=0}^{n_{j_0}-1} \lambda_{j_0,l}\phi_{j_0,j}(x_i) + \sum_{j=j_0}^{j-1}\sum_{m=0}^{3}\sum_{l=0}^{m_j-1} \gamma_{m,j,l}\psi_{m,j,l}(x_i) \,. \tag{6.6}$$

The main disadvantage of this type of transform is the fact that it depends on the image to be analyzed and thus is affected by rotations, for example.

To overcome this ambiguity, a transformation is introduced in [3] and dealt with using axisymmetric stereographic wavelets obtained from the previous transform. To guarantee a best transposition of plane functions on the sphere, it is possible to apply stereographic projection of radial wavelets. Indeed, consider the transformation $\pi^{-1}: x \to \omega = \pi^{-1}x = (\theta(r), \phi)$, where $\theta(r) = 2\arctan(\frac{r}{2})$. A radial wavelet can be transformed into a spherical wavelet via one rotation $\omega_0 = (\theta_0, \phi_0)$ according to the axis Oy and Oz, respectively.

Such transforms yield with a natural way a convolution product on the sphere. Let $f(\omega)$ be defined on the plane and $\psi(\theta)$ a radial wavelet. We can define the convolution as

$$\phi(\theta) * f(\theta, \phi) = \int_{S^2} d\Omega\psi^*(R_\varrho^{-1}\omega)F(\omega) \,. \tag{6.7}$$

The constructed wavelets are next decomposed by means of spherical harmonics $Y_{l,0}$. Thus the convolution of a function f decomposed as

$$f(\omega) = \sum_{l=0,m<|l|}^{L} \widehat{a_{lm}} Y_{lm}$$

with an axisymmetric filter is evaluated as

$$\phi(\theta) * f(\theta, \phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \widehat{a_{lm}}\widehat{\varphi}_{l,0} Y_{l,m}(\theta, \phi) \,. \tag{6.8}$$

Consider next the dilation operator

$$[D(a)G(\omega)] = \lambda^{\frac{1}{2}}(a, \theta)G(D_a^{-1}\omega) \,, \tag{6.9}$$

where $\lambda^{\frac{1}{2}}(a, \theta)$ is a normalization factor defined by

$$\lambda^{\frac{1}{2}}(a, \theta) = \frac{a^{-1}[1 + \tan^2(\frac{\theta}{2})]}{1 + a^{-2}\tan^2(\frac{\theta}{2})} \tag{6.10}$$

The dilation operator satisfies

$$D_a(\theta, \phi\phi) = (\theta_a(\theta), \phi)\ , \tag{6.11}$$

where $\theta_a(\theta)$ is defined on $\theta \in [0, \pi[\to \theta_a \in [0, \pi[$ by

$$\tan\left(\frac{\theta_a(\theta)}{2}\right) = a \tan\left(\frac{\theta}{2}\right). \tag{6.12}$$

The south pole of the sphere is invariant. Note that the dilation operator is somehow an extension of the classical dilation on the plane to the case of the sphere and it associates with a similar sphere.

In [3], some examples are developed to illustrate the efficiency of the method. A first example was based on the well-known Mexican hat wavelet. On the plane, the radial version of the Mexican hat is defined by

$$\psi(r) = \frac{1}{\sqrt{2\pi}}\left(2 - \left(\frac{r}{R}\right)^2\right)e^{-\frac{r^2}{2R^2}} \tag{6.13}$$

where R is a scaling factor and r is the distance to the center of the wavelet.

Such a function is applied next to introduce a continuous transform adapted to spherical images. Applying the inverse stereographic projection, an extension to a spherical Mexican hat has been developed. Consider

$$\psi_s(\delta) = \frac{1}{\sqrt{2\pi}N_R}\left(1 + \left(\frac{\delta}{2}\right)^2\right)^2\left(2 - \frac{\delta}{R}\right)^2 e^{-\frac{\delta^2}{2R^2}} \tag{6.14}$$

where R is always a scaling factor and N_R is a normalization one,

$$N_R = \left(1 + \frac{R^2}{2} + \frac{R^4}{4}\right)^{\frac{1}{2}} \tag{6.15}$$

and δ is the distance to the tangent point associated with the polar angle θ of the inverse stereographic projection,

$$\delta = 2\tan\left(\frac{\theta}{2}\right)\ . \tag{6.16}$$

The resulting wavelet is a zonal function, which permits one to evaluate spherical harmonics coefficients. It holds in fact that the radial Mexican hat may induce a privileged direction to yield a directional spherical wavelet. Let

$$\psi^{\text{mex}}(\omega) = \sqrt{\frac{2}{\pi}}N(\sigma_x, \sigma_y)\left(1 + \tan^2\left(\frac{\theta}{2}\right)\right)\left[1 - \frac{4\tan^2\frac{\theta}{2}}{\sigma_x^2 + \sigma_y^2}\left(\frac{\sigma^2 y}{\sigma_x^2}\cos^2\phi\right.\right.$$
$$\left.\left. + \frac{\sigma^2 y}{\sigma_x^2}\cos^2\phi\right)\right]e^{-2\tan^2\tan(\frac{\theta}{2})(\cos^2\frac{\phi}{\sigma_x^2} + \sin^2\frac{\phi}{\sigma_y^2})} \tag{6.17}$$

Here, $N(\sigma_x, \sigma_y)$ is a normalization constant,

$$N(\sigma_x, \sigma_y) = (\sigma_x^2 + \sigma_y^2)[\sigma_x\sigma_y(3\sigma_x^4 + 3\sigma_y^4 + 2\sigma_x\sigma_y)]^{\frac{-1}{2}} \tag{6.18}$$

6.5 Wavelets for time-series processing

In [92], a well-adapted method to represent scattered spherical data by multiscale spherical wavelets has been developed yielding efficient algorithms for decomposition and reconstruction. The proposed method is illustrated by numerical examples employed to analyze and compress the surface air temperatures observed at a global network of weather stations.

The method has been applied on scattered data obtained from climatology. The earth is considered a sphere so that a meteorological variable $T(\eta)$, representing the surface air temperature, can be treated as a spherical function. The variable $\eta = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi)$ denotes the unit vector that points to a location on the earth from the center of the sphere, with φ and θ being the latitude and longitude of the location. In practice, the function $T(\eta)$ is of course observed at a finite number of observing sites, denoted by $T_j = T(\eta_j)$ on some grid $j = 1, 2, \dots, J$, which enables one to see it as a time series. We explain briefly the idea developed in [92] yielding a multiresolution analysis on the sphere. Let $N_1 = \{\eta_j;\ j = 1, \dots, J\}$ be the source spacial grid to represent $T(\eta)$, and $N_2 = \{\eta_j;\ j = 1, \dots, K\}$ with some $K < J$ being a smaller network obtained by removing the last $J - K$ instants from N_1. Consider next a representation of the function T by means of a spherical harmonics basis

$$T_1(\eta) = \sum_{j=1}^{J} \beta_{1,j}\psi(\eta.\eta_j)$$

and similarly a representation

$$T_2(\eta) = \sum_{j=1}^{K} \beta_{2,j}\psi(\eta.\eta_j)\,.$$

This leads to approximation spaces

$$V_1 = \mathrm{span}\{\psi(., \eta_j);\ \eta_j \in N_1\} \quad \text{and} \quad V_2 = \mathrm{span}\{\psi(., \eta_j);\ \eta_j \in N_2\}\,.$$

As it is seen, T_2 can be understood as the projection of T_1 on V_2. By denoting $D_1(\eta) = T_1(\eta) - T_2(\eta)$ and the collection of $D_1(\eta)$ by W_1, we observe that $V_1 = V_2 \bigoplus W_1$. The space W_1 represents the details lost by using the smaller grid N_2 instead of N_1 for the description of the time series $T_1(\eta_j)$. This means in other words that any function $T_1(\eta)$ in V_1 can be decomposed as

$$T_1(\eta) = T_2(\eta) + D_1(\eta)\,, \tag{6.19}$$

where $T_2(\eta) \in V_2$ and $D_1(\eta) \in W_1$, which explains the multiresolution analysis.

The decomposition (6.19) above can be naturally generalized to a nested sequence $N_1 \supset N_2 \supset \cdots \supset N_L$ and the corresponding spaces of spherical functions $V_1 \supset V_2 \supset \cdots \supset V_L$ satisfying

$$V_{l+1} = V_l \bigoplus W_l$$

for all l, and consequently, $T_1(\eta)$ will be decomposed as

$$T_l(\eta) = T_l(\eta) + D_{l-1}(\eta) + \cdots + D_1(\eta) \tag{6.20}$$

for $l = 2, \ldots, L$.

This decomposition leads to an efficient recursive algorithm. Indeed, let $g_l(\eta)$ be the vector formed by the spherical wavelets associated with N_l and let $\beta = vec\{\beta_{l,j}\}_{j=1}^{N_l}$ be the vector or spherical wavelets coefficients of $T_l(\eta)$. Then we have the following relations, which explain a decomposition/reconstruction algorithm concept:

$$T_{l+1}(\eta) = \beta_{l+1}^T g_{l+1}(\eta), \quad D_l(\eta) = T_l(\eta) - T_{l+1}(\eta) = \gamma_l^T w_l(\eta)\,,$$

where

$$\beta_l = vec\{\alpha_{l+1}, \gamma_l\}, \quad \beta_{+1} = \alpha_{l+1} + E_l\gamma_l, \quad w_l(\eta) = h_l(\eta) - E_l^T g_{l+1}(\eta)\,.$$

Here, the sequence $h_l(\eta)$ is defined by the relation $g_l(\eta) = \{g_{l+1}(\eta), h_l(\eta)\}$.

In [92], an application of such theory was developed based on a real data set of the average surface air temperatures observed during the period of December 1967 to February 1968. A multiscale spherical wavelets decomposition transformed the time series into multiscale components to discover local anomalies at different scales and a compression of the data was applied based on a subset of wavelet coefficients.

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