Digital Sieves

Epilogue by Ellen Harlizius-Klück

Die Primzahl selbst ist völlig nutzlos. Primzahlen sucht man aus demselben Grund, aus dem man auf Achttausender steigt: Sie sind nun mal da, und es macht Spaß.

GÜNTER ZIEGLER

The sieve is an instrument of order. In a single act of sifting it separates wanted from unwanted material.[1] Accordingly it is a binary instrument and lends its name to an increasing number of algorithms that split digital data into desired and undesired.[2] Spam filters and other digital sieves sort out information according to patterns and rules in order to predict a desired outcome.[3] In the digital humanities such instruments are important for information processing. For example, on archaeological sites the new algorithmic sieves meet the old material ones. Digital sifting of fresco fragments increases the efficiency of matching them and can be done by people unskilled in archaeology thus lowering costs for research projects.[4] Still, the material sifting for finding the fresco pieces in the first place is done by skilled human hands.

Digital sieves are able to sort out the skilled from the unskilled because they rely on mathematical algorithms implemented in the options and choices of a database and the calculating routines of a computer. While fresco fragments are matched by humans through visual recognition and comparison, digital matching is a question of measures and numbers. Kockelman compares such algorithms with gatekeepers that accept certain features and reject others.[5] But on the very basic level, algorithms do not decide by looking at the features of an object; they are not concerned with meaning.

Sieve theory is a mathematical discipline that became important in recent years.[6] The Wikipedia article on sieve theory presents a collection of such algorithms named after their inventors: the sieve of Sundaram, the sieve of Atkin, the Legendre sieve, the Brun sieve, the Turán sieve, the Selberg sieve.[7] Ancestor of all these is the sieve of Eratosthenes, a simple method to determine prime numbers up to a given integer.[8]

The Sieve of Eratosthenes

How does the sieve of Eratosthenes work? Today we arrange numbers on a line from negative to positive infinity. But the sieve only works for positive integers. If such a number has only two divisors, 1 and itself, it is called 'prime'. Thus the number 1 is not prime by definition as it has only one divisor. For applying the sieve of Eratosthenes, we write the positive integers down up to a certain number (let us say 24). According to the usual description, the method of Eratosthenes now crosses out all multiples of 2, that are bigger than 2.

~~1~~, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9, ~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~, 17, ~~18~~, 19, ~~20~~, 21, ~~22~~, 23, ~~24~~...

In the next round we cross out all multiples of 3 that are bigger than 3.

~~1~~, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~...

Then cross out all multiples of the next number that is not crossed out...

~~1~~, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~, 17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~...

... and repeat this operation until no number is left.

You might have realized that there is already nothing left to cross out after we deleted the multiples of 3, because our line of numbers is quite short and most multiples are already deleted. The remaining numbers are the prime numbers up to 24 namely: 2, 3, 5, 7, 11, 13, 17, 19, and 23.

In this way the sieve of Eratosthenes is described in most lexica.[9] It is a description and number representation that fits to modern mathematics and can demonstrate that the series of prime numbers is arbitrary and bears no order. The number line progresses one by one and makes no other distinction between numbers than their position.

What do you need primes for?

Prime numbers cannot be generated from other numbers by multiplication. They were not of much practical interest until recently when they, just because of their non-generativity, became a core subject of cryptography. It is therefore still a mystery, why the Greeks were so interested in factorization of numbers and the theory of primes.

There is one very ancient craft of patterning where primes are of importance, or better: preventing them. In weaving, prime amounts of warp threads would allow no full pattern repeat of any pattern in the fabric to be woven. The whole system of number classification described by Nicomachus and especially the fundamental distinction of odd and even numbers is what weavers constantly work with when structuring and designing their fabric.

This distinction of odd and even numbers is the foundation of the number classification in dyadic arithmetic, a number theory that looks out for cognates more than for calculations. The class of even numbers is divided into even-times-even, odd-times-even, and even-times-odd ones. The existence of the last two classes that, from a point of view of modern mathematics, are identical, can demonstrate that there has been a sense for symmetry and meaning that is not conveyed by numbers today. To the Greeks, numbers demonstrate generative principles of the cosmos. This is the reason why the prime numbers are called prime in the first place: from Latin: *primus* or Greek *protos*, both denoting the first. Primes are not a result of combinations of other numbers. Therefore they are often called the atoms of numbers, an idea that we already find in the work of Nicomachus:

> To be sure, when they are combined with themselves, other numbers might be produced, originating from them as from a fountain or root, wherefore they are called 'prime', because they exist beforehand as the beginnings of the others. For every origin is elementary and incomposite, into which everything is resolved and out of which everything is made, but the origin itself cannot be resolved into anything or constituted out of anything.[10]

It has been stated that the sieve of Eratosthenes is not a very elegant method. Waldal remarks the limited interest if applied to a progressing line of natural numbers. When the primes are collected as the smallest of the remaining numbers of each round of sifting, all composite numbers (all multiples) are sifted out and lumped together extirpating any information on the original relationship (which are multiples of three, which of five etc.).[11] Also, the only source for the algorithm, the *Introduction to Arithmetic* written by Nicomachus of Gerasa probably in the first century AD, has been accused of being "unbearable in length and shallowness."[12] Likewise the denomination as 'sieve', Greek *kóskinon*, has been challenged and it is not clear if this name indeed denotes the regular grid that we associate with this device today. *Kóskinon* is not a very common word in ancient Greek, so there are hardly any sources to compare and get more information on the way a *kóskinon* sifts.[13]

As a mathematical method or algorithm on numbers, the sieve of Eratosthenes has no iconological value. Asking for the factors of numbers has no meaning beyond calculation. This is the reason for the critique that we meet when modern mathematicians refer to ancient mathematicians like Nicomachus. That numbers have been of cosmological importance for the Greeks and especially for the Pythagoreans appears as a mystical approach to numbers that is now obsolete.

This faith in having reached a level of objectivity today that mathematics deserves is the reason why the demonstration of the method of Eratosthenes as documented in Nicomachus is hardly ever presented as it appears in the source. In fact, the method is only applied to odd numbers beginning with 3 and it does not cross out multiples of 3, but every number that has a distance of three steps from 3. The idea is more to apply a rhythmical pattern, than to execute calculations. And this repeated application of patterns generated by a different length of steps is what resembles a sieve mechanism.

Primes and Patterns

Actually the method of Eratosthenes applies a series of sieves and works in the following way. Let us say we are looking for the primes up to 60. We write down all numbers from 2 to 60 and put the 1 away (which is not prime per definition). Now we need to sift out all multiples of 2 as not prime and put the 2 as first prime number on a separate heap (in order so make the order of the sieve visible, the actual sifting structure in the tables below is marked grey and the numbers that are sifted out remain stroked through).

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~
51	~~52~~	53	~~54~~	55	~~56~~	57	~~58~~	59	~~60~~

Now we put the next remaining number 3 on our prime heap and sift out all multiples of 3.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	35	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~
~~51~~	~~52~~	53	~~54~~	55	~~56~~	~~57~~	~~58~~	59	~~60~~

Now we put the next remaining number, the 5, on our prime heap and from the rest sift all multiples of 5.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~

The next number to put on the prime heap is 7. Thus we have collected: 2, 3, 5, 7 and sort out all multiples of 7.

~~1~~	2	3	~~4~~	~~5~~	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~

In fact, most multiples are already sifted out as multiples of smaller numbers so that we only have to delete the 49.

We could repeat this principle or rule now for the numbers 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, and 59, but after sifting all multiples of 7 the work is actually done.

The reader might have observed that the multiples of a number show in the tables as patterns or bands. So in fact the sieve of Eratosthenes is generated by overlapping sieve patterns of increasing numbers. However, the collected prime numbers show no such pattern.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~

Especially not when we display them according to the original report in Nicomachus without multiples of two that are responsible for the vertical stripes or bands (we need to keep the two as the only even prime number and therefore separate it):

2	~~1~~	3	5	7	~~9~~
	11	13	~~15~~	17	19
	~~21~~	23	~~25~~	~~27~~	29
	31	~~33~~	~~35~~	37	~~39~~
	41	43	~~45~~	47	~~49~~
	~~51~~	53	~~55~~	~~57~~	59

Waldal takes advantage of the fact that the numbers 'pattern' according to the system of arrangement and presents the sieve of Eratosthenes as tables with 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18, and 30 columns.

Let us look at his example for 7:

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49
50	51	52	53	54	55	56
57	58	59	60	61	62	63

We might notice that the series of pairs of primes around an even number appear on virtual diagonal lines like 11–13, 17–19, 29–31, (41–43), 59–61. Such pairs are called prime twins and suggest that primes indeed show a sort of pattern.

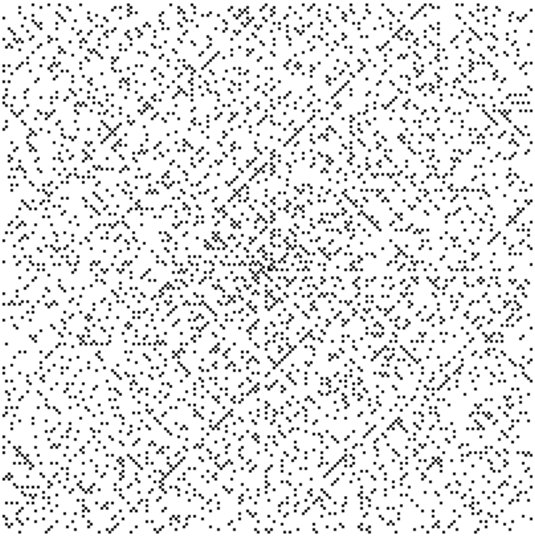
Waldal presents one arrangement with 6 columns that makes this even more visible (beginning with 2):

1	2	3	4	5	6	7
	8	9	10	11	12	13
	14	15	16	17	18	19
	20	21	22	23	24	25
	26	27	28	29	30	31
	32	33	34	35	36	37
	38	39	40	41	42	43
	44	45	46	47	48	49
	50	51	52	53	54	55
	56	57	58	59	60	61

A lot of similar approaches to such 'speaking' visual representations have been made to detect the pattern of primes. Stanislaw Ulam arranged numbers in a meander-like spiral.

```
37 — 36 — 35 — 34 — 33 — 32 — 31
 |                             |
38   17 — 16 — 15 — 14 — 13   30
 |    |                   |    |
39   18    5 —  4 —  3   12   29
 |    |    |         |    |    |
40   19    6    1 —  2   11   28
 |    |    |              |    |
41   20    7 —  8 —  9 — 10   27
 |    |                        |
42   21 — 22 — 23 — 24 — 25 — 26
 |
43 — 44 — 45 — 46 — 47 — 48 — 49 ...
```

The advantage here is that, if you mark each prime as a dot, you can compute huge images showing strange diagonal lines that seem to indicate a pattern *(fig. 37)*.



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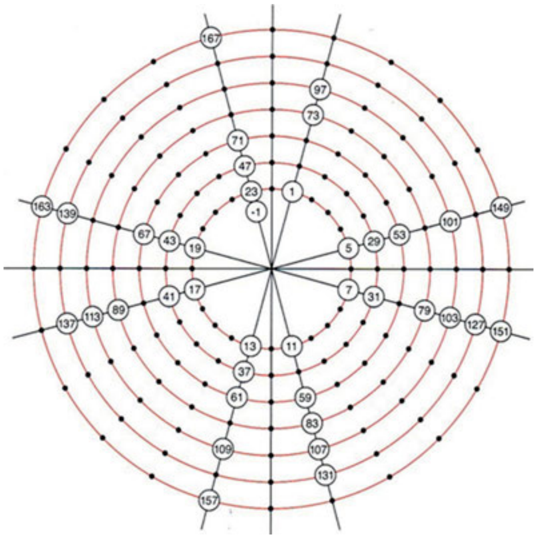
Ulam-spiral showing black dots where prime numbers are

Another famous representation is the prime cross of Peter Plichta (1991–2004) who claims to reveal the secrets of the atoms.[15] He states that the primes have meaning for the structure of the four-dimensional space and of logarithmic primes as background for the three-dimensional space.



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The prime number cross evoking the Maltese cross

Such approaches give new opportunities for number magic by looking for similarities and analogies. The prime number cross is sometimes seen as a Maltese cross *(fig. 38)* and as such finally replaces the former sieve of chastity of Queen Elizabeth I as a modern version of a sieve that is applied to the mantle of Elizabeth II on the same side as the former sieve was depicted *(fig. 39)*.[16] Here iconology comes back into its own as an investigation of symbols in transformation.

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Leonard Boden, *Queen Elizabeth II*, b. 1926. Oil on canvas, 74 × 49 cm, the artist's estate, courtesy of Powys County Council