Damir N. Gainanov
Graphs for Pattern Recognition

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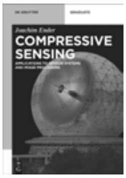

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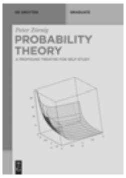

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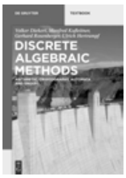

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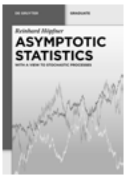

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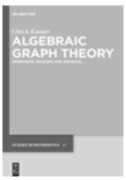

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Damir N. Gainanov

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Infeasible Systems of Linear Inequalities

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Preface

This book deals with mathematical constructions that are foundational in such an important area of *data mining* as *pattern recognition*. A closer look is taken at *infeasible systems of linear inequalities*, whose generalized solutions act as building blocks of geometric decision rules for recognition.

Infeasible systems of linear inequalities proved to be a key object in pattern recognition problems described in geometrical terms thanks to *the committee method*.

Infeasible systems of inequalities represent an important special subclass of *infeasible systems of constraints with monotonicity property* – systems whose multi-indices of feasible subsystems form *abstract simplicial complexes* (*independence systems*), fundamental objects of combinatorial topology. In discrete mathematics, the faces of such complexes are interpreted as zeros of *monotone Boolean functions*. Chapter 1 of the book deals with simplicial complexes and monotone Boolean functions related to common infeasible systems of constraints. The graph-theoretic methods represent a very productive way to study combinatorial and structural properties of infeasible systems of constraints. From the applied point of view, the most important property is the *connectedness* of a specific graph assigned to a family of *maximal feasible subsystems*. For instance, the set of solutions taken one by one for each of the maximal feasible subsystems of an infeasible system, which constitute an *odd cycle* in such a graph, represents a committee for an infeasible system of linear inequalities over $\mathbb{R}^n$ formally describing a pattern recognition problem. Thus, graph-theoretic methods that help us to solve one of the main tasks of committee theory – searching for a committee with the minimal number of elements can be taken as a basis for efficient algorithms of constructing decision rules for pattern recognition. The connectedness of graphs discussed is actually determined by the connectedness of the space $\mathbb{R}^n$; moreover, the connectedness of similar graphs in the context of common topological spaces is also determined by the connectedness of these spaces. The subject matter of Chapter 2 is (hyper)graphs corresponding to facets of common simplicial complexes and to maximal feasible subsystems of infeasible systems of linear inequalities.

Equally interesting results are obtained from an analysis of infeasible systems of linear inequalities by methods of combinatorial geometry. In Chapter 3, the notion of *diagonal* of a polytope, which is traditional for plane geometry, is generalized to multidimensional convex polytopes. A dual correspondence between diagonals and facets of polytopes, on the one hand, and multi-indices of maximal feasible and minimal infeasible subsystems of inequalities, on the other hand, is described. This duality is used, in particular, to obtain different estimates of the number of subsystems.

In Chapter 4, the correspondence between infeasible systems of inequalities and monotone Boolean functions motivates us to construct *algorithms for optimal inference* of functions. Several criteria for optimality of algorithms of inference are considered, and algorithms satisfying these criteria are constructed.

In Chapter 5, the algorithmic approach to constructing an optimal committee of an infeasible system of linear inequalities is considered; it is based on such principal features of graphs as the connectedness and the existence of odd cycles. A brief review of *alternative covers* in the second half of this chapter provides a new look at collective solutions to infeasible systems of constraints.

The aim of this book is to present a mathematical toolset finding an application to the construction of pattern recognition complexes that solve the recognition problem in its geometric setting.

Such complexes of pattern recognition start their work with preprocessing of a training sample, that is, a massive collection of vectors from a high-dimensional feature space. Because the vectors of the training sample are preliminarily divided into groups that partially represent logically uniform classes or categories, they reflect a certain knowledge domain in the boundaries of which every new unclassified vector entering into the complex must be referred to one of the classes. At consecutive stages of preprocessing, the groups from the training sample are aggregated, with the use of hierarchical tree-like structures, into two extended groups that partially represent the corresponding generalized classes. The task of the recognition complex consists in the search for a geometric object that has a relatively simple formal description and, at the same time, strictly separates the vectors from distinct extended groups of the training sample. In the context of the book, the above-mentioned task can be interpreted, for example, as the search for a separating hyperplane in an Euclidean feature space. In practice, information contained in almost any training sample leads to a situation where a unique separating hyperplane cannot be found, because the linear inequality system underlying the problem of the discrimination of the two extended groups turns out to be infeasible. By means of some dimensional increase of the input data, the inequalities become homogeneous; their strictness is motivated by the stability demands that must be satisfied by the decision rules generated by the pattern recognition complex. This is how the infeasible system of homogeneous strict linear inequalities comes to the stage in the contradictory two-class pattern recognition problem, which has to be solved by the complex. The system as a whole has no solution, but any of its feasible subsystems can be solved by the software of the recognition complex that implements modern powerful techniques of linear optimization. The smart committee strategy of the recognition complex consists in the finding of solutions to a few maximal feasible subsystems and in their combining into a committee decision rule which operates with arrangements of separating hyperplanes. On the one hand, such a rule always allows the complex to correctly discriminate the vectors from the two extended groups of the training sample and, on the other hand, it makes it possible to apply the procedure of committee voting to a new vector entering into the complex; the majority decision rule, governed by the committee, refers the new vector to a generalized class. The recognition complex implements various effective techniques for constructing the separating committees, by exploiting specific properties of the (hyper)graphs of the maximal feasible subsystems of infeasible systems of linear inequalities. With the help of these techniques, the complex repeatedly solves the two-class pattern recognition problem for each higher level extended group of vectors from the training sample, adding at every step some committee decision rule to a resulting hierarchical tree-like structure. This structure represents the machine for recognition of new vectors, and it correctly recognizes any vector of the training sample.

This edition is the extended translation of the book *Combinatorial Geometry and Graphs in an Analysis of Infeasible Systems and Pattern Recognition* published by *Nauka*, Moscow, in 2014.

Moscow and *Ekaterinburg* — *Damir N. Gainanov*
October 2016

Contents

Pattern recognition, infeasible systems of linear inequalities, and graphs

Full-function complexes of pattern recognition allow a human or technological user to mine relevant feature data in two main directions that can be considered interconnected, depending on the goals that must be achieved by the complexes.

One direction of data mining in pattern recognition is most often referred to as *unsupervised learning*. The complex deals with a massive collection of vectors whose components represent qualitative or quantitative descriptions of various parameters that are specific for the problem domain of the user. Although some categorical labels could have preliminarily been assigned to those vectors, in order to reflect a knowledge of the domain, the complex treats the data without using any earlier classifying information. Instead, the task consists in an exploratory analysis of the massive amount of high-dimensional vectors from the feature space, which aims at the elucidation of the inner structure of the data cloud. Typically, one is interested in how many relatively dense and isolated subclouds, called *clusters*, can be discovered in the whole data cloud, and how each of them can be given a concise characterization in working terms of the problem domain.

Various strong mathematical mechanisms, as well as heuristics, are involved for preprocessing the input sample of vectors and obtaining a resulting hierarchical picture of the data cloud. Let us mention just two questions that must be answered by the designers of a complex of pattern recognition. How incomplete or missing information on the components of vectors from the feature space should be dealt with? Is there any possibility to artificially decrease the complexity of the data sample by means of an information-preserving map of the sample into a derived feature space of much lower dimension? It is clear that for obtaining concise descriptions of relatively isolated data subclouds, outermost vectors, say the vectors lying on the boundaries of the convex hulls of the subclouds, are most relevant; for this reason certain methods of thinning irrelevant vectors may be provided.

The essential topics in unsupervised learning are the choice of *metrics* that allow the recognition complex to measure the *similarity* or *distance* between vectors and between *clusters* of vectors, and the choice of the presentation format for the cluster hierarchy revealed to the user. It is convenient to visualize the hierarchy with the help of interactively scaled tree-like graphical structures that make it possible to easily reveal information on the cluster membership and on metric intercluster dissimilarities.

Although the exploratory cluster analysis surely plays an important role in data mining, the result of unsupervised learning of the recognition complex should consist in the generation of decision rules, which would allow the complex to refer any new vector of the feature space to a large isolated cluster, thus recognizing the new vector as a representative of a certain category. Such a recognition rule is based on the procedure of comparison of the similarities or distances between the new unclassified vector and the large isolated clusters.

The aim of this book is to present a mathematical toolset finding an application to the construction of pattern recognition complexes that solve the recognition problem, in its *geometric setting*, in the *supervised learning* mode.

Such complexes of pattern recognition begin their work with preprocessing of a *training sample*, that is, a massive collection of vectors from a high-dimensional feature space that are preliminarily divided into groups that partially represent logically uniform *classes* or *categories*. These groups reflect a certain knowledge domain in the boundaries of which every new unclassified vector entering into the complex must be referred to one of the classes.

The variety of approaches to supervised recognition learning includes such universally accepted methodologies as *nearest-neighbor classifiers*, *neural networks*, and *support vector machines*.

At consecutive stages of preprocessing, the groups from the training sample are aggregated, with the use of hierarchical tree-like structures, into two extended groups that partially represent the corresponding generalized classes.

Given an odd integer k, a *k-nearest-neighbor classifier* finds, for a new unclassified vector from the feature space, its k distinct nearest neighbors from the training sample; a majority of these neighbors belongs to one of the extended groups and, as a consequence, that group votes for the referring of the vector to the generalized class represented by the group. A hierarchically organized procedure of making similar k-nearest-neighbor decisions, that is applied to each of the extended subgroups of the training sample, allows the complex to recognize the new vector as a representative of the class partially described by a group from the training sample.

Dealing with an extended subgroup of vectors from the training sample, which is, in turn, divided into two subgroups at some stage of a hierarchical learning process, a *neural network* represents a collection of interconnected layers of neurons. *Neurons* are elementary computational operators that reflect vectors of the feature space to weighted values of a *sigmoid function* taken at certain weighted sums of the components of those vectors. As the result of supervised training, the neural network combines the responses of individual neurons into a decision, based on a mechanism of *thresholds*, which refers a new unclassified vector to some generalized subclass.

A *support vector machine* tries to find, at a step of a hierarchically organized procedure, three parallel hyperplanes of the feature space, namely the *maximal-margin hyperplane* which separates the vectors of two subgroups from the training sample and, at the same time, maximizes the distance between two *margin hyperplanes* containing the nearest vectors of the training sample that belong to different subgroups. The quadratic optimization technique allows the recognition complex to find the maximal-margin hyperplanes (when training subgroups are affinely separable) or to motivate the search for nonlinear separating surfaces (when the subgroups cannot be separated by hyperplanes). The hierarchical collection of the separating hyperplanes and surfaces makes it possible to refer new unclassified vectors from the feature space to some classes partially represented by the vectors of the training sample.

Thus, the task of the recognition complex that implements a supervised learning methodology often consists in the search for a geometric object that has a relatively simple formal description and, at the same time, strictly separates the vectors from distinct extended groups of the training sample.

In the context of the book, the above-mentioned task can be seen as the search for a separating hyperplane in an Euclidean feature space. In practice, information contained in almost any training sample leads to a situation where a unique separating hyperplane cannot be found, because the linear inequality system underlying the problem of the discrimination of the two extended groups turns out to be infeasible. Indeed, let $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ be the two extended groups of vectors from the training sample, processed at some step of the hierarchical supervised learning procedure. These are just two finite sets of vectors of the feature space $\mathbb{R}^{n-1}$. Let us augment every vector from the sets $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ by a new nth component which is equal to 1. We thus obtain two sets $\boldsymbol{B}, \boldsymbol{C} \subset \mathbb{R}^n$, for which the recognition complex tries to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \boldsymbol{b}, \mathbf{x} \rangle > 0, & \boldsymbol{b} \in \boldsymbol{B}\,, \\ \langle \boldsymbol{c}, \mathbf{x} \rangle < 0, & \boldsymbol{c} \in \boldsymbol{C}\,, \end{cases} \tag{1}$$

where $\langle \boldsymbol{b}, \mathbf{x} \rangle$ denotes the standard scalar product $\sum_{k \in [n]} b_{ik}\mathrm{x}_k$, and $[n] := \{1, 2, \dots, n\}$. The *strictness* of these *homogeneous* inequalities is motivated by the stability demands that must be satisfied by the decision rules generated by the pattern recognition complex.

If $\boldsymbol{x}$ is a solution to system (1), then classification of a new vector $\boldsymbol{g} \in \mathbb{R}^n$ (i.e., the referring of $\boldsymbol{g}$ to one of the extended classes partially represented by the sets $\boldsymbol{B}$ and $\boldsymbol{C}$) is performed on the basis of the sign of the scalar product $\langle \boldsymbol{x}, \boldsymbol{g} \rangle$. However, the system under consideration can turn out to be infeasible, and this most frequent case requires the development of special methods of problem-solving.

Even if system (1) as a whole has no solution, any of its feasible subsystems can be solved by the software of the recognition complex that implements techniques of linear optimization.

By means of the passage from system (1) to the infeasible system

$$\begin{cases} \langle \boldsymbol{b}, \mathbf{x} \rangle > 0, & \boldsymbol{b} \in \boldsymbol{B}\,, \\ \langle -\boldsymbol{c}, \mathbf{x} \rangle > 0, & \boldsymbol{c} \in \boldsymbol{C}\,, \end{cases}$$

which we will briefly describe here as the system

$$\{\langle \boldsymbol{a}, \mathbf{x} \rangle > 0 \colon\; \boldsymbol{a} \in \boldsymbol{A}\}\,, \tag{2}$$

the recognition complex deals with the mathematical construction that has the principal feature: if any subsystem, with two inequalities, of system (2) is feasible, then this simple condition guarantees that the recognition complex can involve in its computational arsenal a powerful technique for constructing certain *collective solutions* to infeasible system (2), and further use them as the components of hierarchical decision rules for recognition.

Recall that a *committee* of infeasible system (2) is defined as a finite subset of vectors $\mathcal{K} \subset \mathbb{R}^n$ satisfying the relation

$$|\{\boldsymbol{x} \in \mathcal{K} \colon \ \langle \boldsymbol{a}, \boldsymbol{x} \rangle > 0\}| > \tfrac{1}{2}|\mathcal{K}| \,,$$

for each vector $\boldsymbol{a} \in \boldsymbol{A}$.

Suppose that a committee $\mathcal{K}$ of system (2) is found by the recognition complex. Then an unclassified vector of the feature space $\mathbb{R}^{n-1}$, lifted to the working $(n-1)$-dimensional affine subspace of the space $\mathbb{R}^n$ with the help of the additional nth component 1, can be recognized as an element of the classes, partially represented by the sets $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$, according to the result of the majority voting procedure performed by the members of the committee $\mathcal{K}$.

The smart committee strategy of the recognition complex consists in the finding of solutions to a few *maximal feasible subsystems* (*MFSs*) of system (2), and in their combining into the committee decision rule, which operates with arrangements of separating hyperplanes.

A feasible subsystem of infeasible system (2) is called *maximal* if any additional inequality from the system turns the resulting collection of inequalities into an infeasible subsystem.

If $[m]$ is the set of indices with which the inequalities from infeasible system (2) are marked, then a *multi-index* $T \subseteq [m]$ corresponds to the subsystem composed of the inequalities with the indices from the set T.

If we let $\mathbf{J}$ denote the family of the multi-indices of all maximal feasible subsystems of system (2), then the *graph of MFSs* of system (2) is defined as the graph with the vertex set $\mathbf{J}$; an unordered pair $\{J, J'\} \subset \mathbf{J}$ is an edge of this graph if and only if the multi-indices J and J' cover the index set of system (2), that is, $J \cup J' = [m]$.

The high efficiency of supervised learning algorithms implemented by the recognition complex, which uses the graph of MFSs, is explained by the following three basic facts:

- The graph of MFSs is *connected*.
- The graph of MFSs is *not bipartite*.
- The complement $[m] - J$ of the multi-index $J \in \mathbf{J}$ of any *MFS* of system (2) is the multi-index of a *feasible subsystem*.

Since the graph of MFSs is not bipartite, it contains at least one *cycle of odd length*.

A fundamental result in the committee theory is formulated as follows: if the multi-indices of some MFSs represent the vertex set of a cycle of odd length in the graph of MFSs, then in order to construct a committee, it suffices to take one vector from the open cone of solutions to each MFS from the vertex set of the cycle.

Thus, the problem of constructing a committee with a small number of members can be reduced to the problem of finding a cycle of short odd length in the graph of MFSs. This derived problem is solved by the software of the recognition complex with the help of various strong and heuristic methods.

On the one hand, the obtained committee decision rule always allows the recognition complex to correctly discriminate the vectors from the two extended groups of the training sample and, on the other hand, it makes it possible to apply the procedure of committee voting to a new vector entering into the complex; the majority decision rule, governed by the committee, refers the new vector to a generalized class.

The complex repeatedly solves the two-class pattern recognition problem for each higher level extended group of vectors from the training sample, adding at every step some committee decision rule to a resulting hierarchical tree-like structure.

This structure represents the machine for recognition of new vectors, and it correctly recognizes any vector of the training sample.

1 Infeasible monotone systems of constraints

In discrete mathematics, the following research subjects are of prime importance: Let $\mathfrak{S} := \{\mathfrak{s}_1, \mathfrak{s}_2, \dots, \mathfrak{s}_m\}$ be a finite nonempty *system of constraints* and $[m] := \{1, 2, \dots, m\}$ the set of the *indices* of constraints with which the elements of the set $\mathfrak{S}$ are marked. Assigning to the set $[m]$ the *Boolean lattice* $\mathbb{B}(m)$ of all its subsets partially ordered by set-theoretical inclusion, we call an arbitrary element $B \in \mathbb{B}(m)$ the *multi-index of the subsystem* $\{\mathfrak{s}_i : i \in B\}$ of the system $\mathfrak{S}$; in many studies the shorter term *index of a subsystem* is used. To the relation $A \subseteq B$ of inclusion for the multi-indices $A, B \subseteq [m]$ corresponds the comparison relation $A \preceq B$ for the elements A and B in the lattice $\mathbb{B}(m)$. The set of *atoms* $\mathbb{B}(m)^{(1)} := \{\{1\}, \{2\}, \dots, \{m\}\}$ of the lattice $\mathbb{B}(m)$ is in one-to-one correspondence with the set of constraints $\mathfrak{S}$. The *least element* $\hat{0}$ of the lattice $\mathbb{B}(m)$ is the multi-index of the *empty subsystem* $\emptyset$ of the system $\mathfrak{S}$, and the *greatest element* $\hat{1}$ of the lattice $\mathbb{B}(m)$ is the multi-index $[m]$ of the entire system $\mathfrak{S}$.

Let a map $\pi\colon\ \mathbb{B}(m) \to \mathbf{2}^{\varGamma}$ into the family of subsets of some nonempty set $\varGamma$ be given, with the following properties:

- The empty subsystem of the system $\mathfrak{S}$ is *feasible*, that is,

$$\pi(\hat{0}) \neq \emptyset\,; \tag{1.1}$$

  one usually supposes $\pi(\hat{0}) := \varGamma$.
- Each constraint taken independently is realizable or, in other words, each subsystem consisting of one constraint is *feasible*:

$$B \in \mathbb{B}(m)^{(1)} \Longrightarrow \pi(B) \neq \emptyset\,. \tag{1.2}$$

- Further,

$$A,\ B \in \mathbb{B}(m), \quad A \preceq B \Longrightarrow \pi(A) \supseteq \pi(B)\,, \tag{1.3}$$

  and thus

$$A,\ B \in \mathbb{B}(m) \Longrightarrow \pi(A) \cap \pi(B) \supseteq \pi(A \vee B)\,,$$

  where $A \vee B$ denotes the least upper bound (i.e., the set-union $A \cup B$) of the elements A and B in the lattice $\mathbb{B}(m)$.
- One often considers *infeasible systems* $\mathfrak{S}$ such that

$$\pi(\hat{1}) = \emptyset\,. \tag{1.4}$$

We will call any system of constraints $\mathfrak{S}$, for which the map π and the range family of this map associated with $\mathfrak{S}$ satisfy conditions (1.1)–(1.4), a *finite infeasible monotone system of constraints*.

1.1 Structural and combinatorial properties of infeasible monotone systems of constraints

In this chapter, we particularly describe some properties of constraint systems, which are essentially associated with the *representativity of sets*.

Speaking briefly, the mutual representativity of sets A and B of any kind is related to the answer to the question on the nonemptiness of their intersection $A \cap B$.

The subject of this chapter goes back to the standard problem of *combinatorial optimization*: for a nonempty family $\mathcal{A} := \{A_1, \ldots, A_\alpha\}$ of nonempty and pairwise distinct subsets of a finite *ground set* $\mathrm{V}(\mathcal{A}) := \bigcup_{i=1}^{\alpha} A_i$, to describe, from the structural and combinatorial points of views, the properties of the pair $(\mathcal{A}, \mathfrak{B}(\mathcal{A}))$, where $\mathfrak{B}(\mathcal{A})$ is the family of all *minimal* (by inclusion) *systems of representatives* for $\mathcal{A}$ – several equivalent terms which will be mentioned later are used for naming these constructions; by definition, *any set from* $\mathfrak{B}(\mathcal{A})$ *has a nonempty intersection with each set from the family* $\mathcal{A}$ *and, under the removal of its arbitrary element, lacks this property*.

A subset $B \subseteq \mathrm{V}(\mathcal{A})$ satisfying the condition

$$B \cap A_i \neq \emptyset\,, \quad \forall\, i \in [\alpha]\,, \tag{1.5}$$

is called a *system of representatives*, *blocking set*, *transversal*, or *transversal set* of the family $\mathcal{A}$. From the graph-theoretic point of view, the family $\mathcal{A}$ is the family of *hyperedges of a hypergraph* on the set of *vertices* $\mathrm{V}(\mathcal{A})$ and, in this context, the set B with property (1.5) is called a *vertex cover* of the hypergraph.

Note that the system of representatives B of $\mathcal{A}$ contains as a subset at least one minimal system of representatives.

If a family $\mathcal{A}$ has the property

$$A_i \not\subseteq A_j \quad \forall\, i, j \in [\alpha],\;\; i \neq j\,,$$

(in other words, if the sets from $\mathcal{A}$ are pairwise *incomparable by inclusion*) then the following terms are used for such a family: a *Sperner family*, *clutter*, or *antichain*. Note that for an arbitrary family $\mathcal{A}$, the corresponding family $\mathfrak{B}(\mathcal{A})$ is by definition a Sperner family.

In the theory of combinatorial optimization, the family $\mathfrak{B}(\mathcal{A})$ is called the *blocker* of the family $\mathcal{A}$; its sets $B \in \mathfrak{B}(\mathcal{A})$ are called the *minimal* (by inclusion) *blocking sets*, *minimal transversals*, *minimal transversal sets*, or *minimal systems of representatives* (the last three terms should not be confused with related terms used in an analysis of systems of *distinct* representatives, which are irrelevant to the subject of our research).

The question on the systems of representatives of the family $\mathcal{A} := \{A_1, \ldots, A_\alpha\}$, with the ground set $\mathrm{V}(\mathcal{A}) = [m]$, of nonempty pairwise distinct sets can be posed from several points of views. Recall some of them:

(a) Functions of logical algebra

The problem on systems of representatives is instructive for elucidation of the basic principles of the application of Boolean algebra and of the very effective trick of "the turning elements and sets into propositions."

The *proposition* stating that some set B is a system of representatives of a family $\mathcal{A}$ looks like

$$\left(\bigvee_{\mathrm{x}\in A_1} \text{“}\mathrm{x} \in B\text{”}\right) \wedge \left(\bigvee_{\mathrm{x}\in A_2} \text{“}\mathrm{x} \in B\text{”}\right) \wedge \cdots \wedge \left(\bigvee_{\mathrm{x}\in A_\alpha} \text{“}\mathrm{x} \in B\text{”}\right) = 1\,. \tag{1.6}$$

It is customary to write, for brevity, the character x instead of the proposition "$\mathrm{x} \in B$"; it is thought of as a *Boolean variable* (whose value is 1 when x belongs to the set B, and 0 otherwise.) It is yet convenient to write disjunction as sum, and conjunction as product. Then the proposition "B is a system of representatives of $\mathcal{A}$" of the form (1.6) can be reformulated as follows:

$$\left(\sum_{\mathrm{x}\in A_1} \mathrm{x}\right) \cdot \left(\sum_{\mathrm{x}\in A_2} \mathrm{x}\right) \cdots \left(\sum_{\mathrm{x}\in A_\alpha} \mathrm{x}\right) = 1\,, \tag{1.7}$$

and this equation is satisfied by those tuples of the values of Boolean variables $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_m$ only for which the proposition is true.

In order to find the minimal systems of representatives of $\mathcal{A}$, one transforms the left-hand side of equation (1.7) into the *minimal disjunctive normal form*

$$\prod_{\mathrm{x}\in B_1} \mathrm{x} + \prod_{\mathrm{x}\in B_2} \mathrm{x} + \cdots + \prod_{\mathrm{x}\in B_\beta} \mathrm{x} = 1\,,$$

by removing parentheses and using the absorption law. Such a form is unique. The sets $B_1, B_2, \ldots, B_\beta \subseteq \mathrm{V}(\mathcal{A})$ are the minimal systems of representatives of the family $\mathcal{A}$, and they compose its blocker $\mathfrak{B}(\mathcal{A})$.

(b) $(0,1)$-matrices

One can put in correspondence with a cover $\mathcal{A}$ of the set $[m]$ the binary *incidence matrix* of size $\alpha \times m$ whose (i,j)th entry by definition equals to 1 when $j \in A_i$, and to 0 otherwise.

Incidence matrices serve as the main connecting link between combinatorial problems on the existence and choice, on the one hand, and matrix theory, on the other.

The *width* of an incidence matrix is the minimal number of its columns such that the sum of elements in each row of the submatrix formed by the selected columns is positive. The width of this matrix coincides with the least quantity among the cardinalities of the representative systems of the family $\mathcal{A}$.

(c) Transversal sets (vertex covers) of hypergraphs

In graph theory, as was noticed earlier, the set $\mathrm{V}(\mathcal{A})$ and the family $\mathcal{A}$ are interpreted as a hypergraph on the vertex set $\mathrm{V}(\mathcal{A})$, with the hyperedge family $\mathcal{A}$. If any hyperedge of a hypergraph is of cardinality r then such a hypergraph is called *r-regular*. The 2-regular hypergraphs are called *simple graphs* without isolated vertices.

The *transversal sets* of the hypergraph are precisely the systems of representatives of the hyperedge family $\mathcal{A}$. The minimal cardinality of transversal sets is called the *transversality number* of the hypergraph.

The importance of the problems on the transversality number of hypergraphs comes from the fact that many combinatorial questions can be reformulated in terms of finding the transversality number of some hypergraph.

(d) Bipartite graphs and binary relations

A simple graph $\mathbf{G}(V_1, V_2; \mathcal{E})$ is called *bipartite* if its vertex set $V_1 \dot{\cup} V_2$ is a union of two nonempty disjoint sets (*parts* or *classes*) V_1 and V_2 such that each edge from the family $\mathcal{E}$ is incident to vertices from different classes. If one orients all the edges of $\mathbf{G}(V_1, V_2; \mathcal{E})$ in the direction from V_1 to V_2 then the graph $\mathbf{G}(V_1, V_2; \mathcal{E})$ can be identified with a *binary relation* (some subset of the *Cartesian product* $V_1 \times V_2$) on the classes V_1 and V_2.

Let us put in correspondence with the set family $\mathcal{A}$ under consideration the bipartite graph $\mathbf{G}(\mathrm{V}(\mathcal{A}), \mathcal{A}; \mathcal{E})$ in which, by definition, for all vertices $v \in \mathrm{V}(\mathcal{A})$ and sets $A \in \mathcal{A}$, the inclusions $(v, A) \in \mathcal{E}$ are fulfilled if and only if $v \in A$. The systems of representatives of the family $\mathcal{A}$ are in one-to-one correspondence with subfamilies of the edge family $\mathcal{E}$ such that the sets of elements from the class $\mathcal{A}$, that are incident with them, cover this class.

(e) Systems of distinct representatives

This research topic in combinatorial optimization is the last among those we mention, and it is completely beyond the scope of our consideration. Mathematical constructions related to systems of distinct representatives are *bipartite matchings*, *transversal matroids*, and the *permanents of incidence matrices*.

A subset $B \subseteq \mathrm{V}(\mathcal{A})$ is called a *system of distinct representatives* of the family $\mathcal{A}$ if there is a bijection $\phi\colon B \to [\alpha]$ such that for each element $b \in B$ the inclusion $b \in A_{\phi(b)} \in \mathcal{A}$ holds.

It is evident that the family (possibly, empty) of all systems of distinct representatives of $\mathcal{A}$ is a subfamily of the family of all systems of representatives of $\mathcal{A}$. Note also that any system of distinct representatives (if such systems exist) contains as a subset at least one set from the blocker $\mathfrak{B}(\mathcal{A})$.

Recall the relationship between the combinatorial properties of two partially ordered sets (posets) that arise naturally together with specific partitions of Boolean lattices. Let $\mathbb{B}(m)$ be the Boolean lattice of rank $m \geq 1$, which is again thought of as the lattice of all subsets of the set $[m]$. Recall that this lattice represents the power set $\mathbf{2}^{[m]}$ of the set $[m]$ whose elements (*faces*) are partially ordered by set-theoretic inclusion. As earlier, we denote the least element of the lattice $\mathbb{B}(m)$, which is the empty subset $\emptyset$ of the set $[m]$, by $\hat{0}$. Let $\mathcal{V} := \{V_1, V_2, \ldots, V_\gamma\} \subseteq \mathbb{B}(m) - \{\hat{0}\}$ be some nonempty family of sets equipped with the ordering induced by the partial order $\preceq$ on $\mathbb{B}(m)$. We denote the sets of minimal and maximal elements of the poset $\mathcal{V}$ by $\mathbf{min}\,\mathcal{V}$ and $\mathbf{max}\,\mathcal{V}$, respectively. Recall that the *order ideal* $\mathfrak{I}(\mathcal{V}) = \mathfrak{I}(\mathbf{max}\mathcal{V})$ of the lattice $\mathbb{B}(m)$ generated by a set $\mathcal{V}$ is defined as the subposet $\mathfrak{I}(\mathcal{V}) := \{E \in \mathbb{B}(m)\colon\ \exists V \in \mathcal{V},\ E \preceq V\}$.

Let $\rho\colon\ \mathbb{B}(m) \longrightarrow \{0\}\dot{\cup}[m]$ be the *rank function* on $\mathbb{B}(m)$ that reflects the subsets of $[m]$ to their cardinalities. As earlier, we use the notation $\mathbb{B}(m)^{(1)} := \{D \in \mathbb{B}(m)\colon \rho(D) = 1\} = \{\{1\},\{2\},\ldots,\{m\}\}$ to denote the layer of *atoms* of the lattice $\mathbb{B}(m)$, that is, the family of one-element subsets of the set $[m]$. If $\mathcal{V}$ is an *antichain* or, in other words, if any two elements of the set $\mathcal{V}$ are *incomparable* in $\mathbb{B}(m)$, that is, $\mathcal{V} = \mathbf{min}\,\mathcal{V} = \mathbf{max}\,\mathcal{V}$, then the *unordered* family of sets

$$\Delta := \{F \subseteq [m]\colon\ F \in \mathfrak{I}(\mathcal{V})\} \tag{1.8}$$

is called the *abstract simplicial complex* on the set of *vertices* $\bigcup_{V\in\mathcal{V}} V$, with the family of *facets* $\mathcal{V}$. The sets from the complex Δ are called its *faces*, and the ideal $\mathfrak{I}(\mathcal{V})$ is called the *face poset* of the complex Δ.

The abstract simplicial complex is a fundamental construction in algebraic and combinatorial topology, discrete mathematics, and mathematical cybernetics; in some cases it is called the *independence system*, and in this context one says on the *independent sets* and *bases* of independence systems instead of the faces and facets of complexes.

We built complex (1.8) on the basis of an antichain of the Boolean lattice of subsets of a finite set, and we interpreted the antichain as the facet family. The notion of abstract simplicial complex is often introduced without initial addressing to the Boolean lattices and without using any orderings: Let $\mathcal{A}$ be a Sperner family; the abstract simplicial complex on the *vertex set* $\mathrm{V}(\mathcal{A})$, with the *facet family* $\mathcal{A}$, is defined as the set family

$$\Delta := \{F \subseteq \mathrm{V}(\mathcal{A})\colon\ \exists A \in \mathcal{A},\ F \subseteq A\}\,. \tag{1.9}$$

Let us note once again that an arbitrary abstract simplicial complex Δ is characterized by the property

$$G \in \Delta\,, \quad F \subseteq G \quad \Longrightarrow \quad F \in \Delta\,;$$

in particular, $\emptyset \in \Delta$.

We will often deal with complexes that are reconstructed from the families of their facets, and we will use the notation $\Delta(\mathcal{A})$ to denote the complex with the given facet family $\mathcal{A}$.

Let $\Delta(\mathcal{A}) \subsetneq \mathbf{2}^{[m]}$, that is, $\mathcal{A} \neq \{\hat{1}\}$. Consider the complement $\mathbf{2}^{[m]} - \Delta(\mathcal{A})$ of the family $\Delta(\mathcal{A})$ up to the power set $\mathbf{2}^{[m]}$. If $B \in \mathbf{2}^{[m]} - \Delta(\mathcal{A})$ then $B \nsubseteq A$ or, equivalently, $B \cap ([m] - A) \neq \emptyset$ for all facets $A \in \mathcal{A}$. In other words, such a set B is a system of representatives of the family $\mathcal{A}^{\perp} := \{[m] - A\colon\ A \in \mathcal{A}\}$. From this point of view, the blocker $\mathfrak{B}(\mathcal{A}^{\perp})$ is the set $\mathbf{min}\big(\mathbb{B}(m) - \mathfrak{I}(\mathcal{A})\big)$ of minimal elements of the subposet $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A})$. In addition, the subposet $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A})$ corresponding to the family $\mathbf{2}^{[m]} - \Delta(\mathcal{A})$ carries the structure of an order filter of the lattice $\mathbb{B}(m)$.

Let $\mathcal{W} \subseteq \mathbb{B}(m)$. The *order filter* $\mathfrak{F}(\mathcal{W}) = \mathfrak{F}(\mathbf{min}\ \mathcal{W})$ generated by $\mathcal{W}$ is defined as the subposet $\mathfrak{F}(\mathcal{W}) := \{E \in \mathbb{B}(m) \colon\ \exists W \in \mathcal{W},\ E \succeq W\}$.

Thus, the family $\mathbf{2}^{[m]} - \Delta(\mathcal{A})$ is represented in the lattice $\mathbb{B}(m)$ by the order filter $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A}) = \mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp}))$.

Any antichain $\mathcal{A} \subset \mathbb{B}(m) - \{\hat{1}\}$ then induces the partition of the lattice $\mathbb{B}(m)$ of the form

$$\mathbb{B}(m) = \mathfrak{I}(\mathcal{A}) \,\dot{\cup}\, \mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp}))\,. \tag{1.10}$$

We will below recall some basic combinatorial properties of the pair $(\mathfrak{I}(\mathcal{A}), \mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp})))$.

Let us return to a specific interpretation of the lattice $\mathbb{B}(m)$ mentioned at the beginning of the chapter. Let $[m]$ be the set of indices marking the constraints that form some infeasible monotone system of constraints $\mathfrak{S}$ described by means of (1.1)–(1.4). The monotonicity property prescribed to the system $\mathfrak{S}$ means that *each subsystem of a feasible subsystem from $\mathfrak{S}$ is feasible, and each subsystem containing an infeasible subsystem is also infeasible*.

Let $\mathbf{I}$ be the family of multi-indices of *minimal* (*by inclusion*) *infeasible or irreducible infeasible subsystems – IISs*, and let $\mathbf{J}$ be the family of multi-indices of *maximal* (*by inclusion*) *feasible subsystems – MFSs* of the system $\mathfrak{S}$. A key construction associated with this system is the partition

$$\mathbb{B}(m) = \mathfrak{I}(\mathbf{J}) \,\dot{\cup}\, \mathfrak{F}(\mathbf{I})\,. \tag{1.11}$$

Since

$$\mathbf{I} = \mathfrak{B}(\mathbf{J}^{\perp})\,, \tag{1.12}$$

that is, *the family of the multi-indices of minimal infeasible subsystems is the blocker of the family of complements of the multi-indices of maximal feasible subsystems*, we have $\mathfrak{F}(\mathbf{I}) = \mathfrak{F}(\mathfrak{B}(\mathbf{J}^{\perp}))$.

We will call a system $\mathfrak{S}$ *irreducible* when $\bigcap_{J \in \mathbf{J}} J = \emptyset$, and *reducible* otherwise.

1.2 Abstract simplicial complexes and monotone Boolean functions

Let $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$ again be a finite family of finite and pairwise distinct sets. Recall once again that a set B is called a *system of representatives* of the family $\mathcal{A}$ if for all $i \in [\alpha]$ it holds $B \cap A_i \neq \emptyset$; in the family of the systems of representatives of $\mathcal{A}$ one distinguishes the blocker $\mathfrak{B}(\mathcal{A})$ of this family, that is, the family of all the minimal (by inclusion) systems of representatives of $\mathcal{A}$. A set B by definition belongs to the blocker $\mathfrak{B}(\mathcal{A})$ if and only if the following conditions are satisfied: (1) B is a system of representatives of $\mathcal{A}$; (2) for any element $b \in B$ there exists an index $j \in [\alpha]$ such that $(B - \{b\}) \cap A_j = \emptyset$.

Recall also that a family $\mathcal{A}$ is called a Sperner family if for all indices $i, j \in [\alpha]$, $i \neq j$, the condition $A_i \nsubseteq A_j$ is satisfied.

The following result is a key research tool in discrete mathematics:

Proposition 1.1. *If $\mathcal{A}$ is a Sperner family then*

$$\mathfrak{B}(\mathfrak{B}(\mathcal{A})) = \mathcal{A}\ .$$

Note that the main property of blockers recalled in Proposition 1.1 complements observation (1.12) made with respect to the multi-index families of minimal infeasible and maximal feasible subsystems of infeasible monotone systems of constraints, by the dual result

$$\mathbf{J} = \mathfrak{B}(\mathbf{I})^{\perp}\ . \tag{1.13}$$

If Δ is an abstract simplicial complex and $F \in \Delta$, then the *dimension* $\dim(F)$ of a face F by definition is less than its cardinality by 1: $\dim(F) = |F| - 1$. The *dimension* $\dim \Delta$ of the complex Δ is the quantity $\max\{\dim F\colon\ F \in \Delta\}$. $f_j(\Delta)$ denotes the number of j-dimensional faces of Δ. One calls the ordered collection of the integers $f_j(\Delta)$ the *f-vector* of the complex Δ. By definition, $f_{-1}(\Delta) = 1$, and $f_0(\Delta)$ is the number of vertices of the complex Δ. $\#\Delta$ denotes the total number of faces of the complex Δ.

If A is some subset of the set $[m]$ then we will denote its complement $[m] - A$ by $A^{\perp}$ and, as earlier, for a cover $\mathcal{A} := \{A_1, A_2, \dots, A_\alpha\}$ of the set $[m] = \bigcup_{i=1}^{\alpha} A_i$ we will denote the corresponding family of complements $\{{A_1}^{\perp}, {A_2}^{\perp}, \dots, {A_\alpha}^{\perp}\}$ by $\mathcal{A}^{\perp}$.

Considering an abstract simplicial complex $\Delta(\mathcal{A})$ with facet family $\mathcal{A}$, and the corresponding order ideal $\mathfrak{I}(\mathcal{A})$ of the Boolean lattice $\mathbb{B}(m)$, it often turns out to be convenient to study the structure of the construction $\mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp}))^{\perp}$, closely related to the complement $\mathbb{B}(m) - \mathfrak{I}(\mathcal{A}) = \mathfrak{F}(\mathfrak{B}(\mathcal{A}^{\perp}))$, instead of the complement itself; the above-mentioned construction represents the face poset of the simplicial complex with the facets that are the complements of the minimal systems of representatives of the family $\mathcal{A}^{\perp}$ up to the set $[m]$. Thus, a study of the pair $\big(\Delta(\mathcal{A})\mathbf{2}^{[m]} - \Delta(\mathcal{A})\big)$ is most commonly substituted by a study of the pair of simplicial complexes

$$\big(\,\Delta(\mathcal{A})\,,\ \Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big)\,\big)\,; \tag{1.14}$$

here $\Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big) = \big(\mathbf{2}^{[m]} - \Delta(\mathcal{A})\big)^{\perp}$.

From the combinatorial topological point of view, the set family $\Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big)$ is the complex called the Alexander dual of $\Delta(\mathcal{A})$.

When one digresses from the key aspect of the representativity of sets characterizing the relationship between families (1.14), Alexander duals are traditionally defined as follows:

If $\Delta \subsetneq \mathbf{2}^{[m]}$ is an abstract simplicial complex on the vertex set $[m]$ then the complex $\Delta^{\vee}$, the *Alexander dual* of Δ, is the family

$$\Delta^{\vee} := \{G^{\perp}\colon\ G \subseteq [m],\ G \notin \Delta\}\ .$$

Thus, for a Sperner cover $\mathcal{A}$ of the set $[m]$ such that $\mathcal{A} \neq \{[m]\}$, we have $\Delta(\mathcal{A})^{\vee} = \Delta\big(\mathfrak{B}(\mathcal{A}^{\perp})^{\perp}\big)$.

The following simple observation is related to the basic fact that the number of rank j elements (or, in the language of combinatorial poset theory, the jth *Whitney number of the second kind*), which correspond in the Boolean lattice $\mathbb{B}(m)$ to the j-subsets of the set $[m]$, is the binomial coefficient $\binom{m}{j} := \frac{m!}{j!(m-j)!}$:

Proposition 1.2. *Let $\Delta \subsetneq \mathbf{2}^{[m]}$ be an abstract simplicial complex on the vertex set $[m]$. Then for all j, $-1 \le j \le m-1$, it holds $f_j(\Delta) + f_{m-j-2}(\Delta^\vee) = \binom{m}{j+1}$.*

This observation makes it possible to come to several conclusions. In the first statement we use the *Kronecker delta* $\delta(s,t)$ which is equal, by definition, to 1 when $s = t$, and to 0 when $s \neq t$. The point is that the number of the faces of all dimensions in the complex $\Delta^\vee$ determine the dimension of the complex Δ.

Corollary 1.3. *Let $\Delta \subsetneq \mathbf{2}^{[m]}$ be an abstract simplicial complex on the vertex set $[m]$. Then*

$$\begin{aligned}\dim(\Delta) &= m - \sum_{j=-1}^{m-1} \delta\left(f_j\left(\Delta^\vee\right), \binom{m}{j+1}\right) - 1 \\ &= m - \max\left\{i \in \{0\} \dot\cup [m] :\ f_{i-1}\left(\Delta^\vee\right) = \binom{m}{i}\right\} - 2\,.\end{aligned}$$

The total number of faces in the partition under consideration is equal to the number of sets in the power set $\mathbf{2}^{[m]}$, namely $\#\Delta + \#\Delta^\vee = 2^m$.

Let us denote the number of all feasible and infeasible subsystems, of cardinality k, of an infeasible monotone system $\mathfrak{S}$ by ν_k and τ_k, respectively. We have $\nu_k + \tau_k = f_{k-1}(\Delta(\mathbf{J})) + f_{m-k-1}(\Delta(\mathbf{I}^\perp)) = \binom{m}{k}$, for all $k \in [m]$, and $\#\Delta(\mathbf{J}) + \#\Delta(\mathbf{I}^\perp) = 2^m$.

Speaking of combinatorial tools of the exact enumeration of the faces of complexes with the known structure of their facet families, it is worth recalling that the combinatorial *inclusion–exclusion principle* is formulated, in one of its various forms, as follows:

– The number $\mathrm{N}_k(\mathcal{A})$ of k-subsets of the set $[m]$ containing as a subset at least one set $A_i \in \mathcal{A}$ is
$$\mathrm{N}_k(\mathcal{A}) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{m - \left|\bigcup_{t\in T} A_t\right|}{m-k}\,; \tag{1.15}$$

– the number $\mathrm{N}_k(\mathcal{A}^\perp)$ of k-subsets of the set $[m]$ containing as a subset at least one set ${A_i}^\perp \in \mathcal{A}^\perp$ is
$$\mathrm{N}_k(\mathcal{A}^\perp) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{\left|\bigcap_{t\in T} A_t\right|}{m-k}\,; \tag{1.16}$$

– the number $\mathrm{R}_k(\mathcal{A})$ of k-subsets of the set $[m]$ contained in at least one set $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$ is
$$\mathrm{R}_k(\mathcal{A}) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{\left|\bigcap_{t\in T} A_t\right|}{k}\,; \tag{1.17}$$

– the number $\mathrm{R}_k(\mathcal{A}^\perp)$ of k-subsets of the set $[m]$ contained in at least one set $A_i{}^\perp \in \mathcal{A}^\perp$ is

$$\mathrm{R}_k(\mathcal{A}^\perp) = -\sum_{j\in[\alpha]} (-1)^j \cdot \sum_{\substack{T\subseteq[\alpha]:\\ |T|=j}} \binom{m - |\bigcup_{t\in T} A_t|}{k} . \tag{1.18}$$

If the structure of the family $\mathbf{J}$ of the multi-indices of maximal feasible subsystems (or the structure of the family $\mathbf{I}$ of the multi-indices of minimal infeasible subsystems) of the system $\mathfrak{S}$ is known then the inclusion–exclusion principle allows us to find the number ν_k of all feasible subsystems of cardinality k and the number τ_k of all infeasible subsystems of cardinality k. Let us make use of relations (1.15) and (1.17), taking into account that any feasible subsystem is contained in at least one MFS, and any infeasible subsystem contains at least one IIS. We come to the conclusion:

Proposition 1.4. *Let $\mathfrak{S}$ be a finite infeasible monotone system of constraints, and let $\mathbf{I}$ and $\mathbf{J}$ be the families of the multi-indices of its IISs and MFSs, respectively. Let τ_k and ν_k be the numbers of infeasible subsystems and of feasible subsystems, of cardinality k, respectively. Then*

$$\tau_k = \binom{m}{k} - \nu_k = -\sum_{s\in[\#\mathbf{I}]} (-1)^s \cdot \sum_{\substack{T\subseteq[\#\mathbf{I}]:\\ |T|=s}} \binom{m - |\bigcup_{t\in T} I_t|}{m-k} ,$$

$$\nu_k = \binom{m}{k} - \tau_k = -\sum_{s\in[\#\mathbf{J}]} (-1)^s \cdot \sum_{\substack{T\subseteq[\#\mathbf{J}]:\\ |T|=s}} \binom{|\bigcap_{t\in T} J_t|}{k} . \tag{1.19}$$

If the quantities τ_k and ν_k are known then Corollary 1.3 allows us to determine for the system $\mathfrak{S}$ the extremal sizes of its IISs and MFSs.

Proposition 1.5. *Let $\mathfrak{S}$ be a finite infeasible monotone system of constraints.*

– *The cardinality of the smallest IIS is*

$$\begin{aligned}\min_{I\in\mathbf{I}} |I| &= \sum_{t=0}^{m} \delta\left(\nu_t, \tbinom{m}{t}\right)\\ &= \max\left\{k \in \{0\}\,\dot\cup\,[m] :\ \nu_k = \tbinom{m}{k}\right\} + 1 .\end{aligned}$$

– *The cardinality of the largest MFS is*

$$\begin{aligned}\max_{J\in\mathbf{J}} |J| &= m - \sum_{t=0}^{m} \delta\left(\tau_{m-t}, \tbinom{m}{t}\right)\\ &= m - \max\left\{k \in \{0\}\,\dot\cup\,[m] :\ \tau_{m-k} = \tbinom{m}{k}\right\} - 1 .\end{aligned}$$

Now we will briefly discuss some combinatorial characteristics of hypergraphs, which are put in correspondence with simple graphs by means of representative systems.

Let $\mathbf{H}([m], \mathcal{A})$ be a hypergraph with the vertex set $[m]$ and with the hyperedge family $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$. As usual, $\mathfrak{B}(\mathcal{A})$ denotes the blocker of the family $\mathcal{A}$. For what hypergraphs $\mathbf{H}([m],\mathcal{A})$ are the corresponding hypergraphs $\mathbf{H}([m],\mathfrak{B}(\mathcal{A}))$ finite simple graphs? In order to answer this question, we will use the following auxiliary statement:

Proposition 1.6. *If a family $\mathcal{A}$ is Sperner then $\bigcap_{E\in\mathfrak{B}(\mathcal{A})} E = \emptyset$ if and only if the family $\mathcal{A}$ contains no one-element sets.*

Proof. Let us denote the family $\mathfrak{B}(\mathcal{A})$ by $\mathcal{E}$. In view of Proposition 1.1, $\mathcal{A} = \mathfrak{B}(\mathcal{E})$.

The *sufficiency*: since $\bigcap_{E\in\mathcal{E}} E = \emptyset$, none of the elements x of $\bigcup_{A\in\mathcal{A}} A = \bigcup_{E\in\mathcal{E}} E$ is a system of representatives of $\mathcal{E}$, that is, $\mathcal{A}$ contains no one-element sets.

To prove the *necessity*, suppose to the contrary that $\bigcap_{E\in\mathcal{E}} E =: X \neq \emptyset$. Then any one-element subset $\{x\} \subseteq X$ is a minimal system of representatives of $\mathcal{E}$, that is, $\{x\} \in \mathcal{A}$, a contradiction. □

Let us determine conditions that must be satisfied by a set family $\mathcal{A}$ such that the dimension of the complex $\Delta(\mathfrak{B}(\mathcal{A})^\perp)$ is less than or equal to a given value or, on the contrary, the dimension is greater than the value.

According to Corollary 1.3,

$$\dim \Delta(\mathfrak{B}(\mathcal{A})^\perp) = m - \sum_{j=-1}^{m-1} \delta\left(f_j\,(\Delta(\mathcal{A}^\perp))\,,\,\tbinom{m}{j+1}\right) - 1\;.$$

If this dimension should not be greater than k, then the relation

$$\sum_{j=-1}^{m-1} \delta\left(f_j\,(\Delta(\mathcal{A}^\perp))\,,\,\tbinom{m}{j+1}\right) \geq m - k - 1$$

should hold.

On the other hand, if $\dim \Delta(\mathfrak{B}(\mathcal{A})^\perp) > k$, then the relation

$$\sum_{j=-1}^{m-1} \delta\left(f_j\,(\Delta(\mathcal{A}^\perp))\,,\,\tbinom{m}{j+1}\right) < m - k - 1$$

should hold.

Let us sum up this information as a proposition:

Proposition 1.7. *If $\mathcal{A}$ is a family of nonempty finite and pairwise distinct sets covering the set $[m]$ then, for a fixed k, the following relations hold:*

(1) *If for all j, $-1 \leq j \leq m-k-3$, it holds $f_j\,(\Delta(\mathcal{A}^\perp)) = \binom{m}{j+1}$ then $\dim \Delta(\mathfrak{B}(\mathcal{A})^\perp) \leq k$. In particular, if equality holds for all j, $-1 \leq j \leq m-4$, then the complex $\Delta(\mathfrak{B}(\mathcal{A})^\perp)$ is a simple graph (possibly, with isolated vertices).*

(2) *If $f_{m-k-3}\,(\Delta(\mathcal{A}^\perp)) < \binom{m}{k+2}$ then $\dim \Delta(\mathfrak{B}(\mathcal{A})^\perp) > k$.*

Let us return to consider finite infeasible monotone systems of constraints $\mathfrak{S}$ for which the maps π and the range families $\mathbf{2}^\Gamma$ of these maps, which are put in correspondence with the systems, satisfy conditions (1.1)–(1.4).

For combinatorial analysis of systems $\mathfrak{S}$, a unique relevant property is perhaps the emptiness or nonemptiness of the images $\pi(B)$ of various multi-indices $B \in \mathbb{B}(m)$ of subsystems under the map π. For this reason, the setting of the problem of analyzing the above-mentioned systems in the language of monotone Boolean functions is universally accepted; we will return to the question of optimal *inference* of these functions in Section 4.1.

Let $\mathbf{B}$ denote the two-element set $\{0, 1\}$. The *unit discrete m-dimensional cube* $\mathbf{B}^m$ is equipped with the following relation $\le$ of partial ordering: for two binary tuples $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_m)$ one supposes $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ if and only if $\alpha_i \le \beta_i$, for all $i \in [m]$. The *Boolean function* $\mathfrak{f}\colon \mathbf{B}^m \to \mathbf{B}$ is called *monotone* if $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ implies $\mathfrak{f}(\boldsymbol{\alpha}) \le \mathfrak{f}(\boldsymbol{\beta})$. Any monotone Boolean function induces the partition $\mathbf{B}^m = \mathfrak{f}^{-1}(0) \dot\cup \mathfrak{f}^{-1}(1)$. The set $\mathfrak{f}^{-1}(0)$ is composed of the so-called *zeros* of the function $\mathfrak{f}$, and the set $\mathfrak{f}^{-1}(1)$ is composed of the *units* of this function. The subset $\mathbf{max}\, \mathfrak{f}^{-1}(0)$ of the maximal elements of the subposet $\mathfrak{f}^{-1}(0)$ is called the set of *upper zeros* of the function $\mathfrak{f}$; the subset $\mathbf{min}\, \mathfrak{f}^{-1}(1)$ of the minimal elements of the subposet $\mathfrak{f}^{-1}(1)$ is called the set of *lower units* of the function $\mathfrak{f}$.

With the system $\mathfrak{S}$ can be naturally put in correspondence a monotone Boolean function $\mathfrak{f}$, which is defined as follows:

$$\mathfrak{f}(\boldsymbol{\alpha}) = 0 \Longleftrightarrow \pi\Big(\bigcup_{\{a_i\}\in\mathbb{B}(m)^{(1)}\colon\ \alpha_i=1} \{a_i\}\Big) \neq \emptyset\,,$$

$$\mathfrak{f}(\boldsymbol{\alpha}) = 1 \Longleftrightarrow \pi\Big(\bigcup_{\{a_i\}\in\mathbb{B}(m)^{(1)}\colon\ \alpha_i=1} \{a_i\}\Big) = \emptyset\,.$$

The elements of the set $\mathfrak{f}^{-1}(1)$ are in one-to-one correspondence with the multi-indices of *infeasible* subsystems of the system $\mathfrak{S}$: $\boldsymbol{\alpha}$ is a unit of $\mathfrak{f}$ if and only if the set $\bigcup_{\substack{a_i\in\mathbb{B}(m)^{(1)}\colon\\ \alpha_i=1}} \{a_i\}$ is the multi-index of an infeasible subsystem of $\mathfrak{S}$. In a similar matter, the elements of the set $\mathfrak{f}^{-1}(0)$ are in one-to-one correspondence with the multi-indices of *feasible* subsystems of the system $\mathfrak{S}$: $\boldsymbol{\alpha}$ is a zero of $\mathfrak{f}$ if and only if the set $\bigcup_{\substack{a_i\in\mathbb{B}(m)^{(1)}\colon\\ \alpha_i=1}} \{a_i\}$ is the multi-index of a feasible subsystem of $\mathfrak{S}$. Finally, $\boldsymbol{\alpha}$ is a lower unit of $\mathfrak{f}$ if and only if $\bigcup_{\substack{a_i\in\mathbb{B}(m)^{(1)}\colon\\ \alpha_i=1}} \{a_i\}$ is the multi-index of a minimal infeasible subsystem of the system $\mathfrak{S}$, and $\boldsymbol{\alpha}$ is an upper zero of $\mathfrak{f}$ if and only if $\bigcup_{\substack{a_i\in\mathbb{B}(m)^{(1)}\colon\\ \alpha_i=1}} \{a_i\}$ is the multi-index of a maximal feasible subsystem of the system $\mathfrak{S}$.

Various examples of collections $(\mathfrak{S}, \Gamma, \pi)$ satisfying conditions (1.1)–(1.4) are provided by finite infeasible systems of equations or inequalities, or by mixed systems of equations and inequalities, over vector spaces.

We will especially be interested in collections $(\mathfrak{S}, \mathbb{R}^n, \pi)$, where

$$\mathfrak{S} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0\colon\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n,\ \|\boldsymbol{a}_i\| = 1,\ i \in [m]\} \tag{1.20}$$

is a finite infeasible system, of rank n, of homogeneous strict linear inequalities over the finite dimensional space $\mathbb{R}^n$. $\langle \boldsymbol{a}_i, \mathbf{x}\rangle := \sum_{k\in[n]} a_{ik}\mathrm{x}_k$ denotes here the standard scalar product; $\|\boldsymbol{a}_i\| := \sqrt{\langle \boldsymbol{a}_i, \boldsymbol{a}_i \rangle}$ is the Euclidean norm of the vector $\boldsymbol{a}_i$. The map π by definition assigns to the multi-index $T \in \mathbb{B}(m)$ the open cone of solutions to the subsystem $\{\langle \boldsymbol{a}_t, \mathbf{x} \rangle > 0 : \ t \in T\}$ as follows:

$$\pi(T) : \ = \bigcap_{t \in T} \{\boldsymbol{x} \in \mathbb{R}^n : \ \langle \boldsymbol{a}_t, \boldsymbol{x} \rangle > 0\} \ ,$$
$$\pi(\hat{0}) : \ = \mathbb{R}^n \ . \tag{1.21}$$

The map π defined by (1.21) induces the partition

$$\mathbb{B}(m) = \{T \in \mathbb{B}(m) : \ \pi(T) = \emptyset\} \ \dot{\cup} \{T \in \mathbb{B}(m) : \ \pi(T) \neq \emptyset\}$$

or, in other words, a partition of the form (1.11) of the lattice $\mathbb{B}(m)$ into the ideal $\mathfrak{I}(\mathbf{J})$ and filter $\mathfrak{F}(\mathbf{I})$ generated by the families of the multi-indices of maximal feasible subsystems and of minimal infeasible subsystems, respectively.

Notes

Infeasible systems of constraints are an integral part of the studies in discrete mathematics and mathematical cybernetics [96]; among a wide variety of constructions with the monotonicity property, the most close to the material of the present research are infeasible systems of linear inequalities [39–43, 96].

One can become familiar with basic problems of combinatorial optimization in works [13, 33, 63, 80, 81, 115]. Partially ordered sets and, in particular, Boolean lattices are thoroughly studied in [3, 12, 62, 85, 105, 134].

Mentioning the problem on representative systems in the context of logical functions, we follow [166].

One can become familiar with the studies of (0, 1)-matrices in works [111, 112, 123, 138, 139]. We give a short description of the link of the problem on covers with (0, 1)-matrices, following [137].

We speak of transversal sets of hypergraphs, following [104]. See, for example, [3] on bipartite graphs and binary relations.

A thorough review of works devoted to the transversality number of hypergraphs is given in [46]. One can become familiar with the studies of systems of distinct representatives in works [102, 103]. Matching theory is presented in [87]. Information on transversal matroids can be found in [3, 13]. See also [5, 7, 13, 122, 135] on the mentioned and related questions. A survey of the studies of permanents is given in [106]; the connection between permanents and systems of distinct representatives is discussed in [122].

The fundamental statement from Proposition 1.1 was proved in [36, 82, 83].

Abstract simplicial complexes are described in detail in [25, 116, 121, 132, 133].

Books [86, 90, 110, 155] are devoted to Boolean functions. Minimization of Boolean functions in the class of disjunctive normal forms is discussed in many works, see [10, 125, 151, 154, 160–162]; a review of this research can be found in [113, 127].

See [3, 9, 72, 122, 134] on the combinatorial inclusion–exclusion principle.

The literature of (hyper)graph theory is enormous, we mention here just a few books: [13, 28, 32, 67, 104, 136, 147, 156, 166].

The setting of the problem of analyzing infeasible systems in the language of monotone Boolean functions is universally accepted after the appearance of seminal work [159].

2 Complexes, (hyper)graphs, and inequality systems

In many problems of combinatorial optimization, it is necessary to distinguish in an abstract simplicial complex its facet or a collection of facets with some properties, say of maximal dimension. Recall that in our study complexes often serve as combinatorial models of (in)feasible monotone systems of constraints – the multi-indices (or the constructions marked with them) of feasible subsystems represent the faces of complexes. Thus, a complex with a unique facet corresponds to a feasible system, and a complex with several facets emulates an infeasible system. In this context, it is interesting to investigate the structural properties of the facet family of a complex and, in particular, to investigate a specific graph associated with the facet family; this graph describes the *covers* of the vertex set by *facet pairs*. For this construction, one traditionally uses the term *graph of an independence system*.

The *hypergraph of an independence system* arises when considering the *covers* of the vertex set of a complex by *arbitrary subfamilies of facets*.

In the algorithmic context, those situations are of interest where the graphs of independence systems are connected because the connectedness of such graphs can be efficiently used for constructing algorithms of determining facet families.

In this chapter, sufficient conditions of the connectedness of graphs provided by specific classes of complexes are discussed. Special attention will be paid to the study of the *graph of maximal feasible subsystems* (*graph of MFSs*) of a finite infeasible system of linear inequalities. In Section 5.1, the connectedness of this graph will serve as the basis of algorithms of designing decision rules in applied problems of pattern recognition.

2.1 The graph of an independence system

In this section, we will use the notation (V, Δ) to denote the abstract simplicial complex Δ, with the facet family $\mathbf{max}\,\Delta$, on the vertex set $V := \bigcup_{H \in \mathbf{max}\,\Delta} H$.

The term *graph of an independence system* $\mathtt{ISG}(V, \Delta)$ is used to refer to a simple graph defined as follows:

– *the vertex set of the graph* $\mathtt{ISG}(V, \Delta)$ *is the facet family* $\mathbf{max}\,\Delta$*;*
– *the edge family of the graph* $\mathtt{ISG}(V, \Delta)$ *is the family of all the unordered pairs of facets* $\{H, H'\} \subseteq \mathbf{max}\,\Delta$ *that cover the vertex set:*

$$H \cup H' = V .$$

Let us denote by $\overrightarrow{\mathtt{ISG}}(V, \Delta)$ the *oriented graph of the independence system*, with the vertex set $\mathbf{max}\,\Delta$, and with the arc family $\vec{\mathcal{E}}$, which is obtained from the graph $(\mathbf{max}\,\Delta, \mathcal{E}) := \mathtt{ISG}(V, \Delta)$ by an assignment of a direction to each edge of the family $\mathcal{E}$.

Given two complexes (V, Δ) and (V', Δ'), a map $\varphi\colon V \to V'$ between their vertex sets is called a *homomorphism* (or a *simplicial map*) if the image $\varphi(F)$ of each face $F \in \Delta$ is a face of Δ', that is, $\varphi(F) \in \Delta'$. If there exists a bijective homomorphism $\mathfrak{i}\colon V \to V'$ such that the inverse map $\mathfrak{i}^{-1}\colon V' \to V$ is also a homomorphism then the complexes (V, Δ) and (V', Δ') are said to be *isomorphic*, and in this case the map $\mathfrak{i}$ is called an *isomorphism*. If the complexes (V, Δ) and (V', Δ') are isomorphic then we will write $(V, \Delta) \simeq (V', \Delta')$ for mentioning this property.

Since the simple graphs are abstract simplicial complexes, the same definitions are relevant to them: a map φ from the vertex set of a simple graph $\mathbf{G}$ into the vertex set of another simple graph $\mathbf{G}'$ is called a *homomorphism* if the images of the end vertices of any edge of the graph $\mathbf{G}$ under the map φ are either the end vertices of an edge of the graph $\mathbf{G}'$ or these images coincide. A homomorphism $\mathfrak{i}$ of $\mathbf{G}$ to $\mathbf{G}'$ is called an *isomorphism* if it is one-to-one and if the inverse map $\mathfrak{i}^{-1}$ is also a homomorphism; if these conditions are satisfied then the graphs $\mathbf{G}$ and $\mathbf{G}'$ are called *isomorphic*.

Isomorphic complexes (V, Δ) and (V', Δ') are evidently assigned isomorphic graphs $\mathtt{ISG}(V, \Delta)$ and $\mathtt{ISG}(V', \Delta')$.

Proposition 2.1. *If there exists a surjective homomorphism $\varphi\colon V \to V'$ of a complex (V, Δ) to a complex (V', Δ'), then there exists a homomorphism of the graph $\mathtt{ISG}(V, \Delta)$ to the graph $\mathtt{ISG}(V', \Delta')$.*

Proof. If the complex (V, Δ) is the power set $\mathbf{2}^V$ of the set V, then the corresponding graph of the independence system $\mathtt{ISG}(V, \Delta)$ is the isolated vertex $\{V\}$, and the complex (V', Δ') represents the power set $\mathbf{2}^{V'}$ because $\varphi(V) = V' \in \Delta'$, and thus the graph $\mathtt{ISG}(V', \Delta')$ is the isolated vertex $\{V'\}$; in this case the map $\{V\} \mapsto \{V'\}$ is the unique homomorphism $\mathtt{ISG}(V, \Delta) \to \mathtt{ISG}(V', \Delta')$.

Assume that (V, Δ) is not the power set $\mathbf{2}^V$, that is, the vertex set of the graph $\mathtt{ISG}(V, \Delta)$ is not a singleton. For each facet $H \in \mathbf{max}\,\Delta$, pick an arbitrary facet $\gamma(H) \in \mathbf{max}\,\Delta'$ of the complex Δ' such that $\gamma(H) \supseteq \varphi(H)$, having thus defined some map $\gamma\colon \mathbf{max}\,\Delta \to \mathbf{max}\,\Delta'$ between the facet families of the complexes under consideration.

- If the graph $\mathtt{ISG}(V, \Delta)$ is edgeless then the map γ is its homomorphism to the graph $\mathtt{ISG}(V', \Delta')$.
- If the graph $\mathtt{ISG}(V, \Delta)$ is not edgeless then for each of its edge $\{H_1, H_2\}$ we have $\gamma(H_1) \cup \gamma(H_2) \supseteq \varphi(H_1) \cup \varphi(H_2) = \varphi(H_1 \cup H_2) = \varphi(V) = V'$; this means

$$\gamma(H_1) \cup \gamma(H_2) = V' ,$$

  that is,
  - either $\gamma(H_1) = \gamma(H_2)$, the complex Δ' is the power set $\mathbf{2}^{V'}$ of the set V', and the graph $\mathtt{ISG}(V', \Delta')$ represents the isolated vertex $\{V'\}$,
  - or $\{\gamma(H_1), \gamma(H_2)\}$ is an edge of the graph $\mathtt{ISG}(V', \Delta')$ corresponding to the complex (V', Δ') which is not the power set $\mathbf{2}^{V'}$;

therefore, the map γ is a homomorphism from $\mathtt{ISG}(V, \Delta)$ to $\mathtt{ISG}(V', \Delta')$. ☐

Let us now address the question on the representation of an arbitrary graph as the graph of an independence system.

Proposition 2.2. *Any finite simple graph* $\mathbf{G}$ *is isomorphic to the graph of some independence system.*

Proof. If the graph $\mathbf{G}$ represents an isolated vertex then it is isomorphic to the graph of the independence system $(V, \mathbf{2}^V)$, for any nonempty set V.

Let $\overline{\mathbf{G}} := (V, \mathcal{E})$ be the graph with vertex set V, $|V| > 1$, and edge family $\mathcal{E}$, which is the *complement* of the graph $\mathbf{G}$. Let us consider the common collection $A := V \cup \mathcal{E}$ of vertices and edges, which is the family of nonempty faces of the complex $\overline{\mathbf{G}}$, and denote by A_v the set A after the removal of the vertex v and all the edges that are incident with v (the end vertices of the mentioned edges, different from v, are not removed). For any two distinct vertices v and u of the graph $\overline{\mathbf{G}}$, the sets A_v and A_u are incomparable by inclusion because $v \in A_u - A_v$ and $u \in A_v - A_u$; in other words, $\{A_v : \ v \in V\}$ is a Sperner family.

Let us define an independence system (A, Δ) on the vertex set A as follows: some collection $F \subseteq A$ of vertices and edges of the graph $\overline{\mathbf{G}}$ is a face of the complex (A, Δ), $F \in \Delta$, if and only if there is a vertex $w \in V$ for which $F \subseteq A_w$; that is, (A, Δ) is the complex with the facet family $\mathbf{max}\,\Delta = \{A_v : \ v \in V\}$, and $\#\mathbf{max}\,\Delta = |V|$. Two distinct facets A_u and A_v of the complex (A, Δ) cover the set A if and only if the pair $\{u, v\}$ is not an edge of the graph $\overline{\mathbf{G}}$, $\{u, v\} \notin \mathcal{E}$. Thus, the map $V \to \mathbf{max}\,\Delta$, $v \mapsto A_v$, is an isomorphism of the graph $\mathbf{G}$ onto the graph of the independence system $\mathtt{ISG}(A, \Delta)$. ☐

Corollary 2.3. *Any finite simple graph* $\mathbf{G} = (V, \mathcal{E})$ *with vertex set* V *and edge family* $\mathcal{E}$ *is the graph of an independence system* $\mathtt{ISG}(A, \Delta)$ *associated with a complex* (A, Δ) *and, besides,* $2|A| \le |V|^2 + |V| - 2\#\mathcal{E}$.

Let us determine the following partial order relation on the family of ordered pairs of subsets of a finite nonempty set V: $(V_1, V_2) \preceq (V_1', V_2')$ when $V_1 \subseteq V_1'$ and $V_2 \subseteq V_2'$, or $V_1 \subseteq V_2'$ and $V_2 \subseteq V_1'$.

We will use the following auxiliary statement whose proof is given on page 25, it follows the proof of Proposition 2.5:

Proposition 2.4. *Let* (V, Δ) *be a complex such that for the corresponding graph of the independence system* $(\mathbf{max}\,\Delta, \mathcal{E}) := \mathtt{ISG}(V, \Delta)$ *one has* $\#\mathcal{E} > 1$. *If there exists a partition of the edge family* $\mathcal{E} = \mathcal{C} \dot\cup \mathcal{D}$ *into nonempty subfamilies* $\mathcal{C}$ *and* $\mathcal{D}$ *such that none of the edges from* $\mathcal{C}$ *is adjacent to an edge from* $\mathcal{D}$*, then there do not exist edges* $\vec{c} \in \vec{\mathcal{C}}$ *and* $\vec{d} \in \vec{\mathcal{D}}$ *of the oriented graph* $(\mathbf{max}\,\Delta, \vec{\mathcal{E}}) := \overrightarrow{\mathtt{ISG}}(V, \Delta)$ *and nonempty vertex subsets* $V_1, V_2 \subseteq V$ *for which* $V_1 \cup V_2 = V$ *and* $\vec{c} \succeq (V_1, V_2) \preceq \vec{d}$.

Let some finite nonempty multifamily $\mathcal{V} := \{v_i := (X_i, X_i') : \ i \in [m]\}$ of the ordered pairs of subsets of a nonempty set X be given. Define for this multifamily the operation of intersection of subset pairs: $(X_{i_1}, X_{i_1}') \cap (X_{i_2}, X_{i_2}') := (X_{i_1} \cap X_{i_2}, X_{i_1}' \cap X_{i_2}')$.

Let us consider a complex $(V, \Delta_\cap)$ for which, by definition

$$F \in \Delta_\cap \iff \bigcap_{v \in F} v \neq (\emptyset, \emptyset)\ . \tag{2.1}$$

We now turn to the graphs of independence systems whose connectedness is induced by the connectedness of topological spaces.

Proposition 2.5. *Let V be some finite multifamily of ordered pairs $v_i := (\boldsymbol{Z}_i, \boldsymbol{Z}'_i)$, $i \in [m]$, of closed subsets $\boldsymbol{Z}_i \subset \boldsymbol{Z}$ and $\boldsymbol{Z}'_i \subset \boldsymbol{Z}$ of a connected topological space $\boldsymbol{Z}$, that cover the space: $\boldsymbol{Z}_i \cup \boldsymbol{Z}'_i = \boldsymbol{Z}$.*

If $\#\mathbf{max}\,\Delta_\cap > 1$ then the graph of the independence system $\mathtt{ISG}(V, \Delta_\cap)$ is connected.

Proof. Let us first show that the graph $(\mathbf{max}\,\Delta_\cap, \mathcal{E}) := \mathtt{ISG}(V, \Delta_\cap)$ has no isolated vertices. Let $H \in \mathbf{max}\,\Delta_\cap$ be an arbitrary facet of the complex under consideration. Since $H \in \Delta_\cap$, we have $(\boldsymbol{A}, \boldsymbol{A}') := \bigcap_{v \in H} v \neq (\emptyset, \emptyset)$, by convention (2.1). Specifically, let us suppose $\boldsymbol{A} \neq \emptyset$. Fix some element $\boldsymbol{a} \in \boldsymbol{A}$. It follows from the maximality of H that for any pair $(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V - H$, we have $\boldsymbol{a} \notin \boldsymbol{Z}_i$ and, because of $\boldsymbol{Z}_i \cup \boldsymbol{Z}'_i = \boldsymbol{Z}$, the inclusion $\boldsymbol{a} \in \boldsymbol{Z}'_i$ holds, that is, $V - H \in \Delta_\cap$. Thus, there exists a facet $H' \in \mathbf{max}\,\Delta_\cap$, $H' \supseteq V - H$, such that $H \cup H' = V$, that is, the vertex H is not an isolated vertex of the graph $\mathtt{ISG}(V, \Delta_\cap)$. If $\#\mathbf{max}\,\Delta_\cap = 2$, then the proposition is proved.

Suppose that $\#\mathbf{max}\,\Delta_\cap > 2$ and, as a consequence, $\#\mathcal{E} > 1$.

We will use the oriented graph $(\mathbf{max}\,\Delta_\cap, \vec{\mathcal{E}}) := \overrightarrow{\mathtt{ISG}}(V, \Delta_\cap)$ of the independence system $(V, \Delta_\cap)$, with the vertex set $\mathbf{max}\,\Delta_\cap$ and the edge family $\vec{\mathcal{E}}$ that represents the pairs, ordered in an arbitrary way, of end vertices of the edges of the graph $\mathtt{ISG}(V, \Delta_\cap)$.

Let us assign to an arbitrary element $\boldsymbol{z} \in \boldsymbol{Z}$ faces $F_{\boldsymbol{z}} \subset V$ and $F'_{\boldsymbol{z}} \subset V$ of the complex $\Delta_\cap$, defined as follows: $F_{\boldsymbol{z}} := \{(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V\colon\ \boldsymbol{z} \in \boldsymbol{Z}_i\}$ and $F'_{\boldsymbol{z}} := \{(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V\colon \boldsymbol{z} \in \boldsymbol{Z}'_i\}$; note that they form a cover $F_{\boldsymbol{z}} \cup F'_{\boldsymbol{z}}$ of the vertex set V of the complex $\Delta_\cap$. Thus, for an element $\boldsymbol{z} \in \boldsymbol{Z}$ there is an arc $\vec{e} := (H, H') \in \vec{\mathcal{E}}$ for which

$$F_{\boldsymbol{z}} \subseteq H \in \mathbf{max}\,\Delta_\cap \quad \text{and} \quad F'_{\boldsymbol{z}} \subseteq H' \in \mathbf{max}\,\Delta_\cap$$

or

$$F_{\boldsymbol{z}} \subseteq H' \in \mathbf{max}\,\Delta_\cap \quad \text{and} \quad F'_{\boldsymbol{z}} \subseteq H \in \mathbf{max}\,\Delta_\cap\ ,$$

that is,

$$(F_{\boldsymbol{z}}, F'_{\boldsymbol{z}}) \preceq \vec{e}\ . \tag{2.2}$$

Assume that the graph $\mathtt{ISG}(V, \Delta_\cap)$, with no isolated vertices, is disconnected, that is, there exists a partition $\mathcal{E} = \mathcal{C} \dot\cup \mathcal{D}$ of its edge family into nonempty subfamilies $\mathcal{C}$ and $\mathcal{D}$ such that none of the edges from $\mathcal{C}$ is adjacent to an edge from $\mathcal{D}$.

In the space $\boldsymbol{Z}$, distinguish its subsets

$$\boldsymbol{Y} := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ \exists \vec{c} \in \vec{\mathcal{C}},\ (F_{\boldsymbol{z}}, F'_{\boldsymbol{z}}) \preceq \vec{c}\}\ ,$$
$$\boldsymbol{Y}' := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ \exists \vec{d} \in \vec{\mathcal{D}},\ (F_{\boldsymbol{z}}, F'_{\boldsymbol{z}}) \preceq \vec{d}\}\ .$$

Since $\mathcal{E} = \mathcal{C} \dot\cup \mathcal{D}$, it follows from (2.2) that

$$\boldsymbol{Y} \cup \boldsymbol{Y}' = \boldsymbol{Z} . \tag{2.3}$$

Further, we have $\boldsymbol{Y} \cap \boldsymbol{Y}' = \emptyset$ because, in the contrary case, for any $\boldsymbol{x} \in \boldsymbol{Y} \cap \boldsymbol{Y}'$ there would exist arcs $\vec{c} \in \vec{\mathcal{C}}$ and $\vec{d} \in \vec{\mathcal{D}}$ such that $\vec{c} \succeq (F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \preceq \vec{d}$, a contradiction with Proposition 2.4.

For any subset $U \subseteq V$ of the vertex set of the complex $\Delta_\cap$, define subsets

$$\mathbf{T}(U) := \bigcap_{\substack{\boldsymbol{Z}_i \subseteq \boldsymbol{Z}:\\ (\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in U}} \boldsymbol{Z}_i \quad \text{and} \quad \mathbf{T}'(U) := \bigcap_{\substack{\boldsymbol{Z}'_i \subseteq \boldsymbol{Z}:\\ (\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in U}} \boldsymbol{Z}'_i$$

of the space $\boldsymbol{Z}$, with $\mathbf{T}(\emptyset) = \mathbf{T}'(\emptyset) := \boldsymbol{Z}$. Let us show that

$$\boldsymbol{Y} = \bigcup_{\boldsymbol{x} \in \boldsymbol{Y}} (\mathbf{T}(F_{\boldsymbol{x}}) \cap \mathbf{T}'(F'_{\boldsymbol{x}})) . \tag{2.4}$$

For this, it suffices to show that

$$\mathbf{T}(F_{\boldsymbol{x}}) \cap \mathbf{T}'(F'_{\boldsymbol{x}}) \subseteq \boldsymbol{Y} , \tag{2.5}$$

for any element $\boldsymbol{x} \in \boldsymbol{Y}$.

Assume that inclusion (2.5) does not hold for some element $\boldsymbol{a} \in \boldsymbol{Y}$. Then there exists an element $\boldsymbol{b} \in \boldsymbol{Y}' \cap \mathbf{T}(F_{\boldsymbol{a}}) \cap \mathbf{T}'(F'_{\boldsymbol{a}})$. As a consequence, there exist arcs $\vec{c} \in \vec{\mathcal{C}}$ and $\vec{d} \in \vec{\mathcal{D}}$ such that $\vec{d} \succeq (F_{\boldsymbol{b}}, F'_{\boldsymbol{b}}) \succeq (F_{\boldsymbol{a}}, F'_{\boldsymbol{a}}) \preceq \vec{c}$, a contradiction with Proposition 2.4. Thus, relations (2.4) and (2.5) hold. It follows from the closedness of $\mathbf{T}(U)$ and $\mathbf{T}'(U)$, for $U \subseteq V$, and from the finiteness of V, taking into account (2.4), that $\boldsymbol{Y}$ is also closed. Let us show that $\boldsymbol{Y} \neq \emptyset$. Let $\vec{c} := (H_1, H_2) \in \vec{\mathcal{C}} \neq \emptyset$. We have $(\boldsymbol{A}, \boldsymbol{A}') := \bigcap_{v \in H_1} v \neq (\emptyset, \emptyset)$. Specifically, suppose $\boldsymbol{A} \neq \emptyset$ and pick some element $\boldsymbol{x} \in \boldsymbol{A}$. Then

$$F_{\boldsymbol{x}} = H_1 . \tag{2.6}$$

It follows from the maximality of H_1 that for $(\boldsymbol{Z}_i, \boldsymbol{Z}'_i) \in V - H_1$ we have $\boldsymbol{x} \in \boldsymbol{Z}'_i$, that is,

$$F'_{\boldsymbol{x}} \supseteq V - H_1 . \tag{2.7}$$

Relations (2.6) and (2.7) imply

$$(F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \succeq (H_1, V - H_1) . \tag{2.8}$$

Assume that $\boldsymbol{Y} = \emptyset$. Since $\boldsymbol{Y} \cup \boldsymbol{Y}' = \boldsymbol{Z}$, we have $\boldsymbol{x} \in \boldsymbol{Y}'$. It follows from the definition of the set $\boldsymbol{Y}'$ that there exists an arc $\vec{d} \in \vec{\mathcal{D}}$ such that $(F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \preceq \vec{d}$. Then, taking into account (2.8), we obtain

$$\vec{\mathcal{C}} \ni \vec{c} := (H_1, H_2) \succeq (H_1, V - H_1) \preceq (F_{\boldsymbol{x}}, F'_{\boldsymbol{x}}) \preceq \vec{d} \in \vec{\mathcal{D}} ,$$

a contradiction with Proposition 2.4. Thus, $\boldsymbol{Y} \neq \emptyset$.

It can be proved analogously that the set $\boldsymbol{Y}'$ is nonempty and closed. Thus, we have partitioned the connected space $\boldsymbol{Z}$ into two nonempty disjoint closed subsets $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ – such a contradiction proves the *weak connectedness* of the oriented graph of the independence system $\overrightarrow{\mathtt{ISG}}(V, \Delta_\cap)$ and, as a consequence, the *connectedness* of the underlying undirected graph $\mathtt{ISG}(V, \Delta_\cap)$, thus completing the proof. □

Proof of Proposition 2.4. Assume the converse: let there exist arcs $\vec{c} = (C_1, C_2) \in \vec{\mathcal{C}}$, $\vec{d} = (D_1, D_2) \in \vec{\mathcal{D}}$ of the graph $\overrightarrow{\mathtt{ISG}}(V, \Delta)$, and nonempty subsets $V_1, V_2 \subseteq V$ such that $V_1 \cup V_2 = V$ and $\vec{c} \succeq (V_1, V_2) \preceq \vec{d}$. Without loss of generality, we will suppose that $C_1 \supseteq V_1$, $C_2 \supseteq V_2$, and $V_1 \subseteq D_1$, $V_2 \subseteq D_2$. Since V_1 and V_2 cover the set V then moreover $C_1 \cup D_2 = V$ and thus $e := \{C_1, D_2\} \in \mathcal{E}$. Specifically, suppose $e \in \mathcal{C}$. Then $e \cap d = \{D_2\}$, that is, the edges $e \in \mathcal{C}$ and $d \in \mathcal{D}$ of the graph $\mathtt{ISG}(V, \Delta)$ are adjacent, a contradiction. □

Let $\mathbf{F}(\boldsymbol{Z}) := \{f_i\colon\ i \in [m]\}$, $m > 1$, be a finite system of real continuous functions $f_i\colon\ \boldsymbol{Z} \to \mathbb{R}$ over a *connected* topological space $\boldsymbol{Z}$. Let us consider three classes of complexes $\Delta_\geq$, $\bar{\Delta}$ and $\Delta_>$, on the vertex set $\mathbf{F}(\boldsymbol{Z})$, defined via their nonempty faces F as follows:

– $F \in \big(\mathbf{F}(\boldsymbol{Z}), \Delta_\geq\big)$ if and only if

$$\bigcap_{f \in F} \{(\alpha_f, \boldsymbol{z}) \in (\mathbb{R} - \{0\}) \times \boldsymbol{Z}\colon\ \alpha_f f(\boldsymbol{z}) \geq 0\} \neq \emptyset$$

or, equivalently, the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0\colon f \in F\}$ is feasible for some factor $\alpha_F^\star \in \mathbb{R} - \{0\}$;

– $F \in \big(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta}\big)$ if and only if $\bigcap_{f \in F}\big(\overline{\mathbf{C}_>(f)}, \overline{\mathbf{C}_<(f)}\big) \neq (\emptyset, \emptyset)$, where $\overline{\mathbf{C}_>(f)}$ and $\overline{\mathbf{C}_<(f)}$ denote the closures of the sets

$$\mathbf{C}_>(f) := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ f(\boldsymbol{z}) > 0\} \quad \text{and} \quad \mathbf{C}_<(f) := \{\boldsymbol{z} \in \boldsymbol{Z}\colon\ f(\boldsymbol{z}) < 0\}\,,$$

respectively;

– $F \in \big(\mathbf{F}(\boldsymbol{Z}), \Delta_>\big)$ if and only if

$$\bigcap_{f \in F} \{(\alpha_f, \boldsymbol{z}) \in \mathbb{R} \times \boldsymbol{Z}\colon\ \alpha_f f(\boldsymbol{z}) > 0\} \neq \emptyset$$

or, equivalently, the inequality system $\{\alpha_F f(\boldsymbol{z}) > 0\colon f \in F\}$ is feasible for some factor $\alpha_F^\star \in \mathbb{R}$.

Remark 2.6. *If F is a nonempty face of the complex $\Delta_\geq$ on the vertex set $\mathbf{F}(\boldsymbol{Z})$, and F' is a face of this complex such that $F' \supseteq F$ (if in particular F' is a facet containing F), that is, the subsystem $\{\alpha_{F'} f(\boldsymbol{z}) \geq 0\colon\ f \in F'\}$ of the (in)feasible system $\{\alpha_{\mathbf{F}(\boldsymbol{Z})} f(\boldsymbol{z}) \geq 0\colon\ f \in \mathbf{F}(\boldsymbol{Z})\}$ is feasible for some factor $\alpha_{F'}^\star$, then the system*

$$\{\alpha_F f(\boldsymbol{z}) \geq 0\colon\ f \in F\}$$

is also feasible when $\alpha_F = \alpha_{F'}^\star$; an analogous observation is true for the faces F of the complex $\Delta_>$. □

Let us comment on the definitions of the complexes $\Delta_{\geq}$ and $\Delta_{>}$ by the example of the complex $\Delta_{\geq}$ and an *infeasible* system of inequalities

$$\{f(\boldsymbol{z}) \geq 0 \colon\ f \in \mathbf{F}(\boldsymbol{Z})\}\ . \tag{2.9}$$

Let Γ be a complex whose faces are the *feasible* subsystems of system (2.9); in other words, if the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0 \colon\ f \in F\}$ is feasible for the factor $\alpha_F^\star := 1$ then $F \in \Gamma$. If $H \in \mathbf{2}^{\mathbf{F}(\boldsymbol{Z})} - \Gamma$ is an *infeasible* subsystem of system (2.9) then $H \in \Delta_{\geq} - \Gamma$ if and only if the inequality system $\{\alpha_H h(\boldsymbol{z}) \geq 0 \colon\ h \in H\}$ is feasible for some factor $\alpha_H^\star \in \mathbb{R} - \{0, 1\}$. If $G \subseteq H$, $G \neq \emptyset$, then, according to Remark 2.6, the subset G is a face of the complex $\Delta_{\geq}$ because the system $\{\alpha_H^\star g(\boldsymbol{z}) \geq 0 \colon\ g \in G\}$ is feasible. Thus,

$$\Delta_{\geq} = \Gamma \cup \Delta\big(\mathbf{max}\,(\Delta_{\geq} - \Gamma)\big)\ .$$

For a nonempty face F of the complex $\Delta_{\geq}$, denote by $\boldsymbol{A}_F^\star$ the set of all the nonzero real factors $\alpha_F^\star$ for which the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0 \colon\ f \in F\}$ is feasible. If $\mathcal{H}(F) := \{H \in \mathbf{max}\,\Delta_{\geq} \colon\ H \supseteq F\}$ is the subfamily of facets of $\Delta_{\geq}$ that contain the face F, then the inequality system $\{\alpha_F f(\boldsymbol{z}) \geq 0 \colon\ f \in F\}$ turns out to be feasible for any factor

$$\alpha_F^\star \in \bigcap_{H \in \mathcal{H}(F)} \boldsymbol{A}_H^\star \subseteq \bigcup_{H \in \mathcal{H}(F)} \boldsymbol{A}_H^\star \subseteq \boldsymbol{A}_F^\star\ .$$

We proceed by investigating the connectedness of the graphs of independence systems that correspond to the complexes of the three above-defined classes.

Proposition 2.7. *If* $\#\mathbf{max}\,\Delta_{\geq} > 1$ *then the graph of the independence system* $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq})$ *is connected.*

Proof. Let us consider a multifamily $V := \big\{(\mathbf{C}_{\geq}(f), \mathbf{C}_{\leq}(f)) \colon\ f \in \mathbf{F}(\boldsymbol{Z})\big\}$ of ordered pairs whose sets $\mathbf{C}_{\geq}(f)$ and $\mathbf{C}_{\leq}(f)$ are defined as follows:

$$\mathbf{C}_{\geq}(f) := \{\boldsymbol{z} \in \boldsymbol{Z} \colon\ f(\boldsymbol{z}) \geq 0\} \quad \text{and} \quad \mathbf{C}_{\leq}(f) := \{\boldsymbol{z} \in \boldsymbol{Z} \colon\ f(\boldsymbol{z}) \leq 0\}\ .$$

The definition of the complex $(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq})$ and definition (2.1) of the complex $(V, \Delta_{\cap})$ imply that they are isomorphic, $(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq}) \simeq (V, \Delta_{\cap})$; an isomorphism is provided by the map $\mathbf{F}(\boldsymbol{Z}) \to V$, $f \mapsto (\mathbf{C}_{\geq}(f), \mathbf{C}_{\leq}(f))$. Since the set V satisfies the conditions of Proposition 2.5, the graph of the independence system $\mathtt{ISG}(V, \Delta_{\cap})$ is connected and, as a consequence, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{\geq})$, isomorphic to the graph $\mathtt{ISG}(V, \Delta_{\cap})$, is also connected. ☐

If $\boldsymbol{X}$ is a subset of the space $\boldsymbol{Z}$ then we will use the notation $\mathrm{Fr}\,(\boldsymbol{X})$ to denote its boundary in $\boldsymbol{Z}$.

Proposition 2.8. *If* $\#\mathbf{max}\,\bar{\Delta} > 1$ *and the set* $f^{-1}(0)$ *is nowhere dense for each function* $f \in \mathbf{F}(\boldsymbol{Z})$, *then the graph of the independence system* $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ *is connected.*

Proof. The multifamily $V := \{(\overline{\mathbf{C}_>(f)}, \overline{\mathbf{C}_<(f)}) : f \in \mathbf{F}(\boldsymbol{Z})\}$ satisfies the conditions of Proposition 2.5: indeed, $\overline{\mathbf{C}_>(f)} \cup \overline{\mathbf{C}_<(f)} = \boldsymbol{Z}$, for each continuous function $f \in \mathbf{F}(\boldsymbol{Z})$ because $f^{-1}(0) \subseteq \mathrm{Fr}(\mathbf{C}_>(f)) \cup \mathrm{Fr}(\mathbf{C}_<(f))$ and the preimage $f^{-1}(0)$ is by condition nowhere dense in $\boldsymbol{Z}$. The map $\mathbf{F}(\boldsymbol{Z}) \to V, f \mapsto (\overline{\mathbf{C}_>(f)}, \overline{\mathbf{C}_<(f)})$, is an isomorphism of the complexes $(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ and $(V, \Delta_\cap)$; according to Proposition 2.5, the graph of the independence system $\mathtt{ISG}(V, \Delta_\cap)$ is connected and thus the isomorphic graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ is also connected. □

Proposition 2.9. *Let* $\mathbf{F}(\boldsymbol{Z})$ *be a system of continuous functions, together with the corresponding complex* $(\mathbf{F}(\boldsymbol{Z}), \Delta_>)$*, such that* $\#\mathbf{max}\,\Delta_> > 1$*. If the sets* $f^{-1}(0)$ *are nowhere dense for each function* $f \in \mathbf{F}(\boldsymbol{Z})$*, then for each facet* $H \in \mathbf{max}\,\Delta_>$ *the inequality system*

$$\begin{cases} \alpha f(\boldsymbol{z}) > 0\,, & \text{if } f \in H\,, \\ -\alpha f(\boldsymbol{z}) > 0\,, & \text{if } f \in \mathbf{F}(\boldsymbol{Z}) - H\,, \end{cases} \tag{2.10}$$

where $\alpha \in \mathbb{R} - \{0\}$*, is feasible.*

Proof. By the hypothesis of the proposition, the system

$$\{\alpha f(\boldsymbol{z}) > 0 :\ f \in H\} \tag{2.11}$$

is feasible; its nonempty set of solutions $\boldsymbol{S} \subset \boldsymbol{Z}$ is open. The maximality of H implies that for each function $f \in \mathbf{F}(\boldsymbol{Z}) - H$ and for any point $\boldsymbol{z} \in \boldsymbol{S}$ it holds

$$\alpha f(\boldsymbol{z}) \le 0\,. \tag{2.12}$$

Since the preimage $f^{-1}(0)$ is nowhere dense for any function $f \in \mathbf{F}(\boldsymbol{Z})$, it follows from (2.12) that the constraint system

$$\{-\alpha g(\boldsymbol{z}) > 0 :\ \boldsymbol{z} \in \boldsymbol{S}\} \tag{2.13}$$

is feasible for each function $g \in \mathbf{F}(\boldsymbol{Z}) - H$ and, besides, its solutions represent an open subset of the solution set $\boldsymbol{S}$ to system (2.11) – denote it by $\boldsymbol{S}_g$; thus, $\boldsymbol{S}_g$ is the solution set to the feasible system

$$\begin{cases} \alpha f(\boldsymbol{z}) > 0\,, & f \in H\,, \\ \alpha(-g(\boldsymbol{z})) > 0\,. \end{cases}$$

Without loss of generality we will suppose that $\mathbf{F}(\boldsymbol{Z}) - H = \{f_1, f_2, \ldots, f_k\}$. If $k = 1$ then $\mathbf{F}(\boldsymbol{Z}) - H = \{g\}$, and the proposition is proved. If $k > 1$ then let us consider each function $g \in \{f_2, f_3, \ldots, f_k\}$ one by one. Repeating the above argument, we see that for each index i, $2 \le i \le k$, the subsystem

$$\begin{cases} \alpha f(\boldsymbol{z}) > 0\,, & f \in H \dot\cup \{f_1, \ldots, f_{i-1}\}\,, \\ \alpha(-f_i(\boldsymbol{z})) > 0 \end{cases}$$

is feasible. As a consequence, the initial system (2.10) is also feasible. □

Corollary 2.10. *Under the hypothesis of Proposition* 2.9*,*
(i) $H \in \mathbf{max}\,\Delta_{>} \implies \mathbf{F}(\boldsymbol{Z}) - H \in \Delta_{>}$,
(ii) *the graph of the independence system* $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$ *has no isolated vertices.*

Proposition 2.11. *Suppose that* $\#\mathbf{max}\,\Delta_{>} > 1$. *If for the system* $\mathbf{F}(\boldsymbol{Z})$ *the sets* $h^{-1}(0)$ *are nowhere dense for all functions* $h \in \mathbf{F}(\boldsymbol{Z})$*, and the condition*

$$f, g \in \mathbf{F}(\boldsymbol{Z}) \,, \;\; f \neq g \quad \implies \quad f^{-1}(0) \cap g^{-1}(0) = \emptyset \tag{2.14}$$

is satisfied, then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$ *is connected.*

Proof. Recall that, according to Proposition 2.8, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ is connected. Let us show that $\Delta_{>} = \bar{\Delta}$.

The inclusion $\Delta_{>} \subseteq \bar{\Delta}$ holds: indeed, let F be a nonempty face of the complex $\Delta_{>}$, that is, for some real factor $\alpha^{\star}_{F}$ the inequality system $\{\alpha^{\star}_{F} f(\boldsymbol{z}) > 0 \colon\; f \in F\}$ is feasible. Then $\bigcap_{f \in F} \overline{\mathbf{C}_{>}(f)} \neq \emptyset$ and thus $F \in \bar{\Delta}$.

Let us prove the reverse inclusion $\Delta_{>} \supseteq \bar{\Delta}$. Let $F \in \bar{\Delta}$, that is, $\bigcap_{f \in F}\Big(\overline{\mathbf{C}_{>}(f)}, \overline{\mathbf{C}_{<}(f)}\Big) \neq (\emptyset, \emptyset)$. Specifically, suppose that $\bigcap_{f \in F} \overline{\mathbf{C}_{>}(f)} \neq \emptyset$.

Since $\overline{\mathbf{C}_{>}(f)} = \mathbf{C}_{>}(f) \dot{\cup} \mathrm{Fr}(\mathbf{C}_{>}(f))$ and each function f is by condition continuous, the inclusion $\mathrm{Fr}(\mathbf{C}_{>}(f)) \subseteq f^{-1}(0)$ holds.

Taking into account condition (2.14), we have

$$\begin{aligned}\bigcap_{f \in F} \overline{\mathbf{C}_{>}(f)} &= \bigcap_{f \in F}\Big(\mathbf{C}_{>}(f) \dot{\cup} \mathrm{Fr}(\mathbf{C}_{>}(f))\Big) \\ &= \Big(\bigcap_{f \in F} \mathbf{C}_{>}(f)\Big) \cup \bigcup_{f \in F}\Big(\mathrm{Fr}(\mathbf{C}_{>}(f)) \;\cap \bigcap_{g \in F \colon\; g \neq f} \mathbf{C}_{>}(g)\Big) \neq \emptyset \,. \end{aligned} \tag{2.15}$$

If the intersection $\bigcap_{f \in F} \mathbf{C}_{>}(f)$ in expression (2.15) is nonempty then $F \in \Delta_{>}$.

If there exists a function $f \in F$ such that

$$\mathrm{Fr}(\mathbf{C}_{>}(f)) \;\cap \bigcap_{g \in F,\; g \neq f} \mathbf{C}_{>}(g) \neq \emptyset \tag{2.16}$$

in expression (2.15), then $\bigcap_{g \in F} \mathbf{C}_{>}(g) \neq \emptyset$. Thus, $F \in \Delta_{>}$ and, taking into account the above argument, $\bar{\Delta} = \Delta_{>}$. Since the graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \bar{\Delta})$ is connected, the isomorphic graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$is also connected. □

Proposition 2.12. *Suppose that* $\#\mathbf{max}\,\Delta_{>} > 1$*, and for the system* $\mathbf{F}(\boldsymbol{Z})$ *the sets* $h^{-1}(0)$ *are nowhere dense for each function* $h \in \mathbf{F}(\boldsymbol{Z})$*. If the set*

$$\boldsymbol{Z}' := \boldsymbol{Z} - \bigcup_{\substack{f, g \in \mathbf{F}(\boldsymbol{Z}) \colon \\ f \neq g}} (f^{-1}(0) \cap g^{-1}(0))$$

is connected then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_{>})$ *is connected.*

Proof. Let us consider the collection $\mathbf{F}(\boldsymbol{Z}') := \{f\,|_{\boldsymbol{Z}'} : f \in \mathbf{F}(\boldsymbol{Z})\}$ of the maps from $\mathbf{F}(\boldsymbol{Z})$ restricted to $\boldsymbol{Z}'$.

Let $\Delta'_>$ be a simplicial complex on the vertex set $\mathbf{F}(\boldsymbol{Z}')$ whose nonempty faces are by definition those subsets

$$F' := \{(f' := f\,|_{\boldsymbol{Z}'}):\ f \in F\} \subseteq \mathbf{F}(\boldsymbol{Z}') \tag{2.17}$$

corresponding to the sets $F \subseteq \mathbf{F}(\boldsymbol{Z})$ for which the inequality systems

$$\{\alpha_{F'} f'(\boldsymbol{z}) > 0:\ f' \in F'\} \tag{2.18}$$

are feasible for some factors $\alpha^\star_{F'} \in \mathbb{R}$.

For the nonempty subsets $F' \subseteq \mathbf{F}(\boldsymbol{Z}')$ defined in (2.17), we have

$$F' \in \Delta'_> \iff F \in \Delta_> . \tag{2.19}$$

The *sufficiency* is evident: indeed, if system (2.18) is feasible for some factor $\alpha^\star_{F'}$ then the system $\{\alpha^\star_{F'} f(\boldsymbol{z}) > 0 : f \in F\}$ is also feasible.

The *necessity*. Let $F \in \Delta_>$, that is, the system $\{\alpha_F f(\boldsymbol{z}) > 0:\ f \in F\}$ is by definition feasible for some factor $\alpha^\star_F \in \mathbb{R}$; specifically, suppose that $\alpha^\star_F > 0$. If $\boldsymbol{z}^\star$ is a solution to the system $\{\alpha^\star_F f(\boldsymbol{z}) > 0:\ f \in F\}$ then, because of the continuity of the functions $f \in \mathbf{F}(\boldsymbol{Z})$, elements of some neighborhood $\boldsymbol{O}_{\boldsymbol{z}^\star}$ of the solution $\boldsymbol{z}^\star$ are also solutions to the system. Since the sets $f^{-1}(0)$ are nowhere dense for each function $f \in \mathbf{F}(\boldsymbol{Z})$, we have $\boldsymbol{O}_{\boldsymbol{z}^\star} - \bigcup_{f\in F} f^{-1}(0) \neq \emptyset$. As a consequence, there exists an element $\boldsymbol{z}' \in \boldsymbol{Z}'$ such that $f'(\boldsymbol{z}') > 0$, for each function $f' := f\,|_{\boldsymbol{Z}'}$, that is, $F' \in \Delta'_>$. Thus, $\Delta(\mathbf{F}(\boldsymbol{Z}'), \Delta'_>) \simeq \Delta(\mathbf{F}(\boldsymbol{Z}), \Delta_>)$.

According to Proposition 2.11, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}'), \Delta'_>)$ is connected; therefore, the isomorphic graph $\mathtt{ISG}(\mathbf{F}(\boldsymbol{Z}), \Delta_>)$ is also connected. □

We proceed by considering the graphs of independence systems that are associated with finite collections $\mathbf{F}(\mathbb{R}^n) := \{f_i :\ i \in [m]\}$, $m > 1$, of polynomial functions $f_i : \mathbb{R}^n \to \mathbb{R}$ on the real Euclidean space $\mathbb{R}^n$, $n > 1$.

For a nonempty tuple $F \subseteq \mathbf{F}(\mathbb{R}^n)$, we will use the notation $\boldsymbol{V}(F) := \bigcap_{f\in F}\{\boldsymbol{z} \in \mathbb{R}^n:\ f(\boldsymbol{z}) = 0\}$.

Remark 2.13. *If $f, g : \mathbb{R}^n \to \mathbb{R}$ are two relatively prime polynomials then the set $\boldsymbol{V}(f, g)$ has the topological dimension at most $n - 2$.*

In order to see that it suffices to consider the situation where the polynomials f and g are irreducible – in this case the sets $\boldsymbol{V}(f)$ and $\boldsymbol{V}(g)$ are algebraic varieties, that is, these sets are irreducible. Since $g \neq \lambda f$, for any factor $\lambda \in \mathbb{R}$, the strict inclusion $\boldsymbol{V}(f, g) \subsetneq \boldsymbol{V}(f)$ holds. Thus, the algebraic dimension of the set $\boldsymbol{V}(f, g)$ does not exceed $n - 2$; recall that the topological dimension $\boldsymbol{V}(f, g)$ coincides with its algebraic dimension. □

Proposition 2.14. *If $\mathbf{F}(\mathbb{R}^n)$ is a collection of pairwise relatively prime polynomials then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is connected.*

Proof. According to Remark 2.13, the sets $f^{-1}(0) \cap g^{-1}(0)$ have the topological dimension at most $n-2$, for any distinct polynomials $f, g \in \mathbf{F}(\mathbb{R}^n)$. As a consequence, the complement $\mathbb{R}^n - \bigcup_{\substack{f,g \in \mathbf{F}(\mathbb{R}^n):\\ f \neq g}} (f^{-1}(0) \cap g^{-1}(0))$ is connected; therefore, Proposition 2.12 allows us to come to the conclusion that the graph $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is connected. □

Let us consider some corollaries of the above argument; two of them follow from Proposition 2.7:

Corollary 2.15. *Let $(\boldsymbol{V}, \Delta)$ be the complex whose vertex set $\boldsymbol{V}$ is a finite subset of points on the sphere $\mathbb{S}^{n-1}$, and a nonempty subset $F \subseteq \boldsymbol{V}$ is a face if and only if the set F is contained in a closed hemisphere.*

The graph of the independence system $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is connected.

Proof. Let us assign to each point $\boldsymbol{v} \in \boldsymbol{V}$ the linear functional $f_{\boldsymbol{v}}\colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{z} \mapsto \langle \boldsymbol{v}, \boldsymbol{z} \rangle$, and consider the collection $\mathbf{F}(\mathbb{R}^n) := \{f_{\boldsymbol{v}}\colon \boldsymbol{v} \in \boldsymbol{V}\}$ of all such functionals. Since the space $\mathbb{R}^n$ is connected, the graph $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_\geq)$ is also connected in accordance with Proposition 2.7. Recall that a nonempty subset F of the set $\boldsymbol{V}$ is contained in a closed hemisphere if and only if the inequality system $\{\langle \boldsymbol{v}, \mathbf{z} \rangle \geq 0\colon \boldsymbol{v} \in F\}$ is feasible, that is, $\{f_{\boldsymbol{v}}\colon \boldsymbol{v} \in F\} \in \Delta_\geq$. As a consequence, $(\boldsymbol{V}, \Delta) \simeq (\mathbf{F}(\mathbb{R}^n), \Delta_\geq)$ and the graph of the independence system $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is connected. □

Corollary 2.16. *Let $(\boldsymbol{V}, \Delta)$ be the complex whose vertex set $\boldsymbol{V} := \{\boldsymbol{P}_i\colon i \in [m]\}$, $m > 1$, is a finite family of closed half-spaces of the space $\mathbb{R}^n$ with a nonempty intersection, and a nonempty subfamily $F \subseteq \boldsymbol{V}$ is a face if and only if the polyhedron $\bigcap_{\boldsymbol{P} \in F} \boldsymbol{P}$ is unbounded.*

The graph of the independence system $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is connected.

Proof. Let us represent each half-space $\boldsymbol{P}_i$ in the form $\boldsymbol{P}_i := \{\boldsymbol{z} \in \mathbb{R}^n\colon \langle \boldsymbol{v}_i, \boldsymbol{z} \rangle \geq b_i\}$, where $\boldsymbol{v}_i \in \mathbb{S}^{n-1}$ and $b_i \in \mathbb{R}$. Since for a nonempty index subset $L \subseteq [m]$ the polyhedron $\bigcap_{i \in L} \boldsymbol{P}_i$ is unbounded if and only if the point set $\{\boldsymbol{v}_i\colon i \in L\}$ is contained in a closed hemisphere, the statement follows from Corollary 2.15. □

We conclude this section by pointing out at two corollaries of Proposition 2.11:

Corollary 2.17. *If $\mathbf{F}(\mathbb{R}^n) := \{f_i\colon i \in [m]\}$ is a collection of linear functionals $f_i\colon \mathbb{R}^n \to \mathbb{R}$ and $f_i \neq -\lambda f_j$, for any distinct indices $i, j \in [m]$ and for positive factors $\lambda \in \mathbb{R}$, then the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is connected.*

Proof. Suppose that $f_i = \lambda f_j$ for some distinct indices $i, j \in [m]$ and for a real factor $\lambda > 0$. In this case, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is isomorphic to the graph $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n) - \{f_j\}, \Delta'_>)$ that corresponds to the complex $\Delta'_>$ on the vertex set $\mathbf{F}(\mathbb{R}^n) - \{f_j\}$. In this context, we can content ourselves with a discussion of the case when the functionals from $\mathbf{F}(\mathbb{R}^n)$ satisfy the condition $f_i \neq \lambda f_j$, for $i, j \in [m]$ and $\lambda > 0$. Taking into account that, analogously, $f_i \neq -\lambda f_j$, for $i, j \in [m]$ and $\lambda > 0$, we come to the conclusion that the hypothesis of Proposition 2.12 holds and, as a consequence, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is connected. □

Corollary 2.18. *Let* $(\boldsymbol{V}, \Delta)$ *be the complex whose vertex set* $\boldsymbol{V}$ *is a finite subset of points on the sphere* $\mathbb{S}^{n-1}$, *and a nonempty subset* $F \subseteq \boldsymbol{V}$ *is a face of* Δ *if and only if the set* F *is contained in an open hemisphere.*

If the set $\boldsymbol{V}$ *does not contain antipodes then the graph of the independence system* $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ *is connected.*

Proof. Let us assign to each point $\boldsymbol{v} \in \boldsymbol{V}$ the linear functional $f_{\boldsymbol{v}}\colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{z} \mapsto \langle \boldsymbol{v}, \boldsymbol{z} \rangle$, and consider the collection $\mathbf{F}(\mathbb{R}^n) := \{f_{\boldsymbol{v}}\colon\ \boldsymbol{v} \in \boldsymbol{V}\}$ of all such functionals. The complex $(\boldsymbol{V}, \Delta)$ is isomorphic to the complex $(\mathbf{F}(\mathbb{R}^n), \Delta_>)$, and the lack of antipodes in $\boldsymbol{V}$ implies the fulfillment of conditions $f_{\boldsymbol{v}} \neq -\lambda f_{\boldsymbol{w}}$, for any points $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$ and real factors $\lambda > 0$. According to Corollary 2.17, the graph of the independence system $\mathtt{ISG}(\mathbf{F}(\mathbb{R}^n), \Delta_>)$ is connected; as a consequence, the isomorphic graph $\mathtt{ISG}(\boldsymbol{V}, \Delta)$ is also connected. ☐

2.2 The hypergraph of an independence system

The so-called hypergraph of an independence system is a natural generalization of the notion of graph of this system:

The *hypergraph of the independence system* $\mathtt{ISH}(V, \Delta)$ corresponding to an abstract simplicial complex (V, Δ), with the facet family $\mathbf{max}\,\Delta$ and with the vertex set $V := \bigcup_{H \in \mathbf{max}\,\Delta} H$, is the hypergraph defined as follows:

- *the vertex set of the hypergraph* $\mathtt{ISH}(V, \Delta)$ *is the facet family* $\mathbf{max}\,\Delta$ *of the complex* Δ*;*
- *the hyperedge family of the hypergraph* $\mathtt{ISH}(V, \Delta)$ *is the family of all the unordered collections of facets* $\mathcal{H} \subseteq \mathbf{max}\,\Delta$ *of the complex* Δ *that cover the vertex set of the complex:*
$$\bigcup_{H \in \mathcal{H}} H = V\,.$$

Recall that any finite simple graph is isomorphic to the graph of some independence system – see Proposition 2.2. Similarly, the family of hypergraphs that are isomorphic to the hypergraphs $\mathtt{ISH}(V, \Delta)$ is quite large.

In this section, we will consider finite infeasible monotone systems of constraints $\mathfrak{S} := \{\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_m\}$ and the corresponding maps $\pi\colon\ \mathbb{B}(m) \to \mathbf{2}^{\Gamma}$, where Γ is some nonempty set, such that

$$\pi(\{\emptyset\}) = \Gamma\,, \quad \pi([m]) = \emptyset\,;$$
$$\{i\} \in \mathbb{B}(m)^{(1)} \quad \Longrightarrow \quad \boldsymbol{D}_i := \pi(i) \neq \emptyset\,;$$
$$A,\ B \in \mathbb{B}(m) \quad \Longrightarrow \quad \pi(A) \cap \pi(B) = \pi(A \cup B)\,;$$

we studied similar systems in Chapter 1.

In other words, any such system $\mathfrak{S}$ represents an infeasible system of constraints

$$\mathbf{x} \in \boldsymbol{D}_i \subset \Gamma\,, \quad i \in [m]\,. \tag{2.20}$$

Let us denote the family of the multi-indices of MFSs of system (2.20) as usual by $\mathbf{J}$. The family of the multi-indices of all feasible subsystems of this system represents the abstract simplicial complex $\Delta(\mathbf{J})$ on the vertex set $[m]$, with the facet family $\mathbf{max}\,(\Delta(\mathbf{J})) := \mathbf{J}$.

Proposition 2.19. *A hypergraph $\Lambda := (\{v_1, \ldots, v_p\}, \mathcal{E})$, with the vertex set $\{v_1, \ldots, v_p\}$, $p > 1$, and with hyperedge family $\mathcal{E}$, is isomorphic to the hypergraph of the independence system* $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ *that corresponds to system* (2.20) *for some relevant quantity m, if and only if the hyperedge family $\mathcal{E}$, partially ordered by inclusion, represents an order filter in the Boolean lattice $\mathbb{B}(p)$ of subsets of the set $\{v_1, \ldots, v_p\}$ and, besides,*

$$\mathcal{E} \subseteq \mathbb{B}(p) - \{\mathbb{B}(p)^{(1)}\} . \tag{2.21}$$

Proof. The *necessity*. Suppose that the hypergraph Λ with pairwise distinct hyperedges is isomorphic to the hypergraph of the independence system $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ that corresponds to system (2.20); we denote an isomorphism providing this correspondence by $\varphi\colon\ \{v_1, \ldots, v_p\} \to \mathbf{J}$.

Let us show that inclusion (2.21) holds. Indeed, since by convention $p > 1$, system (2.20) is infeasible and thus the hypergraph $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ has no *loops*; as a consequence, the isomorphic hypergraph Λ also has no loops – in other words, $\mathcal{E} \cap \mathbb{B}(p)^{(1)} = \emptyset$.

Let us verify that the hyperedge family $\mathcal{E}$ is an order filter. Consider an arbitrary hyperedge, say $U := \{v_1, \ldots, v_k\} \in \mathcal{E}$, such that $U \neq \{v_1, \ldots, v_p\}$, and consider some hyperedge $W := \{v_1, \ldots, v_k, \ldots, v_s\} \supsetneq U$ containing it as a subset; we have $\varphi(U) = \{J_{i_1}, \ldots, J_{i_k}\}$ and $\varphi(W) = \{J_{i_1}, \ldots, J_{i_k}, \ldots, J_{i_s}\}$, where $J_{i_1}, \ldots, J_{i_s}$ are the multi-indices of some maximal feasible subsystems of system (2.20). Since φ is an isomorphism, the family $\varphi(U)$ of the multi-indices of MFSs is a hyperedge of the hypergraph $\mathtt{ISH}([m], \Delta(\mathbf{J}))$, that is, according to the definition of this hypergraph, $\bigcup_{e=1}^{k} J_{i_e} = [m]$; moreover, $\bigcup_{e=1}^{s} J_{i_e} = [m]$ and thus the family $\varphi(W)$ is a hyperedge of the hypergraph $\mathtt{ISH}([m], \Delta(\mathbf{J}))$. Since the inverse map φ^{-1} is also an isomorphism, the family $W = \varphi^{-1}(\varphi(W))$ is a hyperedge of the hypergraph Λ.

The *sufficiency*. Let us show that if the hyperedge family $\mathcal{E}$ of the hypergraph Λ, partially ordered by inclusion, represents an order filter in the Boolean lattice $\mathbb{B}(p)$, and condition (2.21) is satisfied, then there exist an integer m and sets $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_m \subset \mathbb{N}$ such that the hypergraph of the independence system $\mathtt{ISH}([m], \Delta(\mathbf{J}))$ corresponding to system (2.20) is isomorphic to the hypergraph Λ.

Recall that, by the hypothesis, $p > 1$, and consider the family

$$\mathcal{F} := \mathbb{B}(p) - \{\mathcal{E} \cup \{\hat{0}\}\} ,$$

that is the order ideal $\mathbb{B}(p) - \mathcal{E}$ with the minimal element removed. Since the hypergraph Λ has no loops, the finite nonempty family $\mathcal{F} := \{F_1, \ldots, F_m\}$ contains all one-element subsets of the set $\mathbb{B}(p)^{(1)} := \{v_1, \ldots, v_p\}$. Suppose

$$J_k := \{i \in [m]\colon\ v_k \notin F_i\}$$

for all $k \in [p]$. The sets, constructed in such a way, satisfy the following conditions:

$$\emptyset \neq J_k \subset [m] \,, \tag{2.22}$$

$$k_1, k_2 \in [p],\ k_1 \neq k_2 \implies J_{k_1} - J_{k_2} \neq \emptyset \,, \tag{2.23}$$

$$\bigcup_{k \in L} J_k = [m] \iff \{v_k \in \mathbb{B}(p)^{(1)} :\ k \in L\} \in \mathcal{E} \,,\ \emptyset \neq L \subset [p] \,. \tag{2.24}$$

Let us show that relation (2.24) holds. Consider an arbitrary proper subset L of the set $[p]$.

Let a set $\{v_k \in \mathbb{B}(p)^{(1)} :\ k \in L\}$ be a hyperedge of the hypergraph Λ. Assume that $\bigcup_{k \in L} J_k \neq [m]$. Then there exists an index $i_0 \in [m]$ such that for all elements $k \in L$ we have $v_k \in F_{i_0}$. Thus, $F_{i_0} \supseteq \{v_k \in \mathbb{B}(p)^{(1)} :\ k \in L\} \in \mathcal{E}$, a contradiction with the fact that $\mathcal{E}$ is an order filter in $\mathbb{B}(p)$. We have verified that $\{v_k \in \mathbb{B}(p)^{(1)} :\ k \in L\} \in \mathcal{E} \implies \bigcup_{k \in L} J_k = [m]$.

Now suppose that $\bigcup_{k \in L} J_k = [m]$. Since for each index $i \in [m]$ we have $\{v_k \in \mathbb{B}(p)^{(1)} :\ k \in L\} \cap J_i \neq \emptyset$, then $\hat{0} \neq \{v_k \in \mathbb{B}(p)^{(1)} :\ k \in L\} \notin \mathcal{F}$; thus, $\bigcup_{k \in L} J_k = [m] \implies \{v_k \in \mathbb{B}(p)^{(1)} :\ k \in L\} \in \mathcal{E}$.

Conditions (2.22)–(2.24) guarantee that the sets $J_1, \dots, J_p$ are the multi-indices of MFSs of some constraint system. Indeed, for each $j \in [p]$, suppose

$$\boldsymbol{D}_j := \{i \in [p] :\ j \in J_i\} \,. \tag{2.25}$$

The sets $J_1, \dots, J_p$ only are the multi-indices of MFSs of system (2.20), with the sets $\boldsymbol{D}_j$ determined by relation (2.25). Indeed, for each J_i and for any $j \in J_i$, by construction, the inclusion $i \in \boldsymbol{D}_j$ holds; as a consequence, $\bigcap_{j \in J_i} \boldsymbol{D}_j \neq \emptyset$. On the other hand, let $\bigcap_{j \in L} \boldsymbol{D}_j \neq \emptyset$, where $L \neq \emptyset$. By construction, there exists a number $i \in [p]$ such that $i \in \boldsymbol{D}_j$ for each $j \in L$; thus, $L \subseteq J_i$.

By using the bijection $\varphi \colon \{v_1, \dots, v_p\} \to \mathbf{J}$, $v_k \mapsto J_k$, we verify that, because of condition (2.24), the map φ is an isomorphism of the hypergraphs Λ and $\mathtt{ISH}([m], \Delta(\mathbf{J}))$. □

2.3 The graph of maximal feasible subsystems of an infeasible system of linear inequalities

In this section, we study the graphs of independence systems associated with the complexes (of the multi-indices) of the feasible subsystems of infeasible systems of linear inequalities.

We will investigate the properties of the finite infeasible system

$$\mathfrak{S} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 :\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m];\ i_1 \neq i_2 \Rightarrow \boldsymbol{a}_{i_1} \neq -\boldsymbol{a}_{i_2}\} \tag{2.26}$$

of homogeneous strict linear inequalities, of rank n, over the real Euclidean space $\mathbb{R}^n$, whose set of determining vectors $\boldsymbol{A}(\mathfrak{S}) := \{\boldsymbol{a}_i :\ i \in [m]\}$ contains no pairs of antipodes.

Following the argument presented on page 18, we assign to system (2.26) the map

$$\pi\colon\ \mathbb{B}(m) \to \mathbf{2}^{\mathbb{R}^n}\,, \quad \hat{0} \neq T \mapsto \bigcap_{t\in T}\{\boldsymbol{x} \in \mathbb{R}^n\colon\ \langle \boldsymbol{a}_t, \boldsymbol{x}\rangle > 0\}\,,$$
$$\hat{0} \mapsto \mathbb{R}^n\,, \tag{2.27}$$

putting in correspondence with each nonempty multi-index $T \in \mathbb{B}(m)$ of the subsystem $\{\langle \boldsymbol{a}_t, \mathbf{x}\rangle > 0\colon\ t \in T\}$ of the system $\mathfrak{S}$ the open cone of its solutions. Since we will consider ordered pairs of subsets of the space $\mathbb{R}^n$ associated with these cones, we also use the synonymous notation $\mathbf{C}_>(T) := \pi(T)$ resembling the notation used in Section 2.1. The linear subspaces $\bigcap_{t\in T}\{\boldsymbol{x} \in \mathbb{R}^n\colon\ \langle \boldsymbol{a}_t, \boldsymbol{x}\rangle = 0\}$ will be denoted by $\mathbf{H}(T)$. For brevity, we use the notation $\mathbf{C}_>(i)$, instead of $\mathbf{C}_>(\{i\})$, for open half-spaces, and $\mathbf{H}(i)$, instead of $\mathbf{H}(\{i\})$, for hyperplanes; $\mathbf{C}_<(T) := -\mathbf{C}_>(T)$.

Let T be the multi-index of a feasible subsystem of the system $\mathfrak{S}$, and $L \subseteq T$ a multi-index such that for any index $i \in [m] - L$ the strict inclusion $\mathbf{H}(L\,\dot\cup\{i\}) \subsetneq \mathbf{H}(L)$ holds; in this case the *face*, open with respect to the subspace $\mathbf{H}(L)$, $\mathbf{H}(L)\cap\mathbf{C}_>(T-L)$ of the closed cone $\overline{\mathbf{C}_>(T)}$ will be denoted by $\mathcal{F}(L, T)$. If $\dim\mathbf{H}(L) = r$ then the face $\mathcal{F}(L, T)$ of the cone $\overline{\mathbf{C}_>(T)}$ is called *r-dimensional*.

Let us denote by $[\boldsymbol{x}, \boldsymbol{y}] := \operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\}$ the closed segment connecting the points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$; $(\boldsymbol{x}, \boldsymbol{y}) := [\boldsymbol{x}, \boldsymbol{y}] - \{\boldsymbol{x}, \boldsymbol{y}\}$ is the corresponding open segment.

Let us denote by $\mathbf{J}$ the family of the multi-indices of MFSs of system (2.26). The abstract simplicial complex $\Delta(\mathbf{J})$, with the facet family $\mathbf{J}$, on the vertex set $[m]$, is the family of the multi-indices of all feasible subsystems of the system $\mathfrak{S}$.

With system (2.26) is put in correspondence a specific graph-theoretic construction from the common family of the graphs of independence systems defined in Section 2.1:

The *graph* $\mathtt{MFSG}(\mathfrak{S})$ *of maximal feasible subsystems* (*the graph of MFSs*) of the system $\mathfrak{S}$ is defined as the graph

$$\mathtt{MFSG}(\mathfrak{S}) := \mathtt{ISG}\big([m], \Delta(\mathbf{J})\big)$$

of the independence system that corresponds to the complex $\big([m], \Delta(\mathbf{J})\big)$. Thus, by definition,

– *the vertex set of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *is the family* $\mathbf{J}$ *of the multi-indices of MFSs of the system* $\mathfrak{S}$*;*
– *the edge family of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *is the family of all the unordered pairs* $\{J, J'\}$ *of the multi-indices of MFSs of the system* $\mathfrak{S}$ *that cover the index set of the inequalities of the system:*

$$J \cup J' = [m]\,.$$

Theorem 2.20. *The graph* $\mathtt{MFSG}(\mathfrak{S})$ *of maximal feasible subsystems of system* (2.26) *is connected.*

Proof. Let $(\boldsymbol{V}, \Delta)$ be the abstract simplicial complex on the vertex set $\boldsymbol{V} := \boldsymbol{A}(\mathfrak{S})$ representing the vectors that define system (2.26); a nonempty subset $F \subseteq \boldsymbol{V}$ is by definition a face of Δ if and only if the set F is contained in an open hemisphere of the unit sphere $\mathbb{S}^{n-1}$.

Using Corollary 2.18 and the isomorphism $[m] \to \boldsymbol{V}$, $i \mapsto \boldsymbol{a}_i$, of the complex $([m], \Delta(\mathbf{J}))$ onto the complex $(\boldsymbol{V}, \Delta)$, considered in Corollary 2.18, we verify that the graph $\mathtt{MFSG}(\mathfrak{S})$ is connected. □

It will be shown below that the problem of extracting MFSs of the system $\mathfrak{S}$, as well as the properties of the graph of MFSs of the system $\mathfrak{S}$, play a significant role in the solving of pattern recognition problems in their geometric setting.

When constructing algorithms of extracting MFSs of the system $\mathfrak{S}$, it is important to know those properties of its graph of MFSs that characterize the neighborhoods of vertices, say some estimates for the degrees of vertices. We will denote the neighborhood of the vertex J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ by $\mathcal{N}(J_s)$.

It is useful to keep in mind the following properties of convex polyhedral cones:

Lemma 2.21. (i) *If $M \subset [m]$ and $\mathbf{C}_>(M) \neq \emptyset$, then $\mathbf{H}(M) \subset \overline{\mathbf{C}_>(M)}$.*
(ii) *If $L, M \subset [m]$, $\mathbf{C}_>(L) \neq \emptyset$ and $\mathbf{C}_>(M) \neq \emptyset$, then*
  (1) $\overline{\mathbf{C}_>(L)} \cap \mathbf{C}_>(M) \neq \emptyset \Longleftrightarrow \mathbf{C}_>(L) \cap \mathbf{C}_>(M) \neq \emptyset$ *and*
  (2) $\mathbf{H}(L) \cap \mathbf{C}_>(M) \neq \emptyset \Longrightarrow \mathbf{C}_>(L) \cap \mathbf{C}_>(M) \neq \emptyset$.

Lemma 2.22. *For the multi-index $J_s \in \mathbf{J}$ of an arbitrary maximal feasible subsystem of the system $\mathfrak{S}$ the inclusion $-\mathbf{C}_>(J_s) \subsetneqq \mathbf{C}_>([m] - J_s)$ holds.*

Proof. Let us fix some vector $\boldsymbol{x}^* \in \mathbf{C}_>(J_s)$ and show that the inclusion $-\boldsymbol{x}^* \in \mathbf{C}_>([m] - J_s)$ holds. Assume the converse: let $j^* \in [m] - J_s$ be an index such that $-\boldsymbol{x}^* \notin \mathbf{C}_>(j^*)$. In this case, the relations $\boldsymbol{x}^* \in \overline{\mathbf{C}_>(j^*)}$ and $\overline{\mathbf{C}_>(j^*)} \cap \mathbf{C}_>(J_s) \neq \emptyset$ imply $\mathbf{C}_>(j^*) \cap \mathbf{C}_>(J_s) \neq \emptyset$, a contradiction with the maximality of the subsystem with the multi-index J_s. Thus, $-\mathbf{C}_>(J_s) \subseteq \mathbf{C}_>([m] - J_s)$. The equality $-\mathbf{C}_>(J_s) = \mathbf{C}_>([m] - J_s)$ is impossible because the set $\boldsymbol{A}(\mathfrak{S})$ of vectors determining the system (2.26) by convention contains no pairs of antipodes. □

Lemma 2.23. *Suppose that each subsystem of cardinality $k + 1$, where $2 \leq k \leq n - 1$, of the system $\mathfrak{S}$ is feasible. Given the multi-index $J_s \in \mathbf{J}$ of some of its maximal feasible subsystem, let us consider an arbitrary $(n - k)$-dimensional face $\mathcal{F}(L, J_s)$ of the closed cone $\overline{\mathbf{C}_>(J_s)}$.*
(i) *The inclusion $-\mathcal{F}(L, J_s) \subset \mathbf{C}_>\big([m] - J_s\big)$ holds, and*
(ii) $\mathbf{C}_>(L) \cap \mathbf{C}_>([m] - J_s) \neq \emptyset$, *that is, the subsystem $\big\{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 \colon \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n; i \in L \cup ([m] - J_s)\big\}$ of the system $\mathfrak{S}$ is feasible.*

Proof. (i) Suppose to the contrary that there exists an index $j^* \in [m] - J_s$ such that $-\mathcal{F}(L, J_s) \not\subset \mathbf{C}_>(j^*)$. Using Lemma 2.22, we have $-\mathcal{F}(L, J_s) \subset -\overline{\mathbf{C}_>(J_s)} \subseteq \overline{\mathbf{C}_>\big([m] - J_s\big)} \subseteq \overline{\mathbf{C}_>(j^*)}$. Thus, the two cases are only possible:
(1) $-\mathcal{F}(L, J_s) \cap \mathbf{C}_>(j^*) \neq \emptyset$ and $-\mathcal{F}(L, J_s) \cap \mathbf{H}(j^*) \neq \emptyset$;
(2) $-\mathcal{F}(L, J_s) \cap \mathbf{C}_>(j^*) = \emptyset$ and, as a consequence, $-\mathcal{F}(L, J_s) \subset \mathbf{H}(j^*)$.

In the first case, let us pick some points $\boldsymbol{x} \in -\mathcal{F}(L, J_s) \cap \mathbf{C}_>(j^*)$ and $\boldsymbol{y} \in -\mathcal{F}(L, J_s) \cap \mathbf{H}(j^*)$. Under $\lambda > 0$ the inclusion $\boldsymbol{z} := -\lambda\boldsymbol{x} + (1+\lambda)\boldsymbol{y} \in -\mathbf{C}_>(j^*)$ holds. Since $\boldsymbol{x}, \boldsymbol{y} \in -\mathcal{F}(L, J_s)$ and the set $-\mathcal{F}(L, J_s)$ is convex and open with respect to $\mathbf{H}(L)$, then moreover $\boldsymbol{z} \in -\mathcal{F}(L, J_s) \subseteq -\overline{\mathbf{C}_>(J_s)}$ for a sufficiently small $\lambda > 0$. Thus, $\boldsymbol{z} \in -\mathbf{C}_>(j^*) \cap -\overline{\mathbf{C}_>(J_s)}$ for a sufficiently small $\lambda > 0$; as a consequence, $-\mathbf{C}_>(j^*) \cap -\overline{\mathbf{C}_>(J_s)} \neq \emptyset$, a contradiction with the maximality of the feasible subsystem with the multi-index J_s.

Consider the second case: $-\mathcal{F}(L, J_s) \subset \mathbf{H}(j^*)$. Since the set $-\mathcal{F}(L, J_s)$ is open with respect to the subspace $\mathbf{H}(L)$, then $\mathbf{H}(L) \subseteq \mathbf{H}(j^*)$, that is, the rank of the subsystem with the multi-index $L \cup \{j^*\}$ equals the rank of the subsystem with the multi-index L, namely k. Since any subsystem, with $k+1$ inequalities, of the system $\mathfrak{S}$ is feasible, the rank k subsystem with the multi-index $L \cup \{j^*\}$ is also feasible, that is, $\mathbf{C}_>(L \cup \{j^*\}) \neq \emptyset$.

By definition, $\mathcal{F}(L, J_s) \subset \mathbf{C}_>(J_s - L)$; on the other hand, using Lemma 2.21 (i) for the multi-index $L \cup \{j^*\}$, we have $\mathcal{F}(L, J_s) \subset \mathbf{H}(L) = \mathbf{H}(L \cup \{j^*\}) \subseteq \overline{\mathbf{C}_>(L \cup \{j^*\})}$. Thus, $\mathbf{C}_>(J_s - L) \cap \overline{\mathbf{C}_>(L \cup \{j^*\})} \supseteq \mathcal{F}(L, J_s)$ or $\mathbf{C}(J_s \cup \{j^*\}) \neq \emptyset$, a contradiction with the maximality of the feasible subsystem with the multi-index J_s. This proves the inclusion $-\mathcal{F}(L, J_s) \subset \mathbf{C}_>([m] - J_s)$.

(ii) Since $-\mathcal{F}(L, J_s) \subset \mathbf{H}(L) \subseteq \overline{\mathbf{C}_>(L)}$, then $\overline{\mathbf{C}_>(L)} \cap \overline{\mathbf{C}_>([m] - J_s)} \supseteq -\mathcal{F}(L, J_s)$, and thus $\mathbf{C}_>(L) \cap \mathbf{C}_>([m] - J_s) \neq \emptyset$. □

Lemma 2.24. *If any subsystem, of cardinality n, of the system $\mathfrak{S}$ is feasible then for the multi-index $J_s \in \mathbf{J}$ of each of its maximal feasible subsystem and for an arbitrary collection of $|\mathcal{N}(J_s)|$ representatives $\{\boldsymbol{y}_t \in \mathbf{C}_>(J_t) : J_t \in \mathcal{N}(J_s)\}$ the inclusion $-\overline{\mathbf{C}_>(J_s)} \subseteq \operatorname{pos}\{\boldsymbol{y}_t \in \mathbf{C}_>(J_t) : J_t \in \mathcal{N}(J_s)\}$ holds.*

Proof. Let $\{\mathcal{F}(L_k, J_s) : k \in [l]\}$ be the set of all one-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$, that is, by definition, $\dim \mathbf{H}(L_k) = 1$ and $L_k \subset J_s$ for each $k \in [l]$; let $\{\boldsymbol{x}_k \in \mathcal{F}(L_k, J_s) - \{\mathbf{0}\} : k \in [l]\}$ be an arbitrary collection of l representatives of these faces. Recall that

$$\operatorname{pos}\{\boldsymbol{x}_k : k \in [l]\} = \overline{\mathbf{C}_>(J_s)}\,. \tag{2.28}$$

Let us assign to each representative $\boldsymbol{x}_k$, $k \in [l]$, a vector $\boldsymbol{z}_k \in -\mathbf{C}_>(J_s)$ and the multi-index of a MFS $J_{t_k} \in \mathcal{N}(J_s)$ such that $-\boldsymbol{x}_k \in (\boldsymbol{y}_{t_k}, \boldsymbol{z}_k)$. For this, let us choose the multi-index J_{t_k} in such a way that the inclusion $J_{t_k} \supseteq L_k \cup ([m] - J_s)$ holds. We can do that because, in accordance with Lemma 2.23 (ii), $\mathbf{C}_>(L_k \cup ([m] - J_s)) \neq \emptyset$ for all $k \in [l]$. By Lemma 2.21 (i), $-\boldsymbol{x}_k \in -\mathcal{F}(L_k, J_s) \subset \mathbf{H}(L_k) \subseteq \overline{\mathbf{C}_>(L_k)}$. By the choice of the number t_k, we have $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(J_{t_k}) \subseteq \mathbf{C}_>(L_k)$.

Let us suppose $\boldsymbol{z}_k := -\lambda\boldsymbol{y}_{t_k} + (1+\lambda)(-\boldsymbol{x}_k)$ and show that $\boldsymbol{z}_k \in -\mathbf{C}_>(J_s)$ for a sufficiently small parameter $\lambda > 0$.

Indeed, for any $\lambda > 0$ we have $\langle \boldsymbol{a}_i, \boldsymbol{z}_k \rangle = -\lambda\langle \boldsymbol{a}_i, \boldsymbol{y}_{t_k} \rangle + (1+\lambda)\langle \boldsymbol{a}_i, -\boldsymbol{x}_k \rangle$, for each $i \in L_k$; that is, $\langle \boldsymbol{a}_i, \boldsymbol{z}_k \rangle < 0$, because $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(L_k)$ and $-\boldsymbol{x}_k \in \mathbf{H}(L_k)$.

Further, for each $i \in J_s - L_k$ we have $\langle \boldsymbol{a}_i, \boldsymbol{z}_k \rangle = \langle \boldsymbol{a}_i, -\boldsymbol{x}_k \rangle + \lambda(\langle \boldsymbol{a}_i, -\boldsymbol{x}_k \rangle - \langle \boldsymbol{a}_i, \boldsymbol{y}_{t_k} \rangle) < 0$ for a sufficiently small $\lambda > 0$ because $-\boldsymbol{x}_k \in \mathbf{H}(L_k) \cap -\mathbf{C}_>(J_s - L_k)$. By the choice of $\boldsymbol{z}_k$, the inclusion $-\boldsymbol{x}_k \in [\boldsymbol{y}_{t_k}, \boldsymbol{z}_k]$ holds and, because of $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(L_k)$ and $\boldsymbol{z}_k \in -\mathbf{C}_>(L_k)$, we have $-\boldsymbol{x}_k \in (\boldsymbol{y}_{t_k}, \boldsymbol{z}_k)$.

Let us pick an arbitrary index $k^* \in [m] - J_s$. Denote $\boldsymbol{H}^* := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_{k^*}, \boldsymbol{x}\rangle = 1\}$. According to Lemma 2.23, $-\boldsymbol{x}_k \in -\mathcal{F}(L_k, J_s) \subset \mathbf{C}_>([m] - J_s)$ and, as a consequence, $-\overline{\mathbf{C}_>(J_s)} = \operatorname{pos}\{-\boldsymbol{x}_k : k \in [l]\} \subset \mathbf{C}_>([m] - J_s) \subset \mathbf{C}_>(k^*)$. Besides, for each $k \in [l]$ we have $\boldsymbol{y}_{t_k} \in \mathbf{C}_>(J_{t_k}) \subset \mathbf{C}_>([m] - J_s) \subseteq \mathbf{C}_>(k^*)$. Thus, for each $k \in [l]$ the inequalities $\langle \boldsymbol{a}_{k^*}, -\boldsymbol{x}_k\rangle > 0$ and $\langle \boldsymbol{a}_{k^*}, \boldsymbol{y}_{t_k}\rangle > 0$ are fulfilled; therefore, without loss of generality we can suppose that the vectors $-\boldsymbol{x}_k$ and $\boldsymbol{y}_{t_k}$, $k \in [l]$, are chosen by their norms in such a way that they belong to the hyperplane $\boldsymbol{H}^*$. But this means that the inclusions $\boldsymbol{z}_k \in \boldsymbol{H}^*$, $k \in [l]$, also hold. Besides, we have

$$\boldsymbol{H}^* \cap -\overline{\mathbf{C}_>(J_s)} = \operatorname{conv}\{-\boldsymbol{x}_k : k \in [l]\},$$
$$-\boldsymbol{x}_k, \boldsymbol{z}_k \in \operatorname{conv}\{-\boldsymbol{x}_k : k \in [l]\}, \quad -\boldsymbol{x}_k \in (\boldsymbol{y}_{t_k}, \boldsymbol{z}_k), \quad k \in [l]. \tag{2.29}$$

It follows from a geometric argument that the following statement is true: If $\boldsymbol{E} \subseteq \mathbb{R}^n$, as well as $\boldsymbol{x}, \boldsymbol{z} \in \operatorname{conv}\boldsymbol{E}$ and $\boldsymbol{x} \in (\boldsymbol{y}, \boldsymbol{z})$, then $\operatorname{conv}\boldsymbol{E} \subseteq \operatorname{conv}\left((\boldsymbol{E} \cup \{\boldsymbol{y}\}) - \{\boldsymbol{x}\}\right)$. Using this observation and relations (2.29), let us write down the following chain of inclusions:

$$\begin{aligned} &\operatorname{conv}\{-\boldsymbol{x}_1, \dots, -\boldsymbol{x}_l\} \\ &\quad \subseteq \operatorname{conv}\{\boldsymbol{y}_{t_1}, -\boldsymbol{x}_2, \dots, -\boldsymbol{x}_l\} \subseteq \cdots \subseteq \operatorname{conv}\{\boldsymbol{y}_{t_1}, \boldsymbol{y}_{t_2}, \dots, \boldsymbol{y}_{t_{l-1}}, -\boldsymbol{x}_l\} \\ &\qquad \subseteq \operatorname{conv}\{\boldsymbol{y}_{t_1}, \boldsymbol{y}_{t_2}, \dots, \boldsymbol{y}_{t_l}\} \subseteq \operatorname{conv}\{\boldsymbol{y}_t : J_t \in \mathcal{N}(J_s)\}. \end{aligned} \tag{2.30}$$

Note that (2.28) and (2.30) imply $-\overline{\mathbf{C}_>(J_s)} = \operatorname{pos}\{-\boldsymbol{x}_1, \dots, -\boldsymbol{x}_l\} \subseteq \operatorname{pos}\{\boldsymbol{y}_t : J_t \in \mathcal{N}(J_s)\}$, and the lemma is proved. □

As mentioned earlier, the degrees of the vertices of the graph of MFSs of system (2.26) are important parameters. We present two estimates which will be augmented later by Proposition 2.36.

Theorem 2.25. *Let $J_s \in \mathbf{J}$ be the multi-index of some maximal feasible subsystem of system* (2.26).
(i) *The degree of the vertex J_s in its graph* $\mathtt{MFSG}(\mathfrak{S})$ *is at least two:* $|\mathcal{N}(J_s)| \geq 2$.
(ii) *If each subsystem, of cardinality n, of system* (2.26) *is feasible then the degree of the vertex J_s in its graph* $\mathtt{MFSG}(\mathfrak{S})$ *is at least n:* $|\mathcal{N}(J_s)| \geq n$.

Proof. (i) Let $\{\mathcal{F}(L_k, J_s) : k \in [d]\}$ be the set of all $(n-1)$-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$. According to Lemma 2.22, $\mathbf{C}_>([m] - J_s) \neq \emptyset$ and, as a consequence, there exists a maximal feasible subsystem of the system $\mathfrak{S}$ with a multi-index $J_p \supset [m] - J_s$.

Since J_s and J_p are the multi-indices of distinct MFSs, we have $\mathbf{C}_>(J_s) \cap \mathbf{C}_>(J_p) \neq \emptyset$. Then there exists the index k^* of an $(n-1)$-dimensional face $\mathcal{F}(L_{k^*}, J_s)$ of the cone $\mathbf{C}_>(J_s)$ such that $L_{k^*} \subsetneq J_p$, because otherwise the inclusion $L_k \subseteq J_p$ would hold for each of d such faces, and this would imply the relations $\mathbf{C}_>(J_p) \subseteq \mathbf{C}_>(\bigcup_{k=1}^{d} L_k) = \mathbf{C}_>(J_s)$ and $\mathbf{C}_>(J_p) \cap \mathbf{C}_>(J_s) = \mathbf{C}_>(J_p) \neq \emptyset$.

In the system $\mathfrak{S}$ every subsystem with two inequalities is by convention feasible, and this implies, according to Lemma 2.23 (ii), that for the $(n-1)$-dimensional face $\mathcal{F}(L_{k^*}, J_s)$ we have $\mathbf{C}_>\big(L_{k^*} \cup ([m]-J_s)\big) \neq \emptyset$; as a consequence, there exists a maximal feasible subsystem with a multi-index $J_t \supseteq L_{k^*} \cup ([m]-J_s)$, $J_t \neq J_p$, because $L_{k^*} \subset J_t$ and $L_{k^*} \subsetneq J_p$, that is, $\mathcal{N}(J_s) \supseteq \{J_p, J_t\}$ and thus $|\mathcal{N}(J_s)| \geq 2$.

(ii) Let us pick for the open cone $\mathbf{C}_>(J_t)$ of solutions to each maximal feasible subsystem, with a multi-index $J_t \in \mathcal{N}(J_s)$, one representative $\boldsymbol{y}_t \in \mathbf{C}_>(J_t)$. According to Lemma 2.24, the inclusion $-\overline{\mathbf{C}_>(J_s)} \subseteq \operatorname{pos}\{\boldsymbol{y}_t \in \mathbf{C}_>(J_t)\}$ holds. Since the cone $\overline{\mathbf{C}_>(J_s)}$ is n-dimensional, the relation $|\mathcal{N}(J_s)| \geq n$ holds. □

Lemma 2.26. *If $J_s \in \mathbf{J}$ and $J_t \in \mathbf{J}$ are the multi-indices of two distinct maximal feasible subsystems of system* (2.26) *then $J_s \cap J_t \neq \emptyset$.*

Proof. Suppose to the contrary that $J_s \cap J_t = \emptyset$. Let $\mathcal{F}(L, J_t)$ be an $(n-1)$-dimensional face of the cone $\overline{\mathbf{C}_>(J_t)}$. Since the set $\boldsymbol{A}(\mathfrak{S})$ of vectors that determine the system $\mathfrak{S}$ by convention contains no pairs of antipodes, we have $\mathbf{C}_>([m]-J_t) \cap \mathbf{C}_>(L) \neq \emptyset$, according to Lemma 2.23 (ii), that is, the subsystem of the system $\mathfrak{S}$, with the multi-index $([m]-J_t) \cup L$, is feasible; since $L \neq \emptyset$ and $L \cap J_s \neq \emptyset$, this contradicts the maximality of J_s. □

Lemma 2.27. *Let a partition $\mathbf{J} = \mathbf{J}' \dot\cup \mathbf{J}''$, $\#\mathbf{J}' > 0$, $\#\mathbf{J}'' > 0$, of the family of the multi-indices of MFSs of system* (2.26) *be given, and let $J' \in \mathbf{J}'$ and $J'' \in \mathbf{J}''$ be the multi-indices of maximal feasible subsystems such that $|J' \cap J''| = \max_{L \in \mathbf{J}',\, M \in \mathbf{J}''} |L \cap M|$. Then the subsystem of the system $\mathfrak{S}$, with the multi-index $([m]-J') \cup ([m]-J'')$, is feasible.*

Proof. By Lemma 2.26, $J' \cap J'' \neq \emptyset$. For the multi-index $J := J' \cap J''$ and for any indices $j' \in J' - J$ and $j'' \in J'' - J$, we have

$$\mathbf{C}_>\big(J \cup \{j'\} \cup \{j''\}\big) = \emptyset\,. \tag{2.31}$$

Indeed, suppose to the contrary that $\mathbf{C}_>\big(J \cup \{j'\} \cup \{j''\}\big) \neq \emptyset$ for some indices $j' \in J' - J$ and $j'' \in J'' - J$. Then there exists a maximal feasible subsystem with a multi-index $J^* \supseteq J \cup \{j'\} \cup \{j''\}$. Let $J^* \in \mathbf{J}'$. Then $J'' \in \mathbf{J}''$ and $|J^* \cap J''| \geq |J \cup \{j''\}| = |J| + 1$, a contradiction with the maximal cardinality $|J|$ of the multi-index J. The case $J^* \in \mathbf{J}''$, analogously, leads to a contradiction. These contradictions verify (2.31).

Fix two vectors $\boldsymbol{x} \in \mathbf{C}_>(J')$ and $\boldsymbol{y} \in \mathbf{C}_>(J'')$. Let us define the sets

$$\boldsymbol{C}' := \bigcup_{j' \in J'-J} \mathbf{C}_>(J \cup \{j'\}) \quad \text{and} \quad \boldsymbol{C}'' := \bigcup_{j'' \in J''-J} \mathbf{C}_>(J \cup \{j''\})\,.$$

Note that the inclusions

$$\mathbf{C}_>(J') \subset \boldsymbol{C}' \subset \mathbf{C}_>(J)\,, \quad \mathbf{C}_>(J'') \subset \boldsymbol{C}'' \subset \mathbf{C}_>(J)\,, \quad [\boldsymbol{x}, \boldsymbol{y}] \subset \mathbf{C}_>(J)\,. \tag{2.32}$$

hold.

We also have $[\boldsymbol{x}, \boldsymbol{y}] \cap \overline{\boldsymbol{C}'} = [\boldsymbol{x}, \boldsymbol{z}_1]$ and $[\boldsymbol{x}, \boldsymbol{y}] \cap \overline{\boldsymbol{C}''} = [\boldsymbol{z}_2, \boldsymbol{y}]$, for some vectors $\boldsymbol{z}_1, \boldsymbol{z}_2 \in [\boldsymbol{x}, \boldsymbol{y}]$.

Suppose that $[\boldsymbol{x}, \boldsymbol{z}_1] \cap [\boldsymbol{z}_2, \boldsymbol{y}] \neq \emptyset$, and fix a vector $\boldsymbol{z}^* \in [\boldsymbol{x}, \boldsymbol{z}_1] \cap [\boldsymbol{z}_2, \boldsymbol{y}] \subset \overline{\boldsymbol{C}'} \cup \overline{\boldsymbol{C}''}$. It follows from the definition of the sets $\overline{\boldsymbol{C}'}$ and $\overline{\boldsymbol{C}''}$ that $\boldsymbol{z}^* \in \overline{\mathbf{C}_>(J \cup \{j'\})} \cap \overline{\mathbf{C}_>(J \cup \{j''\})} \subseteq \overline{\mathbf{C}_>(j')} \cap \overline{\mathbf{C}_>(j'')}$ for some indices $j' \in J' - J$ and $j'' \in J'' - J$. Since the set $\boldsymbol{A}(\mathfrak{S})$ of vectors determining system (2.26) by convention contains no pairs of antipodes, we have $\mathbf{C}_>(j') \cap \mathbf{C}_>(j'') \neq \emptyset$. Let us pick a vector $\boldsymbol{v} \in \mathbf{C}_>(j') \cap \mathbf{C}_>(j'')$. Since $[\boldsymbol{x}, \boldsymbol{y}] \in \mathbf{C}_>(J)$, the inclusion $\boldsymbol{z}^* \in \mathbf{C}_>(J)$ holds; but then $\boldsymbol{z}^* + \varepsilon\boldsymbol{v} \in \mathbf{C}_>(J') \cap \mathbf{C}_>(j') \cap \mathbf{C}_>(j'')$ for a sufficiently small $\varepsilon > 0$, a contradiction with (2.31); as a consequence, $[\boldsymbol{x}, \boldsymbol{z}_1] \cap [\boldsymbol{z}_2, \boldsymbol{y}] = \emptyset$ and thus $(\boldsymbol{z}_1, \boldsymbol{z}_2) \neq \emptyset$ and $(\boldsymbol{z}_1, \boldsymbol{z}_2) \cap (\overline{\boldsymbol{C}'} \cup \overline{\boldsymbol{C}''}) = \emptyset$. Consider a vector $\boldsymbol{w} \in (\boldsymbol{z}_1, \boldsymbol{z}_2)$; for each index $j' \in J' - J$ we have $\boldsymbol{w} \in -\mathbf{C}_>(j')$, because otherwise $\boldsymbol{w} \in \overline{\mathbf{C}_>(j')} \cap \overline{\mathbf{C}_>(J)} = \overline{\mathbf{C}_>(\{j'\} \cup J)} \subseteq \overline{\boldsymbol{C}'}$. Analogously, for each index $j'' \in J'' - J$ the inclusion $\boldsymbol{w} \in -\mathbf{C}_>(j'')$ holds. Thus,

$$\boldsymbol{w} \in -\mathbf{C}_>\big((J' - J) \cup (J'' - J)\big) . \tag{2.33}$$

According to Lemma 2.22, $\boldsymbol{x} \in -\mathbf{C}_>([m] - J')$ and $\boldsymbol{y} \in -\mathbf{C}_>([m] - J'')$. If $([m] - J') \cap ([m] - J'') \neq \emptyset$ then we have

$$\boldsymbol{w} \in [\boldsymbol{x}, \boldsymbol{y}] \subset -\mathbf{C}_>\big(([m] - J') \cap ([m] - J'')\big) . \tag{2.34}$$

Besides, the equality

$$\big(([m] - J') \cap ([m] - J'')\big) \cup (J' - J) \cup (J'' - J) = ([m] - J') \cup ([m] - J'') \tag{2.35}$$

holds. It follows from (2.33)–(2.35) that $\boldsymbol{w} \in \mathbf{C}_>\big(([m] - J') \cup ([m] - J'')\big) \neq \emptyset$, and the lemma is proved. □

Theorem 2.28. *The graph* $\mathtt{MFSG}(\mathfrak{S})$ *of system* (2.26) *contains at least one cycle of odd length.*

Proof. Suppose to the contrary that the graph $\mathtt{MFSG}(\mathfrak{S})$ contains no cycles of odd length. In this case $\mathtt{MFSG}(\mathfrak{S})$ is bipartite, that is, there exists a partition $\mathbf{J} = \mathbf{J}' \dot{\cup} \mathbf{J}''$, $\#\mathbf{J}' > 0$, $\#\mathbf{J}'' > 0$, of its vertex set such that the vertices from the parts $\mathbf{J}'$ and $\mathbf{J}''$ are not linked by an edge. According to Lemma 2.27, there exist the multi-indices of MFSs $J' \in \mathbf{J}'$ and $J'' \in \mathbf{J}''$ such that the subsystem with the multi-index $([m] - J') \cup ([m] - J'')$ is feasible. As a consequence, there exists a subsystem of the system $\mathfrak{S}$, with a multi-index J, such that $J \supseteq ([m] - J') \cup ([m] - J'')$ and thus the pairs $\{J, J'\}$ and $\{J, J''\}$ are edges of the graph $\mathtt{MFSG}(\mathfrak{S})$. Since $J \in \mathbf{J}' \dot{\cup} \mathbf{J}'' = [m]$, either the edge $\{J, J'\}$ or the edge $\{J, J''\}$ contradicts the assumption that the graph $\mathtt{MFSG}(\mathfrak{S})$ is bipartite. □

For the construction of algorithms of extracting maximal feasible subsystems of system (2.26) it is important to know a characterization of the graphs of MFSs of such systems. The answer to this question is well known in the case of the rank 2 systems $\mathfrak{S}$. Before recalling it, we will describe some additional properties of the systems $\mathfrak{S}$.

Let $L \subset [m]$ be the multi-index of a feasible subsystem of system (2.26), that is, $\mathbf{C}_>(L) \neq \emptyset$. The inequality with an index $l \in L$ is called an *inequality implied by the subsystem* with the multi-index $L - \{l\}$ when $\mathbf{C}_>(L - \{l\}) = \mathbf{C}_>(L)$. Let us denote by

$\mathbf{mmi}(L) \subseteq L$ the inclusion-**m**inimal **m**ulti-**i**ndex of the subsystem of the system with the multi-index L such that $\mathbf{C}_>(\mathbf{mmi}(L)) = \mathbf{C}_>(L)$; in other words, the set $\mathbf{mmi}(L)$ is by definition composed of those indices $i \in [m]$ for which the half-spaces $\mathbf{H}(i)$ represent the linear hulls of the $(n-1)$-dimensional faces of the cone $\overline{\mathbf{C}_>(L)}$.

Remark 2.29. *If $L \subset [m]$ is the multi-index of a feasible subsystem of system* (2.26) *then*
(i) $i \in L - \mathbf{mmi}(L) \Longleftrightarrow \mathbf{C}_>(L) = \mathbf{C}_>(L - \{i\})$;
(ii) $i \in \mathbf{mmi}(L) \Longleftrightarrow \mathcal{F}(\{i\}, L) \neq \emptyset$, $\dim \operatorname{lin}(\mathcal{F}(\{i\}, L)) = n - 1$.

Lemma 2.30. *If J_s and J_t are the multi-indices of two distinct MFSs of system* (2.26) *then*

$$\mathbf{mmi}(J_s) \cap J_t \neq \emptyset\,, \tag{2.36}$$

$$\mathbf{mmi}(J_s) \cap ([m] - J_t) \neq \emptyset\,. \tag{2.37}$$

Proof. Let us prove (2.36): suppose to the contrary that $\mathbf{mmi}(J_s) \cap J_t = \emptyset$. Then $\mathbf{mmi}(J_s) \subseteq [m] - J_t$ and $\mathbf{C}_>(J_s) = \mathbf{C}_>([m] - J_t) \supset -\mathbf{C}_>(J_t) \neq \emptyset$. If $-\mathbf{C}_>(J_t) \cap \mathbf{C}_>(J_s) \neq \emptyset$ then $J_s \cap J_t = \emptyset$, a contradiction with Lemma 2.26.

Relation (2.37) is proved similarly. □

Lemma 2.31. *If J_s and J_t are the multi-indices of MFSs of system* (2.26) *such that $J_s \cup J_t = [m]$, that is, $\{J_s, J_t\}$ is an edge of the graph $\texttt{MFSG}(\mathfrak{S})$, then $\emptyset \neq \mathbf{mmi}(J_s) - J_t \subseteq \mathbf{mmi}([m] - J_t)$.*

Proof. According to Lemma 2.30, $\mathbf{mmi}(J_s) - J_t \neq \emptyset$. Let us consider an arbitrary index $j^* \in \mathbf{mmi}(J_s) - J_t \subseteq J_s - J_t$. Let us show that $j^* \in \mathbf{mmi}([m] - J_t)$. Suppose to the contrary that $j^* \in ([m] - J_t) - \mathbf{mmi}([m] - J_t)$. According to Remark 2.29 (i), $\mathbf{C}_>(([m] - J_t) - \{j^*\}) = \mathbf{C}_>([m] - J_t)$. Since, by definition, $J_s \cup J_t = [m]$, we have $J_s \supseteq [m] - J_t$. Let us represent the multi-index J_s in the form $J_s = (J_s - ([m] - J_t)) \cup ([m]-J_t)$. Then $\mathbf{C}_>(J_s) = \mathbf{C}_>(J_s - ([m] - J_t)) \cap \mathbf{C}_>([m]-J_t)$ and, because of $\mathbf{C}_>([m]-J_t) = \mathbf{C}_>(([m] - J_t) - \{j^*\})$ and $j^* \in [m] - J_t$, we have $\mathbf{C}_>(J_s) = \mathbf{C}_>(J_s - \{j^*\})$, from where, according to Remark 2.29 (i), we obtain $j^* \in J_s - \mathbf{mmi}(J_s)$, a contradiction with the choice of the index j^*. □

Lemma 2.32. *If $\mathfrak{S}$ is a rank two system* (2.26) *over $\mathbb{R}^2$, then the degree of each vertex of its graph $\texttt{MFSG}(\mathfrak{S})$ is 2.*

Proof. Let J_s be an arbitrary vertex of the graph $\texttt{MFSG}(\mathfrak{S})$. Without loss of generality we will suppose that in the graph $\texttt{MFSG}(\mathfrak{S})$ this vertex is adjacent to the vertices $J_1, J_2, \ldots, J_p$. Since, by Theorem 2.25 (i), we have $p \geq 2$, it suffices to show that $p \leq 2$. Since $J_i \supseteq [m] - J_s$ for each index i, $1 \leq i \leq p$, the inclusions

$$\mathbf{C}_>(J_i) \subseteq \mathbf{C}_>([m] - J_s)\,, \quad 1 \leq i \leq p\,,$$

hold. Further, according to Lemma 2.31, $\emptyset \neq \mathbf{mmi}(J_i) - J_s \subseteq \mathbf{mmi}([m] - J_s)$, for each index i, $1 \leq i \leq p$. For each such an index, let us pick an index $t_i \in \mathbf{mmi}(J_i) \cap ([m] - \mathbf{mmi}(J_t)) \neq \emptyset$. Then the inclusions

$$\emptyset \neq \mathcal{F}(\{t_i\}, J_i) \subseteq \mathcal{F}(\{t_i\}, [m] - J_s)\,, \quad 1 \leq i \leq p\,, \tag{2.38}$$

hold. Let us show that the implications

$$i_1 \neq i_2 \quad \Longrightarrow \quad \mathcal{F}(\{t_{i_1}\}, J_{i_1}) \cap \mathcal{F}(\{t_{i_2}\}, J_{i_2}) \neq \emptyset\,, \quad 1 \leq i_1, i_2 \leq p\,, \tag{2.39}$$

are true. Suppose to the contrary that there exist distinct indices i_1 and i_2, $1 \leq i_1, i_2 \leq \#\mathbf{J}$, such that some vector $\boldsymbol{x}^* \in \mathcal{F}(\{t_{i_1}\}, J_{i_1}) \cap \mathcal{F}(\{t_{i_2}\}, J_{i_2})$ can be chosen, $\boldsymbol{x}^* \neq \mathbf{0}$, that is, $\boldsymbol{x}^* \in \mathbf{H}(t_{i_1}) \cap \mathbf{C}_>(J_{i_1} - \{t_{i_1}\}) \cap \mathbf{H}(t_{i_2}) \cap \mathbf{C}_>(J_{i_2} - \{t_{i_2}\})$. Since $n = 2$, then $\mathbf{H}(i_1) = \mathbf{H}(i_2)$ for $i_1 \neq i_2$; this is impossible because each subsystem of the system $\mathfrak{S}$, with two inequalities, is by definition of rank 2.

Since $\emptyset \neq \mathbf{C}_>([m] - J_s) \subset \mathbb{R}^2$, the boundary of the cone $\overline{\mathbf{C}_>([m] - J_s)}$ can be represented as a union of two rays $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ radiating from the point $\mathbf{0}$; thus, $\mathcal{F}(\{j\}, [m] - J_s) \subseteq \boldsymbol{l}_1 \cup \boldsymbol{l}_2$, for each index $j \in \mathbf{mmi}([m] - J_s)$. For each index i, $1 \leq i \leq p$, the nonempty face $\mathcal{F}(\{t_i\}, J_i)$ represents a ray radiating from the point $\mathbf{0}$ and not containing it. It follows from (2.38) that for each number i, $1 \leq i \leq p$, one of the two equalities $\mathcal{F}(\{t_i\}, J_i) = \boldsymbol{l}_1 - \{\mathbf{0}\}$ and $\mathcal{F}(\{t_i\}, J_i) = \boldsymbol{l}_2 - \{\mathbf{0}\}$ only holds, from where it follows, taking into account (2.39), that $p \leq 2$. □

Proposition 2.33. *Some graph is isomorphic to the graph* $\mathtt{MFSG}(\mathfrak{S})$ *of a rank two system* $\mathfrak{S}$ *over* $\mathbb{R}^2$ *if and only if this graph represents a simple cycle of odd length* q, $3 \leq q \leq m$.

Proof. The *necessity*. Let an arbitrary system (2.26), with m inequalities, over $\mathbb{R}^2$ be given. It is well known that the system $\mathfrak{S}$ contains an odd number, not exceeding m, of MFSs. Since, by Theorem 2.20, the graph $\mathtt{MFSG}(\mathfrak{S})$ is connected and, by Lemma 2.32, the degree of each of its vertex equals 2, the graph $\mathtt{MFSG}(\mathfrak{S})$ represents a simple cycle of odd length $\#\mathbf{J}$.

The *sufficiency* is verified by explicit constructing, for an arbitrary odd q, $3 \leq q \leq m$, a rank 2 system (2.26), with m inequalities, over $\mathbb{R}^2$. Let each vector $\boldsymbol{a}'_i \in \mathbb{R}^2$, $1 \leq i \leq q$, be obtained by rotating the vector $(0, 1) \in \mathbb{R}^2$ through a counterclockwise angle of $2\pi i/q$. The infeasible system $\mathfrak{S}' := \{\langle \boldsymbol{a}'_i, \mathbf{x}\rangle > 0 \colon\ i \in [q]\}$, composed of q inequalities, has q maximal feasible subsystems with the multi-indices $J_1 := \{1, 2, \ldots, \lfloor q/2\rfloor + 1\}$, $J_2 := \{2, 3, \ldots, \lfloor q/2\rfloor + 2\}$, …, $J_q := \{q, 1, \ldots, \lfloor q/2\rfloor\}$. The corresponding multi-indices $\mathbf{mmi}(J_i)$ describing the one-dimensional faces of the closures of the solution cones to the maximal feasible subsystems are the two-element sets $\mathbf{mmi}(J_1) = \{1, \lfloor q/2\rfloor + 1\}$, $\mathbf{mmi}(J_2) = \{2, \lfloor q/2\rfloor + 2\}$, …, $\mathbf{mmi}(J_q) = \{q, \lfloor q/2\rfloor\}$; note that the index of each inequality of the system $\mathfrak{S}'$ occurs in the multi-indices $\mathbf{mmi}(J_i)$ twice. The graph $\mathtt{MFSG}(\mathfrak{S}')$ is a simple cycle of length q. Let us choose the value of angular deviation ϵ sufficiently small for the augmented inequality system $\mathfrak{S} := \{\mathfrak{S}', \{\langle \boldsymbol{a}_k, \mathbf{x}\rangle > 0 \colon\ k \in [m - q]\}\}$ – in which the determining vector $\boldsymbol{a}_k$ is obtained from the vector $\boldsymbol{a}'_1$ by rotating through an angle of $k\epsilon$ – to differ from the initial system $\mathfrak{S}'$ by a deformation of the solution cone to a unique maximal feasible subsystem. By construction, the resulting system $\mathfrak{S}$, with m inequalities and with q maximal feasible subsystems, has the graph $\mathtt{MFSG}(\mathfrak{S})$ isomorphic to the graph $\mathtt{MFSG}(\mathfrak{S}')$ and representing a simple cycle of odd length q. □

Recall that a shortest simple path connecting two vertices of a graph is called a *geodesic*, and the *diameter* of the graph is defined as the length of its largest geodesic.

According to Proposition 2.33, the graph of MFSs of a rank 2 system (2.26) has the following properties: any its edge belongs to a simple cycle of length not exceeding m; the graph contains a simple cycle of odd length not exceeding m; the diameter of the graph does not exceed $\lfloor \frac{m}{2} \rfloor$.

It turns out that the graphs of MFSs of the systems $\mathfrak{S}$, of any rank n, have analogous properties. Before verifying this claim, let us turn to the half-spaces defined by the inequalities of system (2.26), and prove an auxiliary statement.

Let us consider the family

$$\{(\mathbf{C}_{>}(i), \mathbf{C}_{<}(i)) : \ i \in [m]\} \tag{2.40}$$

of the ordered pairs of open half-spaces associated with system (2.26), and consider the abstract simplicial complex $([m], \Delta^{>})$, on the vertex set $[m]$, defined for the nonempty subfamilies $F \subset [m]$ as follows:

$$F \in \Delta^{>} \quad \Longleftrightarrow \quad \bigcap_{f \in F} (\mathbf{C}_{>}(f), \mathbf{C}_{<}(f)) \neq (\emptyset, \emptyset) .$$

If $\boldsymbol{S}$ is a subspace of the space $\mathbb{R}^n$ then the complex $([m], \Delta^{>}_{\boldsymbol{S}})$ is defined in a similar way:

$$F \in \Delta^{>}_{\boldsymbol{S}} \quad \Longleftrightarrow \quad \bigcap_{f \in F} (\mathbf{C}_{>}(f), \mathbf{C}_{<}(f)) \cap (\boldsymbol{S}, \boldsymbol{S}) \neq (\emptyset, \emptyset) ; \tag{2.41}$$

in particular, $([m], \Delta^{>}) := ([m], \Delta^{>}_{\mathbb{R}^n})$.

Lemma 2.34. *Suppose that each subsystem of cardinality k, where $3 \leq k \leq n$, of system* (2.26) *of rank $n \geq 3$ has rank k.*

Consider an arbitrary family $\{J_{s_1}, J_{s_2}, \ldots, J_{s_r}\} \subset \mathbf{J}$ of the multi-indices of its MFSs, where $1 \leq r \leq \min\{k, n-1\}$.

There exists an $(n-1)$-dimensional subspace $\boldsymbol{R} \subset \mathbb{R}^n$ satisfying the condition

$$L \subset [m], \ |L| = r \quad \Longrightarrow \quad \dim(\boldsymbol{R} \cap \mathbf{H}(L)) = n - r - 1 .$$

In the graph of the independence system $\mathtt{ISG}([m], \Delta^{>})$ there are vertices $J^*_{t_1}$, $J^*_{t_2}, \ldots, J^*_{t_r}$ such that $J^*_{t_i} = J_{s_i}$, $1 \leq i \leq r$.

There exists a homomorphism $\psi \colon \mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^{>})$ of the graphs of independence systems such that for each facet $J^* \in \mathbf{max}\,\Delta^{>}_{\boldsymbol{R}}$ the inclusion $\psi(J^*) \supseteq J^*$ holds, and for each i, $1 \leq i \leq r$, the relations $\psi(J^*_{t_i}) = J_{s_i}$ and $\psi^{-1}(J_{s_i}) = \{J^*_{t_i}\}$ hold.

Proof. Let us define, for any vector $\boldsymbol{x} \in \mathbb{R}^n$, a continuous map $\varphi_{\boldsymbol{x}} \colon \mathbb{R}^n \to \mathbb{R}^n$, $\boldsymbol{y} \mapsto \langle \boldsymbol{y}, \boldsymbol{y} \rangle \boldsymbol{x} - \langle \boldsymbol{x}, \boldsymbol{y} \rangle \boldsymbol{y}$. Let us pick unique representatives $\boldsymbol{x}_i \in \mathbf{C}_{>}(J_{s_i})$, $1 \leq i \leq r$. Since $r \leq n-1$, there exists a vector $\boldsymbol{z}^*$ such that $\|\boldsymbol{z}^*\| = 1$ and $\langle \boldsymbol{x}_i, \boldsymbol{z}^* \rangle = 0$, $1 \leq i \leq r$. Then $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}^*) = \boldsymbol{x}_i$, $1 \leq i \leq r$, and thus $\boldsymbol{z}^* \in \bigcap_{1 \leq i \leq r} \varphi^{-1}_{\boldsymbol{x}_i}(\boldsymbol{x}_i) \subset \boldsymbol{V} := \bigcap_{1 \leq i \leq r} \varphi^{-1}_{\boldsymbol{x}_i}(\mathbf{C}_{>}(J_{s_i})) \neq \emptyset$.

The set $\boldsymbol{V}$ is open because of the continuity of the maps $\varphi_{\boldsymbol{x}_i}$ and the openness of the cones $\mathbf{C}_>(J_{s_i})$, $1 \le i \le r$. Family (2.40) satisfies the condition

$$L \subset [m],\ |L| = r \quad \Longrightarrow \quad \dim \mathbf{H}(L) = n - r\,. \tag{2.42}$$

For an arbitrary multi-index $L \subseteq [m]$, let us denote by $(\mathbf{H}(L))^{\perp}$ the orthogonal complement of the subspace $\mathbf{H}(L)$ up to $\mathbb{R}^n$; thus, $\mathbb{R}^n = \mathbf{H}(L)\oplus(\mathbf{H}(L))^{\perp}$. Since $2 \le r \le n-1$, the set $\boldsymbol{U} := \bigcup_{|L|=r}(\mathbf{H}(L))^{\perp}$ is nowhere dense in $\mathbb{R}^n$; as a consequence, $\boldsymbol{V} \cap (\mathbb{R}^n - \boldsymbol{U}) \neq \emptyset$. Let us pick a vector $\boldsymbol{z} \in \boldsymbol{V} \cap (\mathbb{R}^n - \boldsymbol{U})$, $\boldsymbol{z} \neq \mathbf{0}$. Suppose $\boldsymbol{R} := \{\boldsymbol{x} \in \mathbb{R}^n\colon\ \langle \boldsymbol{z}, \boldsymbol{x}\rangle = 0\}$. Since for any multi-index $L \subseteq [m]$, $|L| = r$, we have $\boldsymbol{z} \notin \boldsymbol{U}$, that is, $\boldsymbol{z} \notin (\mathbf{H}(L))^{\perp}$, then $\boldsymbol{R} \not\supseteq \mathbf{H}(L)$, $L \subseteq [m]$, $|L| = r$, from where, taking into account (2.42), we obtain

$$L \subseteq [m],\ |L| = r \quad \Longrightarrow \quad \dim(\boldsymbol{R} \cap \mathbf{H}(L)) = (n-1) - r\,. \tag{2.43}$$

Consider the family

$$\{(\mathbf{C}_>(i), \mathbf{C}_<(i)) \cap (\boldsymbol{R}, \boldsymbol{R})\colon\ i \in [m]\} \tag{2.44}$$

over the $(n-1)$-dimensional subspace $\boldsymbol{R}$, and the corresponding abstract simplicial complex $\Delta^>_{\boldsymbol{R}}$ defined in (2.41). Since $\boldsymbol{z} \in \boldsymbol{V} := \bigcap_{1\le i\le r} \varphi^{-1}_{\boldsymbol{x}_i}(\mathbf{C}_>(J_{s_i}))$, for each i, $1 \le i \le r$, the inclusions $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}) \in \mathbf{C}_>(J_{s_i})$ hold. Besides, $\langle \varphi_{\boldsymbol{x}_i}(\boldsymbol{z}), \boldsymbol{z}\rangle = 0$, $1 \le i \le r$, that is, $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}) \in \boldsymbol{R}$, $1 \le i \le r$. Thus, $\varphi_{\boldsymbol{x}_i}(\boldsymbol{z}) \in \boldsymbol{R} \cap \mathbf{C}_>(J_{s_i})$, $1 \le i \le r$, and comparing families (2.40) and (2.44) verifies that the subset $J_{s_i} \subset [m]$ is a facet of the complex $\Delta^>_{\boldsymbol{R}}$, that is, in the graph $\mathtt{ISG}([m], \Delta^>_{\boldsymbol{R}})$ there are vertices $J^*_{t_1}, J^*_{t_2}, \ldots, J^*_{t_r}$ such that $J_{s_i} = J^*_{t_i}$, $1 \le i \le r$. For any subsets $F \subset [m]$, the implications $\bigcap_{f\in F}(\mathbf{C}_>(f), \mathbf{C}_<(f)) \cap (\boldsymbol{R}, \boldsymbol{R}) \neq (\emptyset, \emptyset) \Rightarrow \bigcap_{f\in F}(\mathbf{C}_>(f), \mathbf{C}_<(f)) \neq (\emptyset, \emptyset)$ are true and thus, according to Proposition 2.1, there exists a homomorphism $\psi\colon \mathtt{ISG}([m], \Delta^>_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^>)$ such that $\psi(J^*_i) \supseteq J_i$, for each facet $J^*_i \in \mathbf{max}\,\Delta^>_{\boldsymbol{R}}$. The maximality of the sets J_{s_i} and $J^*_{t_i}$, and the equalities $J_{s_i} = J^*_{t_i}$ for all i, $1 \le i \le r$, imply that $\psi(J^*_{t_i}) = J_{s_i}$ and $\psi^{-1}(J_{s_i}) = \{J^*_{t_i}\}$, $1 \le i \le r$. ☐

Proposition 2.35. (i) *Any edge of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *of system* (2.26) *belongs to a simple cycle* (*and, as a consequence, the graph* $\mathtt{MFSG}(\mathfrak{S})$ *has no bridges*) *of length not exceeding* m*;*
(ii) *the graph* $\mathtt{MFSG}(\mathfrak{S})$ *contains a simple cycle of odd length not exceeding* m*;*
(iii) *the diameter of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *does not exceed* $\lfloor \frac{m}{2} \rfloor$.

Proof. (i) The proof is by induction on the rank n of system (2.26). The statement is true for $n = 2$ because, according to Proposition 2.33, the graph $\mathtt{MFSG}(\mathfrak{S})$ represents a simple cycle of odd length not exceeding m.

Let us assume that the statement is true for an arbitrary system (2.26) of rank $n-1$ over $\mathbb{R}^{n-1}$, where $n \ge 3$. Let us consider family (2.40) of the ordered pairs of subspaces associated with the rank n system $\mathfrak{S}$ over $\mathbb{R}^n$; this family satisfies condition (2.42). Let us consider an arbitrary edge of the graph $\mathtt{MFSG}(\mathfrak{S})$; without loss of generality we will suppose that this edge is the pair $\{J_1, J_2\}$. Let us set $r := 2 \le \min\{2, n-1\}$ and use Lemma 2.34 for the r multi-indices J_1 and J_2 of MFSs of the system $\mathfrak{S}$ with the corresponding family (2.40). Let $\boldsymbol{R}$ be an $(n-1)$-dimensional subspace of the space $\mathbb{R}^n$; it is assigned to family (2.44) of subspace pairs and the graph of the independence system $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$. According to Lemma 2.34, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exist vertices $J^*_{t_1}$ and $J^*_{t_2}$ such that $J^*_{t_1} = J_1$ and $J^*_{t_2} = J_2$. Since $J^*_{t_1} \cup J^*_{t_2} = J_1 \cup J_2 = [m]$, the pair $\{J^*_{t_1}, J^*_{t_2}\}$ is an edge of the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$. Then by the induction hypothesis, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exists a simple cycle $(J^*_{t_1}, J^*_{t_2}, \dots, J^*_{t_p}, J^*_{t_1})$, $3 \le p \le m$, containing the edge $\{J^*_{t_1}, J^*_{t_2}\}$. Let $\psi\colon \mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^{>})$ be the homomorphism mentioned in Lemma 2.34. Let us consider the sequence $(\psi(J^*_{t_2}), \psi(J^*_{t_3}), \dots, \psi(J^*_{t_p}), \psi(J^*_{t_1}))$ of the vertices of the graph $\mathtt{ISG}([m], \Delta^{>})$, isomorphic to the graph $\mathtt{MFSG}(\mathfrak{S})$, which is a $(\psi(J^*_{t_2}) \leftrightarrow \psi(J^*_{t_1}))$-path because of the conservation of vertex adjacency under the homomorphism $\psi\colon \mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^{>})$. Since $\psi^{-1}(\psi(J^*_{t_1})) = \{J^*_{t_1}\}$, $\psi^{-1}(\psi(J^*_{t_2})) = \{J^*_{t_2}\}$, and $J^*_{t_1}, J^*_{t_2} \notin \{J^*_{t_3}, \dots, J^*_{t_p}\}$, then $\psi(J^*_{t_1}), \psi(J^*_{t_2}) \notin \{\psi(J^*_{t_3}), \dots, \psi(J^*_{t_p})\}$.

Let us consider the two possible cases: (1) $\psi(J^*_{t_3}) = \psi(J^*_{t_p})$ and (2) $\psi(J^*_{t_3}) \neq \psi(J^*_{t_p})$.

In the case 1, we have the simple cycle $(\psi(J^*_{t_1}), \psi(J^*_{t_2}), \psi(J^*_{t_3}), \psi(J^*_{t_1})) = (J_1, J_2, \psi(J^*_{t_3}), J_1)$, of length three, containing the edge $\{J_1, J_2\}$.

In the case 2, one can distinguish in the path $(\psi(J^*_{t_3}), \dots, \psi(J^*_{t_p}))$ a simple chain $(\psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p}))$, where $(\psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p})) \subseteq \{\psi(J^*_{t_3}), \psi(J^*_{t_4}), \dots, \psi(J^*_{t_p})\}$, and thus $\psi(J^*_{t_1}), \psi(J^*_{t_2}) \notin (\psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p}))$. Since $k + 2 \le p - 2$, the sequence $(\psi(J^*_{t_1}), \psi(J^*_{t_2}), \psi(J^*_{t_3}), \psi(J^*_{q_1}), \dots, \psi(J^*_{q_k}), \psi(J^*_{t_p}), \psi(J^*_{t_1}))$ represents a simple cycle of length $k + 4 \le p \le m$, and it contains the edge $\{\psi(J^*_{t_1}), \psi(J^*_{t_2})\} = \{J_1, J_2\}$, as was to be proved.

(ii) Recall that the existence of a cycle of odd length in the graph $\mathtt{MFSG}(\mathfrak{S})$ was proved in Theorem 2.28. We will show here, by using a similar argument, that there exists such a cycle of length not exceeding m. In the case of a rank 2 system (2.26) the graph $\mathtt{MFSG}(\mathfrak{S})$ represents such a cycle – see Proposition 2.33. Assume by induction that the statement is true for all systems of rank $n-1 \ge 2$. Let us fix in $\mathbb{R}^n$ an arbitrary $(n-1)$-dimensional subspace $\boldsymbol{R}$; then, as in the proof of statement (i), let us turn to Lemma 2.34 in the situation where $r := 2 \le \min\{2, n-1\}$. Let us choose in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ some cycle $(J^*_{t_1}, J^*_{t_2}, \dots, J^*_{t_p}, J^*_{t_1})$ of odd length p such that $3 \le p \le m$ which is contained in the graph by the induction hypothesis. We will use the denotations $\mathbf{Z}^* := \{J^*_{t_1}, J^*_{t_2}, \dots, J^*_{t_p}\}$ and $\mathbf{Z} := \{\psi(J^*_{t_1}), \psi(J^*_{t_2}), \dots, \psi(J^*_{t_p})\}$ and show that the induced subgraph $\mathtt{ISG}([m], \Delta^{>})\langle\mathbf{Z}\rangle$ of the graph $\mathtt{ISG}([m], \Delta^{>})$ contains a simple cycle of odd length not exceeding m.

First, assume that the subgraph $\mathtt{ISG}([m], \Delta^{>})\langle\mathbf{Z}\rangle$ contains no simple cycles of odd length at all, that is, it is bipartite. Then there exists a partition $\mathbf{Z} = \mathbf{Z}_1 \dot\cup \mathbf{Z}_2$, $\#\mathbf{Z}_1 > 0$, $\#\mathbf{Z}_2 > 0$, such that the induced subgraphs $\mathtt{ISG}([m], \Delta^{>})\langle\mathbf{Z}_1\rangle$ and $\mathtt{ISG}([m], \Delta^{>})\langle\mathbf{Z}_2\rangle$ are edgeless. In this case, we get the partition of the family $\mathbf{Z}^*$ into two nonempty subfamilies $\psi^{-1}(\mathbf{Z}_1) \cap \mathbf{Z}^*$ and $\psi^{-1}(\mathbf{Z}_2) \cap \mathbf{Z}^*$. Under the homomorphism ψ, the edge $\{J^*_{t_i}, J^*_{t_j}\}$ is mapped onto the edge $\{\psi(J^*_{t_i}), \psi(J^*_{t_j})\}$; indeed, the situation where $\psi(J^*_{t_i}) = \psi(J^*_{t_j})$ is impossible – that would mean that $\psi(J^*_{t_i}) \cup \psi(J^*_{t_j}) \neq [m]$, a contradiction with the condition according to which the map ψ is a homomorphism. Then the edgelessness of the subgraphs $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z}_1 \rangle$ and $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z}_2 \rangle$ implies that the subgraphs $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})\langle \psi^{-1}(\mathbf{Z}_1) \cap \mathbf{Z}^* \rangle$ and $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})\langle \psi^{-1}(\mathbf{Z}_2) \cap \mathbf{Z}^* \rangle$ are also edgeless, but this is impossible because the subgraph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})\langle \mathbf{Z}^* \rangle$ contains a simple cycle of odd length. As a consequence, the subgraph $\mathtt{ISG}([m], \Delta^{>})\langle \mathbf{Z} \rangle$, as well as the graph $\mathtt{ISG}([m], \Delta^{>})$ itself which is isomorphic to the graph $\mathtt{MFSG}(\mathfrak{S})$, contains a simple graph of odd length not exceeding m, because $|\mathbf{Z}| \le |\mathbf{Z}^*| \le p \le m$, as was to be proved.

(iii) It was mentioned in the proof of (i) and (ii) that the statement is true for any rank 2 system $\mathfrak{S}$ – see Proposition 2.33. Assume that it is true for any system $\mathfrak{S}$ of rank $n-1$, where $n-1 \ge 2$. Let us consider family (2.40) and the corresponding graph of the independence system $\mathtt{ISG}([m], \Delta^{>})$ that is isomorphic to the graph $\mathtt{MFSG}(\mathfrak{S})$. Let J_{s_1} and J_{s_2} be arbitrary vertices of the graph $\mathtt{ISG}([m], \Delta^{>})$. Let us show that they are linked by a simple chain of length not exceeding $\lfloor \frac{m}{2} \rfloor$. Taking into account that family (2.40) satisfies restriction (2.42), let us set $r := 2 \le \min\{2, n-1\}$ and use Lemma 2.34 for family (2.40) and for the distinguished vertices J_{s_1} and J_{s_2} of the graph $\mathtt{ISG}([m], \Delta^{>})$. Let $\boldsymbol{R}$ be an $(n-1)$-dimensional subspace of the space $\mathbb{R}^n$, and (2.44) the corresponding family of subspace pairs, which in its turn is assigned the graph of the independence system $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$. Let $\psi\colon \mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}}) \to \mathtt{ISG}([m], \Delta^{>})$ be the graph homomorphism mentioned in Lemma 2.34; according to this lemma, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exist vertices $J^*_{t_1}$ and $J^*_{t_2}$ such that $J^*_{t_1} = J_{s_1}$, $J^{*t_2 = J_{s_2}}$, and $\psi(J^*_{t_1}) = J_{s_1}$, $\psi(J^{*t_2}) = J_{s_2}$. By the induction hypothesis, in the graph $\mathtt{ISG}([m], \Delta^{>}_{\boldsymbol{R}})$ there exists a simple $(J^*_{t_1} \leftrightarrow J^*_{t_2})$-chain of length $p \le \lfloor \frac{m}{2} \rfloor$. Under the homomorphism ψ, this chain is mapped onto a $\big(\psi(J^*_{t_1}) \leftrightarrow \psi(J^*_{t_2})\big)$-walk in the graph $\mathtt{ISG}([m], \Delta^{>})$ containing $p \le \lfloor \frac{m}{2} \rfloor$ edges. One can distinguish in this walk a simple $\big(\psi(J^*_{t_1}) \leftrightarrow \psi(J^*_{t_2})\big)$-chain containing at most $p \le \lfloor \frac{m}{2} \rfloor$ edges and linking the vertices J_{s_1} and J_{s_2}, because $\psi(J^*_{t_1}) = J_{s_1}$ and $\psi(J^*_{t_2}) = J_{s_2}$; this completes the proof of the statement. ☐

When constructing algorithms of extracting maximal feasible subsystems of system (2.26), with the use of its graph $\mathtt{MFSG}(\mathfrak{S})$, those properties of the graph $\mathtt{MFSG}(\mathfrak{S})$ play a significant role which characterize its type of connectedness. It turns out that the graph of MFSs of the system $\mathfrak{S}$ conditionally has a connectedness type, which is stronger than just the connectedness certified by Theorem 2.20.

We will need an auxiliary statement that complements Theorem 2.25 and whose proof is given on page 47; it touches on the degrees of vertices in $\mathtt{MFSG}(\mathfrak{S})$.

Proposition 2.36. *Suppose that for some k, $1 \le k \le n-1$, each subsystem with $k+1$ inequalities of system* (2.26) *is feasible. Then the degree of any vertex J_s in the graph* $\mathtt{MFSG}(\mathfrak{S})$ *is at least $k+1$.*

Proposition 2.37. *If the rank of each subsystem with 3 inequalities of system* (2.26) *equals 3, then its graph* $\mathtt{MFSG}(\mathfrak{S})$ *is 2-connected.*

Proof. According to Proposition 2.36, the graph $\mathtt{MFSG}(\mathfrak{S})$ has at least four vertices: $\#\mathbf{J} \ge 4$; it suffices to show that $\mathtt{MFSG}(\mathfrak{S})$ does not contain a cutvertex, that is a vertex $J \in \mathbf{J}$ such that the subgraph $\mathtt{MFSG}(\mathfrak{S})\langle \mathbf{J} - \{J\} \rangle$ is disconnected.

Suppose to the contrary that the graph of MFSs of system (2.26) has a cutvertex, say the vertex J_q, that is, the graph $\mathtt{MFSG}(\mathfrak{S})\langle\mathbf{J} - \{J_q\}\rangle$ is disconnected. Without loss of generality we will suppose that the vertices J_1 and J_2 belong in the subgraph $\mathtt{MFSG}(\mathfrak{S})\langle\mathbf{J} - \{J_q\}\rangle$ to distinct connected components. According to relation (2.37) of Lemma 2.30, $\mathbf{mmi}(J_1) \cap ([m] - J_q) \neq \emptyset$ and $\mathbf{mmi}(J_2) \cap ([m] - J_q) \neq \emptyset$. Let us choose arbitrary indices $j_1 \in \mathbf{mmi}(J_1) \cap ([m] - J_q)$ and $j_2 \in \mathbf{mmi}(J_2) \cap ([m] - J_q)$. For the $(n-1)$-dimensional faces $\mathfrak{F}(\{j_1\}, J_1)$ and $\mathfrak{F}(\{j_2\}, J_2)$ of the cones $\overline{\mathbf{C}_>(J_1)}$ and $\overline{\mathbf{C}_>(J_2)}$, respectively, we have

$$\mathbf{H}(J_1) \cap \mathbf{C}_>(J_1 - \{j_1\}) \neq \emptyset \quad \text{and} \quad \mathbf{H}(J_2) \cap \mathbf{C}_>(J_2 - \{j_2\}) \neq \emptyset \,. \tag{2.45}$$

Let us consider the families

$$\{(\mathbf{C}_>(i) \cap \mathbf{H}(j_1), -\mathbf{C}_>(i) \cap \mathbf{H}(j_1)) : \ i \in [m] - \{j_1\}\} \,,$$
$$\{(\mathbf{C}_>(i) \cap \mathbf{H}(j_2), -\mathbf{C}_>(i) \cap \mathbf{H}(j_2)) : \ i \in [m] - \{j_2\}\}$$

and

$$\{(\mathbf{C}_>(i) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2), -\mathbf{C}_>(i) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2)) : \ i \in [m] - \{j_1, j_2\}\} \,; \tag{2.46}$$

they are assigned the abstract simplicial complexes $\Delta^>_{\mathbf{H}(j_1)}$, $\Delta^>_{\mathbf{H}(j_2)}$ and $\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)}$, respectively, defined in (2.41).

Since the implications

$$|L| = 3 \quad \Longrightarrow \quad \dim \mathbf{H}(L) = n - 3 \tag{2.47}$$

are by condition true for all multi-indices $L \subseteq [m]$, we conclude with the help of Theorem 2.20 that the graphs of the independence systems $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)})$ and $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_2)})$, isomorphic to the graphs of MFSs of some rank $n-1$ linear inequality systems, are connected.

Let us show that the graph of the independence system $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)})$ is not edgeless. Let J_s be some facet of the complex $\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)}$. Fix an arbitrary vector $\boldsymbol{x}^* \in \mathbf{C}_>(J_s) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2)$. Then $-\boldsymbol{x}^* \in \mathbf{C}_>\big(([m] - \{j_1, j_2\}) - J_s\big) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2)$, because otherwise, under the assumption that $-\boldsymbol{x}^* \in -\overline{\mathbf{C}_>(j_0)}$ for some index $j_0 \in ([m] - \{j_1, j_2\}) - J_s$, we would obtain $\mathbf{C}_>(J_s) \cap \mathbf{C}_>(j_0) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2) \neq \emptyset$, taking into account (2.47) – a contradiction with the maximality of the feasible subsystem with the multi-index J_s. As a consequence, the subfamily, with the multi-index $([m] - \{j_1, j_2\}) - J_s$, of family (2.46) has an intersection different from $(\emptyset, \emptyset)$, and thus there exists a facet J_t of the complex $\Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)}$ such that $J_t \supset ([m] - \{j_1, j_2\}) - J_s$. Thus, $\{J_s, J_t\}$ is an edge of the graph $\mathtt{ISG}([m], \Delta^>_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)})$.

Note that, according to Lemma 2.21 (ii), the implications

$$L \subseteq [m] - \{j_1\}\,,\ \ \mathbf{C}_{>}(L) \cap \mathbf{H}(j_1) \neq \emptyset \implies \mathbf{C}_{>}(L \cup \{j_1\}) \neq \emptyset\,,$$

$$L \subseteq [m] - \{j_2\}\,,\ \ \mathbf{C}_{>}(L) \cap \mathbf{H}(j_2) \neq \emptyset \implies \mathbf{C}_{>}(L \cup \{j_2\}) \neq \emptyset\,,$$

$$L \subseteq [m] - \{j_1, j_2\}\,,\ \ \mathbf{C}_{>}(L) \cap \mathbf{H}(j_1) \cap \mathbf{H}(j_2) \neq \emptyset \implies \mathbf{C}_{>}(L \cup \{j_1, j_2\}) \neq \emptyset$$

hold.

These implications imply that there exist maps $\psi_1 : \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)} \to \mathbf{max}\,\Delta^{>}$, $\psi_2 : \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)} \to \mathbf{max}\,\Delta^{>}$, $\psi_3 : \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} \to \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)}$ and $\psi_4 : \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} \to \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)}$ such that the relations

$$\begin{aligned} J \in \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)} &\implies \psi_1(J) \supseteq J \cup \{j_1\}\,, \\ J \in \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)} &\implies \psi_2(J) \supseteq J \cup \{j_2\}\,, \\ J \in \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} &\implies \psi_3(J) \supseteq J \cup \{j_1\}\,, \\ J \in \mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)} &\implies \psi_4(J) \supseteq J \cup \{j_2\} \end{aligned} \tag{2.48}$$

hold.

The maps ψ_1, ψ_2, ψ_3, and ψ_4 are homomorphisms of the corresponding graphs of independence systems. Since the graphs of the independence systems $\mathtt{ISG}([m], \Delta^{>}_{\mathbf{H}(j_1)})$ and $\mathtt{ISG}([m], \Delta^{>}_{\mathbf{H}(j_2)})$ are connected, the subgraphs $\mathtt{ISG}([m], \Delta^{>})\langle\psi_1(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)})\rangle$ and $\mathtt{ISG}([m], \Delta^{>})\langle\psi_2(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)})\rangle$ are also connected.

Now let $\{J_s, J_t\}$ be an edge of the graph $\mathtt{ISG}([m], \Delta^{>}_{\mathbf{H}(j_1)\cap\mathbf{H}(j_2)})$, that is, $J_s \cup J_t = [m] - \{j_1, j_2\}$; then $\psi_3(J_s) \cup \psi_4(J_t) = [m]$ and thus $\psi_1(\psi_3(J_s)) \cup \psi_2(\psi_4(J_t)) = [m]$. As a consequence, the vertex $\psi_1(\psi_3(J_s)) \in \psi_1(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)})$ is adjacent in the graph $\mathtt{ISG}([m], \Delta^{>})$ to the vertex $\psi_2(\psi_4(J_t)) \in \psi_2(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)})$, and thus the subgraph $\mathtt{ISG}([m], \Delta^{>})\langle\psi_1(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)})\rangle$ is connected. According to (2.48), any vertex from $\psi_1(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)})$ contains either the index j_1 or the index j_2, and because of $j_1, j_2 \notin J_q$, we have $J_q \notin \psi_1(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)})$, that is, the vertices from $\psi_1(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)})$ belong to the same connected component of the subgraph $\mathtt{ISG}([m], \Delta^{>})\langle\mathbf{J} - \{J_q\}\rangle$. It follows from (2.45) that there exist facets $J_{t_1} \in \mathtt{ISG}([m], \Delta^{>}_{\mathbf{H}(j_1)})$ and $J_{t_2} \in \mathtt{ISG}([m], \Delta^{>}_{\mathbf{H}(j_2)})$ such that $J_{t_1} \supseteq J_1 - \{j_1\}$ and $J_{t_2} \supseteq J_2 - \{j_2\}$. Because of the maximality of the feasible subsystems with the multi-indices J_1 and J_2, we see, taking into account (2.48), that $\psi(J_{t_1}) = J_1$ and $\psi(J_{t_2}) = J_2$; thus, $J_1, J_2 \in \psi_1(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_1)}) \cup \psi_2(\mathbf{max}\,\Delta^{>}_{\mathbf{H}(j_2)})$, a contradiction with the assumption that the vertices J_1 and J_2 in the graph $\mathtt{ISG}([m], \Delta^{>})\langle\mathbf{J} - \{J_q\}\rangle$ belong to distinct connected components. This contradiction proves the proposition. □

Proof of Proposition 2.36. As a matter of fact, we will present the proofs of several independent statements that lead to the result formulated in the proposition:

(A) *Suppose that each subsystem with* $k + 1$ *inequalities, where* $1 \leq k \leq n - 1$*, of system* (2.26) *is feasible; let* $J_s \in \mathbf{J}$ *be the multi-index of some of its MFS, and* $\mathcal{F}(L, J_s)$ *an arbitrary* $(n - k)$*-dimensional face of the cone* $\overline{\mathbf{C}_{>}(J_s)}$.

The inclusion

$$-\mathfrak{F}(L, J_s) \subset \mathbf{C}_>([m] - J_s) \tag{2.49}$$

and the relation $\mathbf{C}_>([m] - J_s) \cap \mathbf{C}_>(L) \neq \emptyset$ *hold.*

▷ *Proof.* Let us first show that inclusion (2.49) holds. Suppose to the contrary that there exists an index $j_0 \in [m] - J_s$ such that $-\mathfrak{F}(L, J_s) \not\subset \mathbf{C}_>(j_0)$. Recall that the complement, up to the index set $[m]$, of the multi-index of any maximal feasible subsystem of the system $\mathfrak{S}$ is the multi-index of a *feasible* subsystem. Hence $-\mathfrak{F}(L, J_s) \subset -\overline{\mathbf{C}_>(J_s)} \subseteq \overline{\mathbf{C}_>([m] - J_s)} \subseteq \overline{\mathbf{C}_>(j_0)}$. Thus, two cases are only possible: (1) $-\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j_0) \neq \emptyset$ and (2) $-\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j_0) = \emptyset$ and, as a consequence, $-\mathfrak{F}(L, J_s) \subset \mathbf{H}(j_0)$.

(1) Let us consider the first case, and pick two vectors $\boldsymbol{x}^* \in -\mathfrak{F}(L, J_s) \cap \mathbf{C}_>(j_0)$ and $\boldsymbol{y}^* \in -\mathfrak{F}(L, J_s) \cap \mathbf{H}_>(j_0)$. Under $\lambda > 0$ we have the inclusion $\boldsymbol{z}^* := -\lambda \boldsymbol{x}^* + (1 + \lambda)\boldsymbol{y}^* \in -\mathbf{C}_>(j_0)$. Since $\boldsymbol{x}^*, \boldsymbol{y}^* \in -\mathfrak{F}(L, J_s)$ and the set $-\mathfrak{F}(L, J_s)$ is convex and open with respect to the subspace $\mathbf{H}(L)$, then also $\boldsymbol{z}^* \in -\mathfrak{F}(L, J_s) \subseteq -\overline{\mathbf{C}_>(J_s)}$, for a sufficiently small $\lambda > 0$. Thus, $\boldsymbol{z}^* \in -\mathbf{C}_>(j_0) \cap -\mathbf{C}_>(J_s)$. As a consequence, $\mathbf{C}_>(j_0) \cap \mathbf{C}_>(J_s) \neq \emptyset$; this contradicts the maximality of the feasible subsystem with the multi-index J_s.

(2) Let us consider the second case, that is, $-\mathfrak{F}(L, J_s) \subset \mathbf{H}(j_0)$. Since the set $-\mathfrak{F}(L, J_s)$ is open with respect to the subspace $\mathbf{H}(L)$, then $\mathbf{H}(L) \subseteq \mathbf{H}(j_0)$, that is, the rank of the subsystem with the multi-index $L \cup \{j_0\}$ equals the rank of the subsystem with the multi-index L, namely k. Since in the system $\mathfrak{S}$ each subsystem with $k+1$ inequalities is by convention feasible, the rank k subsystem with the multi-index $L \cup \{j_0\}$ is also feasible, that is, $\mathbf{C}_>(L \cup \{j_0\}) \neq \emptyset$. By definition, $\mathfrak{F}(L, J_s) \subset \mathbf{C}_>(J_s - L)$; on the other hand, by applying Lemma 2.21 (i) to the multi-index $L \cup \{j_0\}$, we obtain $\mathfrak{F}(L, J_s) \subset \mathbf{H}(L) = \mathbf{H}(L \cup \{j_0\}) \subseteq \overline{\mathbf{C}_>(L \cup \{j_0\})}$. Thus, $\mathbf{C}_>(J_s - L) \cap \overline{\mathbf{C}_>(L \cup \{j_0\})} \supseteq \mathfrak{F}(L, J_s) \neq \emptyset$ or $\mathbf{C}_>(J_s \cup \{j_0\}) \neq \emptyset$; this contradicts the maximality of the feasible subsystem with the multi-index J_s.

Thus, inclusion (2.49) is proved.

Since $-\mathfrak{F}(L, J_s) \subset \mathbf{H}(L) \subseteq \overline{\mathbf{C}_>(L)}$, then $\overline{\mathbf{C}_>(L)} \cap \mathbf{C}_>([m] - J_s) \supseteq -\mathfrak{F}(L, J_s) \neq \emptyset$, and thus $\mathbf{C}_>(L) \cap \mathbf{C}_>([m] - J_s) \neq \emptyset$, and this completes the proof of statement (A). ◁

(B) *Let* $\mathbf{A} := \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p\} \subset \mathbb{R}^n$, $p \geq 2$; *pick two vectors* $\boldsymbol{x}^*, \boldsymbol{z}^* \in \operatorname{pos} \mathbf{A}$, $\boldsymbol{x}^* \neq \boldsymbol{z}^*$, $\boldsymbol{x}^* \in (\boldsymbol{y}^*, \boldsymbol{z}^*)$. *Let* $\boldsymbol{b} \in \mathbb{R}^n$ *be a vector such that* $\langle \boldsymbol{y}^*, \boldsymbol{b} \rangle > 0$ *and* $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \geq \mathbf{0}$, *for each vector* $\boldsymbol{a} \in \mathbf{A}$.
Then $\operatorname{pos} \mathbf{A} \subseteq \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$.

▷ *Proof.* Let us show that $\boldsymbol{z}^* \in \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$.

If $\boldsymbol{z}^* \in \mathbf{A}$ then $\boldsymbol{z}^* \in \mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\}$ because $\boldsymbol{z}^* \neq \boldsymbol{x}^*$.

If $\boldsymbol{x}^* \notin \mathbf{A}$ then $\operatorname{pos}(\mathbf{A} - \{\boldsymbol{x}^*\}) = \operatorname{pos} \mathbf{A}$ and $\boldsymbol{z}^* \in \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$.

Now let $\boldsymbol{z}^* \notin \mathbf{A}$ and $\boldsymbol{x}^* \in \mathbf{A}$; specifically, let us set $\boldsymbol{x}^* = \boldsymbol{a}_1$. The expression

$$\boldsymbol{z}^* = \alpha_1 \boldsymbol{x}^* + \sum_{i=2}^{p} \alpha_i \boldsymbol{a}_i\,, \quad \alpha_i \geq 0\,, \quad 1 \leq i \leq p\,, \tag{2.50}$$

is true.

If $\alpha_1 = 0$ then $\boldsymbol{z}^* \in \operatorname{pos}(\mathbf{A} - \{\boldsymbol{x}^*\}) \subseteq \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$.

Suppose $\alpha_1 > 0$. Since $\boldsymbol{x}^* = \mu \boldsymbol{z}^* + (1-\mu)\boldsymbol{y}^*$, for some μ, $0 < \mu < 1$, then (2.50) implies that

$$\boldsymbol{z}^*(1 - \alpha_1\mu) = \alpha_1(1-\mu)\boldsymbol{y}^* + \sum_{i=2}^{p} \alpha_i \boldsymbol{a}_i\,, \quad \alpha_1 > 0\,, \quad 0 < \mu < 1\,. \tag{2.51}$$

By multiplying scalarly both sides of expression (2.51) by the vector $\boldsymbol{b}$, we obtain

$$(1-\alpha_1\mu)\langle \boldsymbol{z}^*, \boldsymbol{b}\rangle = \alpha_1(1-\mu)\langle \boldsymbol{y}^*, \boldsymbol{b}\rangle + \sum_{i=2}^{p} \alpha_i \langle \boldsymbol{a}_i, \boldsymbol{b}\rangle\,. \tag{2.52}$$

The right-hand side of (2.52) is positive because $\alpha_1(1-\mu)\langle \boldsymbol{y}^*, \boldsymbol{b}\rangle > 0$, and for all i, $2 \leq i \leq p$, the relations $\langle \boldsymbol{a}_i, \boldsymbol{b}\rangle \geq 0$ are fulfilled.

Since $\boldsymbol{z}^* \in \operatorname{pos}\mathbf{A}$, we have $\langle \boldsymbol{z}^*, \boldsymbol{b}\rangle \geq 0$ and, as a consequence, $1 - \alpha_1\mu > 0$. By dividing both sides of (2.51) by $1-\alpha_1\mu$, we obtain $\boldsymbol{z}^* \in \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$; this inclusion implies that $\boldsymbol{x}^* \in (\boldsymbol{z}^*, \boldsymbol{y}^*) \subset \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$ and, as a consequence, $\operatorname{pos}\mathbf{A} \subseteq \operatorname{pos}(\mathbf{A} \cup \{\boldsymbol{y}^*\} - \{\boldsymbol{x}^*\})$. Statement (B) is proved. ◁

Recall two definitions. Let $\boldsymbol{M} \subset \mathbb{R}^n$ be a convex body. A point $\boldsymbol{x}^* \in \operatorname{bd}\boldsymbol{M}$ is said to be *lightened from the outside* by a source $\boldsymbol{y}^* \in \mathbb{R}^n$ if there exists a point $\boldsymbol{z}^* \in \operatorname{int}\boldsymbol{M}$ such that $\boldsymbol{x}^* \in (\boldsymbol{y}^*, \boldsymbol{z}^*)$. The set $\boldsymbol{B} \subset \operatorname{bd}\boldsymbol{M}$ is *lightened from the outside* by a source family $\boldsymbol{N} \subset \mathbb{R}^n$ when each point is lightened from the outside by at least one source $\boldsymbol{y}^* \in \boldsymbol{N}$.

(C) *Let us consider system* (2.26). *Let $M \subset [m]$, $M \neq \emptyset$, $\mathbf{C}_>(M) \neq \emptyset$, $L \neq \emptyset$, and let $\mathcal{F}(L, M)$ be a face of dimension r, $0 \leq r \leq n-1$, of the cone $\overline{\mathbf{C}_>(M)}$. A point $\boldsymbol{x}^* \in \mathcal{F}(L, M)$ is lightened from the outside by a source $\boldsymbol{y}^* \in \mathbb{R}^n$ if and only if $\boldsymbol{y}^* \in -\mathbf{C}_>(L)$.*

▷ *Proof*. The *necessity*. Let the point $\boldsymbol{x}^* \in \mathcal{F}(L, M) \neq \emptyset$ be lightened from the outside by the source $\boldsymbol{y}^*$, that is, there exists a point $\boldsymbol{z}^* \in \mathbf{C}_>(M) \subseteq \mathbf{C}_>(L)$ such that $\boldsymbol{x}^* \in (\boldsymbol{y}^*, \boldsymbol{z}^*)$, that is, $\boldsymbol{x}^* = \alpha \boldsymbol{z}^* + (1-\alpha)\boldsymbol{y}^*$, where $0 < \alpha < 1$. Further, $\boldsymbol{x}^* \in \mathbf{H}(L) \cap \mathbf{C}_>(M - L)$, $\boldsymbol{z}^* \in \mathbf{C}_>(M)$; as a consequence, for $\boldsymbol{y}^* = \frac{1}{1-\alpha}\boldsymbol{x}^* - \frac{\alpha}{1-\alpha}\boldsymbol{z}^*$ we have $\langle \boldsymbol{a}_i, \boldsymbol{y}^*\rangle = \frac{1}{1-\alpha}\langle \boldsymbol{a}_i, \boldsymbol{x}^*\rangle - \frac{\alpha}{1-\alpha}\langle \boldsymbol{a}_i, \boldsymbol{z}^*\rangle < 0$, for each index $i \in L$, that is, $\boldsymbol{y}^* \in -\mathbf{C}_>(L)$.

The *sufficiency*. Let $\boldsymbol{y}^* \in -\mathbf{C}_>(L)$, $\boldsymbol{x}^* \in \mathbf{H}(L) \cap \mathbf{C}_>(M - L)$. Let us consider a vector $\boldsymbol{z}^* := -\varepsilon\boldsymbol{y}^* + (1+\varepsilon)\boldsymbol{x}^*$, where $\varepsilon > 0$. Then $-\varepsilon\boldsymbol{y}^* \in \mathbf{C}_>(L)$ and $(1+\varepsilon)\boldsymbol{x}^* \in \mathbf{H}(L) \cap \mathbf{C}_>(M - L) \subseteq \overline{\mathbf{C}_>(L)}$; as a consequence, $\boldsymbol{z}^* \in \mathbf{C}_>(L)$ for any $\varepsilon > 0$, and $\boldsymbol{z}^* \in \mathbf{C}_>(M - L)$ for a sufficiently small $\varepsilon > 0$; therefore, $\boldsymbol{z}^* \in \mathbf{C}_>(M)$ for a sufficiently small $\varepsilon > 0$. Since $\boldsymbol{x}^* = \frac{1}{1+\varepsilon}\boldsymbol{z}^* + \frac{\varepsilon}{1+\varepsilon}\boldsymbol{y}^*$, where $\varepsilon > 0$, we have $\boldsymbol{x}^* \in (\boldsymbol{y}^*, \boldsymbol{z}^*)$, that is, the point $\boldsymbol{x}^*$ is lightened from the outside by the source $\boldsymbol{y}^*$. ◁

(D) *Suppose that for some k, $1 \le k \le n-1$, each subsystem with $k+1$ inequalities of system* (2.26) *is feasible; let J_s be the multi-index of some of its maximal feasible subsystem, q the degree of the vertex J_s in the graph* $\mathtt{MFSG}(\mathfrak{S})$, $\mathcal{N}(J_s) := \{J_{s_1}, \dots, J_{s_q}\}$ *the neighborhood of the vertex J_s in the graph* $\mathtt{MFSG}(\mathfrak{S})$, *and* $\boldsymbol{y}_i^* \in -\mathbf{C}_>(J_{s_i})$ *some representatives of the sets* $-\mathbf{C}_>(J_{s_i})$, $1 \le i \le q$.
Each $(n-k)$-dimensional face $\mathcal{F}(L, J_s)$ of the cone $\overline{\mathbf{C}_>(J_s)}$ is lightened from the outside by the source family $\boldsymbol{N} := \{\boldsymbol{y}_i^* :\ 1 \le i \le q\}$.

▷ *Proof*. Under the hypothesis of statement (D) for an $(n-k)$-dimensional face $\mathcal{F}(L, J_s)$ of the cone $\overline{\mathbf{C}_>(J_s)}$, we see that $\mathbf{C}_>([m] - J_s) \cap \mathbf{C}_>(L) \neq \emptyset$, in accordance with statement (A). As a consequence, there exists a MFS with a multi-index $J_p \supseteq ([m] - J_s) \cup L$, and since $J_p \cup J_s = [m]$, then $J_p \in \{J_{s_i} :\ 1 \le i \le q\}$. Specifically, let $J_p = J_{s_1}$. Then $\boldsymbol{y}_{s_1}^* \in -\mathbf{C}_>(J_{s_1}) = -\mathbf{C}_>(J_p) \subseteq -\mathbf{C}_>(L)$ and, according to statement (C), the face $\mathcal{F}(L, J_s)$ is lightened from the outside by the source $\boldsymbol{y}_{s_1}^* \in \boldsymbol{N}$; this completes the proof. ◁

(E) *Let us consider system* (2.26). *Let* $\emptyset \neq M \subset [m]$, $\mathbf{C}_>(M) \neq \emptyset$, $r := \dim \mathbf{H}(M) < n-1$, *and suppose that all $(r+1)$-dimensional faces of the cone $\overline{\mathbf{C}_>(M)}$ are lightened from the outside by a source family $\boldsymbol{N}$ and, besides, there exists a vector $\boldsymbol{b} \in \mathbb{R}^n$ such that* $\langle \boldsymbol{y}^*, \boldsymbol{b} \rangle > 0$, *for each* $\boldsymbol{y}^* \in \boldsymbol{N}$, *and* $\langle \boldsymbol{c}, \boldsymbol{b} \rangle \ge 0$, *for each* $\boldsymbol{c} \in \overline{\mathbf{C}_>(M)}$.
Then $\overline{\mathbf{C}_>(M)} \subseteq \operatorname{pos}(\mathbf{H}(M) \cup \boldsymbol{N})$.

▷ *Proof*. Let vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_r \in \mathbf{H}(M)$ represent a basis of the space $\mathbf{H}(M)$, and let us consider the vector $\boldsymbol{b}_0 := -\boldsymbol{b}_1 - \boldsymbol{b}_2 - \cdots - \boldsymbol{b}_r$. We set $\boldsymbol{B} := \{\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_r\}$. Let us choose, for each $(r+1)$-dimensional face $\mathcal{F}(L, M)$ of the cone $\overline{\mathbf{C}_>(M)}$, one representative $\boldsymbol{x}^* \in \mathcal{F}(L, M)$, and let us form the set $\{\boldsymbol{x}_j :\ 1 \le j \le l\}$ containing precisely these points; let l be the number of the $(r+1)$-dimensional faces of the cone $\overline{\mathbf{C}_>(M)}$; note that $l > 0$ because $\dim \mathbf{H}(M) = r$ and $\mathbf{C}_>(M) \neq \emptyset$. It is well known that

$$\mathbf{C}_>(M) = \operatorname{pos}\left(\boldsymbol{B} \cup \{\boldsymbol{x}_j^* :\ 1 \le j \le l\}\right) . \tag{2.53}$$

By convention, the set $\{\boldsymbol{x}_j^* :\ 1 \le j \le l\}$ is lightened from the outside by the source family $\boldsymbol{N}$. Let us assign to each point $\boldsymbol{x}_j^*$, $1 \le j \le l$, a source $\boldsymbol{y}_j^*$ that lighten $\boldsymbol{x}_j^*$ from the outside. Then it follows from the definition that for each number j, $1 \le j \le l$, there exists a point $\boldsymbol{z}_j^* \in \mathbf{C}_>(M)$ such that

$$\boldsymbol{x}_j^* \in (\boldsymbol{z}_j^*, \boldsymbol{y}_j^*) . \tag{2.54}$$

By the hypothesis of the statement, we have

$$\langle \boldsymbol{c}, \boldsymbol{b} \rangle \ge 0 , \quad \forall \boldsymbol{c} \in \mathbf{C}_>(M) \quad \text{and} \quad \langle \boldsymbol{y}_j^*, \boldsymbol{b} \rangle > 0 , \quad 1 \le j \le l . \tag{2.55}$$

Besides,

$$\boldsymbol{x}_j^* \neq \boldsymbol{y}_j^* , \quad 1 \le i, j \le l \quad \text{and} \quad \boldsymbol{x}_j^* \notin \mathbf{H}(M) , \quad 1 \le i \le l . \tag{2.56}$$

By using statement (C) and relations (2.53), (2.54), and (2.56), let us write down the chain of inclusions

$$\begin{aligned} &\operatorname{pos}(\mathbf{H}(M) \cup \{\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \dots, \boldsymbol{x}_{l-1}^*, \boldsymbol{x}_l^*\}) \subseteq \operatorname{pos}(\mathbf{H}(M) \cup \{\boldsymbol{y}_1^*, \boldsymbol{x}_2^*, \dots, \boldsymbol{x}_{l-1}^*, \boldsymbol{x}_l^*\}) \\ &\subseteq \cdots \subseteq \operatorname{pos}(\mathbf{H}(M) \cup \{\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \dots, \boldsymbol{y}_{l-1}^*, \boldsymbol{x}_l^*\}) \subseteq \operatorname{pos}(\mathbf{H}(M) \cup \{\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \dots, \boldsymbol{y}_{l-1}^*, \boldsymbol{y}_l^*\}) \\ &\subseteq \operatorname{pos}(\mathbf{H}(M) \cup \boldsymbol{N}) . \end{aligned} \tag{2.57}$$

Relations (2.53) and (2.57) imply the inclusion $\overline{\mathbf{C}_>(M)} \subseteq \operatorname{pos}(\mathbf{H}(M) \cup \boldsymbol{N})$, which completes the proof. ◁

Let J_s be the multi-index of an arbitrary maximal feasible subsystem of system (2.26), q the degree of the vertex J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ of that system, $\mathcal{N}(J_s) := \{J_{s_1}, \dots, J_{s_q}\}$ the neighborhood of the vertex J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$, and $\boldsymbol{y}_i^* \in -\mathbf{C}_>(J_{s_i})$ some representatives of the open cones $-\mathbf{C}_>(J_{s_i})$, $1 \le i \le q$. Let r_0 be the least integer such that all r_0-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$ are lightened from the outside by the source set $\{\boldsymbol{y}_i^* : 1 \le i \le q\}$. Since, according to statement (D), all $(n-k)$-dimensional faces of the cone $\overline{\mathbf{C}_>(J_s)}$ are lightened from the outside by the source family $\{\boldsymbol{y}_i^* : 1 \le i \le q\}$, then $r_0 \le n-k \in [n-1]$.

The following assertion is true:

(F) *Under the hypothesis of Proposition* 2.36, *there exists a cone* $\overline{\mathbf{C}_>(M)}$, $M \subseteq J_s$, *such that* $\dim \mathbf{H}(M) = r_0 - 1$, *all* r_0*-dimensional faces of the cone* $\overline{\mathbf{C}_>(M)}$ *are lightened from the outside by a source family* $\boldsymbol{N} := \{\boldsymbol{y}_i^* : 1 \le i \le q\}$, *and there exists a vector* $\boldsymbol{b} \in \mathbb{R}^n$ *such that* $\langle \boldsymbol{y}^*, \boldsymbol{b}\rangle > 0$ *for each* $\boldsymbol{y}^* \in \boldsymbol{N}$, *and* $\langle \boldsymbol{c}, \boldsymbol{b}\rangle \ge 0$ *for each* $\boldsymbol{c} \in \mathbf{C}_>(M)$.

▷ *Proof.* According to Lemma 2.26, $J_{s_i} \cap J_s \ne \emptyset$, for each i, $1 \le i \le q$; therefore, $\overline{\mathbf{C}_>(J_{s_i})} \cap -\mathbf{C}_>(J_s) \ne \emptyset$, $1 \le i \le q$. As a consequence, $\{\boldsymbol{y}_i^* : 1 \le i \le q\} \cap -\mathbf{C}_>(J_s) = \emptyset$, and therefore, because of statement (C), the face $\mathcal{F}(J_s, J_s) = \mathbf{H}(J_s)$ of dimension $\dim \mathbf{H}(J_s)$ is not lightened from the outside by the source family $\{\boldsymbol{y}_i^* : 1 \le i \le q\}$. Since the face $\mathcal{F}(J_s, J_s) = \mathbf{H}(J_s)$ is of dimension that is minimal among all faces of the cone $\overline{\mathbf{C}_>(M)}$, then $r_0 > \dim \mathbf{H}(J_s)$ and, by the definition of the quantity r_0, there exists an (r_0-1)-dimensional face $\mathcal{F}(M, J_s)$ of the cone $\overline{\mathbf{C}_>(J_s)}$, which is not lightened from the outside by the source family $\{\boldsymbol{y}_i^* : 1 \le i \le q\}$. Let us show that the cone $\overline{\mathbf{C}_>(M)}$ satisfies statement (E). Let us show that

$$\mathcal{F}(M, J_s) \cap -\mathbf{C}_>([m] - J_s) = \emptyset . \tag{2.58}$$

Assume the converse. Then, because of $\mathcal{F}(M, J_s) \subset \mathbf{H}(M)$, we have $\mathbf{H}(M) \cap \mathbf{C}_>([m]-J_s) \ne \emptyset$ and, according to Lemma 2.21 (ii), $\mathbf{C}_>(M) \cap \mathbf{C}_>([m]-J_s) \ne \emptyset$, from where we conclude that there exists a maximal feasible subsystem with a multi-index $J_p \supseteq M \cup ([m]-J_s)$. Since $J_p \cup J_s = [m]$, then $J_p \in \{J_{s_1}, \dots, J_{s_q}\}$. Specifically, let $J_p = J_{s_1}$. Then the face $\mathcal{F}(M, J_s)$ is lightened from the outside by a source $\boldsymbol{y}_{s_1}^* \in -\mathbf{C}_>(J_{s_1}) \subseteq -\mathbf{C}_>(M)$; this contradicts the choice of the face $\mathcal{F}(M, J_s)$. This contradiction proves (2.58).

On the other hand, $\mathcal{F}(M, J_s) \subset \overline{\mathbf{C}_>(J_s)} \subset -\overline{\mathbf{C}_>([m]-J_s)}$. Taking into account (2.58), we have $\mathcal{F}(M, J_s) \subseteq -\overline{\mathbf{C}_>([m]-J_s)} - (-\mathbf{C}_>([m]-J_s))$. Since $-\overline{\mathbf{C}_>([m]-J_s)} - (-\mathbf{C}_>([m]-J_s)) \subset \bigcup_{j\in[m]-J_s} \mathbf{H}(j)$, the inclusion $\mathcal{F}(M, J_s) \subset \bigcup_{j\in[m]-J_s} \mathbf{H}(j)$ holds, and thus there exists an index $j_0 \in [m] - J_s$ such that $\mathcal{F}(M, J_s) \cap \mathbf{H}(j_0) \neq \emptyset$. Let us show that $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) = \emptyset$. Suppose to the contrary that $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) \neq \emptyset$. Then we derive from the relations $\mathcal{F}(M, J_s) \cap \mathbf{H}(j_0) = \mathbf{H}(M) \cap \mathbf{C}_>(J_s - M) \cap \mathbf{H}(j_0) \neq \emptyset$ and $\mathbf{C}_>(M) \cup \{j_0\} \neq \emptyset$, in accordance with Lemma 2.21 (ii), that $\mathbf{C}_>(M) \cap \mathbf{C}_>(J_s - M) \cap \mathbf{C}_>(j_0) = \mathbf{C}_>(J_s \cup \{j_0\}) \neq \emptyset$, $j_0 \in [m] - J_s$ – a contradiction with the maximality of the feasible subsystem with the multi-index J_s; thus, $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) = \emptyset$. Since $\mathbf{C}_>(M) \cap \mathbf{C}_>(j_0) = \emptyset$ and the sets $\mathbf{C}_>(M)$ and $\mathbf{C}_>(j_0)$ are open in $\mathbb{R}^n$, then $\mathbf{C}_>(M) \cap \overline{\mathbf{C}_>(j_0)} = \emptyset$; as a consequence, $\mathbf{C}_>(M) \subset -\mathbf{C}_>(j_0)$. Since $\mathbf{C}_>(M) \subset -\mathbf{C}_>(j_0)$ and $-\mathbf{C}_>(J_{s_i}) \subset -\mathbf{C}_>([m] - J_s) \subseteq -\mathbf{C}_>(j_0)$, $1 \le i \le q$, the vector $\boldsymbol{b} \in \mathbb{R}^n$ in statement (F) can be replaced with the vector $-\boldsymbol{a}_{j_0}$ for which $-\mathbf{C}_>(j_0) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle -\boldsymbol{a}_{j_0}, \boldsymbol{x}\rangle > 0\}$.

Now suppose $L_0 \subset M$, and let $\mathcal{F}(L_0, M)$ be an arbitrary r_0-dimensional face of the cone $\overline{\mathbf{C}_>(M)}$; such faces in the cone $\overline{\mathbf{C}_>(M)}$ do exist because $\dim \mathbf{H}(M) = r_0 - 1$ and $\mathbf{C}_>(M) \neq \emptyset$. Let us show that then $\mathcal{F}(L_0, J_s)$ represents an r_0-dimensional face of the cone $\overline{\mathbf{C}_>(J_s)}$, thus, $\mathcal{F}(L_0, J_s) = \mathbf{H}(L_0) \cap \mathbf{C}_>(J_s - L_0) \neq \emptyset$. Let us pick two vectors $\boldsymbol{x}^* \in \mathcal{F}(M, J_s) = \mathbf{H}(M) \cap \mathbf{C}_>(J_s - M) = \mathbf{H}(L_0) \cap \mathbf{H}(M - L_0) \cap \mathbf{C}_>(J_s - M) \neq \emptyset$ and $\boldsymbol{y}^* \in \mathcal{F}(L_0, M) = \mathbf{H}(L_0) \cap \mathbf{C}_>(M - L_0) \neq \emptyset$. Since $L_0 \subset M \subset [m]$, then for the vector $\boldsymbol{z}^* := \boldsymbol{x}^* + \varepsilon \boldsymbol{y}^*$ we have $\boldsymbol{z}^* \in \mathbf{H}(L)$, for any ε; $\boldsymbol{z}^* \in \mathbf{C}_>(M - L_0)$ for any $\varepsilon > 0$, and $\boldsymbol{z}^* \in \mathbf{C}_>([m] - M)$ for a sufficiently small ε; as a consequence, $\boldsymbol{z}^* \in \mathbf{H}(L_0) \cap \mathbf{C}_>(M - L_0) \cap \mathbf{C}_>([m] - M) = \mathbf{H}(L_0) \cap \mathbf{C}_>(J_s - L_0) = \mathcal{F}(L_0, J_s) \neq \emptyset$ for a sufficiently small $\varepsilon > 0$. Since $\mathcal{F}(L_0, J_s)$, being an r_0-dimensional face of the cone $\overline{\mathbf{C}_>(J_s)}$, is lightened from the outside by the source family $\{\boldsymbol{y}_i^* : 1 \le i \le q\}$, according to statement (C), $-\mathbf{C}_>(L_0) \cap \{\boldsymbol{y}_i^* : 1 \le i \le q\} \neq \emptyset$; as a consequence, according to statement (C), the face $\mathcal{F}(L_0, M)$ of the cone $\overline{\mathbf{C}_>(M)}$ is lightened from the outside by the source family $\{\boldsymbol{y}_i^* : 1 \le i \le q\}$; this completes the proof of statement (F). ◁

Now we can complete the proof of Proposition 2.36. For the cone $\overline{\mathbf{C}_>(M)}$ from statement (F), according to statement (E), we have $\overline{\mathbf{C}_>(M)} \subseteq \operatorname{pos}\left(\mathbf{H}(M) \cup \{\boldsymbol{y}_i^* : 1 \le i \le q\}\right)$. Since $\dim \mathbf{H}(M) = r_0 - 1$ and the cone $\overline{\mathbf{C}_>(M)}$ is n-dimensional, then the rank of the set $\{\boldsymbol{y}_i^* : 1 \le i \le q\}$ is at least $n - (r_0 - 1)$; as a consequence, $q \ge n - (r_0 - 1)$, and $q \ge k + 1$ because $r_0 \le n - k$. Proposition 2.36 is proved. □

Recall some basic properties of 2-connected graphs:

Proposition 2.38. *Let* $\mathbf{G}$ *be a simple connected graph. Then the following assertions are equivalent:*
(1) *the graph* $\mathbf{G}$ *is* 2*-connected;*
(2) *any two vertices of the graph* $\mathbf{G}$ *belong to some simple cycle;*
(3) *any vertex and any edge of the graph* $\mathbf{G}$ *belong to some simple cycle;*
(4) *any two edges belong to a simple cycle;*

(5) *for any two vertices a and b, and for any edge E, there exists a simple $(a \leftrightarrow b)$-chain, containing E;*
(6) *for any three vertices a, b and c, there exists a simple $(a \leftrightarrow b)$-chain going through c.*

The following statement summarizes Propositions 2.37 and 2.38:

Corollary 2.39. *If the rank of each subsystem with three inequalities of system* (2.26) *equals* 3*, then the following assertions are equivalent:*
(1) *The multi-indices of any two MFSs J_{i_1} and J_{i_2} of the system $\mathfrak{S}$ belong to a simple cycle of the graph* $\mathtt{MFSG}(\mathfrak{S})$*;*
(2) *the multi-index of any MFS J_{i_1} and the multi-indices of any two MFSs J_{i_2} and J_{i_3}, such that $J_{i_2} \cup J_{i_3} = [m]$, belong to a simple cycle of the graph* $\mathtt{MFSG}(\mathfrak{S})$*;*
(3) *the multi-indices of any four MFSs $J_{i_1}, J_{i_2}, J_{i_3}$, and J_{i_4}, such that $J_{i_1} \cup J_{i_2} = J_{i_3} \cup J_{i_4} = [m]$, belong to a simple cycle of the graph* $\mathtt{MFSG}(\mathfrak{S})$*;*
(4) *for the multi-indices of any two MFSs J_{i_1} and J_{i_2}, and for the multi-indices of any pair of MFSs J_{i_3} and J_{i_4}, such that $J_{i_3} \cup J_{i_4} = [m]$, there exists a simple $(J_{i_1} \leftrightarrow J_{i_2})$-chain of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *containing the pair of multi-indices J_{i_3} and J_{i_4};*
(5) *for the multi-indices of any three MFSs J_{i_1}, J_{i_2}, and J_{i_3}, there exists a simple $(J_{i_1} \leftrightarrow J_{i_2})$-chain of the graph* $\mathtt{MFSG}(\mathfrak{S})$ *going through the vertex J_{i_3}.*

2.4 The hypergraph of maximal feasible subsystems of an infeasible system of linear inequalities

An analysis of covers of the index set of the inequalities of system (2.26) by *arbitrary* families of the multi-indices of its MFSs leads to a natural generalization of the notion of graph of MFSs, which represented the research subject in Section 2.3:

The *hypergraph* $\mathtt{MFSH}(\mathfrak{S})$ *of maximal feasible subsystems* (*hypergraph of MFSs*) of the system $\mathfrak{S}$ is defined as follows:
– *the vertex set of the hypergraph* $\mathtt{MFSH}(\mathfrak{S})$ *is the family* $\mathbf{J}$ *of the multi-indices of MFSs of the system $\mathfrak{S}$;*
– *the hyperedge family of the hypergraph* $\mathtt{MFSH}(\mathfrak{S})$ *is the family of all the unordered collections $\mathfrak{J} \subseteq \mathbf{J}$ of the multi-indices of MFSs of the system $\mathfrak{S}$ that cover the index set of the inequalities of the system:*

$$\bigcup_{J \in \mathfrak{J}} J = [m] .$$

The properties of the hypergraph of MFSs of a *rank* 2 system $\mathfrak{S}$ *over* $\mathbb{R}^2$ are well studied and thus augment the information from Proposition 2.33 on the graph of MFSs of this system:

Let $\mathbf{J} := \{J_1, \ldots, J_{\mathfrak{q}}\}$; recall that the number $\mathfrak{q}$ of maximal feasible subsystems of the system $\mathfrak{S}$ is odd, that is, $\mathfrak{q} = 2t + 1$ for some t. Let us consider a $\{0, 1\}$-matrix $\boldsymbol{M}$ of size $m \times \mathfrak{q}$ whose (j, i)th entry m_{ji} by definition is

$$m_{ji} := \begin{cases} 1\,, & \text{if } j \in J_i\,, \\ 0\,, & \text{if } j \notin J_i\,. \end{cases}$$

Let us re-index the inequalities and multi-indices of MFSs of the system $\mathfrak{S}$ in such a way that the matrix $\boldsymbol{M}$ gets a presentation suitable for consideration. For this to be done, let us assign to each inequality of the system $\mathfrak{S}$ a directing unit vector $\boldsymbol{c}_j$ of the line $\{\mathbf{x} \in \mathbb{R}^2 :\ \langle \boldsymbol{a}_j, \mathbf{x}\rangle = 0\}$, by choosing between the two possible vectors the unique vector such that, when moving along the line in the direction prescribed by it, the half-plane $\{\mathbf{x} \in \mathbb{R}^2 :\ \langle \boldsymbol{a}_j, \mathbf{x}\rangle > 0\}$ is left on the right-hand side. Without loss of generality, we will suppose that the multi-indices $J_1, \ldots, J_{\mathfrak{q}}$ of maximal feasible subsystems of the system $\mathfrak{S}$ are indexed in the ascending order with respect to the polar angles of the corresponding vectors $\boldsymbol{c}_j$, supposing that the index 1 is assigned to the directing vector of the left boundary of the solution cone to the maximal feasible subsystem with the multi-index J_1.

Under the chosen numeration of the inequalities and multi-indices of MFSs of the system $\mathfrak{S}$, the matrix $\boldsymbol{M}$ obtains the following form:

$$\boldsymbol{M} := \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ \multicolumn{9}{c}{\dotfill} \\ 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 \\ \multicolumn{9}{c}{\dotfill} \\ 1 & 1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 \\ \multicolumn{9}{c}{\dotfill} \\ \multicolumn{9}{c}{\dotfill} \\ 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 \\ \multicolumn{9}{c}{\dotfill} \\ 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

The index of each inequality is included in the multi-indices of precisely $t + 1$ MFSs; since in the matrix $\boldsymbol{M}$ there are precisely $\mathfrak{q} = 2t + 1$ pairwise distinct rows, then the indices of the inequalities that compose the system $\mathfrak{S}$, are partitioned into $\mathfrak{q}$ equivalence classes: the inequalities with indices j_1 and j_2 are included in the multi-indices of the same MFSs if and only if they are representatives of the same class (respectively when the rows of the matrix $\boldsymbol{M}$, with the indices j_1 and j_2, coincide.) Let us index the equivalence classes of the inequalities of the system $\mathfrak{S}$, in natural order, by the integers $1, \ldots, \mathfrak{q}$.

Let us consider the hypergraph $(\mathbf{J}, \mathcal{E}) := \mathtt{MFSH}(\mathfrak{S})$, on the vertex set $\mathbf{J}$, with the hyperedge family $\mathcal{E}$, of a rank 2 system $\mathfrak{S}$. For investigating the structure of the hyperedge family $\mathcal{E}$, it suffices to leave for consideration the index of one inequality for each equivalence class, by considering – instead of the initial matrix $\boldsymbol{M}$ – the square $\{0, 1\}$-matrix $\boldsymbol{M}'$ of size $\mathfrak{q} \times \mathfrak{q}$:

$$
\boldsymbol{M}' := \begin{array}{cc}
 & \begin{array}{ccccccccc} & & & & t+1 & & & & \end{array} \\
\begin{array}{c} \\ \\ \\ \\ t+1 \\ \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & 1 \\
1 & 1 & 0 & \ddots & \ddots & 0 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 & 1 \\
1 & \ddots & \ddots & \ddots & 1 & 0 & \ddots & \ddots & 0 \\
0 & 1 & \ddots & \ddots & \ddots & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\
0 & \ddots & \ddots & 1 & \ddots & \ddots & \ddots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1
\end{pmatrix}
\end{array} .
$$

According to the description of the matrix $\boldsymbol{M}'$, the vertex J_1 is, for example, included into two-element hyperedges $\{J_1, J_{t+1}\}$ and $\{J_1, j_{t+2}\}$. The following assertion poses a condition on the indices of the vertices that are included in some vertex subset $U \subset \mathbf{J}$, necessary and sufficient for the inclusion $U \in \mathcal{E}$ to hold. When we write $U := \{J_{i_1}, J_{i_2}, \ldots, J_{i_s}\}$ below we will mean that the ordering $i_1 < i_2 < \cdots < i_s$ is the case.

Proposition 2.40. *A vertex subset $U := \{J_{i_1}, \ldots, J_{i_s}\}$ of the hypergraph $(\mathbf{J}, \mathcal{E}) :=$ $\mathtt{MFSH}(\mathfrak{S})$ is its hyperedge, that is, $U \in \mathcal{E}$, if and only if for each $k \in [s]$ the condition*

$$(i_{(k \pmod s))+1} - i_k) \pmod{\mathfrak{q}} \le t + 1$$

is satisfied.

Proof. The *sufficiency* follows from the structure off the matrix $\boldsymbol{M}'$.

The *necessity*. Let $U \in \mathcal{E}$. Then $\bigcup_{k=1}^{s} J_{i_k} = [m]$, by the definition of the hypergraph $\mathtt{MFSH}(\mathfrak{S})$. Let us show that, for example, $i_2 - i_1 \le t + 1$. Note that into the maximal feasible subsystem with a multi-index J_k are included all the inequalities-representatives of the classes only with the numbers: k, $(k \pmod{\mathfrak{q}})+1, \ldots, (k+(t-1)) \pmod{\mathfrak{q}} + 1$. Let us consider an arbitrary inequality with an index τ from the class $((i_1 + t) \pmod{\mathfrak{q}}) + 1$. We have $\tau \notin J_{i_1}$ and, as a consequence, there exists $k \in \{2, 3, \ldots, s\}$ such that $\tau \in J_{i_k}$. Thus, all the inequalities from the specified class are included in the MFS with the multi-index J_{i_k}, that is, either $i_k = ((i_1 + t) \pmod{\mathfrak{q}}) + 1$ or there exists $e \in \{0, 1, \ldots, t-1\}$ such that $((i_k + e) \pmod{\mathfrak{q}}) + 1 = ((i_1 + t) \pmod{\mathfrak{q}}) + 1$. In the first case, $i_k - 1 = (i_k - 1) \pmod{\mathfrak{q}} = (i_1 + t) \pmod{\mathfrak{q}}$, hence $i_k - i_1 = (i_k - i_1) \pmod{\mathfrak{q}} = (t+1) \pmod{\mathfrak{q}} = t+1$. In the second case, $i_k - i_1 = (i_k - i_1) \pmod{\mathfrak{q}} = (t-e) \pmod{\mathfrak{q}} = t - e \le t$. Since $i_2 - i_1 \le i_k - i_1$ then $i_2 - i_1 \le t + 1$. The proposition is proved. ☐

In particular, it follows from Proposition 2.40 that if the number $\mathfrak{q} := \#\mathbf{J}$ of maximal feasible subsystems of the system $\mathfrak{S}$ is quite large, then the hyperedge family $\mathcal{E}$ of the hypergraph $\mathtt{MFSH}(\mathfrak{S})$ has hyperedges that contain no two-element hyperedges as subsets.

Notes

The graphs of independence systems that describe the covers of the vertex sets of any abstract simplicial complexes by pairs of their facets – the research subject in Section 2.1 – were studied in works [48, 53]. They are a natural generalization of the notion of the graph of MFSs of a finite infeasible system of linear inequalities.

Homomorphisms (simplicial maps) of simplicial complexes are basic tools of combinatorial topology, see, for example, [121, 130, 132].

In Section 2.1, we use the standard terminology (open and closed sets, connectedness, density, continuous maps, and so on) of general topology which is presented in various texts; we will point out at just a few sources: [4, 6, 11, 22, 38, 44, 73, 77].

The connectedness regarded from different points of view is one of the most important sides of the description of graphs of general kind, see [13, 28, 32, 67, 104, 136, 147, 156, 166].

Formulating Remark 2.13, we use the information on algebraic varieties from [23].

The notion of hypergraph of an independence system discussed in Section 2.2 reproduces the construction of the hypergraph of maximal feasible subsystems of an infeasible system of constraints, see, for example, [76, 98] and references mentioned in these surveys. Proposition 2.19 is a reformulated Theorem 3.1 from [76] which in its turn is borrowed from [74].

Linear inequality systems are a fundamental subject of pure and applied mathematical research, see, for example, [17, 39–43, 146]; recall that in Section 2.3 we address an analysis of the combinatorial properties of a particular class of infeasible systems of the form (2.26) because of their significance in the simulation of the contradictory problems of pattern recognition.

The notion of graph of MFSs of a finite infeasible system of linear inequalities was introduced by the second and third authors of work [58] as a result of a generalization of constructions presented in [148]. Works in [47, 51, 58] are devoted to a detailed study of the properties of the graph of MFSs.

The proof of Theorem 2.20 describes one of several approaches to the verification of the connectedness of the graph of MFSs of infeasible system (2.26). The first result in this direction was the derivation, by the second author of work [58], of Theorem 2.20 from the assertion, proved by him, on the connectedness of the square of the graph of MFSs. Another justification of the statement is provided by the proof of Theorem 2 from [58]. The proof of Theorem 2.20 given in Section 2.3 follows the works [48, 53].

Theorem 2.25 (i) is proved in work [148] and reproduced in [58] as Theorem 1.

The proof of Theorem 2.28 is, in particular, based on the well-known observation that a graph is bipartite if and only if all its simple cycles are of even length, see, for example, [67, 165].

One component of the proof of Proposition 2.33 is an important proposition presented in [94], according to which the number of MFSs of an infeasible rank 2 system (2.26) over $\mathbb{R}^2$ is odd and it does not exceed m.

In the proof of Proposition 2.35 (iii) we mention the observation verified in [165], according to which in a walk between two vertices of a graph one can distinguish a simple chain linking them.

In section (A) of the proof of Proposition 2.36, we use the assertion that the complement of any MFS of infeasible system (2.26) is feasible; the proof of this fact is presented in [148].

The terminology related to the lightening of convex bodies, which we recall on page 49 is borrowed from [20].

The list of basic properties of bipartite graphs presented in Proposition 2.38 is the content of Theorem 34.1 from [104].

The properties of the hypergraph of MFSs of an infeasible rank 2 system (2.26) over $\mathbb{R}^2$, described in Section 2.4, are borrowed from survey [76]. Proposition 2.40 reproduces Proposition 3.1 from [76], which in its turn can be found in [74].

3 Polytopes, positive bases, and inequality systems

In this chapter, we study some combinatorial and structural properties of convex polytopes, positive bases of vector spaces, and infeasible systems of linear inequalities.

It was shown in the previous chapters that infeasible systems of linear inequalities inherit their significant properties from other fundamental mathematical constructions. For example, being infeasible systems with the monotonicity property, systems of linear inequalities can additionally be described in the language of abstract simplicial complexes. The connectedness of the graphs of maximal feasible subsystems of such systems follows from the fundamental topological property of the connectedness of the space $\mathbb{R}^n$.

Infeasible systems of linear inequalities are our main subject of consideration in the present and subsequent chapters. Vector spaces and convex polytopes play in a sense a subordinate role for the following reason: the combinatorial properties of rank n infeasible systems with m homogeneous strict liner inequalities over $\mathbb{R}^n$ can be investigated via the properties of an m-point subset of the space $\mathbb{R}^{m-n-1}$. Moreover, under appropriate circumstances, the properties of inequality systems turn out to be equivalent to the properties of $(m - n - 1)$-dimensional convex polytopes. In an analysis of infeasible systems of linear inequalities by means of a study of polytopes, the notions of faces and diagonals of polytopes play an important role. In an investigation of the construction of the family of minimal infeasible subsystems, it is important to understand the structure of positive bases of $\mathbb{R}^n$.

3.1 Faces and diagonals of convex polytopes

In plane geometry, the notion of *diagonal* plays a role similar to that played by the notion of side of a polygon. In higher dimensions, sides of polygons are generalized to faces playing an important role in problems of combinatorial classification of polytopes. The notion of diagonal deserving the same attention has less successful d-dimensional fate.

In this section, we consider three possible generalizations of the notion of *diagonal* to an arbitrary d-dimensional case. Each of them can be taken as the basis of some combinatorial classification of polytopes, and in the case of the so-called G-diagonals one obtains a classification agreeing with the conventional one defined by the structure of faces.

We use the standard notation: aff for the affine hull, pos for the positive hull, conv for the convex hull, ri for the relative interior, rbd for the relative boundary, vert for the vertex set, and dim for the affine dimension.

Three notions of diagonals and their relationships

We consider convex and bounded polytopes only. The polytopes of dimension d are called, for brevity, d-polytopes. Speaking of the faces of a polytope, we always mean *proper faces*, that is, the faces different from the empty set and the entire polytope.

Recall that a *cyclic d-polytope* is the convex hull of a finite m-subset, $m > d$, of points of the *moment curve* $(t, t^2, \dots, t^d)$, where $t \in \mathbb{R}$, $t \neq 0$.

A *k-neighborly polytope* is a polytope such that any of its k-subsets of vertices are the vertex sets of some faces. It is well known that the cyclic d-polytope is $\lfloor \frac{d}{2} \rfloor$-neighborly.

We will say that a polytope $\mathcal{P}$ is obtained from polytopes $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ by the operation of *cross*, and use in this case the notation $\mathcal{P} = \mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$, when $\dim \operatorname{aff} \mathcal{P} = \sum_{j \in [k]} \dim \operatorname{aff} \mathcal{P}_j$ and the intersection $\bigcap_{j \in [k]} \operatorname{ri} \mathcal{P}$ is nonempty and, as a consequence, it represents a unique point.

Let us consider two auxiliary statements:

Lemma 3.1. *If $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$, then the faces of the polytope $\mathcal{P}$ are precisely all the sets of the form $\boldsymbol{F} = \operatorname{conv}(\boldsymbol{F}_{j_1} \cup \boldsymbol{F}_{j_2} \cup \cdots \cup \boldsymbol{F}_{j_s})$, where $\boldsymbol{F}_j$ is a face of $\mathcal{P}_j$; here $\boldsymbol{F}_j \neq \mathcal{P}_j$, $j \in \{j_1, j_2, \dots, j_s\} \subset [k]$.*

Proof. Since $\mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k = (\mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_{k-1}) \perp \mathcal{P}_k$, it suffices to prove the lemma for the case $k = 2$. Let $\boldsymbol{F}_i$ be faces of the polytopes $\mathcal{P}_i$, $i \in [2]$, and thus of the polytope $\mathcal{P}$. Let us prove that $\operatorname{conv}(\boldsymbol{F}_1 \cup \boldsymbol{F}_2)$ is also a face of $\mathcal{P}$. Let $\boldsymbol{H}_i$ be a supporting hyperplane of $\mathcal{P}_i$ in $\operatorname{aff} \mathcal{P}_i$, $\boldsymbol{F}_i = \boldsymbol{H}_i \cap \mathcal{P}_i$, $i \in [2]$. Note that $\dim \operatorname{aff}(\boldsymbol{H}_1 \cup \boldsymbol{H}_2) = d - 1$. Let $\boldsymbol{H} := \operatorname{aff}(\boldsymbol{H}_1 \cup \boldsymbol{H}_2)$. Taking into account that $\boldsymbol{H}$ is a hyperplane in $\operatorname{aff} \mathcal{P}_i$, we see that $\boldsymbol{H}_i = \boldsymbol{H} \cap \operatorname{aff} \mathcal{P}_i$, $i \in [2]$; thus, the hyperplane $\boldsymbol{H}$ supports both $\mathcal{P}_1$ and $\mathcal{P}_2$ and, because of $\operatorname{ri} \mathcal{P}_1 \cap \operatorname{ri} \mathcal{P}_2 \neq \emptyset$, it also supports $\mathcal{P}$. It is clear that $\operatorname{conv}(\boldsymbol{F}_1 \cup \boldsymbol{F}_2) = \boldsymbol{H} \cap \mathcal{P}$. Now suppose that $\boldsymbol{F}$ is a face of $\mathcal{P}$, and $\boldsymbol{H}$ is a hyperplane supporting $\mathcal{P}$ in $\mathbb{R}^d$, such that $\boldsymbol{F} = \boldsymbol{H} \cap \mathcal{P}$. Let $V_i := \operatorname{vert}(\boldsymbol{F} \cap \mathcal{P}_i)$, $i \in [2]$. We see that $\boldsymbol{F}_i = \operatorname{conv} V_i$ is a face of $\mathcal{P}_i$ because $\operatorname{conv} V_i = \boldsymbol{H} \cap \mathcal{P}_i$, and $\mathcal{P}_i \not\subset \boldsymbol{H}$ because $\{z\} = \operatorname{ri} \mathcal{P}_1 \cap \operatorname{ri} \mathcal{P}_2 \subset \operatorname{ri} \mathcal{P} \not\subset \boldsymbol{H}$. ☐

Lemma 3.2. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be polytopes; let $\boldsymbol{H}_1 := \operatorname{aff} \mathcal{P}_1$ and $\boldsymbol{H}_2 := \operatorname{aff} \mathcal{P}_2$ be skew planes, and $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are faces of the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Then $\operatorname{conv}(\boldsymbol{F}_1 \cup \boldsymbol{F}_2)$ is a face of the polytope $\operatorname{conv}(\mathcal{P}_1 \cup \mathcal{P}_2)$.*

Proof. Let $d_1 := \dim \boldsymbol{H}_1$ and $d_2 := \dim \boldsymbol{H}_2$. Let us choose a d_i-simplex $\mathcal{S}_i \supset \mathcal{P}_i$ containing the set $\boldsymbol{F}_i = \operatorname{rbd} \mathcal{S}_i \cap \mathcal{P}_i$ inside of some of its face $\boldsymbol{Q}_i$, $i \in [2]$. The statement now follows from the observation that $\operatorname{conv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ is a simplex, and $\operatorname{conv}(\boldsymbol{Q}_1 \cup \boldsymbol{Q}_2)$ is its face. ☐

Let $\mathcal{P}$ be a polytope, $D \subsetneqq \operatorname{vert} \mathcal{P}$. We will say that the set D (or the set $\operatorname{conv} D$ – it will be clear from the context) is

- *an A-diagonal, if $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$, but for any proper subset $D' \subsetneqq D$ the set $\operatorname{conv} D'$ is a face of the polytope $\mathcal{P}$;*

– *a* G-*diagonal, if* $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$, *but any proper subset* $D' \subsetneq D$ *lies in some proper face of the polytope* $\mathcal{P}$*;*
– *an* F-*diagonal, if* $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} = \operatorname{ri} \operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$.

We will denote the family of all A-, G- and F-diagonals of the polytope $\mathcal{P}$ by $\mathcal{D}_{\mathrm{A}}(\mathcal{P})$, $\mathcal{D}_{\mathrm{G}}(\mathcal{P})$, and $\mathcal{D}_{\mathrm{F}}(\mathcal{P})$, respectively; the family of all r-dimensional diagonals (or r-diagonals) will be denoted by $\mathcal{D}^r_{\mathrm{A}}(\mathcal{P})$, $\mathcal{D}^r_{\mathrm{G}}(\mathcal{P})$, and $\mathcal{D}^r_{\mathrm{F}}(\mathcal{P})$, respectively. The following statement follows immediately from the definition of diagonals:

Proposition 3.3. *For* A-, G-, *and* F-*diagonals of a* d*-polytope* $\mathcal{P}$ *the following relations hold:*

$$\mathcal{D}^r_{\mathrm{A}}(\mathcal{P}) \subset \mathcal{D}^r_{\mathrm{G}}(\mathcal{P}) \subset \mathcal{D}^r_{\mathrm{F}}(\mathcal{P}), \quad r \in [d-1]\,; \tag{3.1}$$

$$\mathcal{D}^0_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}^0_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^0_{\mathrm{F}}(\mathcal{P}) = \mathcal{D}^d_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}^d_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^d_{\mathrm{F}}(\mathcal{P}) = \emptyset\,; \tag{3.2}$$

$$\mathcal{D}^1_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{F}}(\mathcal{P})\,. \tag{3.3}$$

Another observation on the relationship between the notions of F- and G-diagonals:

Proposition 3.4. *An* F-*diagonal* D *of a polytope* $\mathcal{P}$ *is a* G-*diagonal if and only if* $\mathcal{P}$ *is a simplex.*

Proposition 3.5. *Let* $\mathcal{P}$ *be a pyramid, namely* $\mathcal{P} := \operatorname{conv}(\{v\} \cup \mathcal{P})$, *for some polyhedral basis* $\mathcal{P}'$ *and a point* $v \notin \operatorname{aff} \mathcal{P}'$. *Then*

$$\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \emptyset\,; \tag{3.4}$$

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \{D = \{v\} \cup D' : \; D' \in \mathcal{D}_{\mathrm{G}}(\mathcal{P}')\}\,; \tag{3.5}$$

$$\mathcal{D}_{\mathrm{F}}(\mathcal{P}) = \{D = \{v\} \cup D' : \; D' \in \mathcal{D}_{\mathrm{F}}(\mathcal{P}')\}\,. \tag{3.6}$$

Proof. If D is a diagonal of the polytope $\mathcal{P}$, of any type, then $v \in D$, because otherwise the convex hull $\operatorname{conv} D$ would lie in a face of $\mathcal{P}$. By a similar argument, in all three cases the relation

$$(D - \{v\}) \cap \operatorname{ri} \mathcal{P}' \neq \emptyset \tag{3.7}$$

holds.

Now in the case (3.4) the set $\operatorname{conv}(D - \{v\})$, being a face of the polytope $\mathcal{P}$, must coincide with $\mathcal{P}'$, that is, $\operatorname{conv} D = \mathcal{P}$, and thus $D = \operatorname{vert} \mathcal{P}$, a contradiction.

For (3.5), the set $D' = D - \{v\}$ is inclusion-minimal with respect to property (3.7) if and only if D is minimal with respect to the property $\operatorname{conv} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$; relation (3.5) is thus proved.

As to relation (3.6), it should be mentioned that the fulfillment of $\operatorname{rbd} D' \cap \operatorname{ri} \mathcal{P}' \neq \emptyset$ is equivalent to the fulfillment of $\operatorname{rbd} D \cap \operatorname{ri} \mathcal{P} \neq \emptyset$. □

Proposition 3.6. *Let* $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$. *Then*

$$\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \bigcup_{j\in[k]} \mathcal{D}_{\mathrm{A}}(\mathcal{P}_j) \cup \{\mathcal{P}_i : \ \mathcal{P}_i \textit{ is a simplex}\}\,; \tag{3.8}$$

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \bigcup_{j\in[k]} \mathcal{D}_{\mathrm{G}}(\mathcal{P}_j) \cup \{\mathcal{P}_i : \ \mathcal{P}_i \textit{ is a simplex}\}\,; \tag{3.9}$$

$$\mathcal{D}_{\mathrm{F}}(\mathcal{P}) = \{D \subset \operatorname{vert}\mathcal{P} : \ \operatorname{conv} D = \operatorname{conv} D_{j_1} \perp \operatorname{conv} D_{j_2} \perp \cdots \perp \operatorname{conv} D_{j_m}\}\,, \tag{3.10}$$

where either $D_{j_i} \in \mathcal{D}_{\mathrm{F}}(\mathcal{P}_{j_i})$ *or* $D_{j_i} = \operatorname{vert}\mathcal{P}_{j_i}$*; besides,* $\{j_i : D_{j_i} = \operatorname{vert}\mathcal{P}_{j_i}\} \subsetneq [k]$.

Proof. In the cases (3.8) and (3.9), if D is a diagonal (of the corresponding type) of the polytope $\mathcal{P}$, and $D_j = D \cap \operatorname{vert}\mathcal{P}_j$, then, in view of Lemma 3.1, we have $\operatorname{conv} D_j \cap \operatorname{ri}\mathcal{P} \neq \emptyset$ for some j. Hence, because of the minimality property of diagonals of these types, we obtain $D = D_j$, and D_j is a diagonal of $\mathcal{P}_j$ or, in the case where $\mathcal{P}_j$ is a simplex, $D_j = \mathcal{P}_j$.

In the case (3.10), let us denote $\{\boldsymbol{z}\} := \bigcap_{j\in[k]} \operatorname{ri}\mathcal{P}_j$. For $j \in [l]$, let us denote $D_j := D \cap \operatorname{vert}\mathcal{P}_j \neq \emptyset$, and show that, under $l \geq 2$, we have $\boldsymbol{z} \in \mathbf{ri\,conv}\, \boldsymbol{D_j}$, $j \in [l]$. Indeed, if $\boldsymbol{z} \notin \operatorname{ri}\operatorname{conv} D_1$, then $\operatorname{ri}\operatorname{conv} D_1 \cap \operatorname{ri}\operatorname{conv}(D - D_1) = \emptyset$, $\operatorname{conv} D_1 \subset \operatorname{rbd}\operatorname{conv} D$, and $\operatorname{conv}(D - D_1) \subset \operatorname{rbd}\operatorname{conv} D$. Hence, by the definition of F-diagonals, $\operatorname{conv} D_1 \subset \operatorname{rbd}\mathcal{P}$, $\operatorname{conv} D_2 \subset \operatorname{rbd}(\mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k)$, and $D_1 \subset \boldsymbol{F}_1$, $D_2 \subset \boldsymbol{F}_2$, for some faces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ of the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2 \perp \cdots \perp \mathcal{P}_k$, respectively; but, by Lemma 3.1, the set $\boldsymbol{F} := \operatorname{conv}(\boldsymbol{F}_1 \cup \boldsymbol{F}_2)$ is a face of the polytope $\mathcal{P}$ and it contains D, a contradiction with the inclusion $D \in \mathcal{D}_{\mathrm{F}}(\mathcal{P})$. Thus, under $l \geq 2$, $\operatorname{conv} D = \operatorname{conv} D_1 \perp \operatorname{conv} D_2 \perp \cdots \perp \operatorname{conv} D_l$ and, in view of Lemma 3.1, $\operatorname{rbd}\operatorname{conv} D \subset \operatorname{rbd}\mathcal{P}$ implies the inclusion $\operatorname{rbd}\operatorname{conv} D_j \subset \operatorname{rbd}\mathcal{P}_j$, that is, $D_j \in \mathcal{D}_{\mathrm{F}}(\mathcal{P})$ or $D_j = \mathcal{P}_j$. The case $l = 1$, that is, $D_j = D$, is obvious. □

Proposition 3.7. *Each vertex of an arbitrary polytope, which is not a simplex, is contained in at least one its* G*-diagonal*

We now ascertain the relationships between the following properties:
C_1: a polytope $\mathcal{P}$ is cyclic;
C_2: the vertex set $\operatorname{vert}\mathcal{P}$ is in general position in $\operatorname{aff}\mathcal{P}$;
C_3: the polytope $\mathcal{P}$ is simplicial;
C_4: $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}_{\mathrm{F}}(\mathcal{P})$;
C_5: $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}_{\mathrm{G}}(\mathcal{P})$;
C_6: $\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}_{\mathrm{F}}(\mathcal{P})$.

Proposition 3.8. *Defining, for any polytope* $\mathcal{P}$,

$$\mathrm{c}_{ij} := \begin{cases} 1, & \textit{if } \mathrm{C}_i \Rightarrow \mathrm{C}_i\,, \\ 0, & \textit{otherwise}\,, \end{cases}$$

we have $(\mathrm{c}_{ij})_{i,j\in[6]} = \begin{pmatrix} 1&1&1&1&1&1 \\ 0&1&1&1&1&1 \\ 0&0&1&0&1&0 \\ 0&0&0&1&1&1 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$.

Proof. The binary relation C, where $i\mathrm{C}j \Leftrightarrow \mathrm{c}_{ij} = 1$, is transitive; we will use below this property without special mention.

The equalities $\mathrm{c}_{ii} = 1$ for $i \in [6]$, and $\mathrm{c}_{23} = \mathrm{c}_{45} = \mathrm{c}_{46} = 1$, $\mathrm{c}_{32} = 0$, are easily verified.

Example 3.9. *Let $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2$, where $\mathcal{P}_1$ is a one-dimensional polytope, $\mathcal{P}_2$ is a $(d-1)$-dimensional simplicial polytope, which is not a simplex. It is easily seen that $\mathcal{P}$ is a simplicial polytope, and $\mathcal{P}_2$ is its* F*-diagonal, which is not a* G*-diagonal* (*Propositions* 3.4 *and* 3.6.) ◁

Example 3.9 yields $\mathrm{c}_{36} = 0$, hence $\mathrm{c}_{34} = 0$. Since any vertex subset of a simplex forms its face, it follows immediately from the definitions that $\mathrm{c}_{35} = 1$; hence $\mathrm{c}_{25} = 1$. Taking into account Proposition 3.6, Example 3.9 implies $\mathrm{c}_{56} = \mathrm{c}_{54} = 0$.

Proposition 3.4 yields $\mathrm{c}_{26} = 1$; this observation and the equality $\mathrm{c}_{25} = 1$ imply $\mathrm{c}_{24} = 1$.

Example 3.10. *Let $\mathcal{P}$ be a pyramid whose basis $\mathcal{P}'$ is in general position in $\mathbb{R}^{d-1} :=$* $\operatorname{aff}\mathcal{P}'$. ◁

Because of the above-proved equality $\mathrm{c}_{26} = 1$, we have $\mathcal{D}_{\mathrm{G}}(\mathcal{P}') = \mathcal{D}_{\mathrm{F}}(\mathcal{P}')$, from where, by Propositions 3.5 and 3.7, it follows that $\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}_{\mathrm{F}}(\mathcal{P}) \neq \emptyset$; at the same time, by Proposition 3.5, $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \emptyset$; thus, $\mathrm{c}_{65} = \mathrm{c}_{64} = 0$. It is known that if $\mathcal{P}$ is a cyclic polytope then $\operatorname{vert}\mathcal{P}$ is a set in general position in $\mathbb{R}^d$, that is, $\mathrm{c}_{12} = 1$. We now have $\mathrm{c}_{1j} = 1$, $3 \le j \le 6$. In order to prove that $\mathrm{c}_{21} = 0$, let us consider Example 3.9 in the situation where $\mathcal{P}_2$ is a simplex. In this case, the polytope whose vertices are obviously in general position has the one-dimensional diagonal $\mathcal{P}_1$, and for this reason it is not 2-neighborly, and it cannot be cyclic. We now have $\mathrm{c}_{i1} = 0$, $3 \le i \le 6$.

It remains to show that $\mathrm{c}_{43} = \mathrm{c}_{42} = \mathrm{c}_{53} = \mathrm{c}_{52} = \mathrm{c}_{62} = 0$. For this, it suffices to check the property $\mathrm{c}_{43} = 0$.

Example 3.11. *Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Let us define the sets*

$$\mathcal{P}_1 := \operatorname{conv}\{\pm\boldsymbol{e}_i :\ i \in [d-1]\}\,,$$
$$\mathcal{P}_2 := \operatorname{conv}\Big\{\boldsymbol{v} := \sum_{i\in[d-1]} \alpha_i\boldsymbol{e}_i + \boldsymbol{e}_d :\ \alpha_i \in \{-1,1\},\ i \in [d-1]\Big\}\,,$$
$$\mathcal{P} := \operatorname{conv}(\mathcal{P}_1 \cup \mathcal{P}_2)\,,\ \ \boldsymbol{H}_1 := \operatorname{aff}\mathcal{P}_1\,,\ \ \boldsymbol{H}_2 := \operatorname{aff}\mathcal{P}_2\,.$$

We will call a pair of vertices $\{\boldsymbol{u}, \boldsymbol{v}\}$, $\boldsymbol{u} \in \operatorname{vert}\mathcal{P}_1$, $\boldsymbol{v} \in \operatorname{vert}\mathcal{P}_2$, diagonal *if $\langle\boldsymbol{u}, \boldsymbol{v}\rangle = -1$, that is, for $\boldsymbol{u} = \alpha_k\boldsymbol{e}_k$, $\alpha_k \in \{-1,1\}$, we must have*

$$\boldsymbol{v} = -\alpha_k\boldsymbol{e}_k + \sum_{\substack{i\in[d-1]\,,\\ i\neq k}} \alpha_i\boldsymbol{e}_i + \boldsymbol{e}_d\,.$$

Let us show that the following two assertions are true:

(1) *any diagonal pair of vertices $\{\boldsymbol{u}, \boldsymbol{v}\}$ forms a diagonal of the polytope $\mathcal{P}$ – it is obvious that $\{\boldsymbol{u}, \boldsymbol{v}\} \in \mathcal{D}_{\mathrm{A}}(\mathcal{P}) \subset \mathcal{D}_{\mathrm{G}}(\mathcal{P}) \subset \mathcal{D}_{\mathrm{F}}(\mathcal{P})$;*

(2) *any inclusion-maximal vertex subset of the polytope $\mathcal{P}$, containing no diagonal pairs, forms some face of $\mathcal{P}$.*

In order to prove (1) *assume the converse. Let $\boldsymbol{u} := \boldsymbol{e}_1$, $\boldsymbol{v} := -\boldsymbol{e}_1 + \sum_{i=2}^{d-1} \tilde{\alpha}_i \boldsymbol{e}_i + \boldsymbol{e}_d$, where $\tilde{\alpha}_i \in \{-1, 1\}$, $2 \le i \le d-1$. Further, let $\boldsymbol{\beta} := (\beta_1, \beta_2, \dots, \beta_k)$, and let the equation $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = 1$ determine a supporting hyperplane of $\mathcal{P}$ that contains the points $\boldsymbol{u}$ and $\boldsymbol{v}$. Since $\mathcal{P}$ contains the origin, we have $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle \le 1$ for all $\boldsymbol{x} \in \mathcal{P}$. We thus obtain*

$$\langle \boldsymbol{\beta}, \boldsymbol{v} \rangle = -\beta_1 + \sum_{2 \le i \le d-1} \tilde{\alpha}_i \beta_i + \beta_d = 1 , \tag{3.11}$$

$$\langle \boldsymbol{\beta}, \boldsymbol{u} \rangle = \beta_1 = 1 , \tag{3.12}$$

and for the vertex $\boldsymbol{w} := \boldsymbol{e}_1 + \sum_{2 \le i \le d-1} \tilde{\alpha}_i \boldsymbol{e}_i + \boldsymbol{e}_d$, we have

$$\langle \boldsymbol{\beta}, \boldsymbol{w} \rangle = \beta_1 + \sum_{2 \le i \le d-1} \tilde{\alpha}_i \beta_i + \beta_d \le 1 . \tag{3.13}$$

Taking into account (3.12), *relations* (3.11) *and* (3.13) *lead to a contradiction.*

We now prove assertion (2). *Let $\mathcal{S}$ be an inclusion-maximal subset of vertices of the polytope $\mathcal{P}$ that contains no diagonal pairs. It is clear that it suffices to prove the case where the sets $\mathcal{S} \cap \operatorname{vert} \mathcal{P}_1$ and $\mathcal{S} \cap \operatorname{vert} \mathcal{P}_2$ are both nonempty.*

It follows from the definition of diagonal pairs that $\mathcal{S} \cap \operatorname{vert} \mathcal{P}_1$ contains no pair $\{\boldsymbol{e}_j, -\boldsymbol{e}_j\}$ for any j. Thus, without loss of generality we suppose that

$$\mathcal{S} := \Big\{ \boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_k, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \dots + \boldsymbol{e}_k + \sum_{k+1 \le i \le d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d : \\ \alpha_i \in \{-1, 1\},\ k+1 \le i \le d-1 \Big\} .$$

Let us define $\boldsymbol{\beta} := (\beta_1, \beta_2, \dots, \beta_d)$ as follows:

$$\beta_i := \begin{cases} 1, & \text{if } i \in [k] , \\ 0, & \text{if } k+1 \le i \le d-1 , \\ 1-k, & \text{if } i = d . \end{cases}$$

Note that for all vectors $\boldsymbol{x} \in \mathcal{S}$ the equality $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = 1$ holds.

Now let $\boldsymbol{x} \in \operatorname{vert} \mathcal{P} - \mathcal{S}$. If $\boldsymbol{x} \in \operatorname{vert} \mathcal{P}_1$, then

$$\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = \begin{cases} -1, & \text{if } \boldsymbol{x} = -\boldsymbol{e}_j,\ j \le k , \\ 0, & \text{if } \boldsymbol{x} = \pm \boldsymbol{e}_j,\ k+1 \le j \le d-1 . \end{cases}$$

But if $\boldsymbol{x} \in \operatorname{vert} \mathcal{P}_2 - \mathcal{S}$, that is, $\boldsymbol{x} = \sum_{0 \le i \le d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d$, where $\alpha_i \in \{-1, 1\}$, $i \in [d-1]$, and among $\alpha_1, \alpha_2, \dots, \alpha_k$ there are numbers different from 1, *then $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = \sum_{i \in [k]} \alpha_i + (1-k) < 1$. Thus, for any vector $\boldsymbol{x} \in \operatorname{vert} \mathcal{P} - \mathcal{S}$, we have $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle < 1$, and for any vector $\boldsymbol{x} \in \mathcal{S}$, we have $\langle \boldsymbol{\beta}, \boldsymbol{x} \rangle = 1$, that is, $\operatorname{conv} \mathcal{S}$ is a face of the polytope $\mathcal{P}$.* ◁

Example 3.11 provides a nonsimplicial polytope $\mathcal{P}$ such that $\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{G}}(\mathcal{P}) = \mathcal{D}^1_{\mathrm{A}}(\mathcal{P}) = \mathcal{D}_{\mathrm{A}}(\mathcal{P})$, and it thus proves the equality $\mathrm{c}_{53} = 0$. For the polytope $\mathcal{P}$, we have $\mathcal{D}_{\mathrm{A}}(\mathcal{P}) \neq \mathcal{D}_{\mathrm{F}}(\mathcal{P})$. For example,

$$\mathcal{S} := \Big\{\boldsymbol{e}_1, -\boldsymbol{e}_1, \boldsymbol{e}_1 + \sum_{2\le i\le d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d, -\boldsymbol{e}_1 + \sum_{2\le i\le d-1} \alpha_i \boldsymbol{e}_i + \boldsymbol{e}_d\Big\} \in \mathcal{D}_{\mathrm{F}}(\mathcal{P}) - \mathcal{D}_{\mathrm{A}}(\mathcal{P})\,;$$

this follows from the above-proved assertions (1) and (2).

The following example proves the stronger property $\mathrm{c}_{43} = 0$.

Example 3.12. *The desired polytope $\mathcal{Q}$ is obtained from $\mathcal{P}$ (see Example 3.11) by the replacement of the vertices $\boldsymbol{u}_i \in \operatorname{vert}\mathcal{P}_1$ by sufficiently close vertices $\boldsymbol{w}_i$, $i \in [2(d-1)]$, lying in the hyperplane $\boldsymbol{H}_1$. Let U_i be an ϵ-neighborhood of the point $\boldsymbol{u}_i \in \operatorname{vert}\mathcal{P}$ in the plane $\boldsymbol{H}_1 := \operatorname{aff}\mathcal{P}_1$, and $\boldsymbol{w}_i \in U_i$ an arbitrary point, $i \in [2(d-1)]$. Let us set $\mathcal{Q} := \operatorname{conv}(\mathcal{P}_2 \cup \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{2(d-1)}\})$.*

We will call $\{\boldsymbol{w}_j, \boldsymbol{v}_i\} \subset \operatorname{vert}\mathcal{Q}$ a diagonal pair *for $\mathcal{P}$ if $\{\boldsymbol{u}_j, \boldsymbol{v}_i\}$ is a diagonal pair for the polytope $\mathcal{P}$.*

Note that the properties (1) and (2) of diagonal pairs $\mathcal{P}$ also remain true for diagonal pairs $\mathcal{Q}$ under the condition that $\epsilon > 0$ is sufficiently small. It deserves explanation (concerning (2)) that an inclusion-maximal vertex set containing no diagonal pairs and coinciding with neither $\operatorname{vert}\mathcal{P}_2$ nor $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{2(d-1)}\}$ has the form $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_k, \boldsymbol{w}_1 + \boldsymbol{w}_2 + \dots + \boldsymbol{w}_k + \sum_{k+1\le i\le d-1} \alpha_i \boldsymbol{u}_i + \boldsymbol{e}_d : \alpha_i \in \{-1, 1\}, k+1 \le i \le d-1\}$, that is, it is composed of the points $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_k$ and other points lying in a $(d-k-1)$-dimensional face of the cube $\mathcal{P}_2$, and it thus necessarily lies in some hyperplane of $\mathbb{R}^d$. The latter, because of the small value of ϵ, will be close to the corresponding hyperplane that supports $\mathcal{P}$ and, as a consequence, it also supports $\mathcal{Q}$.

Now we describe an inductive approach to the choice of $\boldsymbol{w}_i \in U_i$, $i \in [2(d-1)]$, which leads to a desired polytope $\mathcal{Q}$.

Suppose $\boldsymbol{w}_1 := \boldsymbol{u}_1$. Let the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_r$ be already chosen. Let us consider all the planes spanned by the subsets of the set $\operatorname{vert}\mathcal{P}_2 \cup \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_r\}$ and not containing $\boldsymbol{H}_1$. By the Baire category theorem, their union does not cover $\boldsymbol{H}_1$. Let us pick for $\boldsymbol{w}_{r+1} \in U_{r+1}$ an arbitrary uncovered point.

Let us now show that the polytope $\mathcal{Q}$ does satisfy the prescribed condition $\mathcal{D}_{\mathrm{F}}(\mathcal{Q}) = \mathcal{D}_{\mathrm{A}}(\mathcal{Q})$.

Let $\mathcal{S} := \{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_l, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$ be an arbitrary F*-diagonal of the polytope $\mathcal{Q}$. In view of the above argument, it must contain a diagonal pair, say the pair $\{\boldsymbol{w}_s, \boldsymbol{w}_t\}$.*

We have chosen the vectors $\boldsymbol{w}_i$, $i \in [2(d-1)]$, in such a way that $\boldsymbol{E}_1 := \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_l\}$ and $\boldsymbol{E}_2 := \operatorname{aff}\{\boldsymbol{v}_1, \boldsymbol{w}_2, \dots \boldsymbol{v}_m\}$ are skew planes. Indeed, let $\boldsymbol{E}_1 := \boldsymbol{w}_1 + \boldsymbol{L}_1$, $\boldsymbol{E}_2 := \boldsymbol{w}_2 + \boldsymbol{L}_2$, where $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are linear subspaces of the space $\mathbb{R}^d$. Suppose to the contrary that $\boldsymbol{L}$ is a one-dimensional linear subspace contained in $\boldsymbol{L}_1 \cap \boldsymbol{L}_2$. Further, let j be a number such that $\operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots \boldsymbol{w}_j\} \not\supset \boldsymbol{w}_1 + \boldsymbol{L} \subset \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_j, \boldsymbol{w}_{j+1}\}$. This means that $\boldsymbol{w}_{j+1} \in \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_j\} + \boldsymbol{L} \subset \operatorname{aff}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_j, \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_m\}$, a contradiction with the choice of $\boldsymbol{w}_{j+1}$.

Now, if $\mathcal{S} \neq \{\boldsymbol{w}_s, \boldsymbol{v}_t\}$ *then, by Lemma* 3.2, $\operatorname{conv}\{\boldsymbol{w}_s, \boldsymbol{v}_t\}$ *is a face of* $\operatorname{conv}\mathcal{S}$ *and, by the definition of* F*-diagonals,* $\operatorname{conv}\{\boldsymbol{w}_s, \boldsymbol{v}_t\}$ *lies in some face of the polytope* $\mathbf{\Omega}$*. The obtained contradiction yields* $\mathcal{S} = \{\boldsymbol{w}_s, \boldsymbol{v}_t\}$*, that is,* $\mathcal{D}_{\mathrm{F}}(\mathcal{Q}) = \mathcal{D}^1_{\mathrm{F}}(\mathcal{Q}) = \mathcal{D}^1_{\mathrm{A}}(\mathcal{Q}) = \mathcal{D}_{\mathrm{A}}(\mathcal{Q})$. ◁

Thus, $\mathrm{c}_{43} = 0$ is established. □

Diagonals and combinatorial classification of polytopes

Two polytopes $\mathcal{P}$ and $\mathbf{\Omega}$, whose face lattices are isomorphic, by definition have the same combinatorial type. The relation “to have the same combinatorial type” is an equivalence relation, and it determines a classification on the set of all polytopes. A similar classification is also possible on the basis of the notion of diagonal.

Proposition 3.13. *Let* $\mathcal{L}$ *be the face lattice of a bounded convex polytope* $\mathcal{P}$ (*which is not a simplex*) *with the atom set* $\mathcal{L}^{\mathrm{a}} := \{\{\boldsymbol{v}\}\colon\ \boldsymbol{v} \in \operatorname{vert}\mathcal{P}\}$ *and the coatom set* $\mathcal{L}^{\mathrm{c}} := \{H \subset \operatorname{vert}\mathcal{P}\colon\ \operatorname{conv} H$ *is a facet of* $\mathcal{P}\}$*. The following relations hold:*

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathfrak{B}\left(\{\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}}\colon\ H \in \mathcal{L}^{\mathrm{c}}\}^{\perp}\right)\,, \tag{3.14}$$

$$\{\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}}\colon\ H \in \mathcal{L}^{\mathrm{c}}\}^{\perp} = \mathfrak{B}(\mathcal{D}_{\mathrm{G}}(\mathcal{P}))\,; \tag{3.15}$$

$$\mathcal{D}_{\mathrm{G}}(\mathcal{P}) = \mathbf{min}\,\mathcal{D}_{\mathrm{F}}(\mathcal{P})\,, \tag{3.16}$$

$$\mathcal{D}_{\mathrm{A}}(\mathcal{P}) = \left\{D \in \mathcal{D}_{\mathrm{G}}(\mathcal{P})\colon\ D - \{\boldsymbol{u}\} \in \mathcal{L},\ \forall \boldsymbol{u} \in D\right\}\,, \tag{3.17}$$

where, as earlier, $\mathfrak{B}(\cdot)$ *denotes the blocker of a set family;* $\mathfrak{I}(H) := \{Y \in \mathcal{L}\colon\ Y \preceq H\}$ *is the order ideal of the lattice* $\mathcal{L}$ *generated by its element* H*;* $\{\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}}\colon\ H \in \mathcal{L}^{\mathrm{c}}\}^{\perp} := \{\mathcal{L}^{\mathrm{a}} - (\mathfrak{I}(H) \cap \mathcal{L}^{\mathrm{a}})\colon\ H \in \mathcal{L}^{\mathrm{c}}\}$.

Proof. Assertions (3.14), (3.16), and (3.17) follow immediately from the definitions. Relation (3.15) follows easily from (3.14), taking into account Proposition 1.1. □

One says that a nonempty family $\mathcal{A} := \{A_1, A_2, \ldots, A_\alpha\}$ of nonempty subsets of a finite set $\bigcup_{i\in[\alpha]} A_i$ is *combinatorially isomorphic* to a family $\mathcal{B} := \{B_1, B_2, \ldots, B_\alpha\}$ of subsets of a set $\bigcup_{i\in[\alpha]} B_i$, if there exists a one-to-one map $\varphi\colon\ \bigcup_{i\in[\alpha]} A_i \to \bigcup_{i\in[\alpha] B_i}$ such that for each $i \in [\alpha]$ it holds $\varphi(A_i) = B_i$.

We will say that *polytopes* $\mathcal{P}$ *and* $\mathbf{\Omega}$ *have the same* A-, G- *or* F*-diagonal types, if the families* $\mathcal{D}_{\mathrm{A}}(\mathcal{P})$ *and* $\mathcal{D}_{\mathrm{A}}(\mathbf{\Omega})$, $\mathcal{D}_{\mathrm{G}}(\mathcal{P})$ *and* $\mathcal{D}_{\mathrm{G}}(\mathbf{\Omega})$, $\mathcal{D}_{\mathrm{F}}(\mathcal{P})$ *and* $\mathcal{D}_{\mathrm{F}}(\mathbf{\Omega})$*, respectively, are combinatorially isomorphic.*

It is obvious that the relation “to have the same diagonal combinatorial type” is also an equivalence relation, and it determines a combinatorial classification on the set of all polytopes.

Given a polytope $\mathcal{P}$, let us denote by $\mathcal{F}(\mathcal{P})$ the class of all the polytopes whose combinatorial type coincides with that of $\mathcal{P}$, and denote by $\mathcal{F}_{\mathrm{A}}(\mathcal{P})$ the class of all the polytopes whose A-diagonal type coincides with that of $\mathcal{P}$; the denotations $\mathcal{F}_{\mathrm{G}}(\mathcal{P})$ and $\mathcal{F}_{\mathrm{F}}(\mathcal{P})$ have the analogous meaning.

Proposition 3.14. *For an arbitrary polytope $\mathcal{P}$ the relations*

$$\mathcal{F}_A(\mathcal{P}) \supset \mathcal{F}(\mathcal{P}) = \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}_F(\mathcal{P})$$

hold.

Proof. Note that if a subset family of a certain type always determines uniquely a subset family of another type by means of operations that are invariant with respect to one-to-one maps, then the combinatorial equivalence on the basis of the combinatorial isomorphism of subset families of the first type implies the combinatorial equivalence on the basis of the combinatorial isomorphism of subset families of the second type; in other words, the combinatorial types defined by subsets of the second type are "wider." In view of the above argument, we obtain from Proposition 3.13:

$$\begin{aligned}
(3.14) \quad &\Longrightarrow \quad \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}_F(\mathcal{P})\,,\\
(3.15) \quad &\Longrightarrow \quad \mathcal{F}(\mathcal{P}) \supset \mathcal{F}_G(\mathcal{P})\,,\\
(3.16) \quad &\Longrightarrow \quad \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}(\mathcal{P})\,,\\
(3.17) \quad &\Longrightarrow \quad \mathcal{F}_A(\mathcal{P}) \supset \mathcal{F}_G(\mathcal{P})\,.
\end{aligned}$$

□

It is known that the facial structure of a simplicial polytope is determined (in our terminology) by the structure of the family of its A-diagonals. We can now interpret this statement as a corollary of the fact that $\mathcal{F}_G(\mathcal{P}) = \mathcal{F}(\mathcal{P})$ and, moreover, for the simplicial polytope $\mathcal{P}$ we have $\mathcal{D}_A(\mathcal{P}) = \mathcal{D}_G(\mathcal{P})$. A stronger assertion is also true:

Proposition 3.15. *If $\mathcal{P}$ is a simplicial polytope then $\mathcal{F}_A(\mathcal{P}) = \mathcal{F}(\mathcal{P}) = \mathcal{F}_G(\mathcal{P})$.*

Proof. In view of the above argument, it suffices to show that if $\mathcal{Q} \in \mathcal{F}_A(\mathcal{P})$ then $\mathcal{Q}$ is simplicial. For this, it suffices to verify that every face of the polytope $\mathcal{Q}$ has at most d vertices. If $V \subset \operatorname{vert}\mathcal{Q}$ determines a face of $\mathcal{Q}$ then V contains no A-diagonals of the polytope $\mathcal{Q}$, therefore $W \subset \operatorname{vert}\mathcal{Q}$ – the image of the set V, under the bijection that realizes a combinatorial isomorphism of the families of A-diagonals, does not contain A-diagonals of the polytope $\mathcal{P}$, and it thus lies in some face of $\mathcal{P}$. Hence, $|W| = |V| \le d$, because of the simpliciality of $\mathcal{P}$.

There are instances of polytopes for which $\mathcal{F}_A(\mathcal{P}) = \mathcal{F}_G(\mathcal{P}) = \mathcal{F}(\mathcal{P}) = \mathcal{F}_F(\mathcal{P})$.

Example 3.16. *Let $\mathcal{P} := \mathcal{P}_1 \perp \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are both simplices. Let us show that $\mathcal{Q} \in \mathcal{F}_A(\mathcal{P})$ implies $\mathcal{Q} \in \mathcal{F}_F(\mathcal{P})$. By Proposition* 3.6 (*relation* (3.8)), $\mathcal{D}_A(\mathcal{P}) = \{\operatorname{vert}\mathcal{P}_1, \operatorname{vert}\mathcal{P}_2\}$. *Let* $\operatorname{vert}\mathcal{Q} = V_1 \cup V_2$, *where V_1, V_2 are* A-*diagonals of $\mathcal{Q}$ corresponding to* A-*diagonals* $\operatorname{vert}\mathcal{P}_1$ *and* $\operatorname{vert}\mathcal{P}_2$ *of the polytope $\mathcal{P}$. It easily follows from the definitions that in this situation we have* $\mathcal{Q} = \operatorname{conv} V_1 \perp \operatorname{conv} V_2$ (*where the convex hulls are both simplices*) *and now, by Proposition* 3.6 (*relation* (3.10)), $\mathcal{D}_F(\mathcal{Q}) = \{V_1, V_2\}$, *that is,* $\mathcal{Q} \in \mathcal{F}_F(\mathcal{P})$. ◁

Thus, the simpliciality of the polytope $\mathcal{P}$ implies that $\mathcal{F}_A(\mathcal{P}) = \mathcal{F}_G(\mathcal{P})$ (Proposition 3.15). Besides (Example 3.16), we have instances of (simplicial) polytopes for which $\mathcal{F}_G(\mathcal{P}) = \mathcal{F}_F(\mathcal{P})$. The latter, though, seems to be rather an exception than a rule, even in the case of simplicial polytopes. At least, Example 3.9 provides simplicial polytopes $\mathcal{P}$ with $\mathcal{F}_G(\mathcal{P}) \neq \mathcal{F}_F(\mathcal{P})$. Indeed, it suffices to move, by a small perturbation, the vertices of the polytope $\mathcal{P}$ into general position for obtaining a polytope $\mathcal{Q}$ with the isomorphic face lattice (i.e., $\mathcal{Q} \in \mathcal{F}_G(\mathcal{P})$), but without F-diagonals that are not G-diagonals – at the same time, $\mathcal{P}$ has such an F-diagonal, namely vert $\mathcal{P}_2$ (i.e., $\mathcal{Q} \notin \mathcal{F}_F(\mathcal{P})$).

A particular case of Example 3.9, a 3-polytope $\mathcal{P}_0$ ($\mathcal{P}_2$ is a planar polygon), allows us to construct a series of examples of polytopes such that in the chain $\mathcal{F}_A(\mathcal{P}) \supset \mathcal{F}_G(\mathcal{P}) \supset \mathcal{F}_F(\mathcal{P})$ both inclusions are strict.

Example 3.17. *Let us consider in the space $\mathbb{R}^d$, where $d \geq 4$, two polytopes with the equal number $2d$ of vertices, namely $\mathcal{Q}_1$, a prism whose basis is a $(d-1)$-simplex, and $\mathcal{P}$, a $(d-3)$-fold pyramid over a 3-polytope $\mathcal{P}_0$. It is easily verified that for both polytopes the families of* A*-diagonals are empty and, thus, $\mathcal{Q}_1 \in \mathcal{F}_A(\mathcal{P})$, however, it is obvious that $\mathcal{Q}_1$ and $\mathcal{P}$ are not combinatorially equivalent and, thus, $\mathcal{Q}_1 \notin \mathcal{F}_G(\mathcal{P})$.* ◁

Let us now take into consideration a polytope $\mathcal{Q}_2$ obtained from $\mathcal{P}$ by the above-mentioned small perturbation of the vertices of $\mathcal{P}_0$ that moves them into general position in $\mathbb{R}^3$ without a change in the facial structure of $\mathcal{P}_0$. By Propositions 3.4 and 3.5, the polytope $\mathcal{Q}_2$ has no F-diagonals different from G-diagonals; at the same time, $\mathcal{P}$ has such a diagonal, a $(d-3)$-fold pyramid over the planar polygon $\mathcal{P}_2$. Thus, $\mathcal{Q}_2 \in \mathcal{F}_G(\mathcal{P})$, but $\mathcal{Q}_2 \notin \mathcal{F}_F(\mathcal{P})$ and, thus, $\mathcal{F}_A(\mathcal{P}) \supsetneq \mathcal{F}_G(\mathcal{P}) \supsetneq \mathcal{F}_F(\mathcal{P})$. □

3.2 Positive bases of linear spaces

A *positive basis* (*PB*) $\boldsymbol{B}$ of a linear space $\boldsymbol{L}$ is defined as an inclusion-minimal subset of $\boldsymbol{L}$ whose positive hull (i.e., the inclusion-minimal convex cone, with the apex at the origin $\mathbf{0} \in \boldsymbol{L}$, which contains $\boldsymbol{B}$) coincides with $\boldsymbol{L}$.

We study positive bases in $\mathbb{R}^n$, in particular, from the point of view of the combinatorial structure of two special subset families, the so-called minimal sub-bases and maximal one-sided subsets.

It is well known that for positive bases $\boldsymbol{B}$ of the space $\mathbb{R}^n$ the inequalities $n+1 \leq |\boldsymbol{B}| \leq 2n$ hold.

A positive basis $\boldsymbol{B}$ of $\mathbb{R}^n$ is called *minimal* when $|\boldsymbol{B}| = n+1$, and *maximal* when $|\boldsymbol{B}| = 2n$.

A subset $\boldsymbol{B}'$ of a positive basis $\boldsymbol{B}$ is called a *sub-basis* of the basis $\boldsymbol{B}$, if $\boldsymbol{B}'$ is a positive basis of the linear hull $\operatorname{lin}\boldsymbol{B}'$ of the set $\boldsymbol{B}'$. A sub-basis $\boldsymbol{B}' \subset \boldsymbol{B}$ is called a *minimal sub-basis*, if $\boldsymbol{B}'$ is a minimal positive basis of $\operatorname{lin}\boldsymbol{B}'$, that is, $\operatorname{pos}\boldsymbol{B}' = \operatorname{lin}\boldsymbol{B}'$ and $|\boldsymbol{B}'| = \dim\operatorname{lin}\boldsymbol{B}' + 1$.

A positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ is called a *strict positive basis* (*SPB*), if for any of its disjoint subsets $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ it holds $\operatorname{pos}\boldsymbol{B}_1 \cap \operatorname{pos}\boldsymbol{B}_2 = \{\mathbf{0}\}$. A positive basis $\boldsymbol{B}$ consisting of $n + r$ points in $\mathbb{R}^n$ is a SPB if and only if there exists a partition $\boldsymbol{B} = \boldsymbol{B}_1 \dot{\cup} \boldsymbol{B}_2 \dot{\cup} \cdots \dot{\cup} \boldsymbol{B}_r$, where $\boldsymbol{B}_1, \boldsymbol{B}_2, \dots, \boldsymbol{B}_r$ are pairwise disjoint minimal sub-bases of the positive basis of $\boldsymbol{B}$. In addition, the space is represented as the direct sum $\mathbb{R}^n = \operatorname{lin}\boldsymbol{B}_1 + \operatorname{lin}\boldsymbol{B}_2 + \cdots + \operatorname{lin}\boldsymbol{B}_r$. In particular, minimal and maximal positive bases are strict positive bases.

In the general case, the following assertion is true:

Proposition 3.18. *Let $\boldsymbol{B}$ be a positive basis consisting of $n + r$ points in $\mathbb{R}^n$. Then there exists a partition $\boldsymbol{B} = \boldsymbol{B}_1 \dot{\cup} \boldsymbol{B}_2 \dot{\cup} \cdots \dot{\cup} \boldsymbol{B}_r$ satisfying the following conditions:*
(1) $|\boldsymbol{B}_i| \geq |\boldsymbol{B}_{i+1}| \geq 2$, $i \in [r]$.
(2) $\operatorname{pos}(\boldsymbol{B}_1 \dot{\cup} \boldsymbol{B}_2 \dot{\cup} \cdots \dot{\cup} \boldsymbol{B}_j)$ *is a linear subspace of dimension* $|\boldsymbol{B}_1 \dot{\cup} \boldsymbol{B}_2 \dot{\cup} \cdots \dot{\cup} \boldsymbol{B}_j| - j$, $j \in [r]$.

We will call the set $\boldsymbol{X} \subset \mathbb{R}^n$ *one-sided* if it is contained entirely in an open half-space bounded by a hyperplane that contains $\mathbf{0}$. The inclusion-maximal one-sided subsets of some set will be called its *maximal one-sided subsets*.

In the study of positive bases, the notion of diagram of a positive basis turned out to be very useful.

Let us consider tuples of vectors $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n+r}) \subset \mathbb{R}^n$ and $\boldsymbol{E} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{n+r}) \subset \mathbb{R}^r$. The tuple $\boldsymbol{E}$ is called a *linear representation* of the tuple $\boldsymbol{B}$ if the $(n+r) \times (n + r)$ matrix $\mathbf{C}$ whose ith row is the vector $(b_{i1}, b_{i2}, \dots, b_{in}, e_{i1}, e_{i2}, \dots, e_{ir})$, where $(b_{i1}, b_{i2}, \dots, b_{in}) =: \boldsymbol{b}_i$ and $(e_{i1}, e_{i2}, \dots, e_{ir}) =: \boldsymbol{e}_i$, is nonsingular, and each of its first n columns is orthogonal to any of its last r columns. If the tuple $\boldsymbol{B}$ is spanned positively by the space $\mathbb{R}^n$ then the set of points from the tuple $\boldsymbol{E}$ is one-sided. Let $\mathbf{H}$ be a hyperplane that strictly separates the one-element set $\{\mathbf{0}\}$ from the convex hull $\operatorname{conv}\boldsymbol{E}$. Let us denote by $\overline{\boldsymbol{b}}_i$ the intersection point of the hyperplane $\mathbf{H}$ and the ray $\operatorname{pos}\{\boldsymbol{e}_i\}$. The tuple $\overline{\boldsymbol{B}} := (\overline{\boldsymbol{b}}_1, \overline{\boldsymbol{b}}_2, \dots, \overline{\boldsymbol{b}}_{n+r})$ is called a *diagram of the positive basis* $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n+r})$. We will use the following properties of the diagrams of positive bases:

Proposition 3.19. (i) *Each point of a diagram $\overline{\boldsymbol{B}} := (\overline{\boldsymbol{b}}_1, \overline{\boldsymbol{b}}_2, \dots, \overline{\boldsymbol{b}}_{n+r})$, which is a vertex of the polytope* $\operatorname{conv}\overline{\boldsymbol{B}}$*, occurs in the tuple $\overline{\boldsymbol{B}}$ at least twice, and* $\dim\overline{\boldsymbol{B}} = r - 1$.
(ii) *Any tuple satisfying conditions listed in* (i) *is a diagram of some positive basis $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_{n+r})$ of $\mathbb{R}^n$.*
(iii) *A set $\boldsymbol{B}$ is a strict positive basis if and only if* $\operatorname{conv}\overline{\boldsymbol{B}}$ *is a simplex, and every point $\overline{\boldsymbol{b}}_i$ from the tuple $\overline{\boldsymbol{B}}$ coincides with one of its vertices.*
(iv) *A subset $\boldsymbol{B}' := \{\boldsymbol{b}_i : i \in I \subseteq [n + r]\}$ of a positive basis $\boldsymbol{B}$ is a minimal sub-basis if and only if in a diagram $\overline{\boldsymbol{B}}$ the subtuple $\overline{\boldsymbol{B}} - \overline{\boldsymbol{B}'}$ coincides with $\boldsymbol{F} \cap \overline{\boldsymbol{B}}$ for some facet $\boldsymbol{F}$ of the polytope* $\operatorname{conv}\overline{\boldsymbol{B}}$.
(v) *A subset $\boldsymbol{B}'$ of a positive basis $\boldsymbol{B}$ is maximal one-sided if and only if the subtuple $\overline{\boldsymbol{B}} - \overline{\boldsymbol{B}'}$ is a* G*-diagonal of the tuple $\overline{\boldsymbol{B}}$.*

We will need below the following auxiliary statement:

Proposition 3.20. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be d-dimensional polytopes, $\mathcal{P}_1 \subset \mathcal{P}_2$, $\operatorname{vert} \mathcal{P}_1 \neq \operatorname{vert} \mathcal{P}_2$. Then a certain proper face of the polytope $\mathcal{P}_1$ has a nonempty intersection with the interior of the polytope $\mathcal{P}_2$.*

The maximal one-sided subsets of a positive basis

Let $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{n+r})$ be a positive basis of the space $\mathbb{R}^n$, and $\{\boldsymbol{B}_i : i \in [k]\}$ the family of maximal one-sided subsets of the set $\boldsymbol{B}$. We will use the notation $\alpha(\boldsymbol{B}) := \max_{i\in[k]} |\boldsymbol{B}_i|$ and $\beta(\boldsymbol{B}) := \min_{i\in[k]} |\boldsymbol{B}_i|$.

Let us characterize the strict positive bases in terms of their one-sided subsets:

Proposition 3.21. *A positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ is a strict positive basis if and only if any one-sided subset $\boldsymbol{B}' \subset \boldsymbol{B}$ contains at most n elements, that is, $\alpha(\boldsymbol{B}) = \beta(\boldsymbol{B}) = n$.*

Proof. The *necessity*. If $\boldsymbol{B}$ is an SPB, then all points of the tuple $\boldsymbol{B}$ are situated in the vertices of the $(r-1)$-dimensional simplex $\operatorname{conv} \overline{\boldsymbol{B}}$; as a consequence, any G-diagonal of the tuple $\boldsymbol{B}$ contains r elements, and therefore (see Proposition 3.19 (v)) any one-sided subset of $\boldsymbol{B}$ contains at most n elements.

The *sufficiency*. Let any one-sided subset of a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ contain at most n elements. Suppose to the contrary that $\boldsymbol{B}$ is not a SPB. Then two cases are possible:
(a) $\operatorname{conv} \overline{\boldsymbol{B}}$ is not a simplex;
(b) $\operatorname{conv} \overline{\boldsymbol{B}}$ is a simplex, and there exists a point $\overline{\boldsymbol{b}} \in \overline{\boldsymbol{B}}$, but $\overline{\boldsymbol{b}} \notin \operatorname{vert} \operatorname{conv} \overline{\boldsymbol{B}}$.

In the case (a), in the tuple $\overline{\boldsymbol{B}}$ there exists a G-diagonal $\overline{\boldsymbol{B}}(N)$, $N \subset [n+r]$, consisting of $k \leq r-1$ points, that is, $|N| = k \leq r-1$. Indeed, let $\overline{\boldsymbol{B}}(M)$ be a tuple, consisting of r points, of affine dimension $r-1 = \dim \overline{\boldsymbol{B}}$, that is, $\operatorname{conv} \overline{\boldsymbol{B}}(M)$ is a simplex. Since $\operatorname{conv} \overline{\boldsymbol{B}}$ is not a simplex, then (see Proposition 3.20) at least one face of the set $\operatorname{conv} \overline{\boldsymbol{B}}(M)$ has a nonempty intersection with the interior of the set $\operatorname{conv} \overline{\boldsymbol{B}}$. As a consequence, this face contains a G-diagonal $\overline{\boldsymbol{B}}(N)$, $N \subset M$, where $|N| < |M| = r$.

In the case (b), let us consider a tuple $\overline{\boldsymbol{B}}(M)$, where $M \subset [n+r]$, consisting of r points, of affine dimension $r-1$, which contains the point $\overline{\boldsymbol{b}}$. Since $\overline{\boldsymbol{b}} \notin \operatorname{vert} \operatorname{conv} \overline{\boldsymbol{B}}$, then (see Proposition 3.20) at least one face of the simplex $\operatorname{conv} \overline{\boldsymbol{B}}(M)$ has a nonempty intersection with the interior of the simplex $\operatorname{conv} \overline{\boldsymbol{B}}$. As a consequence, this face contains a G-diagonal $\overline{\boldsymbol{B}}(N)$, $N \subset M$, where $|N| < |M| = r$.

Thus, in both cases, $\overline{\boldsymbol{B}}$ contains a G-diagonal with less than r points from the tuple and, as a consequence (see Proposition 3.19 (v)), in the positive basis $\boldsymbol{B}$ there exists a maximal one-sided subset of cardinality at least $n+1$, a contradiction. □

Proposition 3.22. *Let $\boldsymbol{B}$ be a positive basis of the space $\mathbb{R}^n$ that consists of $n+r$ points. Then*

$$n \le \alpha(\boldsymbol{B}) \le \begin{cases} n+r-1 & \text{if } r \in [n-1]\,, \\ 0 & \text{if } r = n\,. \end{cases}$$

Moreover, if $r \in [n-1]$ then for each s, such that $n \le s \le n+r-1$, there exists a positive basis of the space $\mathbb{R}^n$ that consists of $n+r$ points, such that $\alpha(\boldsymbol{B}) = s$.

Proof. The inequality $n \le \alpha(\boldsymbol{B}) \le n+r-1$ is obvious. If $r = n$ then $\boldsymbol{B}$ is a maximal positive basis; as a consequence, $\boldsymbol{B}$ is a SPB and, by Proposition 3.21, we have $\alpha(\boldsymbol{B}) = n$.

Now suppose $1 \le r \le n-1$ and $n \le s \le n+r-1$. Let us construct a positive basis, with $n+r$ points, of $\mathbb{R}^n$, such that $\alpha(\boldsymbol{B}) = s$. For this, it suffices to choose for $\overline{\boldsymbol{B}}$ a tuple of points $\overline{\boldsymbol{B}} := (\overline{\boldsymbol{b}}_1, \overline{\boldsymbol{b}}_2, \ldots, \overline{\boldsymbol{b}}_{n+r})$ from $\mathbb{R}^{r-1}$ such that $\operatorname{conv}\overline{\boldsymbol{B}}$ is an $(r-1)$-dimensional simplex whose every vertex occurs in the tuple $\overline{\boldsymbol{B}}$ at least twice, and the remaining $n+r-2r = n-r \ge 1$ points lie in the relative interior of an $(s-n)$-dimensional face of $\operatorname{conv}\overline{\boldsymbol{B}}$. Then all G-diagonals of the tuple $\overline{\boldsymbol{B}}$ have dimension at least $(r-1)-(s-n)$ and, in addition, there exists a G-diagonal of dimension $(r-1)-(s-n)$, that is, it consists of $r-s+n$ points. Then $\alpha(\boldsymbol{B}) = n+r-(r-s+n) = s$. ☐

Proposition 3.23. *Let s and d be nonnegative integers such that $s \le d$. There exists a d-dimensional polytope with $2d-s+1$ vertices, all* G*-diagonals of which are s-dimensional.*

Proof. If $s := 1$ then a polytope with the desired property is the convex hull of the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_d, -\boldsymbol{x}_1, -\boldsymbol{x}_2, \ldots, -\boldsymbol{x}_d\}$, where $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_d\}$ is a linear basis of $\mathbb{R}^d$. We proceed by induction on s. Suppose that in the space $\mathbb{R}^{d-1}$, $d \ge s$, by the induction hypothesis, there exists a polytope $\mathcal{P}$ with $2(d-1)+1-(s-1)$ vertices, all G-diagonals of which are $(s-1)$-dimensional. Let us embed the polytope $\mathcal{P}$ into a hyperplane $\mathbf{H} \in \mathbb{R}^d$ and pick an arbitrary point $\boldsymbol{x} \notin \mathbf{H}$. Then the polytope $\operatorname{conv}(\mathcal{P} \dot\cup \{\boldsymbol{x}\})$, with $2d+1-s$ vertices, has dimension d, and all its G-diagonals are s-dimensional. ☐

Proposition 3.24. *Let $\boldsymbol{B}$ be a positive basis, with $n+r$ points, of $\mathbb{R}^n$. Then*

$$n \le \beta(\boldsymbol{B}) \le \begin{cases} n & \text{if } r \in \{1, 2, n-1, n\}\,, \\ n+r-2 & \text{if } 2 \le r < n-1\,. \end{cases}$$

If $n+r \ge 4(r-1)$ then for any s, such that $n \le s \le n+r-2$, there exists a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$, with $n+r$ points, such that $\beta(\boldsymbol{B}) = s$.

Proof. If $r := 1$ or $r := n$, then the set $\boldsymbol{B}$ is a SPB and, by Proposition 3.21, we have $\beta(\boldsymbol{B}) = n$. Suppose $r := n-1$. Then for a diagram $\overline{\boldsymbol{B}}$ we have $\dim \overline{\boldsymbol{B}} = n-2$, $|\operatorname{vert}\operatorname{conv}\overline{\boldsymbol{B}}| \le \frac{n+r}{2} = n - \frac{1}{2}$, that is, $\operatorname{conv}\overline{\boldsymbol{B}}$ is a simplex. As a consequence, in $\overline{\boldsymbol{B}}$ there exists a G-diagonal consisting of $n-1$ points, and thus $\beta(\boldsymbol{B}) = n$ (see Proposition 3.19 (v)). If $r \ge 2$, then there does not exist a tuple $\overline{\boldsymbol{B}}$, with $n+r$ points, of dimension $\dim \overline{\boldsymbol{B}} = r-1 \ge 1$, such that all its G-diagonals are zero-dimensional. As a consequence, in this case $\beta(\boldsymbol{B}) \le n+r-2$. Since $\operatorname{lin}\boldsymbol{B} = \mathbb{R}^n$, then $\beta(\boldsymbol{B}) \ge n$. Now suppose

$n + r \geq 4(r - 1)$, $n \leq s \leq n + r - 2$ and $r \geq 2$. In the space $\mathbb{R}^d$, where $d := r - 1$, one can construct (see Proposition 3.23) a d-dimensional polytope $\mathcal{P}$, with $2d + 1 - s'$ vertices, all G-diagonals of which are s'-dimensional, for any $s' \in [d]$. Suppose $s' := n + d - s$. Let us construct a tuple $\boldsymbol{E}$ consisting of $n + r$ points of the space $\mathbb{R}^n$, in which every point from the vertex set $\operatorname{vert}\mathcal{P}$ occurs at least twice, and $\operatorname{vert}\operatorname{conv}\boldsymbol{B} = \operatorname{vert}\mathcal{P}$. The tuple $\boldsymbol{E}$ is a diagram of some positive basis $\boldsymbol{B}$ of $\mathbb{R}^n$ (see Proposition 3.19 (ii)). Since all G-diagonals of the tuple $\boldsymbol{E}$ are s'-dimensional, then $\beta(\boldsymbol{B}) = n + r - (s' + 1) = s$ (see Proposition 3.19 (v)). □

Simplicial representation of a positive basis

Proposition 3.25. *Let* $\mathbf{H}$ *be a hyperplane of the space* $\mathbb{R}^n$ *that does not contain the origin* $\mathbf{0}$*, and let* $\boldsymbol{B}', \boldsymbol{B}''$ *be finite subsets of points from* $\mathbf{H}$*. Suppose* $\boldsymbol{B} := \boldsymbol{B}' \cup -\boldsymbol{B}''$*. Then the set* $\boldsymbol{B}$ *is a positive basis of the space* $\mathbb{R}^n$ *if and only if*

$$\dim(\boldsymbol{B}' \cup \boldsymbol{B}'') = n - 1\,, \tag{3.18}$$

$$\operatorname{ri}\operatorname{conv}\boldsymbol{B}' \cap \operatorname{ri}\operatorname{conv}\boldsymbol{B}'' \neq \emptyset\,, \tag{3.19}$$

and for any $\boldsymbol{B}_1' \subset \boldsymbol{B}_1$, $\boldsymbol{B}_2' \subset \boldsymbol{B}_2$, *such that* $\boldsymbol{B}_1' \cup \boldsymbol{B}_2' \neq \boldsymbol{B}_1 \cup \boldsymbol{B}_2$, *conditions* (3.18) *and* (3.19) *are not satisfied simultaneously.*

Proof. It suffices to show that properties (3.18) and (3.19) are equivalent to the claim that the set $\boldsymbol{B}' \cup -\boldsymbol{B}''$ spans positively the space $\mathbb{R}^n$. Suppose to the contrary that properties (3.18) and (3.19) are fulfilled. Assume that $\boldsymbol{B}' \cup -\boldsymbol{B}''$ does not span positively the space $\mathbb{R}^n$. Then $\operatorname{pos}(\boldsymbol{B}' \cup -\boldsymbol{B}'') \neq \mathbb{R}^n$, and there exists a hyperplane $\boldsymbol{\Gamma}$ that supports the convex hull $\operatorname{pos}(\boldsymbol{B}' \cup -\boldsymbol{B}'')$ at the point $\mathbf{0}$. It is clear that $\mathbf{H} \cap \boldsymbol{\Gamma} \neq \emptyset$, because otherwise we would come to a contradiction with the inclusion $\boldsymbol{B}', \boldsymbol{B}'' \subset \mathbf{H}$. Let $\boldsymbol{\Gamma}^+$ and $\boldsymbol{\Gamma}^-$ be two half-spaces bounded by the hyperplane $\boldsymbol{\Gamma}$ and, specifically, let us suppose that $\operatorname{pos}(\boldsymbol{B}' \cup -\boldsymbol{B}'') \subset \boldsymbol{\Gamma}^+$. Suppose $\boldsymbol{E} := \boldsymbol{\Gamma} \cap \mathbf{H}$, $\boldsymbol{E}^+ := \boldsymbol{\Gamma}^+ \cap \mathbf{H}$, $\boldsymbol{E}^- := \boldsymbol{\Gamma}^- \cap \mathbf{H}$. Then for the half-planes $\boldsymbol{E}^+$ and $\boldsymbol{E}^-$ the inclusions $\boldsymbol{B}' \subset \boldsymbol{E}^+$ and $\boldsymbol{B}'' \subset \boldsymbol{E}^-$ hold, that is, the plane $\boldsymbol{E}$ separates the set $\boldsymbol{E}^+$ from the set $\boldsymbol{E}^-$ in the hyperplane $\mathbf{H}$, but this is impossible in view of (3.18) and (3.19).

Now suppose that $\boldsymbol{B}' \cup -\boldsymbol{B}''$ spans positively the space $\mathbb{R}^n$. This implies that the sets $\boldsymbol{B}'$ and $\boldsymbol{B}''$ are not separated in the hyperplane $\mathbf{H}$ by an $(n-2)$-dimensional plane, but this implies (3.18) and (3.19). □

Thus, with a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ can be put in correspondence a pair $(\boldsymbol{B}', \boldsymbol{B}'')$, if one takes an arbitrary hyperplane $\mathbf{H}$, such that $\mathbf{0} \notin \mathbf{H}$, and $\{\gamma\boldsymbol{b} : \gamma \in \mathbb{R}\} \cap \mathbf{H} \neq \emptyset$ for each $\boldsymbol{b} \in \boldsymbol{B}$; then one sets $\boldsymbol{B}' := \{\gamma\boldsymbol{B} : \gamma > 0\} \cap \mathbf{H}$, $\boldsymbol{B}'' := \{-\gamma\boldsymbol{B} : \gamma > 0\} \cap \mathbf{H}$. In such a situation the positive bases $\boldsymbol{B}$ and $\boldsymbol{B}' \cup -\boldsymbol{B}''$ coincide up to positive factors.

We will call the pair $(\boldsymbol{B}', \boldsymbol{B}'')$ a *representation* of the positive basis $\boldsymbol{B}$.

A representation $(\boldsymbol{B}', \boldsymbol{B}'')$ of a positive basis $\boldsymbol{B}$ in which the convex hulls $\operatorname{conv}\boldsymbol{B}'$ and $\operatorname{conv}\boldsymbol{B}''$ are simplices will be called a *simplicial representation*.

Proposition 3.26. *Let $\boldsymbol{B}$ be a positive basis, consisting of $n+r$ points, of $\mathbb{R}^n$. There exists a linear basis $\boldsymbol{B}' \subset \boldsymbol{B}$ of $\mathbb{R}^n$ strictly separated in $\mathbb{R}^n$ from its complement up to $\boldsymbol{B}$ by a hyperplane that contains the origin $\mathbf{0}$. Moreover, the number of such distinct linear bases is at least 2^r.*

Proof. We proceed by induction on n and r. If $n := 1$ and $r := 1$ then the statement is obvious. Now suppose that $n > 1$ and $r > 1$. Let us use a partition $\boldsymbol{B} = \boldsymbol{E}_1 \dot\cup \boldsymbol{E}_2 \dot\cup \cdots \dot\cup \boldsymbol{E}_r$ with the properties guaranteed by Proposition 3.18. Let us denote $\boldsymbol{E} := \boldsymbol{E}_1 \dot\cup \boldsymbol{E}_2 \dot\cup \cdots \dot\cup \boldsymbol{E}_{r-1}$ and $p := |\boldsymbol{B}_r|$, where $p \geq 2$. Note that the convex hull $\operatorname{conv} \boldsymbol{E}_r$ is a simplex; moreover, the intersection $\operatorname{aff} \boldsymbol{E} \cap \operatorname{aff} \boldsymbol{E}_r$ consists of a unique point $\boldsymbol{z} \in \operatorname{ri} \operatorname{conv} \boldsymbol{E}_r$, and the subspace $\boldsymbol{L} := \operatorname{pos} \boldsymbol{B}$ has dimension $n - p - 1$. By the induction hypothesis, there exists a linear basis $\boldsymbol{E}' \subset \boldsymbol{E}$ strictly separable in $\boldsymbol{L}$ from $\boldsymbol{E} - \boldsymbol{E}'$ by some subspace $\boldsymbol{L}_1 \subset \boldsymbol{L}$ of dimension $n - p - 2$. In view of the above argument, any subset $\boldsymbol{E}_r' \subset \boldsymbol{E}_r$, which consists of $p - 1$ points, together with the set $\boldsymbol{E}' \subset \boldsymbol{L}$, form a linear basis of $\mathbb{R}^n$. Let us show that $\boldsymbol{E}' \cup \boldsymbol{E}_r'$ is strictly separated from its complement up to $\boldsymbol{B}$. Let $\mathbf{H}_1$ be a plane of dimension $p - 2$ containing the point $\boldsymbol{z}$ that strictly separates $\boldsymbol{E}_r'$ from $\boldsymbol{E}_r - \boldsymbol{E}_r'$ in $\operatorname{aff} \boldsymbol{E}_r$. Let us consider the hyperplane $\mathbf{H} := \operatorname{aff}(\boldsymbol{L} \cup \mathbf{H}_1) = \operatorname{lin}(\boldsymbol{L} \cup \mathbf{H}_1)$ in $\mathbb{R}^n$, and consider a subspace $\boldsymbol{L}_2$, which is contained in it, of dimension $n - 2$, with the conditions $\boldsymbol{L}_1 \subseteq \boldsymbol{L}_2$, $\boldsymbol{L}_2 \cap \boldsymbol{E} = \emptyset$. By a sufficiently small rotation of the hyperplane $\mathbf{H}$ around $\boldsymbol{L}_2$ in a relevant direction, we obtain a hyperplane $\mathbf{H}^*$, such that $\mathbf{0} \in \mathbf{H}^*$, which strictly separates the linear basis $\boldsymbol{E}' \cup \boldsymbol{E}_r'$ from its complement up to $\boldsymbol{B}$. It remains to note that, by the induction hypothesis, $\boldsymbol{E}'$ can be chosen in at least 2^{r-1} ways, and $\boldsymbol{E}_r'$ can be chosen in precisely $p \geq 2$ ways; thus, there exist at least 2^r desired linear bases. ☐

Proposition 3.27. *Any positive basis of the space $\mathbb{R}^n$ has a simplicial representation.*

Proof. According to Proposition 3.26, one can distinguish in the positive basis $\boldsymbol{B}$ a linear basis $\boldsymbol{B}_0 \subset \boldsymbol{B}$ such that the sets $\boldsymbol{B}_0$ and $\boldsymbol{B} - \boldsymbol{B}_0$ are strictly separated by a hyperplane $\mathbf{H}$ that contains the origin $\mathbf{0}$. Let $\boldsymbol{c}$ be a normal vector of $\mathbf{H}$, and $\langle \boldsymbol{c}, \boldsymbol{b} \rangle > 0$ for all $\boldsymbol{b} \in \boldsymbol{B}_0$, and also $\langle \boldsymbol{c}, \boldsymbol{b} \rangle < 0$ for all $\boldsymbol{b} \in \boldsymbol{B} - \boldsymbol{B}_0$. Suppose $\mathbf{H}_1 := \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{c}, \boldsymbol{x} \rangle = 1\}$. Then the pair $(\boldsymbol{B}', \boldsymbol{B}'')$, where $\boldsymbol{B}' := \{\lambda \boldsymbol{B}_0 : \lambda > 0\} \cap \mathbf{H}_1$, $\boldsymbol{B}'' := \{-\lambda(\boldsymbol{B} - \boldsymbol{B}_0) : \lambda > 0\} \cap \mathbf{H}_1$, is in fact a desired representation. Indeed, since $\boldsymbol{B}_0$ is linear basis, this implies that $\operatorname{conv} \boldsymbol{B}'$ is a simplex of dimension $n-1$. Suppose to the contrary that $\operatorname{conv} \boldsymbol{B}''$ is not a simplex. Let $\boldsymbol{x}_0 \in \operatorname{ri} \operatorname{conv} \boldsymbol{B}' \cap \operatorname{ri} \operatorname{conv} \boldsymbol{B}''$. By Carathéodory's theorem, there exists a subset $\boldsymbol{B}_1'' \subseteq \boldsymbol{B}''$ such that $\boldsymbol{x}_0 \in \operatorname{conv} \boldsymbol{B}_1''$, and $\operatorname{conv} \boldsymbol{B}_1''$ is a simplex, that is, $\boldsymbol{B}_1'' \subsetneq \boldsymbol{B}''$. Since $\dim \boldsymbol{B}' = n - 1$, then $\operatorname{ri} \operatorname{conv} \boldsymbol{B}' \cap \operatorname{ri} \operatorname{conv} \boldsymbol{B}_1'' \neq \emptyset$ and, in addition, $\dim(\boldsymbol{B}_1' \cup \boldsymbol{B}'') = n-1$, $\boldsymbol{B}_1' \cup \boldsymbol{B}_1'' \neq \boldsymbol{B}' \subset \boldsymbol{B}''$. We have come to a contradiction with Proposition 3.25. ☐

Regular positive bases

We will call a positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ *regular* if for some of its simplicial representation $(\boldsymbol{B}', \boldsymbol{B}'')$ the inclusison $\boldsymbol{B}'' \subset \operatorname{conv} \boldsymbol{B}'$ holds.

Proposition 3.28. *Let $\boldsymbol{B}$ be a positive basis, consisting of $n+r$ points, of $\mathbb{R}^n$. The following assertions are equivalent:*
(i) *$\boldsymbol{B}$ is a regular positive basis.*
(ii) *$\beta(\boldsymbol{B}) = n$.*
(iii) *$\boldsymbol{B}$ has precisely r minimal sub-bases.*
(iv) *The set $\operatorname{conv} \overline{\boldsymbol{B}}$ is a simplex.*
(v) *In every minimal sub-basis there are at least two points that are not contained in the other minimal sub-bases.*
(vi) *The family of minimal sub-bases of the positive basis $\boldsymbol{B}$ forms an inclusion-minimal cover of the set $\boldsymbol{B}$.*

Proof. (i) $\Rightarrow$ (iii). Let $(\boldsymbol{B}', \boldsymbol{B}'')$ be a simplicial representation of the positive basis $\boldsymbol{B}$, $\boldsymbol{B}'' \subset \operatorname{conv} \boldsymbol{B}'$, and $\boldsymbol{E} \subseteq \boldsymbol{B}$. Suppose $\boldsymbol{E}^+ := \{\lambda \boldsymbol{E}\colon \lambda > 0\} \cap \boldsymbol{B}'$ and $\boldsymbol{E}^- := \{-\lambda \boldsymbol{E}\colon \lambda > 0\} \cap \boldsymbol{B}''$. It follows from Proposition 3.25 that $\boldsymbol{E}$ is a minimal sub-basis if and only if the sets $\boldsymbol{E}^+$ and $\boldsymbol{E}^-$ are inclusion-minimal with respect to the property $\operatorname{ri} \operatorname{conv} \boldsymbol{E}^+ \cap \operatorname{ri} \operatorname{conv} \boldsymbol{E}^- \neq \emptyset$. Since the vertex set $\boldsymbol{B}'$ of the simplex $\operatorname{conv} \boldsymbol{B}'$ is affinely independent, then for every point $\boldsymbol{b} \in \operatorname{conv} \boldsymbol{B}'$ there exists a unique subset $\boldsymbol{B_b} \subseteq \boldsymbol{B}'$ such that $\boldsymbol{b} \in \operatorname{ri} \operatorname{conv} \boldsymbol{B_b}$; besides, if $\boldsymbol{C} \subsetneq \operatorname{conv} \boldsymbol{B}'$ and $\boldsymbol{x} \in \operatorname{ri} \operatorname{conv} \boldsymbol{C}$, then $\boldsymbol{B_b} \subseteq \boldsymbol{B_x}$ for any $\boldsymbol{b} \in \boldsymbol{C}$. Summarizing the above said, we conclude that the set $\boldsymbol{E} \subseteq \boldsymbol{B}$ is a minimal sub-basis of the regular positive basis $\boldsymbol{B}$ if and only if $\boldsymbol{E}^- = \{\boldsymbol{e}\}$, $\boldsymbol{E}^+ = \boldsymbol{B_e}$, for some vector $\boldsymbol{e} \in \boldsymbol{B}''$. Thus, $\boldsymbol{B}$ has precisely $|\boldsymbol{B}''| = r$ minimal sub-bases.

(iii) $\Rightarrow$ (iv). The set $\operatorname{conv} \overline{\boldsymbol{B}}$ in $\mathbb{R}^{r-1}$ has (see Proposition 3.19 (iv)) precisely r facets and, because of $\operatorname{aff} \overline{\boldsymbol{B}} = \mathbb{R}^{r-1}$, this means that $\operatorname{conv} \overline{\boldsymbol{B}}$ is a simplex.

(iv) $\Rightarrow$ (ii). In the simplex $\operatorname{conv} \overline{\boldsymbol{B}}$ there are G-diagonals consisting of precisely r points (all vertices of the simplex) and, as a consequence (see Proposition 3.19 (v)), in $\boldsymbol{B}$ there is a maximal one-sided subset with n points, that is, $\beta(\boldsymbol{B}) = n$.

(ii) $\Rightarrow$ (i). Let $\boldsymbol{B}'$ be a maximal one-sided subset, and $|\boldsymbol{B}'| = n$. Note that the set $\boldsymbol{B}'$ is linearly independent; it remains to show that $-(\boldsymbol{B} - \boldsymbol{B}') \subset \operatorname{pos} \boldsymbol{B}'$. Suppose to the contrary that $\boldsymbol{b} \in \boldsymbol{B} - \boldsymbol{B}'$ and $-\boldsymbol{b} \notin \operatorname{pos} \boldsymbol{B}'$. Since $\operatorname{pos} \boldsymbol{B}'$ is an acute cone, then there exists a hyperplane that contains the origin $\mathbf{0}$ and strictly separates the vector $-\boldsymbol{b}$ from $\boldsymbol{B}'$, and thus the point $\boldsymbol{b}$ and the set $\boldsymbol{B}'$ lie in the same open half-space, a contradiction with the maximality of the one-sided set $\boldsymbol{B}'$.

(iv) $\Rightarrow$ (v). Let $\boldsymbol{E}$ be a minimal sub-basis of $\boldsymbol{B}$. It is assigned a facet $\boldsymbol{F}$ of the simplex $\operatorname{conv} \overline{\boldsymbol{B}}$ such that $\boldsymbol{b} \in \boldsymbol{E} \iff \overline{\boldsymbol{b}} \notin \boldsymbol{F}$ for any vector $\boldsymbol{b} \in \boldsymbol{B}$ (see Proposition 3.19 (iv)). Let us choose a vector $\boldsymbol{x} \in \operatorname{vert} \operatorname{conv} \overline{\boldsymbol{B}}$ such that $\boldsymbol{x} \notin \boldsymbol{F}$. In the positive basis $\boldsymbol{B}$ there exist two distinct points $\boldsymbol{b}$ and $\boldsymbol{e}$ such that $\overline{\boldsymbol{b}} = \overline{\boldsymbol{e}} = \boldsymbol{x}$ (see Proposition 3.19 (i)). Since the vector $\boldsymbol{x}$ is contained in all other facets of the simplex $\operatorname{conv} \overline{\boldsymbol{B}}$, then the only minimal sub-basis containing the points $\boldsymbol{b}$ and $\boldsymbol{e}$ is the sub-basis $\boldsymbol{E}$, see Proposition 3.19 (iv).

(v) $\Rightarrow$ (vi). Easily verified.

(vi) $\Rightarrow$ (iv). Suppose to the contrary that the set $\operatorname{conv}\overline{\boldsymbol{B}}$ is not a simplex; then the number of its facets N satisfies the inequality $N \geq r + 1$. Because of the minimality of the cover by minimal sub-bases and the characterization of minimal sub-bases in the language of diagrams (see Proposition 3.19 (iv)), any $N - 1 \geq r$ facets of $\operatorname{conv}\overline{\boldsymbol{B}}$ have a nonempty intersection. Hence, by Helly's theorem applied to the space $\mathbb{R}^{r-1}$, we conclude that all N facets have a nonempty intersection, a contradiction. □

Corollary 3.29. *Any SPB is regular.*

Proof. By Proposition 3.21, for some strict positive basis $\boldsymbol{B}$ of the space $\mathbb{R}^n$ the equality $\beta(\boldsymbol{B}) = \alpha(\boldsymbol{B}) = n$ holds. □

Corollary 3.30. *If $n \in [4]$ then any positive basis of the space $\mathbb{R}^n$ is regular. Besides, for an arbitrary n, if $r \in \{1, 2, n-1, n\}$ then any positive basis of the space $\mathbb{R}^n$, consisting of $n + r$ points, is regular.*

Proof. In view of Proposition 3.24, in the listed cases we have $\beta(\boldsymbol{B}) = n$. □

Corollary 3.31. *For each $n \geq 5$, in the space $\mathbb{R}^n$ there exists a positive basis that is not regular.*

Proof. In view of Proposition 3.24, for each $n \geq 5$, in $\mathbb{R}^n$ there exists a positive basis $\boldsymbol{B}$, consisting of $n + 3 \geq 4(3-1)$ points, for which $\beta(\boldsymbol{B}) = n + 1$. □

Proposition 3.32. *Let $\boldsymbol{B}$ and $\boldsymbol{E}$ be positive bases, consisting of $n + r$ points, of the space $\mathbb{R}^n$. The family of minimal sub-bases of the positive basis $\boldsymbol{B}$ is combinatorially isomorphic to the family of minimal sub-bases of the positive basis $\boldsymbol{E}$ if and only if the family of maximal one-sided subsets of the set $\boldsymbol{B}$ is combinatorially isomorphic to the family of maximal one-sided subsets of the set $\boldsymbol{E}$.*

Proof. It suffices to recall that for any positive basis of the space $\mathbb{R}^n$, the family of its minimal sub-bases and the family of its maximal one-sided subsets uniquely determine each other. □

Proposition 3.33. *Let $I_1, I_2, \ldots, I_r \subset [n]$. The family $\{I_1, I_2, \ldots, I_r\}$ forms an inclusion-minimal cover of the set $[n]$ if and only if there exists a regular positive basis $\boldsymbol{B} := \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n, \ldots, \boldsymbol{b}_{n+r}\}$ of the space $\mathbb{R}^n$ with the family of minimal sub-bases $\{\boldsymbol{B}(I_1 \cup \{n+1\}), \boldsymbol{B}(I_2 \cup \{n+2\}), \ldots, \boldsymbol{B}(I_r \cup \{n+r\})\}$, where $\boldsymbol{B}(I') = \{\boldsymbol{b}_i : i \in I'\}$. Besides, $\boldsymbol{B}$ is a strict positive basis if and only if $I_1, I_2, \ldots, I_r$ are pairwise disjoint subsets.*

Proof. The *sufficiency* follows from Proposition 3.28 (v).

The *necessity*. Let $\mathbf{H}$ be a hyperplane in the space $\mathbb{R}^n$ that does not contain the origin $\mathbf{0}$. Let $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$ be the vertices of some simplex in $\mathbf{H}$. For each index $i \in [r]$, let us pick a point $\boldsymbol{e}_i \in \operatorname{ri}\operatorname{conv}\{\boldsymbol{b}_j : j \in I_i\}$. Let us denote $\boldsymbol{B}' := \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ and $\boldsymbol{B}'' := \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r\}$. Then $\operatorname{ri}\operatorname{conv}\boldsymbol{B}' \cap \operatorname{ri}\operatorname{conv}\boldsymbol{B}'' \neq \emptyset$, because $I_1 \cup I_2 \cup \cdots \cup I_r = [n]$. Suppose $\boldsymbol{B}_1' \subseteq \boldsymbol{B}'$ and $\boldsymbol{B}_1'' \subseteq \boldsymbol{B}''$. If $\boldsymbol{B}_1' = \boldsymbol{B}'$ and $\boldsymbol{B}_1'' \neq \boldsymbol{B}''$, then $\operatorname{ri}\operatorname{conv}\boldsymbol{B}_1' \cap \operatorname{ri}\operatorname{conv}\boldsymbol{B}_1'' = \emptyset$, because of the minimality of the cover $\{I_1, I_2, \dots, I_r\}$ of the set $[n]$. Suppose $\boldsymbol{B}'_1 \neq \boldsymbol{B}'$. Then $\boldsymbol{B}'_1$ lies in some facet $\boldsymbol{F}$ of the simplex $\operatorname{conv}\boldsymbol{B}'$. Suppose $\mathbf{H}_1 := \operatorname{aff}\boldsymbol{F}$, where $\dim \mathbf{H}_1 = n-2$. Note that $\operatorname{ri}\operatorname{conv}\boldsymbol{B}'_1 \cap \operatorname{ri}\operatorname{conv}\boldsymbol{B}''_1 \neq \emptyset$ implies that $\boldsymbol{B}'_1 \cup \boldsymbol{B}''_1 \subset \mathbf{H}_1$, that is, $\dim(\boldsymbol{B}'_1 \cup \boldsymbol{B}''_2) = n-2$. Thus, the conditions from Proposition 3.25 are fulfilled; therefore, the set $\boldsymbol{B} := \boldsymbol{B}'_1 \cup -\boldsymbol{B}''_1$ is a positive basis of the space $\mathbb{R}^n$. The pair $(\boldsymbol{B}'_1, \boldsymbol{B}''_1)$ is a simplicial representation of the positive basis $\boldsymbol{B}$, see the proof of Proposition 3.27. The positive basis $\boldsymbol{B}$ is regular, because, by construction, $\boldsymbol{B}'' \subset \operatorname{conv}\boldsymbol{B}'$. Let us denote $\boldsymbol{b}_{n+1} := -\boldsymbol{e}_i$, for $i \in [r]$. The regular positive basis $\boldsymbol{B}$ has the family of minimal subbases $\{\{\boldsymbol{b}_{n+i}\} \cup \{\boldsymbol{b}_j : j \in I_i\} : i \in [r]\}$, as we saw in the proof of the implication (i) $\Rightarrow$ (iii) of Proposition 3.28. The first assertion of the proposition is thus proved. The second assertion of the proposition, which concerns SPBs, is a known fact. ☐

3.3 Polytopes and infeasible systems of inequalities

We continue the study, initiated in Section 2.3, of infeasible systems (2.26) of homogeneous strict linear inequalities of rank r over the real Euclidean space $\mathbb{R}^r$. The subject of our investigation is the system of the more general form

$$\mathsf{S} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 : \ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m]\}, \tag{3.20}$$

with the set of its determining vectors $\boldsymbol{A}(\mathsf{S}) := \{\boldsymbol{a}_i : i \in [m]\}$.

We use the notation $\mathbf{J}$ to denote the family of the multi-indices of maximal feasible subsystems of the system S, and the notation $\mathbf{I}$ to denote the family of the multi-indices of its minimal infeasible subsystems. The characters $\mathfrak{q}$ and $\mathfrak{p}$ denote the number of multi-indices in the families $\mathbf{J}$ and $\mathbf{I}$, respectively; $\mathfrak{q} := \#\mathbf{J}$, $\mathfrak{p} := \#\mathbf{I}$.

We will need two statements:

Lemma 3.34. *System* (3.20) *is infeasible if and only if* $\sum_{i\in[m]} \lambda_i \boldsymbol{a}_i = \mathbf{0}$, *for some nonnegative numbers* $\lambda_1, \dots, \lambda_m$, *at least one of which is not* 0.

Lemma 3.35. *Let* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *be convex sets in* $\mathbb{R}^r$ *such that* $\operatorname{aff}(\boldsymbol{V} \cup \boldsymbol{W}) = \mathbb{R}^r$. *The sets* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *can be separated by a hyperplane if and only if* $\operatorname{ri}\boldsymbol{V} \cap \operatorname{ri}\boldsymbol{W} = \emptyset$.

An inequality $\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0$ of the system S is called *essential* if it does not belong to at least one its MFS. The system S is called *irreducible* if all its inequalities are essential.

Proposition 3.36. *A system* S *is irreducible if and only if the set* $\operatorname{pos}\boldsymbol{A}(\mathsf{S})$ *is a linear subspace.*

Proof. The *necessity.* Let system (3.20) be irreducible. Let us show that the set $\boldsymbol{K} := \operatorname{pos}\boldsymbol{A}(\mathsf{S})$ is a linear subspace. Assume the converse. Let us assign to an index subset $L \subseteq [m]$ the subset $\boldsymbol{A}_L(\mathsf{S}) := \{\boldsymbol{a}_i : i \in L\}$ of the corresponding vectors that define a subsystem, and suppose $J_0 := \{i \in [m] : \ \boldsymbol{a}_i \in \boldsymbol{K} \cap -\boldsymbol{K}\}$ and $J' := [m] - J_0 = \{i \in [m] : \boldsymbol{a}_i \notin \boldsymbol{K} \cap -\boldsymbol{K}\}$. Since the set $\boldsymbol{K} \cap -\boldsymbol{K}$ is a linear subspace, according to our assumption,

$\boldsymbol{K} \neq \boldsymbol{K} \cap -\boldsymbol{K}$, and thus $|J'| > 0$, $\dim(\boldsymbol{K} \cap -\boldsymbol{K}) \le r-1$. Let us denote by $\boldsymbol{K}^*$ the polar of the cone $\boldsymbol{K}$, that is, the set $\boldsymbol{K}^* := \{\boldsymbol{x} \in \mathbb{R}^r : \langle \boldsymbol{g}, \boldsymbol{x} \rangle \ge 0, \forall \boldsymbol{g} \in \boldsymbol{K}\}$. Taking into account the relation $\dim(\boldsymbol{K} \cap -\boldsymbol{K}) + \dim \boldsymbol{K}^* = r$, we have $\dim \boldsymbol{K}^* \ge 1$. For an arbitrary vector $\boldsymbol{b} \in \operatorname{ri} \boldsymbol{K}^*$ the relation $0 \le \langle \boldsymbol{K} \cap -\boldsymbol{K}, \boldsymbol{b} \rangle \le 0$ holds, therefore, $\langle \boldsymbol{A}_{J_0}(\mathsf{S}), \boldsymbol{b} \rangle = 0$.

Let us show that $\langle \boldsymbol{A}_{J'}(\mathsf{S}), \boldsymbol{b} \rangle > 0$. Suppose to the contrary that there exists an index $s \in J'$ such that $\langle \boldsymbol{a}_s, \boldsymbol{b} \rangle = 0$. Since $(\boldsymbol{K} \cap -\boldsymbol{K}) + \operatorname{lin} \boldsymbol{K}^* = \mathbb{R}^r$, then $\boldsymbol{a}_s = \boldsymbol{z}_1 + \boldsymbol{z}_2$, where $\boldsymbol{z}_1 \in \boldsymbol{K} \cap -\boldsymbol{K}$, $\boldsymbol{z}_2 \in \operatorname{lin} \boldsymbol{K}^*$, and $\boldsymbol{z}_2 \neq \boldsymbol{0}$ in view of $\boldsymbol{a}_s \notin \boldsymbol{K} \cap -\boldsymbol{K}$. But then $\langle \boldsymbol{a}_s, \boldsymbol{b} - \epsilon \boldsymbol{z}_2 \rangle = -\epsilon \langle \boldsymbol{z}_2, \boldsymbol{z}_2 \rangle < 0$ for any $\epsilon > 0$ and, in addition, $\boldsymbol{b} - \epsilon \boldsymbol{z}_2 \in \operatorname{lin} \boldsymbol{K}^*$, which contradicts the choice of the point $\boldsymbol{b} \in \boldsymbol{K}^*$. Further, since $\langle \boldsymbol{A}_{J_0}(\mathsf{S}), \boldsymbol{b} \rangle = 0$ and $\langle \boldsymbol{A}_{J'}(\mathsf{S}), \boldsymbol{b} \rangle > 0$, the equality $\sum_{i \in [m]} \lambda_i \boldsymbol{a}_i = 0$ implies the equality $\lambda_i = 0$ for all indices $i \in J'$. This means, with respect to Lemma 3.34, that the inequalities with the indices from J' are included in none of the IISs of the system S; as a consequence, they belong to all MFSs of the system S, a contradiction with the irreducibility of this system.

The *sufficiency*. Let $\boldsymbol{K} := \operatorname{pos} \boldsymbol{A}(\mathsf{S})$ be a linear subspace of $\mathbb{R}^n$. Let us show that system (3.20) is irreducible. It suffices to check that the inequality $\langle \boldsymbol{a}_1, \mathbf{x} \rangle > 0$ is essential. Since $\boldsymbol{K}$ is a linear subspace, then $-\boldsymbol{a}_1 = \sum_{i \in [m]} \lambda_i \boldsymbol{a}_i$ for some factors $\lambda_i \ge 0$, $i \in [m]$. The latter equality can be rewritten in the form $-\boldsymbol{a}_1 = \sum_{i=2}^{m} \lambda'_i \boldsymbol{a}_i$, where $\lambda'_i \ge 0$, $2 \le i \le m$. Let us choose among all such equalities some equality $-\boldsymbol{a}_1 = \sum_{i \in L} \alpha_i \boldsymbol{a}_i$ with the minimal number of indices in the set L. Let us show that the subsystem with the multi-index L of the system S is feasible. Assume the converse. Then, by Lemma 3.34, $\sum_{i \in L} \gamma_i \boldsymbol{a}_i = \boldsymbol{0}$ for some numbers $\gamma_i \ge 0$, $i \in L$, among which at least one number is positive. Suppose $\epsilon := \min\{\frac{\alpha_i}{\gamma_i} : i \in L, \gamma_i > 0\}$. Then

$$-\boldsymbol{a}_1 = \sum_{i \in L} \alpha_i \boldsymbol{a}_i - \sum_{i \in L} \epsilon \gamma_i \boldsymbol{a}_i = \sum_{i \in L} (\alpha_i - \epsilon \gamma_i) \boldsymbol{a}_i ,$$

where $\alpha_i - \epsilon \gamma_i \ge 0$, by the choice of ϵ, for each index $i \in L$, and $\alpha_i - \epsilon \gamma_i = 0$ for some index $i \in L$. But this contradicts the minimality of L. As a consequence, the subsystem with the multi-index L is feasible. On the other hand, the subsystem with the multi-index $L \cup \{1\}$ is infeasible because $-\boldsymbol{a}_1 = \sum_{i \in L} \alpha_i \boldsymbol{a}_i$, where $\alpha_i \ge 0$ for all $i \in L$. Thus, the inequality $\langle \boldsymbol{a}_1, \mathbf{x} \rangle > 0$ is essential because it does not belong to the MFS that contains the feasible subsystem with the multi-index L. The proposition is proved. □

It follows from Proposition 3.36 that the union of two irreducible subsystems of the system S is also its irreducible subsystem; therefore, the system S has an inclusion-maximal irreducible infeasible subsystem; let J_0 be the multi-index of this subsystem. Then the families $\{J_0 \cap J_s : s \in [\mathfrak{q}]\}$ and $\{I_s : s \in [\mathfrak{p}]\}$ are the families of the multi-indices of MFSs and IISs of the irreducible system $\{\langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0, i \in J_0\}$, respectively, if and only if $\{J_s : s \in [\mathfrak{q}]\}$ and $\{I_s : s \in [\mathfrak{p}]\}$ are the families of the multi-indices of MFSs and IISs of system (3.20), respectively. Thus, in the study of the combinatorial properties of the infeasible systems of linear inequalities we can restrict ourselves to the irreducible systems.

Combinatorial properties of polytopes and infeasible systems of linear inequalities

In the combinatorial theory of polytopes, a very useful method for investigating the combinatorial structure of polytopes is to consider their Gale transforms. The Gale transform establishes a relationship between the face structure of an r-dimensional polytope, which has m vertices, with the positive dependence of a certain arrangement of m vectors that lie in the space $\mathbb{R}^{m-r-1}$. In this section, we use the Gale transform to establish a link between the combinatorial properties of infeasible systems (3.20) and those of the facets and diagonals of polytopes; from now on, when considering the diagonals of polytopes, we always mean G-diagonals defined earlier on page 59. For a finite nonempty tuple of points $\boldsymbol{X} \subset \mathbb{R}^r$, we introduce the notion of diagonal as follows:

An inclusion-minimal subtuple $\boldsymbol{D} \subseteq \boldsymbol{X}$, with the property

$$\operatorname{conv} \boldsymbol{D} \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} \neq \emptyset ,$$

is called a *diagonal* of the tuple $\boldsymbol{X}$.

Let us recall relevant definitions. Let a finite sequence of points $\boldsymbol{X} := (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) \subset \mathbb{R}^r$, such that $\operatorname{aff} \boldsymbol{X} \simeq \mathbb{R}^r$, be given. Consider the $(m-r-1)$-dimensional space $\mathbf{K}(\boldsymbol{X})$ of solutions $(\beta_1, \beta_2, \ldots, \beta_m) \in \mathbb{R}^m$ to the following system of homogeneous linear equations:

$$\sum_{i \in [m]} \beta_i \boldsymbol{x}_i = \mathbf{0} , \quad \sum_{i \in [m]} \beta_i = 0 .$$

In the space $\mathbf{K}(\boldsymbol{X})$, fix its arbitrary ordered basis $(\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{m-r-1})$. Let $\boldsymbol{B}(\boldsymbol{X})$ be the $(m-r-1) \times m$ matrix whose rows are the vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{m-r-1}$ of this basis. For each index $i \in [m]$, let us denote by $\boldsymbol{x}_i^*$ the ith column of the matrix $\boldsymbol{B}(\boldsymbol{X})$ regarded as a vector from the space $\mathbb{R}^{m-r-1}$.

The sequence $\boldsymbol{X}^* = (\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_m^*)$ is called a *Gale transform* of the sequence $\boldsymbol{X}$.

A Gale transform is not unique. Since for any two distinct transforms there exists a linear isomorphism of $\mathbb{R}^{m-r-1}$ onto itself that sends the first Gale transform to the second transform, one usually takes an arbitrary basis of $\mathbf{K}(\boldsymbol{X})$ as a Gale transform.

In the general case, the Gale transform can contain coinciding points. Therefore, each of the pairwise distinct points of the Gale transform is assigned its multiplicity, which is the number of its preimages.

Let $L \subseteq [m]$. If $\boldsymbol{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\} \subset \mathbb{R}^r$ then the sequence $\boldsymbol{X}(L) := \{\boldsymbol{x}_i \in \boldsymbol{X} \colon i \in L\}$ is called a *cofacet* of the sequence $\boldsymbol{X}$, if $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{aff} \boldsymbol{X}([m] - L) = \emptyset$.

Recall some basic properties of the Gale transform.

Proposition 3.37. *Let a sequence of points $\boldsymbol{X} := (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) \subset \mathbb{R}^r$ be given, $\operatorname{aff} \boldsymbol{X} = \mathbb{R}^r$. Let $\boldsymbol{X}^* := (\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_m^*) \subset \mathbb{R}^{m-r-1}$ be a Gale transform of the tuple $\boldsymbol{X}$.*
(i) *If $\sum_{i \in [m]} \boldsymbol{x}_i = \mathbf{0}$ then the tuple $\boldsymbol{X}$ is a Gale transform of the sequence $\boldsymbol{X}^*$.*
(ii) $\sum_{i \in [m]} \boldsymbol{x}_i^* = \mathbf{0}$, $\operatorname{lin} \boldsymbol{X}^* = \mathbb{R}^{m-r-1}$, $\operatorname{pos} \boldsymbol{X}^* = \mathbb{R}^{m-r-1}$.
(iii) $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \boldsymbol{X}^*$.
(iv) *A subtuple $\boldsymbol{X}(L)$ is a coface of the tuple $\boldsymbol{X}$ if and only if* $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \boldsymbol{X}^*(L)$.

It is customary to formulate some properties of the Gale transform in terms of Gale diagrams.

A *Gale diagram* $\mathfrak{G}(\mathcal{P})$ of a bounded convex r-polytope $\mathcal{P} \subset \mathbb{R}^r$, with m vertices $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$, is the sequence of points $(\mathfrak{g}(\boldsymbol{x}_1), \mathfrak{g}(\boldsymbol{x}_2), \ldots, \mathfrak{g}(\boldsymbol{x}_m)) \in \mathbb{R}^{m-r-1}$ defined as follows: $\mathfrak{g}(\boldsymbol{x}_i) := \mathbf{0}$ for $\boldsymbol{x}_i^* = \mathbf{0}$, and $\mathfrak{g}(\boldsymbol{x}_i) := \frac{\boldsymbol{x}_i^*}{\|\mathbf{x}_i^*\|}$ for $\boldsymbol{x}_i^* \neq \mathbf{0}$.

Thus, a Gale diagram consists of a finite sequence of points from the set $\mathbb{S}^{m-r-2} \cup \{\mathbf{0}\}$, where $\mathbb{S}^{m-r-2}$ is the $(m-r-2)$-dimensional unit sphere centered at the origin $\mathbf{0}$.

Given a subsequence $\boldsymbol{V} \subseteq \operatorname{vert} \mathcal{P}$, $\mathfrak{G}(\boldsymbol{V})$ denotes the subset of the Gale diagram $\mathfrak{G}(\mathcal{P})$ that corresponds to the tuple $\boldsymbol{V}$.

Corollary 3.38. (i) *A set $\boldsymbol{X} \subset \operatorname{vert} \mathcal{P}$ is a coface of the vertex tuple of a polytope $\mathcal{P}$ if and only if $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \mathfrak{G}(\boldsymbol{X})$.*

(ii) *A set of points $\boldsymbol{X} := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\}$ represents the vertex set of some r-polytope $\mathcal{P}$ if and only if*
 (a) *either $\mathfrak{g}(\boldsymbol{x}_i) = \mathbf{0}$ for all $\boldsymbol{x}_i \in \boldsymbol{X}$, that is, when $\mathcal{P}$ is an r-simplex, or*
 (b) *for any open half-space $\mathbf{C}_>$ of $\mathbb{R}^{m-r-1}$, such that $\overline{\mathbf{C}_>} \ni \mathbf{0}$, the condition $|\{i \in [m]\colon \mathfrak{g}(\boldsymbol{x}_i) \in \mathbf{C}_>\}| \geq 2$ is satisfied.*

(iii) *If $\boldsymbol{F}$ is a face of the vertex tuple of a polytope $\mathcal{P}$, and $\boldsymbol{Z} := \operatorname{vert} \mathcal{P} - \operatorname{vert} \boldsymbol{F}$ is the corresponding coface, then* $\operatorname{ri} \operatorname{conv} \mathfrak{G}(\boldsymbol{Z}) \ni \mathbf{0}$.

(iv) *A polytope $\mathcal{P}$ is simplicial if and only if for each hyperplane $\mathbf{H}$ containing the origin $\mathbf{0}$, it holds $\mathbf{0} \notin \operatorname{ri} \operatorname{conv} (\mathfrak{G}(\mathcal{P}) \cap \mathbf{H})$.*

(v) *A polytope $\mathcal{P}$ is an r-faced pyramid if and only if in its Gale diagram the origin $\mathbf{0}$ has multiplicity r.*

We will need the following statement:

Lemma 3.39. *Suppose $\boldsymbol{X} := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m\} \subset \mathbb{R}^r$. The inclusion $\mathbf{0} \in \operatorname{ri} \operatorname{conv} X$ holds if and only if there exist coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$ such that $\sum_{i \in [m]} \lambda_i \boldsymbol{x}_i = \mathbf{0}$.*

Lemma 3.40. *Let $\boldsymbol{X} := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$ be a sequence of points in the space $\mathbb{R}^r$, $\operatorname{aff} \boldsymbol{X} = \mathbb{R}^r$, and $\boldsymbol{C} := (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$ the sequence of points in the space $\mathbb{R}^{m-r-1}$ such that $\boldsymbol{C} = \boldsymbol{X}^*$. The subsystem, with a multi-index L, of the system $\{\langle \boldsymbol{c}_i^*, \mathbf{x}\rangle > 0\colon \mathbf{x} \in \mathbb{R}^{m-r-1};\ i \in [m]\}$ is infeasible if and only if $\boldsymbol{X}(L)$ contains a nonempty coface of $\boldsymbol{X}$.*

Proof. By Lemma 3.34, the subsystem $\{\langle \boldsymbol{c}_i^*, \mathbf{x}\rangle > 0\colon \mathbf{x} \in \mathbb{R}^{m-r-1},\ i \in L\}$ is infeasible if and only if there exist nonnegative factors λ_k, $k \in L$, at least one of which is nonzero, such that $\sum_{k \in L} \lambda_k \boldsymbol{a}_k = \mathbf{0}$. By Lemma 3.39, the latter is possible if and only if $\mathbf{0} \in \operatorname{ri} \operatorname{conv} \boldsymbol{X}^*(L')$, where $\emptyset \neq L' \subseteq L$; this proves the statement, in view of Proposition 3.37 (iv). ☐

The following auxiliary assertion that concerns arbitrary point tuples was the basis of the definition of G-diagonals in the case of the vertex tuples of convex polytopes, see page 59.

Lemma 3.41. *Consider a tuple* $\boldsymbol{X} := (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$ *of points in* $\mathbb{R}^r$ *such that* $\operatorname{aff} \boldsymbol{X} = \mathbb{R}^r$. *A subtuple* $\boldsymbol{X}(L)$ *is a diagonal of the tuple* $\boldsymbol{X}$ *if and only if* $\boldsymbol{X}(L)$ *is contained in none of the faces of the tuple* $\boldsymbol{X}$, *and every subtuple* $\boldsymbol{X}(L')$, *where* $\emptyset \neq L' \subset L$, *is contained in at least one face of the tuple* $\boldsymbol{X}$.

Proof. It suffices to show that the subtuple $\boldsymbol{X}(L)$ is included in some face of the tuple $\boldsymbol{X}$ if and only if $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$.

Suppose that $\boldsymbol{X}(L) \subset \boldsymbol{X}(M)$, for some face $\boldsymbol{X}(M)$ of the tuple $\boldsymbol{X}$. For the face $\boldsymbol{X}(M)$, we by definition have $\operatorname{aff} \boldsymbol{X}(M) \cap \operatorname{conv} \boldsymbol{X}([m] - M) = \emptyset$. Using Lemma 3.35, one can verify that $\operatorname{conv} \boldsymbol{X}(M) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$, as a consequence, for the tuple $\boldsymbol{X}(L)$ the equality $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$ also holds.

Now let the relation $\operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X} = \emptyset$ hold for some subtuple $\boldsymbol{X}(L)$. Then, by Lemma 3.35, there exists a hyperplane $\mathbf{H}$ that separates the sets $\operatorname{conv} \boldsymbol{X}(L)$ and $\operatorname{conv} \boldsymbol{X}$. Let us suppose $M := \{i \in [m] : \ \boldsymbol{x}_i \in \mathbf{H}\}$. Note that $\operatorname{aff} \boldsymbol{X}(M) \cap \operatorname{conv} \boldsymbol{X}([m] - M) \subseteq \mathbf{H} \cap \operatorname{conv}([m] - M) = \emptyset$, that is, the subtuple is a face of the tuple $\boldsymbol{X}$. Further, $\operatorname{conv} \boldsymbol{X}(L) \subset \mathbf{H}$ and, thus, $\boldsymbol{X}(L) \subset \boldsymbol{X}(M)$. □

Another auxiliary assertion that we will use is as follows:

Lemma 3.42. *Let* $\boldsymbol{X}(L)$ *be a diagonal of a tuple* $\boldsymbol{X} := (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$ *of points in the space* $\mathbb{R}^r$. *Then the convex hull* $\operatorname{conv} \boldsymbol{X}(L)$ *is a simplex.*

Proof. Let us consider an arbitrary point $\boldsymbol{x}^* \in \operatorname{conv} \boldsymbol{X}(L) \cap \operatorname{ri} \operatorname{conv} \boldsymbol{X}$. It follows from Carathéodory's theorem on the representability of points in the convex hull of a subset from $\mathbb{R}^r$ that $\boldsymbol{x}^* \in \operatorname{conv} \boldsymbol{X}(L')$, for some subset $L' \subseteq L$ of cardinality $|L'| = \dim \boldsymbol{X}(L) + 1$. The assumption that the hull $\operatorname{conv} \boldsymbol{X}(L)$ is not a simplex contradicts the minimality of the subtuple $\boldsymbol{X}(L)$ because $|L'| = \dim \boldsymbol{X}(L) + 1 < |L|$. □

Theorem 3.43. *Let an irreducible infeasible system of linear inequalities* S, *and a sequence* $\boldsymbol{B} := (\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m)$ *of points, of affine dimension* $d := m - r - 1$, *in* $\mathbb{R}^d$, *be given, such that* $\boldsymbol{B}^* := (\boldsymbol{b}_1^*, \boldsymbol{b}_2^*, \dots, \boldsymbol{b}_m^*) = (\lambda_1 \boldsymbol{a}_1, \lambda_2 \boldsymbol{a}_2, \dots, \lambda_m \boldsymbol{a}_m)$ *for some factors* $\lambda_1, \lambda_2, \dots, \lambda_m > 0$. *The following statements are true:*
(i) *A set* $I \subset [m]$ *is the multi-index of some IIS of the system* S *if and only if its complement* $[m] - I$ *is the multi-index of a facet of the tuple* $\boldsymbol{B}$.
(ii) *A set* $J \subset [m]$ *is the multi-index of some MFS of the system* S *if and only if its complement* $[m] - J$ *is the multi-index of a diagonal of the tuple* $\boldsymbol{B}$.

Proof. Since all factors λ_k are positive, it suffices to consider, when proving the theorem, the system

$$\{\langle \boldsymbol{b}_i^*, \mathbf{x} \rangle > 0 : \ \mathbf{x} \in \mathbb{R}^r, \ i \in [m]\} \tag{3.21}$$

instead of system (3.20). Let I be the multi-index of an IIS of system (3.21). It follows from Lemma 3.40 that I is the multi-index of an inclusion-minimal nonempty coface of the tuple $\boldsymbol{B}$, that is, the complement $[m] - I$ is the multi-index of a facet of the tuple $\boldsymbol{B}$. The converse assertion is proved by applying Lemma 3.40 in the opposite direction.

Let us denote by $\mathbf{I}$ the family of the multi-indices of IISs of system (3.21), and by $\mathbf{F}$ the family of the multi-indices of facets of the tuple $\boldsymbol{B}$. A subset $J \subset [m]$ is the multi-index of a MFS of system (3.21) if and only if the complement $[m] - J$ is an inclusion-minimal subset of the set $[m]$, such that $([m] - J) \cap I \neq \emptyset$, for the multi-index $I \in \mathbf{I}$ of any IIS of system (3.21).

On the other hand, by Lemma 3.41, the complement $[m] - J$ is the multi-index of a diagonal of the tuple $\boldsymbol{B}$ if and only if the set $[m] - J$ is an inclusion-minimal subset of the set $[m]$, such that $([m] - J) \cap ([m] - F) \neq \emptyset$, for the multi-index F of any facet of the tuple $\boldsymbol{B}$.

Since $\mathbf{I} = \{[m] - F\colon\ F \in \mathbf{F}\}$, it follows from the above argument that J is the multi-index of a MFS of system (3.20) if and only if the complement $[m] - J$ is the multi-index of a diagonal of the tuple $\boldsymbol{B}$. □

Corollary 3.44. (i) *The family* $\mathbf{I}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of all IISs of some irreducible infeasible system* (3.20), *of rank* r, *over* $\mathbb{R}^r$ *if and only if the family* $\mathbf{I}^{\perp} := \{[m] - I\colon\ I \in \mathbf{I}\}$ *is the family of the multi-indices of all facets of some tuple of* m *points, of affine dimension* $d := m - r - 1$, *in the space* $\mathbb{R}^d$.

(ii) *The family* $\mathbf{J}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of all MFSs of some irreducible infeasible system* (3.20), *of rank* r, *over* $\mathbb{R}^r$ *if and only if the family* $\mathbf{J}^{\perp} := \{[m] - J\colon\ J \in \mathbf{J}\}$ *is the family of the multi-indices of diagonals of some tuple of* m *points, of affine dimension* $d := m - r - 1$, *in the space* $\mathbb{R}^d$.

We now turn to a study of the properties of infeasible systems of linear inequalities

$$\mathsf{S}_2 := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0\colon\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^r;\ \|\boldsymbol{a}_i\| = 1,\ i \in [m]\}\ , \tag{3.22}$$

of the form (3.20), whose set of determining vectors $\boldsymbol{A}(\mathsf{S}_2) := \{\boldsymbol{a}_i\colon\ i \in [m]\}$ satisfies the following structural condition: *for any open half-space* $\mathbf{C}_{>} \subset \mathbb{R}^r$, *bounded by a codimension one linear subspace, the condition*

$$|\{\boldsymbol{a} \in \boldsymbol{A}(\mathsf{S}_2)\colon\ \boldsymbol{a} \in \mathbf{C}_{>}\}| \geq 2 \tag{3.23}$$

is satisfied.

Proposition 3.45. *For system* (3.22, 3.23), *the inclusion* $\mathbf{0} \in \operatorname{ri}\operatorname{conv} \boldsymbol{A}(\mathsf{S}_2)$ *holds.*

Proof. Suppose to the contrary that $\mathbf{0} \notin \operatorname{ri}\operatorname{conv} \boldsymbol{A}(\mathsf{S}_2)$. Then there exists a subset $\boldsymbol{A}' \subset \boldsymbol{A}(\mathsf{S}_2)$ such that $\dim\operatorname{aff} \boldsymbol{A}' = r - 1$, and $\boldsymbol{A}(\mathsf{S}_2)$ is contained in a closed half-space bounded by the hyperplane $\operatorname{aff} \boldsymbol{A}'$ because the impossibility of the mentioned inclusion would mean that the tuple $\boldsymbol{A}(\mathsf{S}_2)$ is contained in some open hemisphere of the unit sphere $\mathbb{S}^{r-1}$ and, as a consequence, the feasibility of the system S_2. But the inclusion $\boldsymbol{A}(\mathsf{S}_2)$ into a closed hemisphere of the sphere $\mathbb{S}^{r-1}$ contradicts condition (3.23). □

Proposition 3.46. *System of linear inequalities* (3.22, 3.23) *is irreducible.*

Proof. Suppose to the contrary that, according to Proposition 3.36, the set $\operatorname{pos}\boldsymbol{A}(\mathsf{S}_2)$ is not a linear subspace. But this is possible (see the proof of Proposition 3.45) if and only if $\operatorname{pos}\boldsymbol{A}(\mathsf{S}_2)$ is some closed half-space $\mathbf{C}_{\geq}$ bounded by a codimension one linear subspace $\mathbf{H}$ of $\mathbb{R}^n$; but this contradicts condition (3.23). □

Propositions 3.39 and 3.45 imply the following statement:

Corollary 3.47. *For a system* S_2 *of the form* (3.22, 3.23), *there exists a tuple of positive factors* λ_i, *such that the origin* $\mathbf{0}$ *is a convex combination of the vector tuple* $\boldsymbol{A}(\mathsf{S}_2)$ *with these coefficients, that is,* $\mathbf{0} = \sum_{i\in[m]} \lambda_i \boldsymbol{a}_i$, $\lambda_i > 0$, $\sum_{i\in[m]} \lambda_i = 1$.

The following proposition clarifies the link between the properties of systems (3.22, 3.23) and those of convex polytopes.

Proposition 3.48. *Let* S_2 *be a system of the form* (3.22, 3.23).
- (i) *A family* $\mathbf{I}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of IISs of the system* S_2 *if and only if the family* $\mathbf{I}^{\perp} := \{[m] - I\colon\ I \in \mathbf{I}\}$ *is the family of the multi-indices of facets of some bounded convex* $(m-r-1)$*-polytope with* m *vertices.*
- (ii) *A family* $\mathbf{J}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of MFSs of the system* S_2 *if and only if the family* $\mathbf{J}^{\perp} := \{[m] - J\colon\ J \in \mathbf{J}\}$ *is the family of the multi-indices of diagonals of some bounded convex* $(m-r-1)$*-polytope with* m *vertices.*

The mentioned polytope is not a pyramid; in particular, it is not a simplex.

Proof. Let us associate with the system S_2 a modified system $\mathsf{S}_2' := \{\langle \lambda_i \boldsymbol{a}_i, \mathbf{x}\rangle > 0\colon\ \boldsymbol{a}_i \in \boldsymbol{A}(\mathsf{S}_2)\}$ such that the coefficients λ_i satisfy the conditions from Corollary 3.47. Since all coefficients λ_i are positive, the sets of solutions to feasible subsystems, with the same multi-indices, for the systems S_2 and S_2' coincide. Since $\sum_{i\in[m]} \lambda_i \boldsymbol{a}_i = \mathbf{0}$, in accordance with Proposition 3.37 (i), the tuple $\{\lambda_1\boldsymbol{a}_1, \lambda_2\boldsymbol{a}_2, \ldots, \lambda_m\boldsymbol{a}_m\}$ is a Gale transform of the tuple $\{\lambda_1\boldsymbol{a}_1, \lambda_2\boldsymbol{a}_2, \ldots, \lambda_m\boldsymbol{a}_m\}^*$. By Corollary 3.38 (ii)(b), and according to condition (3.23), we obtain $\{\lambda_1\boldsymbol{a}_1, \lambda_2\boldsymbol{a}_2, \ldots, \lambda_m\boldsymbol{a}_m\}^*$ is the vertex tuple of a bounded convex $(m-r-1)$-polytope in $\mathbb{R}^{m-r-1}$. The proof of assertions (i) and (ii) is completed by applying Corollary 3.44.

Recall that the vector tuple $\boldsymbol{A}(\mathsf{S}_2)$, by convention, does not contain the origin $\mathbf{0}$; therefore, in accordance with Corollaries 3.38 (ii)(a) and 3.38 (v), the tuple $\{\lambda_1\boldsymbol{a}_1, \lambda_2\boldsymbol{a}_2, \ldots, \lambda_m\boldsymbol{a}_m\}^*$ cannot be the vertex tuple of a pyramid. □

Combinatorially dual systems of linear inequalities

Proposition 3.49. *Let* S_2 *be a rank* r *infeasible system* (3.22) *of* m *homogeneous strict linear inequalities that has* $\mathfrak{p}$ *minimal infeasible subsystems. Let* $\mathbf{J}$ *and* $\mathbf{I}$ *be the families of the multi-indices of all its MFSs and IISs, respectively.*

The system S_2 *satisfies condition* (3.23) *if and only if there exists a rank* $r + \mathfrak{p} - m$ *infeasible system* S_2^0 (*whose families of the multi-indices of all MFSs and IISs are denoted by* $\mathbf{J}^0$ *and* $\mathbf{I}^0$, *respectively*) *of* $\mathfrak{p}$ *homogeneous strict linear inequalities, that has m minimal infeasible subsystems, such that*

(i) *for each multi-index* $I \in \mathbf{I}$ *of an IIS, there exists the index* $t \in [\mathfrak{p}]$ *of an inequality of the system* S_2^0 *such that* $\#\{I^0 \in \mathbf{I}^0 :\ t \in I^0\} = |I|$;

(ii) *for each multi-index* $I^0 \in \mathbf{I}^0$ *of an IIS, there exists the index* $t \in [m]$ *of an inequality of the system* S_2 *such that* $\#\{I \in \mathbf{I} :\ t \in I\} = |I|$;

(iii) *for each multi-index* $J \in \mathbf{J}$ *of a MFS, there exists a family* $\mathcal{M} \subset \mathbf{I}^0$ *of the multi-indices of IISs,* $\#\mathcal{M} = m - |J|$, *such that* $\bigcup_{I \in \mathcal{M}} I = [\mathfrak{p}]$;

(iv) *for each multi-index* $J^0 \in \mathbf{J}^0$ *of a MFS, there exists a family* $\mathcal{M} \subset \mathbf{I}$ *of the multi-indices of IISs,* $\#\mathcal{M} = \mathfrak{p} - |J|$, *such that* $\bigcup_{I \in \mathcal{M}} I = [m]$.

Proof. Proposition 3.48 puts in correspondence with the system S_2 a convex $(m-r-1)$-polytope $\mathcal{P}$ with m vertices and $\mathfrak{p}$ facets. In turn, for this polytope there exists a *dual* $(m-r-1)$-polytope $\mathcal{P}^0$ with $\mathfrak{p}$ vertices and m facets; the face lattices of the polytopes $\mathcal{P}$ and $\mathcal{P}^0$ are anti-isomorphic. The proof is completed by reapplying of Proposition 3.48, to the polytope $\mathcal{P}^0$; and, besides, we need to show that the set of vectors determining the system S_2^0 does not contain the origin $\mathbf{0}$. Suppose to the contrary that it is not the case. Then, in accordance with Corollary 3.38 (v), the polytope $\mathcal{P}^0$ is a pyramid. But the pyramid $\mathcal{P}^0$ is a polytope which is dual to the pyramid $\mathcal{P}$; thus, the set $\boldsymbol{A}(\mathsf{S}_2)$ of vectors determining the system S_2 contains the origin $\mathbf{0}$, a contradiction with the hypothesis of the proposition. □

Systems of linear inequalities and simplicial/simple polytopes. The Dehn–Sommerville relations. Bounds for the number of subsystems

In this section, in addition to the linear inequality systems of the form (2.26), (3.20), and (3.22, 3.23), we consider the inequality system S_2, whose description is given in (3.22, 3.23), which satisfies one of the following new conditions:

$$\textit{every subsystem of rank at most } r-1 \textit{ is feasible}; \tag{3.24}$$

$$\textit{each inequality belongs to } \mathfrak{p} + r - m + 1 \textit{ IISs}. \tag{3.25}$$

We will show below, in Proposition 3.52, that a system of the form (3.22, 3.23, 3.24) can equivalently be defined as a system of the form (3.22, 3.23), such that every of its minimal infeasible subsystem is composed of $r + 1$ inequalities.

The next statement follows immediately from definitions:

Proposition 3.50. *If in the set* $A(\mathsf{S}_2)$ *of vectors, which define a rank* 2 *system* S_2 *of the form* (3.22, 3.23), *there are no antipodal pairs then the system* S_2 *satisfies condition* (3.24).

Proposition 3.51. *If* S_2 *is a system of the form* (3.22, 3.23, 3.24) *then the following assertions are true:*

(i) *A family* $\mathbf{I}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of IISs of the system* S_2 *if and only if the family* $\mathbf{I}^{\perp} := \{[m] - I\colon\ I \in \mathbf{I}\}$ *is the family of the multi-indices of facets of some bounded convex simplicial* $(m - r - 1)$*-polytope with* m *vertices.*

(ii) *A family* $\mathbf{J}$ *of subsets of the set* $[m]$ *is the family of the multi-indices of MFSs of the system* S_2 *if and only if the family* $\mathbf{J}^{\perp} := \{[m] - J\colon\ J \in \mathbf{J}\}$ *is the family of the multi-indices of diagonals of some bounded convex simplicial* $(m - r - 1)$*-polytope with* m *vertices.*

The mentioned polytope is not a simplex.

Proof. First, it is necessary to repeat the argument that we used when proving Proposition 3.48. Taking into account that, for any hyperplane $\mathbf{H}$ that contains the origin $\mathbf{0}$, we have $\mathbf{0} \notin \operatorname{conv}(\boldsymbol{A}(\mathsf{S}_2) \cap \mathbf{H})$, the proof is completed by applying Corollary 3.38 (iv). □

Proposition 3.52. *A system of the form* (3.22, 3.23) *satisfies condition* (3.24) *if and only if all its minimal infeasible subsystems have the same cardinality* $r + 1$.

Given a system S_2 *of the form* (3.22, 3.23, 3.24)*, the number of all its infeasible subsystems, of cardinality* k*, which contain a fixed IIS, is equal to* $\binom{m-r-1}{m-k}$.

Proof. Since in accordance with Proposition 3.51, all index sets $[m] - I$, where $I \in \mathbf{I}$, are the multi-indices of the vertex tuples of facets of a simplicial $(m - r - 1)$-polytope, they all have the same cardinality $m - r - 1$. As a consequence, any multi-index I of a minimal infeasible subsystem has cardinality $r + 1$. □

The next two statements show that the number of feasible and infeasible subsystems, of different cardinalities, of a system S_2 of the form (3.22, 3.23, 3.24) obey special relations.

Proposition 3.53. *Let* S_2 *be a system of the form* (2.26,3.23,3.24)*; let* ν_i *and* τ_i *be the numbers of its feasible and infeasible subsystems, of cardinality* i*, respectively.*

The following relations (*where* x *is a formal variable*) *hold:*

$$\begin{cases} \nu_j = \binom{m}{j}, \ \text{if } 0 \le j \le r\,, \\ \nu_{m-1} = \nu_m = 0\,, \\ \sum_{j=r+1}^{m} \left(\binom{m}{j} - \nu_j\right)(\mathrm{x} - 1)^{m-j} = \sum_{j=r+1}^{m} (-1)^{j-r-1} \left(\binom{m}{j} - \nu_j\right) \mathrm{x}^{m-j}\,. \end{cases}$$

We will call these relations the Dehn–Sommerville equations for the feasible subsystems of the system S_2. *The substitution in these relations of* $\binom{m}{j} - \nu_j$ *by* τ_j *leads to the* Dehn–Sommerville equations for the infeasible subsystems of the system S_2.

Proof. In accordance with Proposition 3.51, the index sets $[m] - J$, where $J \in \mathbf{J}$, are the multi-indices of diagonals of the vertex tuple of a simplicial $(m - r - 1)$-polytope. We will regard the family of its diagonals as a family of subsets of the atom set of its face lattice $\mathcal{L}$; this lattice is of rank $\rho(\mathcal{L}) = m - r$, and its atoms are the vertices of the polytope under consideration. Let n_t be the number of the t-subsets of the atom set of the lattice $\mathcal{L}$ that contain, as a subset, at least one diagonal. Let W_j be the number of rank j elements of the lattice $\mathcal{L}$; here $W_0 = 1$ and $W_1 = m$. In other words, W_j denotes the number of faces with j vertices. We have

$$n_t = \begin{cases} \binom{W_1}{t} - W_t = \binom{m}{t} - W_t, & \text{if } 0 \le t \le m - r - 1\,, \\ \binom{W_1}{t} = \binom{m}{t}, & \text{if } m - r \le t \le m\,. \end{cases}$$

It is clear that $n_0 = n_1 = 0$.
Further, $\nu_k = n_{m-k}$, therefore,

$$\nu_k = \begin{cases} \binom{m}{k}, & \text{if } 0 \le k \le r\,, \\ \binom{m}{k} - W_{m-k}, & \text{if } r + 1 \le k \le m\,. \end{cases}$$

Let us consider the case $r+1 \le k \le m$ in more detail; in this situation, $\nu_k = \binom{m}{k} - W_{m-k}$. The Dehn–Sommerville equations for the Whitney numbers of the second kind W_i of the lattice $\mathcal{L}$ are as follows:

$$\sum_{i=0}^{\rho(\mathcal{L})-1} W_i(\mathrm{x} - 1)^i = \sum_{i=0}^{\rho(\mathcal{L})-1} (-1)^{\rho(\mathcal{L})-i-1} W_i \mathrm{x}^i\,,$$

or, in our case,

$$\sum_{i=0}^{m-r-1} W_i(\mathrm{x} - 1)^i = \sum_{i=0}^{m-r-1} (-1)^{m-r-i-1} W_i \mathrm{x}^i, \; W_0 = 1, \; W_1 = m\,.$$

Let us equivalently rewrite the latter expression in the form

$$\sum_{j=r+1}^{m} W_{m-j}(\mathrm{x} - 1)^{m-j} = \sum_{j=r+1}^{m} (-1)^{j-r-1} W_{m-j} \mathrm{x}^{m-j}, \; W_0 = 1, \; W_1 = m\,.$$

By substituting W_{m-j} by $\binom{m}{j} - \nu_j$, we complete the proof. □

As an illustration, we present the solutions to several initial Dehn–Sommerville equations.

Corollary 3.54. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24), *and* ν_i *the number of its feasible subsystems of cardinality* i. *Then*
(i) *if* $m = r + 3$, *then* $\nu_{r+1} = \binom{m}{2} - m$;
(ii) *if* $m = r + 4$, *then*

$$\begin{aligned} \nu_{r+1} &= \binom{m}{3} - 2m + 4\,, \\ \nu_{r+2} &= \binom{m}{2} - 3m + 6\,; \end{aligned}$$

(iii) *if* $m = r + 5$, *then*

$$\begin{aligned} v_{r+2} &= 2v_{r+1} - 2\binom{m}{4} + \binom{m}{3}\,, \\ v_{r+3} &= -v_{r+1} + \binom{m}{4} - \binom{m}{2} + m\,. \end{aligned}$$

Corollary 3.55. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24), *and* τ_i *the number of its infeasible subsystems of cardinality* i. *Then the following relations hold:*

(i) *if* $r + 1 \le i \le m$, *then*

$$\tau_i = \sum_{j=r+1}^{m} (-1)^{j-r-1} \binom{m-j}{m-i} \tau_j\,;$$

(ii) *if* $k \in [\lfloor \frac{m-r}{2} \rfloor]$, *then*

$$\sum_{j=r+k}^{m} (-1)^{m-j+1} \binom{j-r-1}{k-1} \tau_j = \sum_{j=m-k+1}^{m} (-1)^{r+j} \binom{j-r-1}{m-r-k} \tau_j\,.$$

Proof. The above relations follow immediately from Proposition 3.51 and from the assertions: if $\mathcal{L}$ is the face lattice of a simplicial polytope, then

(i) the Dehn–Sommerville equations for $\mathcal{L}$ are equivalent to

$$W_i = \sum_{j=0}^{\rho(\mathcal{L})-1} (-1)^{\rho(\mathcal{L})-j-1} \binom{j}{i} W_j\,,$$

for $i \in [\rho(\mathcal{L}) - 1]$;

(ii) the Dehn–Sommerville equations for $\mathcal{L}$ are equivalent to

$$\sum_{j=0}^{k-1} (-1)^{\rho(\mathcal{L})+j} \binom{\rho(\mathcal{L})-j-1}{\rho(\mathcal{L})-k} W_j = \sum_{j=0}^{\rho(\mathcal{L})-k} (-1)^{j+1} \binom{\rho(\mathcal{L})-j-1}{k-1} W_j\,,$$

for $k \in [\lfloor \frac{\rho(\mathcal{L})}{2} \rfloor]$. □

Proposition 3.56. *A system* S_2 *of the form* (3.22, 3.23) *satisfies condition* (3.24) *if and only if its combinatorially dual system* S_2^0 *satisfies condition* (3.25).

Proof. The proof is analogous to that of Proposition 3.49. Since the polytope $\mathcal{P}$, which is put in correspondence with the system S_2 by the Gale transform, is simplicial, then its dual polytope $\mathcal{P}^0$ is simple. Therefore, the system S_2^0 satisfies (3.25). □

Proposition 3.57. *Let a system* S_2*of the form* (3.22, 3.23) *satisfy condition* (3.25), *and let* $I_1, I_2, \ldots, I_{m-r-k-1} \in \mathbf{I}$, *where* $0 \le k \le m - r - 2$, *be the multi-indices of some of its IISs. Let us suppose* $I := \bigcup_{j\in[m-r-k-1]} I_j$. *If* $I \neq [m]$, *then the chosen multi-indices* $I_1, I_2, \ldots, I_{m-r-k-1} \in \mathbf{I}$ *only are precisely those families of the multi-indices of IISs of the system* S_2 *whose union is* I.

Proof. A system S_2 of the form (3.22, 3.23) satisifes condition (3.25) if and only if the multi-index of each of its minimal infeasible subsystem is the complement, up to $[m]$, of the multi-index of a facet of some simple $(m-r-1)$-polytope $\mathcal{P}$ with m vertices. Now the proposition follows from the next observation: let $\mathcal{P}$ be a simple d-polytope, and $F_1, F_2, \dots, F_{d-k}$ its facets, where $0 \le k \le d-1$. Let us suppose $F := \bigcap_{i\in[d-k]} F_i$, and assume that $F \ne \emptyset$. Then F represents a k-dimensional face of the polytope $\mathcal{P}$, and the facets $F_1, F_2, \dots, F_{d-k}$ are precisely those faces of $\mathcal{P}$ that contain F. □

Proposition 3.58. *A system* S_2 *of the form* (3.22, 3.23) *satisifes condition* (3.25) *if and only if for any of its IIS, with a multi-index* $I \in \mathbf{I}$, *there exist precisely* $m-r-1$ *multi-indices of IISs* $I_1, I_2, \dots, I_{m-r-1}$ *such that for each* $j \in [m-r-1]$ *it holds* $|I \cup I_j| = r+2$.

Proof. According to Propositions 3.51 and 3.56, the system S_2 satisfies condition (3.24) if and only if the multi-index of any of its minimal infeasible subsystem is the complement, up to $[m]$, of the multi-index of some facet of a simplicial $(m-r-1)$-polytope $\mathcal{P}$ with m vertices, which is dual to a simple $(m-r-1)$-polytope $\mathcal{P}^0$. But $\mathcal{P}^0$ is simple if and only if each of its vertex is incident to precisely $m-r-1$ one-dimensional faces. The one-dimensional faces of the polytope $\mathcal{P}^0$ are anti-isomorphic to the $(m-r-2)$-dimensional simplices of $\mathcal{P}$, from where the proof follows. □

The Dehn–Sommerville relations presented in Proposition 3.53 can be reformulated, according to combinatorial duality described in Propositions 3.49 and 3.56, for systems of the form (3.22, 3.23, 3.25) as follows:

Proposition 3.59. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.25), *let* $\mathbf{I}$ *be the family of the multi-indices of its IISs, and* $\mathfrak{n}_i$ *the number of those subfamilies* $\{I_1, I_2, \dots, I_i\} \subseteq \mathbf{I}$, *for which* $\bigcup_{j\in[i]} I_j = [m]$. *Let us suppose* $\mathfrak{n}_0 := 0$. *Then the following relations hold* x *is a formal variable:*

$$\begin{cases} \mathfrak{n}_0 = \mathfrak{n}_1 = 0\,, \\ \mathfrak{n}_k = \binom{\mathfrak{p}}{k},\ \textit{if}\ m-r \le k \le \mathfrak{p}\ , \\ \sum_{j=r+\mathfrak{p}-m+1}^{\mathfrak{p}} \left(\binom{\mathfrak{p}}{j} - \mathfrak{n}_{\mathfrak{p}-j}\right)(\mathrm{x}-1)^{\mathfrak{p}-j} \\ \qquad = \sum_{j=r+\mathfrak{p}-m+1}^{\mathfrak{p}} (-1)^{j-r-\mathfrak{p}+m-1} \left(\binom{\mathfrak{p}}{j} - \mathfrak{n}_{\mathfrak{p}-j}\right)\mathrm{x}^{\mathfrak{p}-j}\,. \end{cases}$$

Since any subset of vertices of a face of a simplicial polytope is also the vertex set of some of its face, Proposition 3.51 makes it possible to estimate the number of subsystems of different cardinalities in systems of the form (3.22, 3.23, 3.24):

Proposition 3.60. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24); *let* ν_k *and* τ_k *denote the numbers of its feasible and infeasible subsystems, of cardinality k, respectively. Suppose*

$$\Phi_j(m-r-1, m) := \sum_{i=0}^{\lfloor (m-r-1)/2\rfloor} \binom{i}{j}\binom{m+r+i}{i} + \sum_{i=0}^{\lfloor (m-r-1)/2\rfloor -1} \binom{m-r-i-1}{j}\binom{m+r+i}{i}\,.$$

Then

$$\begin{aligned}\tau_k &\le \Phi_{k-r-1}(m-r-1, m) ,\\ \nu_k &\ge \tbinom{m}{k} - \Phi_{k-r-1}(m-r-1, m) ,\end{aligned}$$

for $r+1 \le k \le m-2$.

Proof. According to Proposition 3.51 (i), a system S_2 of the form (3.22, 3.23) satisifies condition (3.24) if and only if the multi-index of any of its minimal infeasible subsystem is the complement, up to $[m]$, of the multi-index of some facet of a simplicial $(m-r-1)$-polytope $\mathcal{P}$ with m vertices. The proposition follows from the *upper bound theorem* proved by McMullen: if we set, for a simplicial d-dimensional polytope with m vertices,

$$\Phi_j(d, m) := \sum_{i=0}^{\lfloor d/2 \rfloor} \tbinom{i}{j}\tbinom{m-d+i-1}{i} + \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \tbinom{d-i}{j}\tbinom{m-d+i-1}{i} ,$$

then, for $1 \le j \le d-1$, the number of its j-dimensional faces is at most $\Phi_{d-j-1}(d, m)$. □

Proposition 3.61. *Let* S_2 *be a system of the form* (3.22, 3.23, 3.24)*; let* ν_k *and* τ_k *denote the numbers of its feasible and infeasible subsystems, of cardinality* k*, respectively. Suppose*

$$\varphi(m-r-1, m) := \begin{cases} (m-r-2)m - (m-r)(m-r-3), & \textit{if } j = 0 ,\\ \tbinom{m-r-1}{j+1}m - \tbinom{m-r}{j+1}(m-r-j-2), & \textit{if } j \in [m-r-3] . \end{cases}$$

Then

$$\begin{aligned}\tau_k &\ge \varphi_{k-r-1}(m-r-1, m) ,\\ \nu_k &\le \tbinom{m}{k} - \varphi_{k-r-1}(m-r-1, m) ,\end{aligned}$$

for $r+1 \le k \le m-2$.

Proof. The argument is similar to that presented in the proof of Proposition 3.60. The proposition follows from the *lower bound theorem* proved by Barnette: if we set, for a simplicial d-dimensional polytope with m vertices,

$$\varphi_j(d, m) := \begin{cases} (d-1)m - (d+1)(d-2), & \text{if } j = 0 ,\\ \tbinom{d}{j+1}m - \tbinom{d+1}{j+1}(d-j-1), & \text{if } j \in [d-2] , \end{cases}$$

then, for $1 \le j \le d-1$, the number of its j-dimensional faces is at least $\varphi_{d-j-1}(d, m)$. □

Diagonals of cyclic polytopes and MFSs of inequality systems

As noted earlier, one fundamental extremal construction in the problems of combinatorial polytope theory is the *cyclic polytope*. Recall that it is defined as the convex hull of m distinct points on the moment curve $\boldsymbol{x}(t) := (t, t^2, \dots, t^d) \in \mathbb{R}^d$, and it is denoted by $\mathfrak{C}(d, m)$. We will count the number of diagonals of the polytope $\mathfrak{C}(d, m)$ and, as a consequence, we will estimate the number of MFSs of inequality systems. The question on the number of diagonals of the cyclic polytopes is one of the questions of most interest to the combinatorial theory of this important class of polytopes.

Let $\boldsymbol{V} := \{\boldsymbol{x}_i := \boldsymbol{x}(t_i) : i \in [m]\}$ be the vertex tuple of a cyclic polytope $\mathfrak{C}(d, m)$, where $i < j \Rightarrow t_i < t_j$. The polytope $\mathfrak{C}(d, m)$ is simplicial (i.e., any proper face of such a polytope is a simplex) and $\lfloor \frac{d}{2} \rfloor$-neighborly (i.e., the convex hull of any of its $\lfloor \frac{d}{2} \rfloor$ vertices is a face of the polytope). We will call a subtuple $\boldsymbol{X} \subseteq \boldsymbol{V}$ *connected*, if it is of the form $\boldsymbol{X} = (\boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_j)$, for some indices $i \le j$.

Let $\boldsymbol{Y} \subseteq \boldsymbol{V}$. We will call the inclusion-maximal connected subtuples of the tuple $\boldsymbol{Y}$ *components* of the tuple $\boldsymbol{Y}$. A component $\boldsymbol{Y}' \subseteq \boldsymbol{Y}$ with an odd number of elements will be called *odd*, otherwise it will be called *even*. A component $\boldsymbol{Y}' \subseteq \boldsymbol{Y}$ will be called *end* when $\boldsymbol{x}_1 \in \boldsymbol{Y}'$ or $\boldsymbol{x}_m \in \boldsymbol{Y}'$. Any subtuple $\boldsymbol{Y} \subseteq \boldsymbol{V}$ is partitioned into disjoint components. A subtuple $\boldsymbol{Y} \subseteq \boldsymbol{V}$ will be called an (r, s)-tuple if $|\boldsymbol{Y}| = r$ and $\boldsymbol{Y}$ has precisely s odd nonend components. We will denote by $\mathrm{s}(\boldsymbol{Y})$ the number of odd nonend components of the subtuple $\boldsymbol{Y}$. We will use the following characterization of the proper faces of the tuple $\boldsymbol{V}$:

Lemma 3.62. *A tuple $\boldsymbol{X} \subset \boldsymbol{V}$ is a proper face of the tuple $\boldsymbol{V}$ if and only if* $\mathrm{s}(\boldsymbol{X}) \le d - |\boldsymbol{X}|$.

Let us denote by $\boldsymbol{X} \backslash \boldsymbol{x}$ the tuple obtained from the tuple $\boldsymbol{X}$ by removing an element $\boldsymbol{x}$. Since the polytope $\mathfrak{C}(d, m)$ is simplicial, for any proper face $\boldsymbol{X}$ of the tuple $\boldsymbol{V}$ the inclusion $\boldsymbol{X}' \subset \boldsymbol{X}$ implies that the tuple $\boldsymbol{X}'$ is also a (proper) face of the tuple $\boldsymbol{V}$. Thus, the next assertion follows from Lemma 3.41:

Lemma 3.63. *A tuple $\boldsymbol{X} \subset \boldsymbol{V}$ is a diagonal of the tuple $\boldsymbol{V}$ if and only if $\boldsymbol{X}$ is not a proper face of the tuple $\boldsymbol{V}$, but $\boldsymbol{X} \backslash \boldsymbol{x}$ is its proper face, for any $\boldsymbol{x} \in \boldsymbol{X}$.*

Lemma 3.64. *Suppose $m \ge d + 2$. Then any diagonal $\boldsymbol{X}$ of the tuple $\boldsymbol{V}$ contains only one-element components.*

Proof. By Lemma 3.63, the tuple $\boldsymbol{X}$ cannot be a proper face of the tuple $\boldsymbol{V}$, therefore, taking into account Lemma 3.62, we have $\mathrm{s}(\boldsymbol{X}) > d - |\boldsymbol{X}|$. Assume that the tuple $\boldsymbol{X}$ has a component $\boldsymbol{X}_0$ that contains at least two elements. Since, by Lemma 3.42, the convex hull $\operatorname{conv} \boldsymbol{X}$ is a simplex then $|\boldsymbol{X}| \le d + 1$. By the hypothesis of the Lemma, we have $m \ge d + 2$; thus, the component $\boldsymbol{X}_0$ does not contain $\boldsymbol{x}_1$ and $\boldsymbol{x}_m$ simultaneously. Specifically, suppose $\boldsymbol{X}_0 = (\boldsymbol{x}_i, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_j)$, where $i \ne 1$. Let us set $\boldsymbol{x} := \boldsymbol{x}_j$ when the cardinality $|X|$ is even, and $\boldsymbol{x} := \boldsymbol{x}_{i+1}$ when the cardinality $|\boldsymbol{X}|$ is odd. Then it follows from the inequality $\mathrm{s}(\boldsymbol{X}) > d - |\boldsymbol{X}|$ that $\mathrm{s}(\boldsymbol{X}\backslash \boldsymbol{x}) = \mathrm{s}(\boldsymbol{X}) + 1 > d - |\boldsymbol{X}| + 1 = d - |\boldsymbol{X}\backslash \boldsymbol{x}|$, that is, $\boldsymbol{X}\backslash \boldsymbol{x}$ is not a proper face of the tuple $\boldsymbol{X}$, a contradiction with Lemma 3.63. □

Let us denote by $\mathrm{D}(d, m)$ the number of all diagonals, and by $\mathrm{D}_s(d, m)$ the number of diagonals, with s elements, of the tuple $\boldsymbol{V}$ of vertices of a cyclic polytope $\mathfrak{C}(d, m)$.

Proposition 3.65.

$$\mathrm{D}(d, m) = \begin{cases} 1, & \textit{if } m \le d + 1\,, \\ 2\binom{m-k-2}{k} + \binom{m-k-2}{k+1}, & \textit{if } m \ge d + 2 \textit{ and } d = 2k\,, \\ \binom{m-k-2}{k+1} + \binom{m-k-3}{k}, & \textit{if } m \ge d + 2 \textit{ and } d = 2k + 1\,. \end{cases}$$

Proof. If $m \le d + 1$, then $\mathfrak{C}(d, m)$ is a simplex and, as a consequence, the tuple $\boldsymbol{V}$ has the unique diagonal that coincides with $\boldsymbol{V}$.

Suppose $m \ge d + 2$. Since the polytope $\mathfrak{C}(d, m)$ is $\lfloor \frac{d}{2} \rfloor$-neighborly, it follows from Lemma 3.63 that

$$\mathrm{D}_s(d, m) = 0\,, \quad 1 \le s \le \lfloor \tfrac{d}{2} \rfloor\,. \tag{3.26}$$

1. Suppose $d = 2k$. Let us find the number $\mathrm{D}_{k+1}(d, m)$. Since the polytope $\mathfrak{C}(d, m)$ is k-neighborly, it follows from Lemma 3.63 that a subtuple $\boldsymbol{Y} \subset \boldsymbol{V}$ with $k + 1$ elements is a diagonal of the tuple $\boldsymbol{V}$ if and only if $\boldsymbol{Y}$ is not a proper face of the tuple $\boldsymbol{V}$. The latter is possible, according to Lemma 3.62, if and only if $\boldsymbol{Y}$ is a $(k + 1, s)$-tuple and $s > d - (k + 1) = k - 1$, that is, $\boldsymbol{Y}$ consists of $k + 1$ one-element components and, besides, $\boldsymbol{x}_1$ and $\boldsymbol{x}_m$ do not belong to $\boldsymbol{Y}$ simultaneously.

The enumeration of such tuples is reduced to the following problem: on a line, $m - (k + 1)$ black points and $m - k$ white points are chosen; besides, every black point is situated between two white points. It is necessary to enumerate all the tuples with $k + 1$ white points that do not contain two white end points simultaneously. We have

$$\mathrm{D}_{k+1}(d, m) = 2\binom{m-k-2}{k} + \binom{m-k-2}{k+1}\,. \tag{3.27}$$

Let us show that $\mathrm{D}_s(d, m) = 0$, when $s > k + 1$. Suppose to the contrary that there exists a diagonal $\boldsymbol{Y}$ of the tuple $\boldsymbol{V}$ that contains $k + 1 + p$ elements, where $p \ge 1$. By Lemma 3.64, the tuple $\boldsymbol{Y}$ consists of $k + 1 + p$ one-element components. By removing from this tuple the first p elements, we get the tuple $\boldsymbol{Y}'$ consisting of $k+1$ one-element components, besides, $\boldsymbol{x}_1 \notin \boldsymbol{Y}'$. As shown earlier, such a tuple $\boldsymbol{Y}'$ is a diagonal of the tuple $\boldsymbol{V}$; this contradicts the minimality of $\boldsymbol{Y}$. Thus, $\mathrm{D}_s(d, m) = 0$, when $s > k + 1$, from where, taking into account (3.26) and (3.27), we obtain Proposition 3.65 in the case of $d = 2k$.

2. Suppose $d = 2k + 1$. Let us find the number $\mathrm{D}_{k+1}(2k + 1, m)$. Arguing the same way as in the case of $d = 2k$, we verify that a tuple $\boldsymbol{Y}$ with $k + 1$ elements is a diagonal of the tuple $\boldsymbol{V}$ if and only if $\boldsymbol{Y}$ consists of $k + 1$ one-element nonend components. Such tuples can be enumerated as was done earlier, that yields

$$\mathrm{D}_{k+1}(2k + 1, m) = \binom{m-k-2}{k+1}\,. \tag{3.28}$$

Let us find the number $\mathrm{D}_{k+2}(2k+1, m)$. Any diagonal $\boldsymbol{Y}$ with $k+2$ elements of the tuple $\boldsymbol{V}$ consists, by Lemma 3.62, of $k+2$ one-element components. Let us show that, besides, $\boldsymbol{x}_1, \boldsymbol{x}_m \in \boldsymbol{Y}$. Suppose to the contrary that, for example, $\boldsymbol{x}_1 \notin \boldsymbol{Y}$. Then, by removing the last element of the tuple $\boldsymbol{Y}$, we get the tuple $\boldsymbol{Y}'$ consisting of $k+1$ one-element nonend components, that is a diagonal of the tuple $\boldsymbol{V}$, a contradiction with the minimality of $\boldsymbol{Y}$.

Now let $\boldsymbol{Y}$ be a tuple consisting of $k+2$ one-element components, and $\boldsymbol{x}_1, \boldsymbol{x}_m \in \boldsymbol{Y}$. Then $\boldsymbol{Y}$ is a $(k+2, k)$-tuple and, by Lemma 3.62, it is not a proper face of the tuple $\boldsymbol{V}$. By removing any element from $\boldsymbol{Y}$, we obtain a $(k+1, s')$-tuple $\boldsymbol{Y}'$, where $s' \le k \le d-(k+1)$, which is a proper face of the tuple $\boldsymbol{V}$. It follows from Lemma 3.63 that the tuple $\boldsymbol{Y}$ is a diagonal of the tuple $\boldsymbol{V}$. Thus, a tuple $\boldsymbol{Y}$ with $k+2$ elements is a diagonal of the tuple $\boldsymbol{V}$ if and only if it consists of $k+2$ one-element components, and $\boldsymbol{x}_1, \boldsymbol{x}_m \in \boldsymbol{Y}$. The number of such tuples is

$$\mathrm{D}_{k+2}(2k+1, m) = \binom{m-k-3}{k} . \tag{3.29}$$

Let us show that $\mathrm{D}_s(2k+1, m) = 0$ when $s > k+2$. Suppose to the contrary that there exists a diagonal $\boldsymbol{Y}$ of the tuple $\boldsymbol{V}$ that consists of $k+1+p$ elements, where $p \ge 2$. By removing from this tuple the first $p-1$ elements and the last element, we obtain a tuple $\boldsymbol{Y}'$ consisting of $k+1$ one-element nonend components. As we saw earlier, such a tuple is a diagonal of the tuple $\boldsymbol{V}$ that contradicts the minimality of $\boldsymbol{Y}$. Thus, $\mathrm{D}_s(d, m) = 0$ when $s > k+2$, from where, taking into account relations (3.26), (3.28) and (3.29), we obtain Proposition 3.65 in the case of $d = 2k+1$. □

Corollary 3.66. *The maximal number of MFSs of irreducible systems of the form* (3.22, 3.23), *of rank $r \ge 1$, of $m \ge 2$ inequalities is at least*

$$\begin{cases} 2\binom{k-1}{n-1} + \binom{k-1}{n-2} , & \text{if } m+n = 2k+1 , \\ \binom{k-1}{n-1} + \binom{k-2}{n} , & \text{if } m+n = 2k . \end{cases}$$

Notes

The notion for which we use in Section 3.1 the term A-*diagonal* was introduced in [8] under the name *missing face* (or *missing simplex*); the construction called G-*diagonal* was considered in [19] as *minimal diagonal* and, independently, in work [50] under the name *diagonal*; F-*diagonals* were introduced in [45] under the name *diagonals*. See also works [69–71, 108] on the missing faces of polytopes.

The notion of missing face is also standard in the theory of abstract simplicial complexes: let a complex Δ with vertex set V be given. A subset $N \subseteq V$ is called a *missing face* of the complex Δ if $N \notin \Delta$, but any proper subset of the set N is a face of the complex Δ, see, for example, in [25, §2.2].

We say on page 59 that a polytope $\mathcal{P}$ is obtained by the operation of *cross*, borrowing terminology from [35].

Propositions 3.4 and 3.7 are actually proved in [19], in the language of cones.

The Baire category theorem mentioned in Example 3.12 is presented, for example, in [73, Ch. 6].

The assertion that the face structure of a simplicial polytope is determined by the structure of the family of its A-diagonals, mentioned on page 66, is proved in [8, Th. 2.4].

The theory of positive bases of a finite dimensional space is a construction that is well known in combinatorial geometry [31, 118, 119, 129]; see also [29, Ch. 2], [128, Ch. 1].

The statement, presented on page 67, that for a positive basis $\boldsymbol{B}$ of $\mathbb{R}^n$ the inequalities $n + 1 \leq |\boldsymbol{B}| \leq 2n$ hold, is discussed, for example, in work [129].

A set $\boldsymbol{X}$ which is a minimal basis of the space $\operatorname{lin} \boldsymbol{X}$ is called in work [20] a *minimally dependent set*. In our study, we use the term *minimal sub-basis* for emphasizing the origin of the minimally dependent sets under consideration. On the basis of well-known facts – see, for example, in [117, Lemma 2.4] – note that the minimal sub-bases of a positive basis $\boldsymbol{B}$, defined in such a manner, are precisely all its inclusion-minimal sub-bases.

We recall on page 68 that a positive basis $\boldsymbol{B}$, with $n + r$ points, of $\mathbb{R}^n$ is a SPB if and only if there exists a partition $\boldsymbol{B} = \boldsymbol{B}_1 \dot{\cup} \boldsymbol{B}_2 \dot{\cup} \cdots \dot{\cup} \boldsymbol{B}_r$, where $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_r$ are pairwise disjoint minimal sub-bases of the positive basis $\boldsymbol{B}$; see on this subject in works [117, 118, 129].

Proposition 3.18 is proved in [117, 118].

On page 68, we for brevity say that a set $\boldsymbol{X} \subset \mathbb{R}^n$ is *one-sided* if it is contained entirely in an open half-space bounded by a linear hyperplane; note that such a set $\boldsymbol{X}$ is called in work [20] *strict one-sided*.

The notion of *diagram of a positive basis* useful for investigating positive bases was introduced in work [129].

We consider on page 68 a tuple $\boldsymbol{B}$ of vectors that span positively the space $\mathbb{R}^r$, and the corresponding *linear representation* $\boldsymbol{E}$. The fact that the point set from the tuple $\boldsymbol{E}$ is one-sided is mentioned in [129].

Assertions (i)–(iv) of Proposition 3.19 are also proved in work [129].

In the proof of Proposition 3.25, we discuss the impossibility of the separation of the sets $\boldsymbol{E}^+$ and $\boldsymbol{E}^-$ in a hyperplane $\mathbf{H}$ by a plane $\boldsymbol{E}$; we are supported in this argument by the theorem on the separability of convex sets from [64, Ch. 2], see Lemma 3.35; see also, for example, in [14, §8], [26, §III.28], [84, §1.3], [150, §4.5]. See [41, §1.12], [42, §1.12], [146, §II.7] on the separability of polyhedral sets.

The proof of Proposition 3.27 is supported by Carathéodory's theorem; see, for example, in [14, §3.7], [24, §1.2], [64, §2.3], [84, §1.2], [117], [163, Lect. 1] on this classical result of convex analysis.

Another classical statement on convex sets, Helly's theorem, is used in the proof of the implication (vi)⇒(iv) of Proposition 3.28; Helly-type assertions can be found, for example, in [14, §3.7], [24, §1.2], [64, Ch. 2], and [120, §IV.21].

The proof of Proposition 3.33 is completed by making reference to a known fact that can be found, for example, in [117, Th. 2.1].

Lemma 3.34 is presented in [146] and, as noted earlier, Lemma 3.35 can be found in [64, Ch. 2].

The relation $\dim(\boldsymbol{K} \cap -\boldsymbol{K}) + \dim \boldsymbol{K}^* = r$ and the related equality $(\boldsymbol{K} \cap -\boldsymbol{K}) + \operatorname{lin} \boldsymbol{K}^* = \mathbb{R}^r$, mentioned in the proof of Proposition 3.36, are given in [120].

Gale transforms and diagrams of a point tuple are powerful tools of studying in combinatorial geometry and in the theory of polytopes, see, for example, in [15, 60, 65, 91, 100, 156], [92, §5.6], [140, Ch. 5], [152, §3.6]. In the mentioned works, in particular, one can find the properties of Gale transforms and diagrams that are presented in Proposition 3.37 and Corollary 3.38.

Lemma 3.39 is proved in work [60].

We consider combinatorially dual systems of linear inequalities and, in particular, their relationship with simplicial and simple polytopes, and we present the Dehn–Sommerville relations, and estimate the number of subsystems, following [93]; see also [56, 57].

The assertion, used in the proof of Proposition 3.49, on the existence for any convex polytope of its dual polytope (the face lattices of such polytopes are by definition anti-isomorphic) is a fundamental observation of convex analysis; see, for example, in [24, §2.10], [64, §3.4], [163, §2.3]. A polytope dual to a pyramid is also a pyramid of the same dimension [24, §2.10].

The Dehn–Sommerville relations for the Whitney numbers of the second kind, taken as the basis of the proof of Proposition 3.53, are formulated in [134, §3.14].

The equivalent reformulations of the Dehn–Sommerville relations, given in the proof of Corollary 3.55, can be found in [156, §1.5].

The observation, mentioned in the proof of Proposition 3.57, is discussed in [24, Th. 12.14].

In the proof of Proposition 3.58, we recall that each vertex of a simple polytope is incident to the same number of its one-dimensional faces; see on this in [24, Th. 12.12].

The upper bound theorem, proved by P. McMullen, which is taken as the basis of the proof of Proposition 3.60, is reproduced in [24, Corollary 18.3]. The lower bound theorem – see the proof of Proposition 3.61 – was proved by D. Barnette; it is presented in [24, Corollary 19.6].

We recall on page 88 that the cyclic polytopes are simplicial and highly neighborly; see on this, for example, in [64, §4.7], [156, §1.2].

We say on the connected subtuples of the vertex set of a cyclic polytope, following [101]. Lemma 3.62 is also proved in [101].

4 Monotone Boolean functions, complexes, graphs, and inequality systems

The multi-indices of the subsystems of infeasible systems with the monotonicity property and, in particular, the multi-indices of the subsystems of infeasible systems of linear inequalities determine a partition of the Boolean lattice of multi-indices into two subposets that correspond to the feasible and infeasible subsystems. This partition is uniquely determined by the so-called *border* that is the common collection of the multi-indices of maximal feasible and minimal infeasible subsystems; the family of the multi-indices of MFSs is naturally regarded as the facet family of an abstract simplicial complex. In terms of monotone Boolean functions, the multi-indices of MFSs and IISs correspond to the upper zeros and lower units of some Boolean function which is assigned to the inequality system under consideration, see Chapter 1.

In this chapter, we investigate the relationship of the problems of searching for the maximal feasible subsystems of an inequality system with the problem of optimal inference of monotone Boolean functions. Inference lies on the basis of numerous applications; for this reason, we will explain in detail a specific approach to its efficient realization.

4.1 Optimal inference of monotone Boolean functions

Let us recall several constructions and notation which we used earlier in Section 1.2.

For binary tuples $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\boldsymbol{\beta} := (\beta_1, \beta_2, \ldots, \beta_m)$ from the unit discrete m-dimensional cube $\mathbf{B}^m := \{0, 1\}^m$, the ordering $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$, by definition holds if and only if $\alpha_i \leq \beta_i$, for all $i \in [m]$. If $\mathcal{A} \subseteq \mathbf{B}^m$, then $\mathbf{max}\,\mathcal{A}$ and $\mathbf{min}\,\mathcal{A}$ denote the sets of all maximal elements and all minimal elements of the set $\mathbf{max}\,\mathcal{A}$, with respect to that ordering, respectively.

The number of units in a tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$ will be denoted by $|\boldsymbol{\alpha}|$.

We will denote by $\boldsymbol{\alpha} \oplus \boldsymbol{\beta}$ the coordinate-wise summation of the tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ over the set $\mathbf{B}$ equipped with the properties of the finite field $\mathbb{F}_2$ with two elements.

Any monotone Boolean function (MBF) $\mathfrak{f}\colon\ \mathbf{B}^m \to \mathbf{B}$, which is a map for which the implications

$$\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{B}^m,\ \boldsymbol{\alpha} \leq \boldsymbol{\beta} \quad \Longrightarrow \quad \mathfrak{f}(\boldsymbol{\alpha}) \leq \mathfrak{f}(\boldsymbol{\beta}) \tag{4.1}$$

hold, induces the partition $\mathbf{B}^m = \mathfrak{f}^{-1}(0) \,\dot{\cup}\, \mathfrak{f}^{-1}(1)$ of the cube $\mathbf{B}^m$ into the preimages of the elements from the set $\mathbf{B}$. Under such a partitioning, the family

$$\mathfrak{I}(\mathbf{J}) := \left\{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(0)\right\}, \tag{4.2}$$

interpreted as a subset of the Boolean lattice $\mathbb{B}(m)$ of subsets of the index set $[m]$, represents its order ideal generated in $\mathbb{B}(m)$ by the family $\mathbf{J}$ of the inclusion-maximal sets from the family $\{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(0)\}$. The ideal $\mathfrak{I}(\mathbf{J})$ is the face poset of the abstract simplicial complex $\Delta(\mathbf{J})$ with the facet family $\mathbf{J}$.

Similarly, it follows from monotonicity property (4.1) that the family

$$\mathfrak{F}(\mathbf{I}) := \{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(1)\}$$

can be regarded as an order filter of the lattice $\mathbb{B}(m)$ generated by the family $\mathbf{I}$ of inclusion-minimal sets from the family $\{\{j \in [m]\colon\ \alpha_j = 1\}\colon\ \boldsymbol{\alpha} \in \mathfrak{f}^{-1}(1)\}$.

Recall that the set $\mathfrak{f}^{-1}(0)$ consists of the *zeros* of the function $\mathfrak{f}$, and the set $\mathfrak{f}^{-1}(1)$ consists of the *units* of this function. The subset $\mathfrak{Q}(\mathfrak{f}) := \mathbf{max}\, \mathfrak{f}^{-1}(0)$ of maximal elements of the poset $\mathfrak{f}^{-1}(0)$ is the set of *upper zeros* of the function $\mathfrak{f}$; the subset $\mathfrak{P}(\mathfrak{f}) := \mathbf{min}\, \mathfrak{f}^{-1}(1)$ of minimal elements of the poset $\mathfrak{f}^{-1}(1)$ is the set of *lower units* of the function $\mathfrak{f}$.

An upper zero $\boldsymbol{\alpha} \in \mathfrak{Q}(\mathfrak{f})$ of the function $\mathfrak{f}$ is called *maximal* if $|\boldsymbol{\alpha}| = \max_{\boldsymbol{\beta} \in \mathfrak{Q}(\mathfrak{f})} |\boldsymbol{\beta}|$. Dually, a lower unit $\boldsymbol{\alpha} \in \mathfrak{P}(\mathfrak{f})$ of the function $\mathfrak{f}$ is called *minimal* if $|\boldsymbol{\alpha}| = \min_{\boldsymbol{\beta} \in \mathfrak{P}(\mathfrak{f})} |\boldsymbol{\beta}|$.

Let us denote the class of all monotone Boolean functions of m variables by $\mathcal{M}_m$.

Let us assign to the function $\mathfrak{f} \in \mathcal{M}_m$ an *oracle* $\mathcal{O}_\mathfrak{f}$ that is an operator making it possible to compute for an arbitrary point $\boldsymbol{\alpha} \in \mathbf{B}^m$ the value of the function $\mathfrak{f}$ at this point. *Inference* of an *a priori* unknown monotone Boolean function means its reconstruction with the use of the oracle $\mathcal{O}_\mathfrak{f}$. The problem of constructing the algorithms of MBF inference which require the least, in a sense, number of invocations of the oracle, is fundamental.

Given some algorithm G, let $\varphi(G, \mathfrak{f})$ denote the number of its calls of the operator $\mathcal{O}_\mathfrak{f}$ when inferring the function $\mathfrak{f} \in \mathcal{M}_m$. Optimality of the algorithm G, in the sense of the number of invocations of the operator $\mathcal{O}_\mathfrak{f}$, can be ranked, for example, by the following functionals:

$$\varphi(G, m) = \max_{\mathfrak{f} \in \mathcal{M}_m} \varphi(G, \mathfrak{f})\,, \tag{4.3}$$

$$\eta(G, m) = \max_{\mathfrak{f} \in \mathcal{M}_m} \frac{\varphi(G, \mathfrak{f})}{|\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})|}\,, \tag{4.4}$$

$$\eta_1(G, m) = \max_{\mathfrak{f} \in \mathcal{M}_m} \left(\varphi(G, \mathfrak{f}) - |\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})|\right), \tag{4.5}$$

$$\eta_2(G, m) = \sum_{\mathfrak{f} \in \mathcal{M}_m} \varphi(G, \mathfrak{f})\,. \tag{4.6}$$

Let us consider the quantity $\varphi(m) := \min_G \varphi(G, m)$, where the minimum is found over all algorithms G of inference of the MBFs of m variables. Let us use the analogous notation $\eta(m)$, $\eta_1(m)$, and $\eta_2(m)$ for criteria (4.4)–(4.6).

The classical, well-known and well-examined criterion $\varphi(G, m)$, defined in (4.3), is called *Shannon's criterion*. For an optimal, with respect to the criterion $\varphi(G, m)$, algorithm of MBF inference, the relation $\varphi(m) = \binom{m}{\lfloor m/2 \rfloor} + \binom{m}{\lfloor m/2 \rfloor + 1}$ holds.

At the same time, an algorithm which is optimal in the Shannon formulation is inadequate for miscellaneous practical purposes.

Let us analyze the complexity of inference algorithms with respect to the criterion $\eta(G, m)$.

During the inference process, such algorithms must invoke the values of the function on the set $\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})$. This means that $\varphi(G, \mathfrak{f}) \geq |\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})|$, for any algorithm G and for any monotone Boolean function $\mathfrak{f} \in \mathcal{M}_m$. Thus, the criterion $\eta(G, m)$ formalizes the natural requirement according to which the computational effort of the algorithm G when inferring the function $\mathfrak{f}$, measured by the quantity $\varphi(G, \mathfrak{f})$, should be proportional to the objective complexity of the inference problem for the function $\mathfrak{f}$, measured by the quantity $|\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})|$.

The inference process for the function $\mathfrak{f} \in \mathcal{M}_m$, which is realized by Algorithm G, can be described by the sequence

$$G(\mathfrak{f}) = \big(\, g_1(\mathfrak{f}), \mathfrak{f}(g_1(\mathfrak{f})),\ g_2(\mathfrak{f}), \mathfrak{f}(g_2(\mathfrak{f})),\ \ldots,\ g_k(\mathfrak{f}), \mathfrak{f}(g_k(\mathfrak{f}))\,\big)\,,$$

of tuples $g_i(\mathfrak{f}) \in \mathbf{B}^m$, chosen by the algorithm, and of the corresponding values $\mathfrak{f}(g_i(\mathfrak{f})) \in \mathbf{B}$ of the function $\mathfrak{f}$, for $i \in [\varphi(G, \mathfrak{f})]$. In other words, $g_i(\mathfrak{f})$ can be interpreted as sequential calls, by the algorithm G, of the operator $\mathcal{O}_\mathfrak{f}$ during the inference process for the function $\mathfrak{f}$, and $\mathfrak{f}(g_i(\mathfrak{f}))$ are the responses of the operator $\mathcal{O}_\mathfrak{f}$.

We will consider below those algorithms G of MBF inference only, for which the sequence $G(\mathfrak{f})$ is determined for each function $\mathfrak{f} \in \mathcal{M}_m$ uniquely.

Suppose $\mathcal{A} \subseteq \mathbf{B}^m$. Let us denote by $\mathfrak{M}_\mathfrak{f}(\mathcal{A})$ the set of all those points $\boldsymbol{\alpha}$ of the unit cube $\mathbf{B}^m$, at which the values of the function $\mathfrak{f}$ are determined, by the monotonicity property, uniquely by its values at the set $\mathcal{A}$, that is, $\boldsymbol{\alpha} \in \mathfrak{M}_\mathfrak{f}(\mathcal{A})$ if there exists a tuple $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$ such that $\boldsymbol{\beta} \in \mathcal{A} \cap \mathfrak{f}^{-1}(0)$, or a tuple $\boldsymbol{\beta}' \leq \boldsymbol{\alpha}$ such that $\boldsymbol{\beta}' \in \mathcal{A} \cap \mathfrak{f}^{-1}(1)$.

In what follows, we will need, as standard procedures, a routine $\mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha})$ of extracting, on the basis of a point $\boldsymbol{\alpha} \in \mathfrak{f}^{-1}(0)$, an upper zero $\boldsymbol{\alpha}'$ of the function $\mathfrak{f}$, such that $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}$, and a routine $\mathrm{LU}(\mathfrak{f}, \boldsymbol{\alpha})$ of extracting, on the basis of a point $\boldsymbol{\alpha} \in \mathfrak{f}^{-1}(1)$, a lower unit $\boldsymbol{\alpha}'$, such that $\boldsymbol{\alpha} \geq \boldsymbol{\alpha}'$. The routines $\mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha})$ and $\mathrm{LU}(\mathfrak{f}, \boldsymbol{\alpha})$ work in accordance to the standard scheme, computing the values of the function $\mathfrak{f}$ at some tuples with the help of the operator $\mathcal{O}_\mathfrak{f}$.

The routine $\mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha})$

Let a tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$ contain $k < m$ units $\alpha_i = 1$. Let us re-index the zeros in $\boldsymbol{\alpha}$ from left to right. Let us denote by $\boldsymbol{\beta}^i$ the binary tuple containing $m-1$ zeros and a single unit at the position of the ith zero of the tuple $\boldsymbol{\alpha}$. Then the sequence

$$\left(\boldsymbol{\alpha}^1, \mathfrak{f}(\boldsymbol{\alpha}^1),\ \boldsymbol{\alpha}^2, \mathfrak{f}(\boldsymbol{\alpha}^2),\ \ldots,\ \boldsymbol{\alpha}^{m-k}, \mathfrak{f}(\boldsymbol{\alpha}^{m-k})\right)\,, \tag{4.7}$$

where $\boldsymbol{\alpha}^1 := \boldsymbol{\alpha} \oplus \boldsymbol{\beta}^1$ and $\boldsymbol{\alpha}^i := \boldsymbol{\alpha} \oplus \boldsymbol{\beta}^i \oplus (1-\mathfrak{f}(\boldsymbol{\alpha}^1))\boldsymbol{\beta}^1 \oplus (1-\mathfrak{f}(\boldsymbol{\alpha}^2))\boldsymbol{\beta}^2 \oplus \cdots \oplus (1-\mathfrak{f}(\boldsymbol{\alpha}^{i-1}))\boldsymbol{\beta}^{i-1}$, determines the upper zero $\boldsymbol{\alpha}' \in \mathfrak{Q}(\mathfrak{f})$ such that $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}$, namely

$$\boldsymbol{\alpha}' = \mathbf{max}\left\{\{\boldsymbol{\alpha}\} \cup \{\boldsymbol{\alpha}^i :\ i \in [m-k],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 0\}\right\}.$$

The routine LU($\mathfrak{f}, \boldsymbol{\alpha}$)

Let a tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$ contain $k > 0$ units $\alpha_i = 1$. Let us re-index the units in $\boldsymbol{\alpha}$ from left to right. Let us denote by $\boldsymbol{y}^i$ the binary tuple containing $m-1$ zeros and a single unit at the position of the ith unit of the tuple $\boldsymbol{\alpha}$. Then the sequence

$$(\boldsymbol{\alpha}^1, \mathfrak{f}(\boldsymbol{\alpha}^1),\ \boldsymbol{\alpha}^2, \mathfrak{f}(\boldsymbol{\alpha}^2),\ \ldots,\ \boldsymbol{\alpha}^k, \mathfrak{f}(\boldsymbol{\alpha}^k))\ , \tag{4.8}$$

where $\boldsymbol{\alpha}^1 := \boldsymbol{\alpha} \oplus \boldsymbol{y}^1$ and $\boldsymbol{\alpha}^i := \boldsymbol{\alpha} \oplus \boldsymbol{y}^i \oplus \mathfrak{f}(\boldsymbol{\alpha}^1)\boldsymbol{y}^1 \oplus \mathfrak{f}(\boldsymbol{\alpha}^2)\boldsymbol{y}^2 \oplus \cdots \oplus \mathfrak{f}(\boldsymbol{\alpha}^{i-1})\boldsymbol{y}^{i-1}$, determines the lower unit $\boldsymbol{\alpha}' \in \mathfrak{P}(\mathfrak{f})$ such that $\boldsymbol{\alpha}' \le \boldsymbol{\alpha}$, namely

$$\boldsymbol{\alpha}' = \mathbf{min}\,\{\{\boldsymbol{\alpha}\} \cup \{\boldsymbol{\alpha}^i:\ i \in [k],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 1\}\}\ .$$

Sequences (4.7) and (4.8) themselves will be denoted below by UZ($\mathfrak{f}, \boldsymbol{\alpha}$) and LU($\mathfrak{f}, \boldsymbol{\alpha}$). Note that the sequence UZ($\mathfrak{f}, \boldsymbol{\alpha}$) contains $m - |\boldsymbol{\alpha}|$ invocations of the operator $\mathcal{O}_\mathfrak{f}$, and the sequence LU($\mathfrak{f}, \boldsymbol{\alpha}$) contains $|\boldsymbol{\alpha}|$ invocations of the operator $\mathcal{O}_\mathfrak{f}$. By definition we will suppose that the sequence UZ($\mathfrak{f}, \boldsymbol{\alpha}$) is empty when $|\boldsymbol{\alpha}| = m$, and the sequence LU($\mathfrak{f}, \boldsymbol{\alpha}$) is empty when $|\boldsymbol{\alpha}| = 0$.

Let us denote by $G(\mathfrak{f}, \boldsymbol{\alpha})$ the sequence of the form

$$G(\mathfrak{f}, \boldsymbol{\alpha}) := \begin{cases} (\boldsymbol{\alpha}, 0,\ \mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha})), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}) = 0\ , \\ (\boldsymbol{\alpha}, 1,\ \mathrm{LU}(\mathfrak{f}, \boldsymbol{\alpha})), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}) = 1\ . \end{cases}$$

It follows from the definition that the sequence $G(\mathfrak{f}, \boldsymbol{\alpha})$ contains an upper zero $\boldsymbol{\alpha}' \ge \boldsymbol{\alpha}$ of the function $\mathfrak{f}$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 0$, and $G(\mathfrak{f}, \boldsymbol{\alpha})$ contains a lower unit $\boldsymbol{\alpha}' \le \boldsymbol{\alpha}$ of the function $\mathfrak{f}$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 1$. Let us use the common notation $\arg G(\mathfrak{f}, \boldsymbol{\alpha})$ for these elements.

Let us denote by $\mathcal{B}(\mathbf{B}^m)$ the family of all subsets of the unit cube $\mathbf{B}^m$. Given a fixed choice function $\psi\colon \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$, define an algorithm of MBF inference, as follows:

$$G_\psi(\mathfrak{f}) := (G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \ldots, G(\mathfrak{f}, \boldsymbol{\alpha}^k))\ , \tag{4.9}$$

where $\boldsymbol{\alpha}^1 := \psi(\mathbf{B}^1)$ and $\boldsymbol{\alpha}^i := \psi(\mathbf{B}^m - \mathfrak{M}_\mathfrak{f}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\}))$.

The inference process for the function $\mathfrak{f}$ is completed by the algorithm G_ψ when we have $\mathfrak{M}_\mathfrak{f}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [k]\}) = \mathbf{B}^m$. Analyzing definition (4.9) of the sequence $G_\psi(\mathfrak{f})$, we can conclude that

$$\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [\varphi(G_\psi, \mathfrak{f})]\} = \mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})\ , \tag{4.10}$$

$$|\{\boldsymbol{\alpha}^i:\ i \in [\varphi(G_\psi, \mathfrak{f})],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 0\}| = |\mathfrak{Q}(\mathfrak{f})|\ , \tag{4.11}$$

$$|\{\boldsymbol{\alpha}^i:\ i \in [\varphi(G_\psi, \mathfrak{f})],\ \mathfrak{f}(\boldsymbol{\alpha}^i) = 1\}| = |\mathfrak{P}(\mathfrak{f})|\ . \tag{4.12}$$

Proposition 4.1. *For any choice function* $\psi\colon\ \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$*, the inequality* $\eta(G_\psi, m) \le m+1$ *holds.*

Proof. The sequence $G(\mathfrak{f}, \boldsymbol{\alpha}^i)$ contains at most $m + 1$ invocations of the operator $\mathcal{O}_\mathfrak{f}$ when $|\boldsymbol{\alpha}^i| = 0$ or $|\boldsymbol{\alpha}^i| = m$, and at most m invocations otherwise. It follows from the above argument, taking into account (4.10), that

$$\varphi(G_\psi, \mathfrak{f}) \leq m|\mathfrak{Q}(\mathfrak{f})| + m|\mathfrak{P}(\mathfrak{f})| + 2\,. \tag{4.13}$$

For the functions *identically zero* $\mathfrak{f}_0(\boldsymbol{\alpha}) \equiv 0$ and *identically unit* $\mathfrak{f}_1(\boldsymbol{\alpha}) \equiv 1$, we have $\eta(G_\psi, \mathfrak{f}_0) \leq m+1$ and $\eta(G_\psi, \mathfrak{f}_1) \leq m+1$, respectively. If $\mathfrak{f} \notin \{\mathfrak{f}_0, \mathfrak{f}_1\}$ then $|\mathfrak{P}(\mathfrak{f}) \,\dot\cup\, \mathfrak{Q}(\mathfrak{f})| \geq 2$, and the proposition follows from (4.13). $\square$

Proposition 4.2. *Let* $\psi\colon\ \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$ *be an arbitrary choice function,* $\mathfrak{f} \in \mathcal{M}_m$*, and let* $\boldsymbol{\alpha}^i$ *be a fixed item of the sequence* $G_\psi(\mathfrak{f}) = \big(G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \ldots, G(\mathfrak{f}, \boldsymbol{\alpha}^k)\big)$.

(i) *If* $\boldsymbol{\alpha}^i \in \mathbf{min}\,(\mathbf{B}^m - \mathcal{M}_m(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\}))$, $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$, *then* $\boldsymbol{\alpha}^i$ *is a minimal lower unit of the function* $\mathfrak{f}$*, and thus* $\boldsymbol{\alpha}^i = \arg G(\mathfrak{f}, \boldsymbol{\alpha}^i)$.

(ii) *If* $\boldsymbol{\alpha}^i \in \mathbf{min}\,(\mathbf{B}^m - \mathcal{M}_m(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\}))$, $\mathfrak{f}(\boldsymbol{\alpha}^i) = 0$, *then* $\boldsymbol{\alpha}^i$ *is a maximal upper zero of the function* $\mathfrak{f}$*, and thus* $\boldsymbol{\alpha}^i = \arg G(\mathfrak{f}, \boldsymbol{\alpha}^i)$.

Proof. Let us prove assertion (i). Let $\{\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \ldots, \boldsymbol{\beta}^i\}$ be the set of elements from $\mathbf{B}^m$ which are covered by the element $\boldsymbol{\alpha}^i$ in the poset $\mathcal{B}(\mathbf{B}^m)$. Then $\boldsymbol{\beta}^j \in \mathfrak{M}_\mathfrak{f}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\})$, $j \in [l]$, because of the minimality of $\boldsymbol{\alpha}^i = \mathbf{min}\,\{\mathbf{B}^m - \mathfrak{M}_\mathfrak{f}\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\}\}$, and $\mathfrak{f}(\boldsymbol{\beta}^j) = 0$, $j \in [l]$, because otherwise the element $\boldsymbol{\alpha}^i$ would belong to the set $\mathfrak{M}_\mathfrak{f}(\{\arg G(\mathfrak{f}, \boldsymbol{\alpha}^s)\colon\ s \in [i-1]\})$, which contradicts the choice of the element $\boldsymbol{\alpha}^i$ of the sequence $G_\psi(\mathfrak{f})$ in (4.9). It follows from the above argument that $\boldsymbol{\alpha}^i$ is a minimal lower unit of the function $\mathfrak{f}$.

Assertion (ii) is proved similarly. $\square$

Proposition 4.2 is of applied significance because the proposition makes it possible, for some choice functions, to simplify the corresponding algorithms of inference G_ψ. Let us consider two interesting examples of the choice functions. Let ψ be an arbitrary choice function. Suppose, for any $\mathcal{A} \subseteq \mathbf{B}^m$,

$$\psi_0(\mathcal{A}) := \psi(\mathbf{min}\,\mathcal{A})\,, \tag{4.14}$$

$$\psi_1(\mathcal{A}) := \psi(\mathbf{max}\,\mathcal{A})\,. \tag{4.15}$$

Let us consider the sequence $G_{\psi_0}(\mathfrak{f}) := \big(G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \ldots, G(\mathfrak{f}, \boldsymbol{\alpha}^k)\big)$. It follows from Proposition 4.2 that any element $\boldsymbol{\alpha}^i$ from $G_{\psi_0}(\mathfrak{f})$, such that $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$, is a minimal lower unit of the function $\mathfrak{f}$. This means that the subsequence $G(\mathfrak{f}, \boldsymbol{\alpha}^i)$ of the sequence $G_{\psi_0}(\mathfrak{f})$, in the case when $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$, can be substituted by $(\boldsymbol{\alpha}^i, 1)$, without affecting the result of inference. The following algorithm is thus defined:

Algorithm G'_{ψ_0},
with the inference sequence $G'_{\psi_0}(\mathfrak{f}) := \big(G'(\mathfrak{f}, \boldsymbol{\alpha}^1), G'(\mathfrak{f}, \boldsymbol{\alpha}^2), \dots, G'(\mathfrak{f}, \boldsymbol{\alpha}^k)\big)$, where

$$G'(\mathfrak{f}, \boldsymbol{\alpha}^i) := \begin{cases} (\boldsymbol{\alpha}^i, 0, \mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha}^i)), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^i) = 0\,, \\ (\boldsymbol{\alpha}^i, 1), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^i) = 1\,, \end{cases}$$

$$\boldsymbol{\alpha}^1 := \psi_0(\mathbf{B}^m) = (0, 0, \dots, 0)\,,$$

$$\boldsymbol{\alpha}^i := \psi_0(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon\ s \in [i-1]\}))\,,$$

$$\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) := \begin{cases} \arg G(\mathfrak{f}, \boldsymbol{\alpha}^s), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^s) = 0\,, \\ \boldsymbol{\alpha}^s, & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^s) = 1\,. \end{cases}$$

The inference process for the function $\mathfrak{f}$ is completed by Algorithm G' when we have $\mathfrak{M}_{\mathfrak{f}}\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon\ s \in [k]\} = \mathbf{B}^m$.

▷ In the description of Algorithm G_{ψ_0}, let us substitute 1 by 0, and 0 by 1 in all positions, except for $\boldsymbol{\alpha}^1$, and $\mathrm{UZ}(\mathfrak{f}, \boldsymbol{\alpha}^i)$ by $\mathrm{LU}(\mathfrak{f}, \boldsymbol{\alpha}^i)$; we obtain *Algorithm* G'_{ψ_1} that is a modification of Algorithm G_{ψ_1} on the basis of Proposition 4.2.

Proposition 4.3. *Let* $\psi \colon \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$ *be an arbitrary choice function, and let the quantities* ψ_0 *and* ψ_1 *be defined by relations* (4.14) *and* (4.15). *Then*

$$\varphi(G'_{\psi_0}, \mathfrak{f}) \le m|\mathfrak{Q}(\mathfrak{f})| + |\mathfrak{P}(\mathfrak{f})| + 1\,, \tag{4.16}$$

$$\varphi(G'_{\psi_1}, \mathfrak{f}) \le |\mathfrak{Q}(\mathfrak{f})| + m|\mathfrak{P}(\mathfrak{f})| + 1\,. \tag{4.17}$$

Proof. Let us prove inequality (4.16). It follows from the definition of Algorithm G'_{ψ_0} and from Proposition 4.2 that relations (4.10) also hold for the sequence $G'_{\psi_0}(\mathfrak{f})$, from where, taking into account the definition of the sequence $G'(\mathfrak{f}, \boldsymbol{\alpha}^i)$, we obtain (4.16).

Relation (4.17) is proved similarly. □

Thus, the algorithm G'_{ψ_0} is efficient for inferring monotone Boolean functions with a relatively small number of maximal upper zeros.

According to the following proposition, the function $\eta(m)$ is not bounded by a constant, uniformly for all m.

Proposition 4.4. *For the function* $\eta(m)$, *it holds*

$$\max\{2, \log_2 m^{1/2}\} \le \eta(m) \le \lfloor \tfrac{m}{2} \rfloor + 2\,.$$

Proof. Let us prove the lower bound. Let us first show that $\eta(m) \ge \max\{2, \log_2 m^{1/2}\}$. Let G be an arbitrary inference algorithm. Suppose $H(k) := \{\mathfrak{f} \in \mathcal{M}_m \colon\ |\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})| \le k\}$. For $i \in [m]$, define the function $\mathfrak{f}_i \in \mathcal{M}_m$ with the single maximal upper zero $\mathfrak{Q}(\mathfrak{f}_i) = \{(1, 1, \dots, 1, 0, 1, \dots, 1)\}$, where the zero is situated at the ith position, and with the single minimal lower unit $\mathfrak{P}(\mathfrak{f}_i) = \{(0, 0, \dots, 0, 1, 0, \dots, 0)\}$; the unit is situated at the ith position. Thus, $|H(2)| \ge m$. Suppose $l := \max\{\varphi(G, \mathfrak{f}) \colon\ \mathfrak{f} \in H(2)\}$. It follows from the requirement of unambiguous determining of the subsequence $G(\mathfrak{f})$ that the number of MBFs $\mathfrak{f} \in \mathcal{M}_m$, for which $\varphi(G, \varphi) \le l$, does not exceed the number of the binary tuples of length l, that is 2^l. As a consequence, the definition of the quantity l implies that $|H(2)| \ge m$, and we get $l \ge \log_2 m$. Then for some function $\mathfrak{f} \in H(2)$, we have $\varphi(G, \mathfrak{f}) \ge \log_2 m$, and thus $\eta(G, m) \ge \log_2 m^{1/2}$. The inequality $\eta(m) \ge 2$ is proved similarly. The lower bound is verified.

In order to prove the inequality $\eta(m) \le \lfloor \frac{m}{2} \rfloor + 2$, let us present a specific algorithm G of MBF inference, with $\eta(G, m) \le \lfloor \frac{m}{2} \rfloor + 2$. Let us introduce for the cube $\mathbf{B}^m$ another relation $\preceq$ of partial ordering: we set $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ if and only if $|\boldsymbol{\alpha}| \le |\boldsymbol{\beta}|$. Denote by $\mathbf{B}^{m,k}$ the set of all binary tuples $\boldsymbol{\alpha} \in \mathbf{B}^m$ that contain precisely k units. Let us define the choice function $\psi_2 : \mathcal{B}(\mathbf{B}^m) \to \mathbf{B}^m$:

$$\psi_2(\mathcal{A}) := \begin{cases} \boldsymbol{\beta}^1 := \psi(\mathbf{max}_{\preceq}\mathbf{min}_{\preceq}\mathcal{A}), & \text{if } |\boldsymbol{\beta}^1| > \lfloor \frac{m}{2} \rfloor\,, \\ \boldsymbol{\beta}^2 := \psi(\mathbf{min}_{\preceq}\mathbf{max}_{\preceq}\mathcal{A}), & \text{if } |\boldsymbol{\beta}^1| \le \lfloor \frac{m}{2} \rfloor \text{ and } |\boldsymbol{\beta}^2| \le \lfloor \frac{m}{2} \rfloor\,, \\ \boldsymbol{\beta}^3 := \psi(\mathbf{B}^{m,\lfloor \frac{m}{2} \rfloor} \cap \mathcal{A})\,, & \text{if } |\boldsymbol{\beta}^1| \le \lfloor \frac{m}{2} \rfloor \le |\boldsymbol{\beta}^2|\,, \end{cases}$$

where ψ means an arbitrary and fixed choice function for $\mathcal{A} \subseteq \mathbf{B}^m$.

Let us prove that this choice function is well defined, that is, the function $\psi_2(\mathcal{A})$ is defined for any tuple set $\mathcal{A} \subseteq \mathbf{B}^m$. For this, it suffices to show that $|\boldsymbol{\beta}^1| \le \lfloor \frac{m}{2} \rfloor \le |\boldsymbol{\beta}^2|$ implies the relation $\mathbf{B}^{m,\lfloor \frac{m}{2} \rfloor} \cap \mathcal{A} \ne \emptyset$. Let $(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \dots, \boldsymbol{\gamma}^k)$ be an inclusion-maximal chain of the poset $(\mathcal{A}, \le)$. It is evident that $\boldsymbol{\gamma}^1 \in \mathbf{min}\,\mathcal{A}$ and $\boldsymbol{\gamma}^k \in \mathbf{max}\,\mathcal{A}$, therefore, it follows from $|\boldsymbol{\beta}^1| \le \lfloor \frac{m}{2} \rfloor \le |\boldsymbol{\beta}^2|$ and from the definition of the tuples $\boldsymbol{\beta}^1$ and $\boldsymbol{\beta}^2$ that $|\boldsymbol{\gamma}^1| \le |\boldsymbol{\beta}^1| \le \lfloor \frac{m}{2} \rfloor \le |\boldsymbol{\beta}^2| \le |\boldsymbol{\gamma}^k|$; this means that among the elements of the chain $(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \dots, \boldsymbol{\gamma}^k)$ there exists a tuple with $\lfloor \frac{m}{2} \rfloor$ units, that is, $\mathbf{B}^{m,\lfloor \frac{m}{2} \rfloor} \cap \mathcal{A} \ne \emptyset$.

▷ Let us consider the sequence $G_{\psi_2}(\mathfrak{f}) := \big(G(\mathfrak{f}, \boldsymbol{\alpha}^1), G(\mathfrak{f}, \boldsymbol{\alpha}^2), \dots, G(\mathfrak{f}, \boldsymbol{\alpha}^k)\big)$. It follows from Proposition 4.2 and from the definition of the choice function ψ_2 that if $|\boldsymbol{\alpha}^i| > \lfloor \frac{m}{2} \rfloor$ and $f(\boldsymbol{\alpha}^i) = 1$, then $\boldsymbol{\alpha}^i$ is a minimal lower unit, and if $|\boldsymbol{\alpha}^i| < \lfloor \frac{m}{2} \rfloor$ and $\mathfrak{f}(\boldsymbol{\alpha}^i) = 0$, then $\boldsymbol{\alpha}^i$ is a maximal upper zero of the function $\mathfrak{f}$. This means that in these cases we can, without affecting the result of the inference process, substitute in the sequence $G_{\psi_2}(\mathfrak{f})$ its subsequence $G(\mathfrak{f}, \boldsymbol{\alpha}^i)$ by $(\boldsymbol{\alpha}^i, 1)$ or by $(\boldsymbol{\alpha}^i, 0)$, respectively. An *Algorithm* G'_{ψ_2} is thus defined, with the inference sequence $G'_{\psi_2}(\mathfrak{f}) := (G'(\mathfrak{f}, \boldsymbol{\alpha}^1), G'(\mathfrak{f}, \boldsymbol{\alpha}^2), \dots, G'(\mathfrak{f}, \boldsymbol{\alpha}^k))$, where

$$G'(\mathfrak{f}, \boldsymbol{\alpha}) := \begin{cases} \big(\boldsymbol{\alpha}, \mathfrak{f}(\boldsymbol{\alpha})\big), & \text{if } \mathfrak{f}(\boldsymbol{\alpha}) = 1 \text{ and } |\boldsymbol{\alpha}| > \lfloor \frac{m}{2} \rfloor \\ & \text{or } \mathfrak{f}(\boldsymbol{\alpha}) = 0 \text{ and } |\boldsymbol{\alpha}| < \lfloor \frac{m}{2} \rfloor\,, \\ G(\mathfrak{f}, \boldsymbol{\alpha}), & \text{otherwise}\,, \end{cases}$$

$$\boldsymbol{\alpha}^1 := \psi_2(\mathbf{B}^m)\,,$$

$$\boldsymbol{\alpha}^i := \psi_2(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon\ s \in [i-1]\}))\,,$$

$$\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) := \begin{cases} \boldsymbol{\alpha}^s, & \text{if } \mathfrak{f}(\boldsymbol{\alpha}^s) = 1 \text{ and } |\boldsymbol{\alpha}^s| > \lfloor \frac{m}{2} \rfloor \\ & \text{or } \mathfrak{f}(\boldsymbol{\alpha}^s) = 0 \text{ and } |\boldsymbol{\alpha}^s| < \lfloor \frac{m}{2} \rfloor\,, \\ \arg G(\mathfrak{f}, \boldsymbol{\alpha}^s), & \text{otherwise}\,. \end{cases}$$

The Algorithm $G'_{\psi_2}(\mathfrak{f})$ completes inference of the function $\mathfrak{f} \in \mathcal{M}_m$ if and only if $\mathfrak{M}_\mathfrak{f}(\{\arg G'(\mathfrak{f}, \boldsymbol{\alpha}^s) \colon\ s \in [k]\}) = \mathbf{B}^m$. Comparing the sequence G_{ψ_2} to $G'_{\psi_2}(\mathfrak{f})$, we verify that G'_{ψ_2} also satisfies relations (4.10). On the other hand, it follows from the definition of the sequence $G'(\mathfrak{f}, \boldsymbol{\alpha})$ that $G'(\mathfrak{f}, \boldsymbol{\alpha})$ contains at most $\lfloor \frac{m}{2} \rfloor + 2$ invocations of the operator $\mathcal{O}_\mathfrak{f}$, for any $i \in [k]$. It follows from the above argument that $\varphi(G'_{\psi_2}, \mathfrak{f}) = (\lfloor \frac{m}{2} \rfloor + 2)|\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})|$ and, thus, $\eta(G'_{\psi_2}, \mathfrak{f}) \le \lfloor \frac{m}{2} \rfloor + 2$. □

4.2 An inference algorithm for monotone Boolean functions associated with graphs

The study of infeasible systems, whose constraints correspond to the vertices of undirected graphs, and the subsystems with two constraints are feasible (or, on the contrary, infeasible) if and only if the corresponding vertex pairs are edges of the graphs, is of special applied interest.

In this section, with a graph is associated a monotone Boolean function whose zeros correspond to the feasible subsystems of the initial infeasible system of constraints.

Let a simple undirected graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ be given, with the vertex set $V(\mathbf{G}) := \{v_1, \dots, v_n\}$ and the edge family $\mathcal{E}(\mathbf{G}) := \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_p\}$. If $U \subset V(\mathbf{G})$, then $\mathbf{G}\langle U\rangle$ denotes the induced subgraph of the graph $\mathbf{G}$, on the vertex set U. For a vertex $v \in V(\mathbf{G})$, as earlier, $\mathcal{N}(v) \subset V(\mathbf{G})$ denotes the neighborhood of the vertex v in the graph $\mathbf{G}$. For a subset of vertices $U \subseteq V(\mathbf{G})$, we let $\binom{U}{2}$ denote the family of all unordered two subsets of the set U. If $\boldsymbol{x} := (x_1, \dots, x_n) \in \mathbf{B}^n := \{0,1\}^n$, then $\operatorname{supp}(\boldsymbol{x}) := \{i \in [n] \colon\ x_i = 1\}$.

Consider the monotone Boolean function $\mathfrak{f}_\mathbf{G} \colon\ \mathbf{B}^n \to \mathbf{B}$ whose set of *units* $\mathfrak{f}_\mathbf{G}^{-1}(1)$ is defined as follows:

$$\mathfrak{f}_\mathbf{G}(\boldsymbol{x}) := 1 \quad \Longleftrightarrow \quad \#\Big(\mathcal{E}(\mathbf{G}) \cap \Big(\substack{\{v_i \in V(\mathbf{G}) \colon\ i \in \operatorname{supp}(\boldsymbol{x})\} \\ 2}\Big)\Big) \ge 1\,; \tag{4.18}$$

in other words, we suppose $\mathfrak{f}_\mathbf{G}(\boldsymbol{x}) := 1$ if and only if the induced subgraph $\mathbf{G}\langle\{v_i \in V(\mathbf{G}) \colon\ i \in \operatorname{supp}(\boldsymbol{x})\}\rangle$ has at least one edge.

Another monotone Boolean function $\mathfrak{g}_\mathbf{G} \colon\ \mathbf{B}^n \to \mathbf{B}$, which is naturally associated with the graph $\mathbf{G}$, is defined by the set of its *zeros* $\mathfrak{g}_\mathbf{G}^{-1}(0)$ as follows:

$$\mathfrak{g}_\mathbf{G}(\boldsymbol{x}) := 0 \quad \Longleftrightarrow \quad \text{subgraph } \mathbf{G}\langle\{v_i \in V(\mathbf{G}) \colon\ i \in \operatorname{supp}(\boldsymbol{x})\}\rangle \text{ is complete}\,; \tag{4.19}$$

we relate to the complete graphs the empty graph $\mathbf{G}\langle\emptyset\rangle$, and the isolated vertices $\mathbf{G}\langle\{v_i\}\rangle$, $v_i \in V(\mathbf{G})$.

The graph-theoretic construction that establishes the connection between MBFs from (4.18) and (4.19) is the complement of the graph. The *complement* $\overline{\mathbf{G}}$ *of the graph* $\mathbf{G}$ by definition has the vertex set $V(\mathbf{G})$ and the edge family $\binom{V(\mathbf{G})}{2} - \mathcal{E}(\mathbf{G})$. Definitions (4.18) and (4.19) imply the following useful identities:

$$\mathfrak{f}_{\mathbf{G}} = \mathfrak{g}_{\overline{\mathbf{G}}}\,, \quad \mathfrak{f}_{\overline{\mathbf{G}}} = \mathfrak{g}_{\mathbf{G}}\,.$$

If $\mathcal{X} \subseteq \mathbf{B}^n$ is a set of binary tuples of length n, then, as earlier, $\mathbf{max}\,\mathcal{X}$ denotes the subset of maximal elements of $\mathcal{X}$ with respect to the partial order on $\mathbf{B}^n$, and $\mathbf{max}_{\preceq}\mathcal{X}$ denotes the subset of all tuples from $\mathcal{X}$ that have the maximal number of unit components.

Problem 4.5. *For the function* $\mathfrak{f}_{\mathbf{G}}$ *defined in* (4.18), *to find the set*

$$\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}}) := \mathbf{max}\,\mathfrak{f}_{\mathbf{G}}^{-1}(0)$$

of its upper zeros.

Problem 4.6. *For the function* $\mathfrak{f}_{\mathbf{G}}$, *to find the set*

$$\mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$$

of its maximal upper zeros.

An algorithm for finding a maximal upper zero of a monotone Boolean function associated with an undirected graph

Let us consider Problem 4.6, for graphs from a certain class, in more detail.

Proposition 4.7. *Let $v_i \in V(\mathbf{G})$ be a vertex of a graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$, such that for its neighborhood $\mathcal{N}(v_i)$ the induced subgraph $\mathbf{G}\langle\mathcal{N}(v_i)\rangle$ of the graph $\mathbf{G}$ is complete. Then there exists a maximal upper zero $\boldsymbol{x}' \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ of the function $\mathfrak{f}_{\mathbf{G}}$ such that $x'_i = 1$.*

Proof. Let us consider an arbitrary maximal upper zero $\boldsymbol{x} \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ of the function $\mathfrak{f}_{\mathbf{G}}$, and associate with this zero the index set $I := \{s \in [n]\colon\ v_s \in \mathcal{N}(v_i)\}$. It is easy to see that among the elements of the set $I\,\dot\cup\{i\}$ there is at least one index j such that $x_j = 1$, because otherwise we could find a tuple $\boldsymbol{x}' \in \mathbf{B}^n$ such that $x'_i = 1$ and $x'_s = x_s$ for all indices $s \in [n] - \{i\}$. Thus, because of $\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}) = 0$, and by the assumption that $x_s = 0$ for all $s \in I$, the definition of the function $\mathfrak{f}_{\mathbf{G}}$ implies that $\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}') = 0$. This contradicts the maximality of the upper zero $\boldsymbol{x}$, because we obtain the strict inclusion $\mathrm{supp}(\boldsymbol{x}') \supsetneqq \mathrm{supp}(\boldsymbol{x})$ and $\mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}') = \mathfrak{f}_{\mathbf{G}}(\boldsymbol{x}) = 0$. Now let us consider the two possible cases. If $x_i = 1$, then we are done. If $x_i = 0$ and $x_s = 1$ for some index $s \in I$, then for the tuple $\boldsymbol{x}$, one can find the tuple $\boldsymbol{x}' \in \mathbf{B}^n$ (by the rule: $x'_j := x_j$ for all $j \in [n] - \{i, s\}$, $x'_i := 1$, and $x'_s := 0$), which is an upper zero of the function $\mathfrak{f}_{\mathbf{G}}$, in view of the completeness of the induced subgraph $\mathbf{G}\langle\mathcal{N}(v_i)\rangle$, and $|\mathrm{supp}(\boldsymbol{x}')| = |\mathrm{supp}(\boldsymbol{x})|$; we thus obtained a maximal upper zero $\boldsymbol{x}'$ of the function $\mathfrak{f}_{\mathbf{G}}$ such that $x'_i = 1$, as was to be proved. □

For an integer $k \in [n-1]$, we call a vertex $v \in V(\mathbf{G})$ of the graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ a k-vertex, *if $|\mathcal{N}(v)| = k$ and the induced subgraph $\mathbf{G}\langle\mathcal{N}(v)\rangle$ of the graph $\mathbf{G}$ is complete.*

For integers $k, m \in [n-1]$, we call a vertex $v \in V(\mathbf{G})$ of the graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ a (k, m)-vertex, *if $k = |\mathcal{N}(v)|$ and $m = \binom{k}{2} - \#(\mathcal{E}(\mathbf{G}) \cap \binom{\mathcal{N}(v)}{2})$.*

A (k, m)-vertex $v \in V(\mathbf{G})$ of the graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ is its k-vertex when $m = 0$.

On the basis of Proposition 4.7, one can propose an efficient recursive algorithm for solving Problem 4.6, which finishes its work either by the construction of a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$, or by the reduction of Problem 4.6 for the function $\mathfrak{f}_{\mathbf{G}}$ to the new Problem 4.6 for a function $\mathfrak{f}_{\mathbf{G}'}$, where $\mathbf{G}' \subset \mathbf{G}$, that is, by the decrease of the dimension of the problem to be solved.

Given a vertex $v \in V_0 \subseteq V(\mathbf{G})$, denote by $\mathcal{N}(v, V_0) \subset V_0$ the neighborhood of the vertex v in the induced subgraph $\mathbf{G}\langle V_0 \rangle$.

– *Algorithm* 1: Algorithm $A(\mathbf{G}, V_0)$ for finding a maximal upper zero $\boldsymbol{x} := (x_1, \ldots, x_n) \in \mathbf{B}^n$ of the function $\mathfrak{f}_{\mathbf{G}}$.

`Input data:` $\mathbf{G}, V_0$
`Output data:` $V_0, \boldsymbol{x}$
  $x_i = 0,\ i \in [n],\ v_i \in V_0$
  `for each` $v_i \in V_0$ `do`
    `if` v_i is an $|\mathcal{N}(v_i, V_0)|$-vertex in the subgraph $\mathbf{G}\langle V_0 \rangle$ `then`
      $x_i \leftarrow 1$
      $V_0 \leftarrow V_0 - (\{v_i\} \,\dot\cup\, \mathcal{N}(v_i, V_0))$
      $A(\mathbf{G}, V_0)$
    `end of condition`
  `end of loop`

If at the finish of the work of *Algorithm* 1, we get $V_0 = \emptyset$, then, according to Proposition 4.7, the resulting tuple $\boldsymbol{x} \in \mathbf{B}^n$ is a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

However, if at the finish of the work of *Algorithm* 1, we have $V_0 \neq \emptyset$, then for all vertices of the graph $\mathbf{G}\langle V - V_0 \rangle$, we determined the values of some components x_i such that there exists a maximal upper zero $\boldsymbol{x}'$ of the function $\mathfrak{f}_{\mathbf{G}}$ with precisely the same values for these components, that is, $x'_i = x_i$; and yet we achieve the decrease of the dimension of the problem from $|V|$ to $|V_0|$.

Lemma 4.8. *Let two graphs $\mathbf{G}_1 := (V, \mathcal{E}(\mathbf{G}_1))$ and $\mathbf{G}_2 := (V, \mathcal{E}(\mathbf{G}_2))$ be given, with the same vertex set V, and*

$$\mathcal{E}(\mathbf{G}_1) \subseteq \mathcal{E}(\mathbf{G}_2)\,.$$

Then

$$\mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{f}_{\mathbf{G}_2}^{-1}(0) \subseteq \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)\,.$$

Proof. It is clear that $\mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{f}_{\mathbf{G}_2}^{-1}(0)$.

Consider an arbitrary tuple $\boldsymbol{x} \in \mathbf{B}^n$ such that $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_2}^{-1}(0)$. By the definition of the set of zeros $\mathfrak{f}_{\mathbf{G}_2}^{-1}(0)$ of the MBF $\mathfrak{f}_{\mathbf{G}_2}$, we have

$$\#\left(\mathcal{E}(\mathbf{G}_2) \cap \binom{\{v_i:\ i \in \operatorname{supp}(\boldsymbol{x})\}}{2}\right) = 0\,.$$

By the hypothesis of the lemma, we have $\mathcal{E}(\mathbf{G}_1) \subseteq \mathcal{E}(\mathbf{G}_2)$ and $V(\mathbf{G}_1) = V(\mathbf{G}_2)$; as a consequence,

$$\#\left(\mathcal{E}(\mathbf{G}_1) \cap \binom{\{v_i:\ i \in \operatorname{supp}(\boldsymbol{x})\}}{2}\right) = 0\ , \quad \forall \boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_2}^{-1}(0)\ ,$$

and

$$\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)\ . \tag{4.20}$$

Then for any tuples $\boldsymbol{x} \in \mathbf{B}^n$ such that $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_2}^{-1}(0)$, inclusion (4.20) holds, that is,

$$\mathfrak{f}_{\mathbf{G}_2}^{-1}(0) \subseteq \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)\ ,$$

as was to be proved. □

It should be mentioned that

$$\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \nsubseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})\ . \tag{4.21}$$

Indeed, consider the graphs

$$\begin{aligned} \mathbf{G}_1 :&= (V(\mathbf{G}_1),\ \mathcal{E}(\mathbf{G}_1)) = (V, \emptyset)\ , \\ \mathbf{G}_2 :&= (V(\mathbf{G}_2),\ \mathcal{E}(\mathbf{G}_2)) = \left(V, \tbinom{V}{2}\right)\ , \end{aligned}$$

for which we have $V(\mathbf{G}_1) = V(\mathbf{G}_2)$ and $\mathcal{E}(\mathbf{G}_1) \subseteq \mathcal{E}(\mathbf{G}_2)$. The graph $\mathbf{G}_1$ has no edges, therefore, the set of upper zeros of the function $\mathfrak{f}_{\mathbf{G}_1}$ consists of the unique tuple

$$\boldsymbol{x} := (1, 1, \ldots, 1)\ .$$

The graph $\mathbf{G}_2$ is complete; thus, the set of upper zeros of the function $\mathfrak{f}_{\mathbf{G}_2}$ has the form

$$\begin{aligned} \boldsymbol{x}^1 :&= (1, 0, \ldots, 0)\ , \\ \boldsymbol{x}^2 :&= (0, 1, \ldots, 0)\ , \\ &\ \ \vdots \\ \boldsymbol{x}^n :&= (0, 0, \ldots, 1)\ . \end{aligned}$$

Any tuple $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2})$ is a zero of the function $\mathfrak{f}_{\mathbf{G}_1}$, that is,

$$\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \subseteq \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)\ , \quad \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2}) \nsubseteq \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})\ ,$$

as Lemma 4.8 asserts; this justifies (4.21).

Let us define the quantity $\max_0 \mathfrak{f}_{\mathbf{G}} := |\operatorname{supp}(\boldsymbol{x})|$, where $\boldsymbol{x} \in \mathbf{max}_{\preceq} \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$, which is the number of unit components in a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

Corollary 4.9. *Let* $\mathbf{G}_1 := (V, \mathcal{E}_1)$ *and* $\mathbf{G}_2 := (V, \mathcal{E}_2)$ *be graphs such that* $\mathcal{E}_1 \subseteq \mathcal{E}_2$. *Then*

$$\max_0 \mathfrak{f}_{\mathbf{G}_1} \geq \max_0 \mathfrak{f}_{\mathbf{G}_2}\ .$$

Proof. Let $\boldsymbol{x} \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_2})$. According to Lemma 4.8, we have $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)$.

By the definition of the maximal upper zeros of the function, for any tuple $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}_1}^{-1}(0)$ there exists a tuple $\boldsymbol{x}' \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})$ such that $\boldsymbol{x}' \geq \boldsymbol{x}$. Then

$$\max{}_0\mathfrak{f}_{\mathbf{G}_1} = |\mathrm{supp}(\boldsymbol{x}')| \geq |\mathrm{supp}(\boldsymbol{x})| = \max{}_0\mathfrak{f}_{\mathbf{G}_2} ,$$

as was to be proved. □

Proposition 4.10. *Let* $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$ *be a graph in which vertices* v_i *and* v_j *are not adjacent. Then*

$$\max{}_0\mathfrak{f}_{\mathbf{G}} \geq \max{}_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq \max{}_0\mathfrak{f}_{\mathbf{G}} - 1 . \tag{4.22}$$

Proof. The inequality $\max_0\mathfrak{f}_{\mathbf{G}} \geq \max_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$ follows from Corollary 4.9.

Let us prove the inequality $\max_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq \max_0\mathfrak{f}_{\mathbf{G}} - 1$. Let $\boldsymbol{x} := (x_1, \dots, x_n)$ be a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

Case 1

Suppose that $x_i = 0$ and $x_j = 0$. Then $\boldsymbol{x}$ is clearly a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$, and it is a maximal upper zero, because otherwise we would obtain, by definition, that there exists a maximal upper zero $\boldsymbol{x}'$ of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$ such that $\boldsymbol{x}' > \boldsymbol{x}$ and $|\mathrm{supp}(\boldsymbol{x}')| > |\mathrm{supp}(\boldsymbol{x})|$. According to Lemma 4.8, we obtain that $\boldsymbol{x}'$ is a zero of the function $\mathfrak{f}_{\mathbf{G}}$, but this contradicts the maximality of $\boldsymbol{x}$.

Thus, in this case, we have

$$\max{}_0\mathfrak{f}_{\mathbf{G}} = \max{}_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq \max{}_0\mathfrak{f}_{\mathbf{G}} - 1 .$$

Case 2

Suppose that $x_i = 1$ and $x_j = 0$.

If the edge (v_i, v_j) is added, then the tuple $\boldsymbol{x}$ is again a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$ and, as shown earlier, it is also a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$.

Case 3

Suppose that $x_i = 1$ and $x_j = 1$.

If the edge (v_i, v_j) is added, then we obtain that $\boldsymbol{x}$ is not a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$. In this case, we can find a tuple $\boldsymbol{x}'$ for which $x'_s = x_s$ for all $s \in [n] - \{i\}$, and $x'_i = 0$. The tuple $\boldsymbol{x}'$ will be a zero of the function $\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}}$. Besides, by construction,

$$|\mathrm{supp}(\boldsymbol{x}')| = |\mathrm{supp}(\boldsymbol{x})| - 1 .$$

By the definition of the maximal upper zeros of the function, we have

$$\max{}_0\mathfrak{f}_{\mathbf{G}\cup\{(v_i,v_j)\}} \geq |\mathrm{supp}(\boldsymbol{x}')| = |\mathrm{supp}(\boldsymbol{x})| - 1 = \max{}_0\mathfrak{f}_{\mathbf{G}} - 1 ,$$

as was to be proved. □

Corollary 4.11. *For a graph* $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$*, let* $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_t\} \subset \binom{V(\mathbf{G})}{2} - \mathcal{E}(\mathbf{G})$ *be a subfamily of t vertex pairs that are not edges of the graph* $\mathbf{G}$*.*

Then

$$\max{}_0 \mathfrak{f}_{\mathbf{G} \cup \{\boldsymbol{e}_1,\dots,\boldsymbol{e}_t\}} \geq \max{}_0 \mathfrak{f}_{\mathbf{G}} - t\,.$$

Proof. It suffices to apply Proposition 4.10, t times, to the graph $\mathbf{G}$. □

On the basis of Proposition 4.10, one can modify *Algorithm* 1 in such a way that the work of the algorithm will continue until the set of remaining vertices V_0 becomes empty and, besides, a zero $\boldsymbol{x}$ of the function $\mathfrak{f}_{\mathbf{G}}$ will be found, for which, at the same time, we will calculate the estimate $(\max_0 \mathfrak{f}_{\mathbf{G}} - |\mathrm{supp}(\boldsymbol{x})|)$ of the deviation of the number of unit components in the resulting tuple $\boldsymbol{x}$ from the number of unit components in a maximal upper zero of the function $\mathfrak{f}_{\mathbf{G}}$.

– *Algorithm* 2: Algorithm $A_m(\mathbf{G}, V_0)$.

```
Input data: G, V_0, m ∈ [n]
Output data: V_0, Ind, x
  Ind = 0
  for each v_i ∈ V_0 do
    if v_i is an (|N(v_i, V_0)|, m)-vertex in the subgraph G⟨V_0⟩ then
      x_i ← 1
      V_0 ← V_0 − ({v_i} ∪̇ N(v_i, V_0))
      Ind ← 1
  end of loop
```

Algorithm 2 sequentially checks, for the given value of m and for each vertex of the initial set V_0, whether it is an $(|\mathcal{N}(v_i, V_0)|, m)$-vertex. If there are no such vertices, then no operations are performed, and the resulting set V_0 at the finish of the work of the algorithm coincides with the input set V_0, the flag Ind $= 0$, a binary tuple $\boldsymbol{x}$ is not determined. In the case where such a vertex v_i is found, the output set V_0 will be obtained from the input set V_0 by means of the “removal” of the vertex v_i and its neighborhood, Ind $= 1$, and the corresponding component x_i of the tuple $\boldsymbol{x}$ takes the value of 1.

– *Algorithm* 3: Algorithm $B(\mathbf{G}, V_0)$.

```
Input data: G, V_0
Output data: x ∈ f_G^{-1}(0)
  while V_0 ≠ ∅
    m = 0
    Ind = 1
    while (Ind = 1) & V_0 ≠ ∅ do
```

$A_m(\mathbf{G}, V_0)$
Ind $\leftarrow$ Ind($A_m(\mathbf{G}, V_0)$)
`end of loop`

`while` (Ind = 0) & $V_0 \neq \emptyset$ `do`
$m \leftarrow m + 1$
$A_m(\mathbf{G}, V_0)$
Ind $\leftarrow$ Ind($A_m(\mathbf{G}, V_0)$)
`end of loop`
`end of loop`

In the course of the work of *Algorithm* 3, as the result of repeated calls of *Algorithm* 2, the tuple $\boldsymbol{x}$ is formed, which is a zero of the function $\mathfrak{f}_{\mathbf{G}}$.

Proposition 4.12. *Let v_i be a (k, m)-vertex in a graph $\mathbf{G} := (V(\mathbf{G}), \mathcal{E}(\mathbf{G}))$. Then there exists a tuple $\boldsymbol{x}' \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ such that $x'_i = 1$ and*

$$|\mathrm{supp}(\boldsymbol{x}')| \geq \max{}_0 \mathfrak{f}_{\mathbf{G}} - m\ .$$

Proof. Suppose, according to the definition of the (k, m)-vertices, that for $v_i \in V(\mathbf{G})$ we have

$$\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\} := \binom{\mathcal{N}(v_i)}{2} - \left(\mathcal{E}(\mathbf{G}) \cap \binom{\mathcal{N}(v_i)}{2}\right)\ .$$

Then the vertex v_i is a k-vertex in the graph $\mathbf{G}_1$, which is obtained from the graph $\mathbf{G}$ by the addition of the m edges $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ into the neighborhood of the vertex v_i of the graph $\mathbf{G}$ up to the complete induced subgraph $\mathbf{G}_1\langle\mathcal{N}(v_i)\rangle$.

According to Proposition 4.7, there exists a tuple $\boldsymbol{x}$ such that $x_i = 1$ and $\boldsymbol{x} \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}_1})$.

According to Corollary 4.11, for the graph $\mathbf{G}_1$ we have

$$|\mathrm{supp}(\boldsymbol{x})| = \max{}_0 \mathfrak{f}_{\mathbf{G}_1} \geq \max{}_0 \mathfrak{f}_{\mathbf{G}} - m\ .$$

It follows from Lemma 4.8 that $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}}^{-1}(0)$. By the definition of the upper zeros, there exists a tuple $\boldsymbol{x}' \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ such that $\boldsymbol{x}' \geq \boldsymbol{x}$ and, as a consequence,

$$|\mathrm{supp}(\boldsymbol{x}')| \geq |\mathrm{supp}(\boldsymbol{x})| \geq \max{}_0 \mathfrak{f}_{\mathbf{G}} - m\ ,$$

as was to be proved. □

In every next loop of *Algorithm* 1, the search is terminated when some k-vertex is found. Such an approach minimizes the number of operations in the working loop of the algorithm, but it does not necessarily lead to the best solution in the case where $V_0 \neq \emptyset$.

Let us present an *Algorithm* 4, in each next working loop of which the parameters k and m are calculated for every vertex from the current set V_0. *Algorithm* 4 admits a larger number of operations in each working loop, but it can provide a more precise approximation to a maximal upper zero.

– *Algorithm* 4:

`Input data:` $\mathbf{G}$, V_0, $m = 0$
`Output data:` $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$, and m which is the estimate of deviation from $\max_0 \mathfrak{f}_{\mathbf{G}}$
`while` $V_0 \neq \emptyset$
for all vertices $v_i \in V_0 \neq \emptyset$, to calculate the parameters k_i and m_i such that v_i is a (k_i, m_i)-vertex in the graph $\mathbf{G}\langle V_0\rangle$; in the set V_0, to extract the subset $V_0' \subseteq V_0$ of vertices with the minimal values of the parameter m_i. Among the extracted vertices in the set V_0', to find a vertex $v_{i_0} \in V_0'$ with the maximal value of the parameter k_{i_0}
$x_{i_0} \leftarrow 1$
$m \leftarrow m + m_{i_0}$
$V_0 \leftarrow V_0 - (\{v_{i_0}\} \dot\cup \mathcal{N}(v_{i_0}, V_0))$
`end of loop`

Algorithm 4 finds a tuple $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$, for which the precision estimate $\max_0 \mathfrak{f}_{\mathbf{G}} - |\mathrm{supp}(\boldsymbol{x})| \le m$ of the solution is true.

Let us estimate the complexity of *Algorithm* 4.

For each vertex v_i from the current set V_0, it is necessary to find the number of vertices in the neighborhood $\mathcal{N}(v_i, V_0)$ and the number of new edges that should be added into the neighborhood $\mathcal{N}(v_i, V_0)$ for turning the induced subgraph $\mathbf{G}\langle\mathcal{N}(v_i, V_0)\rangle$ into a complete graph. We remove the vertices $v_i \dot\cup \mathcal{N}(v_i, V_0)$ and the edges $\boldsymbol{e}_i \in \mathbf{G}\langle\{v_i\} \dot\cup \mathcal{N}(v_i, V_0)\rangle$ until the current set of vertices V_0 becomes empty. Given the input data $V(\mathbf{G}) = \{v_1, \ldots, v_n\}$ and $\mathcal{E}(\mathbf{G}) = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_p\}$, we obtain the following estimate. The common number of iterations undertaken during the work of *Algorithm* 4 is less than or equal to n; every iteration demands no more than $O(np)$ actions for the computation of the parameters k and m; and no more than $O(p)$ actions are needed for the removal of a vertex and its neighborhood from the current graph. Thus, *Algorithm* 4 has the complexity of $O(n \cdot np + np) = O(n^2 p)$.

Solving the problem of searching for a maximal upper zero

For some applied problems that are reduced to Problem 4.6, either exact results were obtained, or the significant decrease of the dimension of Problem 4.6 was achieved.

Example 4.13. *The graph* $\mathbf{G} := (V := \{v_1, \ldots, v_{22}\}, \mathcal{E})$ *is specified by the incidence lists* $\mathcal{N}(v_i)$ *of its vertices,* $i \in [22]$, $V_0 = V$:

$$\mathcal{N}(v_1) := \{v_2, v_3\}, \ \mathcal{N}(v_2) := \{v_1, v_3\}, \ \mathcal{N}(v_3) := \{v_1, v_2, v_4, v_9\},$$
$$\mathcal{N}(v_4) := \{v_3, v_5, v_6, v_{11}\}, \ \mathcal{N}(v_5) := \{v_4, v_6\}, \ \mathcal{N}(v_6) := \{v_4, v_5, v_7, v_{10}, v_{12}\},$$

$$\mathcal{N}(v_7) := \{v_6, v_8\}\,,\ \mathcal{N}(v_8) := \{v_7, v_{12}, v_{16}, v_{17}\}\,,\ \mathcal{N}(v_9) := \{v_3, v_{11}, v_{13}\}\,,$$
$$\mathcal{N}(v_{10}) := \{v_6, v_{11}, v_{12}, v_{14}, v_{15}\}\,,\ \mathcal{N}(v_{11}) := \{v_4, v_9, v_{10}, v_{14}\}\,,$$
$$\mathcal{N}(v_{12}) := \{v_6, v_8, v_{10}, v_{16}\}\,,\ \mathcal{N}(v_{13}) := \{v_9, v_{14}\}\,,\ \mathcal{N}(v_{14}) := \{v_{10}, v_{11}, v_{13}, v_{15}\}\,,$$
$$\mathcal{N}(v_{15}) := \{v_{10}, v_{14}, v_{16}, v_{20}, v_{21}\}\,,\ \mathcal{N}(v_{16}) := \{v_8, v_{12}, v_{15}, v_{17}, v_{19}\}\,,$$
$$\mathcal{N}(v_{17}) := \{v_8, v_{16}, v_{18}, v_{19}\}\,,\ \mathcal{N}(v_{18}) := \{v_{17}, v_{19}\}\,,$$
$$\mathcal{N}(v_{19}) := \{v_{16}, v_{17}, v_{18}, v_{20}, v_{21}, v_{22}\}\,,\ \mathcal{N}(v_{20}) := \{v_{15}, v_{19}, v_{21}, v_{22}\}\,,$$
$$\mathcal{N}(v_{21}) := \{v_{15}, v_{19}, v_{20}, v_{22}\}\,,\ \mathcal{N}(v_{22}) := \{v_{19}, v_{20}, v_{21}\}\,.$$

Acting in accordance with Algorithm 1*, for each vertex $v_i \in V_0$ we check whether it is a k-vertex in the graph* $\mathbf{G}$*.*
$A(\mathbf{G}, V_0)$*:*
1. v_1 *is a* 2*-vertex* $\Rightarrow\ x_1 \leftarrow 1$*;* $V_0 \leftarrow V_0 - \{v_1, v_2, v_3\}$.
2. v_4 *is not a* 3*-vertex.*
3. v_5 *is a* 2*-vertex* $\Rightarrow\ x_5 \leftarrow 1$*;* $V_0 \leftarrow V_0 - \{v_4, v_5, v_6\}$.
4. v_7 *is a* 1*-vertex* $\Rightarrow\ x_7 \leftarrow 1$*;* $V_0 \leftarrow V_0 - \{v_7, v_8\}$.
5. $v_{9\,(10,11,12,13,14,15,16,17)}$ *is not a* $2\ (4,3,2,2,4,5,4,3)$*-vertex.*
6. v_{18} *is a* 2*-vertex* $\Rightarrow\ x_{18} \leftarrow 1$*;* $V_0 \leftarrow V_0 - \{v_{17}, v_{18}, v_{19}\}$.
7. $v_{9\,(10,11,12,13,14,15,16,20,21)}$ *is not a* $2\ (4,3,2,2,4,5,2,3,3)$*-vertex.*
8. v_{22} *is a* 2*-vertex* $\Rightarrow\ x_{22} \leftarrow 1$*;* $V_0 \leftarrow V_0 - \{v_{20}, v_{21}, v_{22}\}$.

$\boldsymbol{x} = (1,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,1)$ *is a zero of the function* $\mathfrak{f}_{\mathbf{G}}$*,* $\boldsymbol{x} \in \mathfrak{f}_{\mathbf{G}}^{-1}(0)$*; besides, a maximal upper zero* $\boldsymbol{x}' \in \mathbf{max}_{\preceq}\mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ *of the function* $\mathfrak{f}_{\mathbf{G}}$ *has the form*

$$\boldsymbol{x}' = (1,0,0,0,1,0,1,0,x_9,x_{10},x_{11},x_{12},x_{13},x_{14},x_{15},x_{16},0,1,0,0,0,1)\,.$$

Thus, the dimension of the problem was decreased from $|V_0| = 22$ to $|V_0| = |\{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}| = 8$.

For exhausting the vertex set V_0, we follow *Algorithm* 3; among the vertices from the set V_0 we search for (k,m)-vertices (the case of $m = 0$ corresponds to the search for k-vertices, which was undertaken by *Algorithm* 1).

Example 4.14.
$V_0 \neq \emptyset,\ m = 0$*:*
$\mathrm{Ind} = 0 \Rightarrow m \leftarrow m + 1 = 1,\ A_1(\mathbf{G}, V_0)$*:*
v_9 *is a* $(2,1)$*-vertex,* `then`*:* $x_9 \leftarrow 1,\ V_0 \leftarrow V_0 - \{v_9, v_{11}, v_{13}\} \Rightarrow$
$\mathrm{Ind} = 1 \Rightarrow m = 0,\ A_0(\mathbf{G}, V_0)$*:*
$v_{10\,(12)}$ *is not a* $3(2)$*-vertex;*
v_{14} *is a* 2*-vertex,* `then`*:* $x_{14} \leftarrow 1,\ V_0 \leftarrow V_0 - \{v_{10}, v_{14}, v_{15}\}$.
$\mathrm{Ind} = 1 \Rightarrow m = 0,\ A_0(\mathbf{G}, V_0)$*:*
v_{12} *is a* 1*-vertex,* `then`*:* $x_{12} \leftarrow 1,\ V_0 \leftarrow V_0 - \{v_{12}, v_{16}\}$.
$V_0 = \emptyset \Rightarrow$
$\boldsymbol{x}' = (1,0,0,0,1,0,1,0,1,0,0,1,0,1,0,0,0,1,0,0,0,1)$ *is a zero of the function* $\mathfrak{f}_{\mathbf{G}}$*, and it is a maximal upper zero of the function* $\mathfrak{f}_{\mathbf{G}\cup\{(v_{11},v_{13})\}}$*; then, according*

to Proposition 4.10, *the number of unit components in a maximal upper zero of the function* $\mathfrak{f}_{\mathbf{G}}$ *is restricted by the inequality:*

$$\max_0 \mathfrak{f}_{\mathbf{G}} \le \max_0 \mathfrak{f}_{\mathbf{G} \cup \{(v_{11}, v_{13})\}} + 1 = |\mathrm{supp}(\boldsymbol{x}')| + 1 = 9 \,.$$

It is convenient to describe the result of the work of *Algorithm* 3 in the form of Table 4.1. The columns of the table correspond to the current state of the set V_0. We sequentially remove k-vertices and their neighborhoods from the set V_0, associating to the corresponding components x_i the value of 1 in the case where v_i is a k-vertex, and the value of 0 otherwise.

Table 4.2 describes the work of *Algorithm* 4. Every column of the table represents an iteration of *Algorithm* 4; the nonzero elements of a column correspond to the set V_0, and in an ith row the values of k and m are related to the vertex v_i in the current subgraph $\mathbf{G}\langle V_0 \rangle$.

For the resulting tuple $\boldsymbol{x} = (1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1)$ it holds that $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ and $|\mathrm{supp}(\boldsymbol{x})| = 9 \ge \max_0 \mathfrak{f}_{\mathbf{G}} - 1$.

Table 4.1. The result of the work of *Algorithm* 3

m	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	***x***
Ind	1	1	1	1	1	1	0	1	1	
v_1	1	0	0	0	0	0	0	0	0	1
v_2	1	0	0	0	0	0	0	0	0	0
v_3	1	0	0	0	0	0	0	0	0	0
v_4	1	1	0	0	0	0	0	0	0	0
v_5	1	1	0	0	0	0	0	0	0	1
v_6	1	1	0	0	0	0	0	0	0	0
v_7	1	1	1	0	0	0	0	0	0	1
v_8	1	1	1	0	0	0	0	0	0	0
v_9	1	1	1	1	1	1	0	0	0	1
v_{10}	1	1	1	1	1	1	1	0	0	0
v_{11}	1	1	1	1	1	1	0	0	0	0
v_{12}	1	1	1	1	1	1	1	1	0	1
v_{13}	1	1	1	1	1	1	0	0	0	0
v_{14}	1	1	1	1	1	1	1	0	0	1
v_{15}	1	1	1	1	1	1	1	0	0	0
v_{16}	1	1	1	1	1	1	1	1	0	0
v_{17}	1	1	1	1	0	0	0	0	0	0
v_{18}	1	1	1	1	0	0	0	0	0	1
v_{19}	1	1	1	1	0	0	0	0	0	0
v_{20}	1	1	1	1	1	0	0	0	0	0
v_{21}	1	1	1	1	1	0	0	0	0	0
v_{22}	1	1	1	1	1	0	0	0	0	1

Table 4.2. The work of *Algorithm* 4

	k/m	*k/m*	*k/m*	*k/m*	*k/m*	*k/m*	*k/m*	*k/m*	*k/m*	*x*
v_1	2/0	2/0								1
v_2	2/0	2/0								0
v_3	4/5	4/5								0
v_4	4/5	4/5	3/2							0
v_5	2/0	2/0	2/0							1
v_6	5/8	5/8	5/8							0
v_7	2/1	2/1	2/1	1/0						1
v_8	4/4	4/4	4/4	4/4						0
v_9	3/3	3/3	2/1	2/1	2/1	2/1	2/1	2/1		0
v_{10}	5/7	5/7	5/7	4/4	4/4	4/4				0
v_{11}	4/5	4/5	4/5	3/2	3/2	3/2	2/1	1/0		1
v_{12}	4/4	4/4	4/4	3/2	2/1	2/1				1
v_{13}	2/1	2/1	2/1	2/1	2/1	2/1	2/1	1/0	0/0	1
v_{14}	4/4	4/4	4/4	4/4	4/4	4/4	3/3			0
v_{15}	5/8	3/2	3/2	3/2	3/2	3/2	1/0			1
v_{16}	5/7	4/4	4/4	4/4	3/3	2/1				0
v_{17}	4/3	3/2	3/2	3/2	2/1					0
v_{18}	2/0	1/0	1/0	1/0	1/0					1
v_{19}	6/10									0
v_{20}	4/2									0
v_{21}	4/2									0
v_{22}	3/0									1

Earlier, for the tuple $\boldsymbol{x}'$ obtained with the help of *Algorithm* 3, we also obtained that

$$|\mathrm{supp}(\boldsymbol{x}')| = 8 \geq \max_0 \mathfrak{f}_{\mathbf{G}} - 1$$

or, in other words, $\max_0 \mathfrak{f}_{\mathbf{G}} \leq 9$. Since $\boldsymbol{x} \in \mathfrak{Q}(\mathfrak{f}_{\mathbf{G}})$ and $|\mathrm{supp}(\boldsymbol{x})| = 9$, we see that $\max_0 \mathfrak{f}_{\mathbf{G}} = 9$.

4.3 Monotone Boolean functions and inequality systems

The problem of extracting inclusion-maximal feasible subsystems of an infeasible monotone system of constraints is naturally reduced to the problem of inference of monotone Boolean functions.

We will consider here the problem of extracting all the MFSs of an infeasible system of linear inequalities of the form (3.20), described on page 75, that is a rank n system

$$\mathsf{S} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > 0 : \ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n; \ \|\boldsymbol{a}_i\| = 1, \ i \in [m]\}$$

of homogeneous strict linear inequalities over the real Euclidean space $\mathbb{R}^n$.

The reduction to the problem of MBF inference consists in the following. Let $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a binary tuple. Let us pick in $\boldsymbol{\alpha}$ all its unit components. Suppose that their indices are $i_1, i_2, \ldots, i_k$. Consider the subsystem, with the multi-index $\{i_1, i_2, \ldots, i_k\}$, of the system S, and denote this subsystem by $\mathsf{S}(\boldsymbol{\alpha})$. Let us set

$$\mathfrak{f}(\boldsymbol{\alpha}) := \begin{cases} 0, & \text{if } \mathsf{S}(\boldsymbol{\alpha}) \text{ is feasible ,} \\ 1, & \text{if } \mathsf{S}(\boldsymbol{\alpha}) \text{ is infeasible .} \end{cases}$$

The function $\mathfrak{f}$ is monotone and the set of its upper zeros is in one-to-one correspondence with the family of maximal feasible subsystems of the system S.

It turns out that after some modification of the operator $\mathcal{O}_{\mathfrak{f}}$, natural for the class of monotone Boolean functions under consideration, it is possible to present an algorithm of MBF inference which is optimal with respect to all criteria (4.3)–(4.6).

(1) The new operator $\mathcal{O}'_{\mathfrak{f}}$ is to determine the value of the function $\mathfrak{f}(\boldsymbol{\alpha})$ at a given tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$;
(2) if $\mathfrak{f}(\boldsymbol{\alpha}) = 1$, then the operator $\mathcal{O}'_{\mathfrak{f}}$ is to extract one lower unit $\boldsymbol{\alpha}'$ of the function $\mathfrak{f}$, such that $\boldsymbol{\alpha}' \le \boldsymbol{\alpha}$.

This modification is indeed reasonable, because for the class of MBFs under consideration, which are associated with the infeasible systems S, some variants of the well-developed technique of linear programming can be chosen as such an operator.

Let us denote by $\varphi(\mathcal{O}'_{\mathfrak{f}}, G, \mathfrak{f})$ the number of calls of the operator $\mathcal{O}'_{\mathfrak{f}}$ by an algorithm G when inferring a function $\mathfrak{f} \in \mathcal{M}_m$. For any algorithm G of MBF inference, and for any function $\mathfrak{f} \in \mathcal{M}_m$, the inequality $\varphi(\mathcal{O}'_{\mathfrak{f}}, G, \mathfrak{f}) \ge |\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})|$ holds.

Proposition 4.15. *There exists an algorithm G^* of MBF inference, such that $\varphi(\mathcal{O}'_{\mathfrak{f}}, G^*, \mathfrak{f}) = |\mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f})|$ for any function $\mathfrak{f} \in \mathcal{M}_m$.*

Proof. Let us construct such an algorithm G^*. Let us suppose $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}) := \boldsymbol{\alpha}$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 0$, and $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}) := \boldsymbol{\alpha}'$ when $\mathfrak{f}(\boldsymbol{\alpha}) = 1$, where $\boldsymbol{\alpha}'$ is a lower unit of the function $\mathfrak{f}$, determined by the operator $\mathcal{O}'_{\mathfrak{f}}$. By definition, the inference sequence $G^*(\mathfrak{f})$ for Algorithm G^* and for the function $\mathfrak{f} \in \mathcal{M}_m$ is as follows:

$$\begin{aligned} G^*(\mathfrak{f}) &:= \big(\boldsymbol{\alpha}^1, \mathfrak{f}(\boldsymbol{\alpha}^1), \boldsymbol{\alpha}^2, \mathfrak{f}(\boldsymbol{\alpha}^2), \ldots, \boldsymbol{\alpha}^k, \mathfrak{f}(\boldsymbol{\alpha}^k)\big) , \\ \boldsymbol{\alpha}^1 &:= (1, 1, \ldots, 1) \in \mathbf{B}^m , \\ \boldsymbol{\alpha}^i &:= \psi\big(\mathbf{max}\,(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s) \colon\ s \in [i-1]\}))\big) , \end{aligned}$$

where ψ is an arbitrary choice function. The inference process is completed when we have $\mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s) \colon\ s \in [k]\}) = \mathbf{B}^m$, that is,

$$\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s) \colon\ s \in [k]\} \supseteq \mathfrak{Q}(\mathfrak{f}) \,\dot\cup\, \mathfrak{P}(\mathfrak{f}) . \tag{4.23}$$

Further, if $\mathfrak{f}(\boldsymbol{\alpha}^i) = 0$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^i)$ is a maximal upper zero of the function $\mathfrak{f}$ because of the maximality of

$$\boldsymbol{\alpha}^i = \psi(\mathbf{max}\,(\mathbf{B}^m - \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [i-1]\}))) ,$$

in analogy to the argument of Proposition 4.2; if $\mathfrak{f}(\boldsymbol{\alpha}^i) = 1$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha})$ is by definition a minimal lower unit of the function $\mathfrak{f}$. Thus, the inclusion

$$\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [k]\} \subseteq \mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f}) \tag{4.24}$$

holds.

Let us now prove the implication

$$t, p \in [k],\ \ t < p \quad \Longrightarrow \quad \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p) . \tag{4.25}$$

Indeed, if $\mathfrak{f}(\boldsymbol{\alpha}^t) \neq \mathfrak{f}(\boldsymbol{\alpha}^p)$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p)$, by definition. If $\mathfrak{f}(\boldsymbol{\alpha}^t) = \mathfrak{f}(\boldsymbol{\alpha}^p) = 0$ then $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p)$ because $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p) \notin \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [t]\})$. If $\mathfrak{f}(\boldsymbol{\alpha}^t) = \mathfrak{f}(\boldsymbol{\alpha}^p) = 1$ then $\boldsymbol{\alpha}^p \notin \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [t]\})$, and $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p) \leq \boldsymbol{\alpha}^p$ implies $\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^p) \notin \mathfrak{M}_{\mathfrak{f}}(\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^s):\ s \in [t]\})$, that is, $A'_{\mathfrak{f}}(\boldsymbol{\alpha}^t) \neq A'_{\mathfrak{f}}(\boldsymbol{\alpha}^p)$. It follows from relations (4.23)–(4.25) that $k = \varphi(\mathcal{O}'_{\mathfrak{f}}, G^*, \mathfrak{f}) = |\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})|$. □

Note that during the inference process for the function $\mathfrak{f}$, it suffices to store in the memory of a computer system just the set $\{\mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^1), \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^2), \ldots, \mathcal{O}'_{\mathfrak{f}}(\boldsymbol{\alpha}^k)\}$, that is, at most $|\mathfrak{Q}(\mathfrak{f}) \dot\cup \mathfrak{P}(\mathfrak{f})|$ binary tuples of length m.

The algorithm G^* of MBF inference is optimal with respect to criteria (4.3)–(4.6).

Notes

A thorough presentation of the Boolean function theory and of its various applications can be found, for example, in books [30, 86, 90, 110, 153].

Among numerous works devoted to the study of monotone Boolean functions, we point out at review [79], and at book [124] which is devoted to Dedekind's problem on the number of MBFs.

A thorough survey of the state-of-art theory and practice in inference of monotone Boolean functions is given in book [144] and in concise note [143].

In this chapter, we follow in our presentation work [52].

In the typical case, inference of monotone Boolean functions requires asymptotically the twice as less number of invocations of the oracle than in the worst case, see [124, 126].

The algorithm $\varphi(G, m)$ of inference of monotone Boolean functions which is optimal with respect to classical Shannon's criterion is presented in work [66], where it was proved that $\varphi(m) = \binom{m}{\lfloor m/2 \rfloor} + \binom{m}{\lfloor m/2 \rfloor + 1}$.

For various applied problems, the algorithms that are optimal with respect to Shannon's criterion are inadequate. For example, such inference algorithms from works [66] and [131] require at least $\binom{m}{\lfloor m/2 \rfloor}$ calls of the operator $\mathcal{O}_{\mathfrak{f}}$ during the inference process for such simple functions as *identically zero* $\mathfrak{f}_0 \equiv 0$ and *identically unit* $\mathfrak{f}_1 \equiv 1$.

Comparing the algorithms described in Section 4.1 to other known algorithms, let us note that for the algorithms "A_1" from [66] and "A_2" from [131], optimal with respect to Shannon's criterion $\varphi(G, m)$, we have $\eta(A_1, m), \eta(A_2, m) \geq \binom{m}{\lfloor m/2 \rfloor}$. This observation follows from the fact that during the inference process for the function *identically zero* $\mathfrak{f}_0 \equiv 0$, the algorithms "A_1" and "A_2" call the operator $\mathcal{O}_{\mathfrak{f}}$ at least $\binom{m}{\lfloor m/2 \rfloor}$ times, where $\binom{m}{\lfloor m/2 \rfloor}$ is the number of chains in the chain partition, considered in [66, 131], of the unit cube $\mathbf{B}^m$.

The close relationship between the problem of inference of monotone Boolean functions and central problems of combinatorial optimization is well known; for example, it was shown in [78] how the knapsack problem is reduced to inference of some MBF.

The adaptive algorithm of solving the multidimensional knapsack problem, presented in [145], can also be efficiently applied to inference of MBFs with a small number of maximal upper zeros. The efficiency of this algorithm is justified by illustrative inference of the specific MBF, of ten binary variables, with the following five maximal upper zeros:

$$\begin{aligned}
\boldsymbol{\alpha}^1 &= (1, 1, 0, 1, 0, 1, 1, 0, 0, 1)\ ,\\
\boldsymbol{\alpha}^2 &= (1, 0, 1, 0, 0, 1, 0, 1, 0, 0)\ ,\\
\boldsymbol{\alpha}^3 &= (1, 0, 0, 1, 1, 1, 0, 0, 1, 1)\ ,\\
\boldsymbol{\alpha}^4 &= (0, 1, 1, 1, 0, 0, 0, 0, 0, 1)\ ,\\
\boldsymbol{\alpha}^5 &= (0, 0, 0, 0, 1, 0, 0, 1, 0, 0)\ .
\end{aligned}$$

The algorithm from [145] requires 150 invocations of the operator $\mathcal{O}_{\mathfrak{f}}$ during the inference process for this MBF, while the algorithms from [66, 131] that are optimal with respect to Shannon's criterion, require at least 252 invocations of the operator $\mathcal{O}_{\mathfrak{f}}$. Let us estimate the number of invocations of the operator $\mathcal{O}_{\mathfrak{f}}$ by Algorithm G'_{ψ_0} during the inference process for this specific function $\mathfrak{f}$. It is easily checked that the function has 20 minimal lower units. Then, according to Proposition 4.3, we obtain that Algorithm G'_{ψ_0} requires at most 71 invocations of the operator $\mathcal{O}_{\mathfrak{f}}$. Algorithm G'_{ψ_0} admits realization when it suffices to store in the memory of a computer system the upper zeros of the function $\mathfrak{f} \in \mathcal{M}_m$ only. Algorithm G'_{ψ_1} is efficient when inferring MBFs with a relatively small number of minimal lower units. It admits realization when it suffices to store in the memory of a computer system the minimal lower units of the function $\mathfrak{f} \in \mathcal{M}_m$ only.

Algorithms of finding the upper zeros and lower units of monotone Boolean functions, similar to those which we considered in Section 4.1, are used, for example, in [21, 89, 149]. Such algorithms known under the common name *algorithms Find-Border* gained widespread acceptance. They are also discussed in [18, 34, 79, 88, 124, 142, 144].

Advantages and disadvantages of different algorithms of inferring MBFs, as well as perspectives on further investigations of this problem, are thoroughly discussed in book [144]: a new criterion function, minimized over all inference algorithms G that changes seriously our point of view on an assessment of the efficiency of approaches to inference, is as follows: $\min_G \frac{\sum_{\mathfrak{f} \in \mathcal{M}_m} \varphi(G, \mathfrak{f})}{\Psi(m)}$, where $\Psi(m)$ is the number of all monotone Boolean functions over $\mathbf{B}^m$, the quantity which is central for Dedekind's problem; see [141] and further works in this direction.

In Section 4.2, we propose an inference algorithm for monotone Boolean functions associated with graphs, and discuss the related problem of searching for their maximal upper zeros, following article [59].

Justification of reducibility of the problem of extracting maximal feasible subsystems of an infeasible system of linear inequalities S, considered in Section 4.3, to the problem of inference of the corresponding MBF, is presented in seminal work [159].

5 Inequality systems, committees, (hyper)graphs, and alternative covers

A *committee* of a rank n infeasible system (2.26),

$$\mathfrak{S} := \left\{ \langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0 \colon \ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n; \ \|\boldsymbol{a}_i\| = 1, \ i \in [m]; \ i_1 \neq i_2 \Rightarrow \boldsymbol{a}_{i_1} \neq -\boldsymbol{a}_{i_2} \right\},$$

of homogeneous strict linear inequalities over the real Euclidean space $\mathbb{R}^n$, introduced on page 33, is defined as a finite set of vectors $\mathcal{K} \subset \mathbb{R}^n - \{\mathbf{0}\}$ satisfying the relation

$$|\{\boldsymbol{x} \in \mathcal{K} \colon \ \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle > 0\}| > \tfrac{1}{2}|\mathcal{K}|,$$

for every vector $\boldsymbol{a}_i$, $i \in [m]$.

As earlier, we will denote by $\mathbf{J} := \{J_s \subset [m] \colon s \in [\mathfrak{q}]\}$ the family of the multi-indices of all MFSs of the system $\mathfrak{S}$.

Among the techniques that are used in the committee method is the extracting minimal infeasible and maximal feasible subsystems of the inequality system $\mathfrak{S}$. Solutions to feasible subsystems are combined into committee constructions that generalize the notion of *solution* to feasible systems.

The committee method is efficiently applied to the synthesis of decision-making procedures and, in particular, to contradictory problems of pattern recognition: the problem of committee discrimination of the so-called training sample, for the purpose of forming the decision rules of recognition, can be reduced to the following basic *two-class* setting:

Let $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ be finite sets of vectors of the *feature space* $\mathbb{R}^{n-1}$ that compose the above-mentioned *training sample*. By augmenting artificially each vector from the sets $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{C}}$ by a new nth component, equal to 1, we obtain two sets $\boldsymbol{B}, \boldsymbol{C} \subset \mathbb{R}^n$ of *extended vectors* of the training sample.

It is necessary to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{cases} \langle \boldsymbol{a}, \mathbf{x} \rangle > 0, & \boldsymbol{a} \in \boldsymbol{B}, \\ \langle \boldsymbol{a}, \mathbf{x} \rangle < 0, & \boldsymbol{a} \in \boldsymbol{C}. \end{cases} \tag{5.1}$$

Strict inequalities are used here, because from the applied point of view the use of nonstrict inequalities would be too risky and lead to the synthesis of unstable decision rules.

If $\boldsymbol{x}$ is a solution to system (5.1), then classification of a new extended vector $\boldsymbol{g} \in \mathbb{R}^n$ (that is the making reference of $\boldsymbol{g}$ to one of the classes represented partially by the sets $\boldsymbol{B}$ and $\boldsymbol{C}$) is performed on the basis of the sign of the scalar product $\langle \boldsymbol{x}, \boldsymbol{g} \rangle$. However, the system under consideration can turn out to be infeasible, and this most frequent case requires the development of special methods of problem solving.

Unification of two subsystems that compose system (5.1),

$$\begin{cases} \langle \boldsymbol{a}, \mathbf{x} \rangle > 0, & \boldsymbol{a} \in \boldsymbol{B}\,, \\ -\langle \boldsymbol{a}, \mathbf{x} \rangle > 0, & \boldsymbol{a} \in \boldsymbol{C}\,, \end{cases}$$

leads us (after normalizing the determining vectors) to constructions $\mathfrak{S}$ of the form (2.26).

5.1 The graph of MFSs of an infeasible system of linear inequalities and committees

Let L be the multi-index of a feasible subsystem of the system $\mathfrak{S}$, and $\{J_i\colon\ i \in T \subseteq [q]\} \subseteq \mathbf{J}$ the family of some (not necessarily all) multi-indices of MFSs that contain the multi-index L as a subset; in other words, $\{J_i\colon\ i \in T\} \subseteq \{J \in \mathbf{J}\colon\ J \supseteq L\}$. The algorithmic problem of extracting the multi-indices of all MFSs of the system $\mathfrak{S}$ that contain the multi-index L, provided their subfamily $\{J_i\colon\ i \in T\}$ is known, will be called the $(L, \{J_i\colon\ i \in T\})$*-problem for the system* $\mathfrak{S}$ or, for brevity, the $(L, \{J_i\colon\ i \in T\})$*-problem*, when it is clear what system of inequalities is meant. An important specific case of the above problem is the $(\emptyset, \emptyset)$*-problem of extracting the multi-indices of all MFSs* of the system $\mathfrak{S}$.

Let us first consider a combinatorial algorithm of solving this problem which will serve in what follows as the basic mechanism of a graph-combinatorial algorithm.

Combinatorial algorithm of extracting MFSs of an infeasible system of linear inequalities

We denote the algorithm that we describe here by CMB($L, \{J_i \in \mathbf{J}\colon\ i \in T\}$), and by $\{J_i\colon\ i \in T_{\mathrm{pr}}\}$ the family of the multi-indices of MFSs of the system $\mathfrak{S}$ which contain the subfamily $\{J_i\colon\ i \in T\}$, as well as of the multi-indices of all MFSs found by algorithm CMB($L, \{J_i \in \mathbf{J}\colon\ i \in T\}$) to the present moment. Before launching the algorithm, we have $\{J_i \in \mathbf{J}\colon\ i \in T_{\mathrm{pr}}\} = \{J_i\colon\ i \in T\}$.

We will use the following statement:

Proposition 5.1. *For a subfamily* $\{J_i\colon\ i \in T\} \subseteq \mathbf{J}$ *and for the multi-index* L *of a feasible subsystem of the system* $\mathfrak{S}$*, there exists its maximal feasible subsystem with a multi-index* $J_s \supseteq L, \mathbf{J} \ni J_s \notin \{J_i\colon\ i \in T\}$*, if and only if in the blocker* $\mathfrak{B}(\{[m] - J_i\colon\ i \in T\})$ *there exists a minimal system of representatives* M *of the family* $\{[m] - J_i\colon\ i \in T\}$ *such that the subsystem, with the multi-index* $M \cup L$*, of the system* $\mathfrak{S}$ *is feasible.*

Proof. The sufficiency is evident. Let us prove the necessity. Since $J_s \notin \{J_i\colon\ i \in T\}$, we have $J_s \cap ([m] - J_i) \neq \emptyset$ for all indices $i \in T$; as a consequence, there exists a minimal system of representatives $M \subseteq J_s$ of the family $\{[m] - J_i\colon\ i \in T\}$, and the subsystem with the multi-index $M \cup L$ is feasible. □

Algorithm CMB$(L, \{J_i \in \mathbf{J}:\ i \in T\})$

1. To find all the minimal systems of representatives of the family $\{[m] - J_i:\ i \in T_{\mathrm{pr}}\}$, that is to form the blocker $\mathfrak{B}(\{[m] - J_i:\ i \in T_{\mathrm{pr}}\})$.
2. To check feasibility of the subsystems of the system $\mathfrak{S}$ with the multi-indices $M \cup L$, for every multi-index $M \in \mathfrak{B}(\{[m] - J_i:\ i \in T_{\mathrm{pr}}\})$.
3. If all these systems are infeasible then the algorithm finishes because, according to Propositon 5.1, the family $\{J_i:\ i \in T_{\mathrm{pr}}\}$ coincides with $\mathbf{J}$.
4. If there is a multi-index $M \in \mathfrak{B}(\{[m] - J_i:\ i \in T_{\mathrm{pr}}\})$ such that the subsystem with the multi-index $M \cup L$ is feasible, then to augment this subsystem up to a MFS. To add the multi-index of the obtained MFS to the current family $\{J_i:\ i \in T_{\mathrm{pr}}\}$ of the multi-indices of MFSs; go to step 1.

In Algorithm CMB$(L, \{J_i \in \mathbf{J}:\ i \in T\})$, it is necessary to repeatedly solve the problem of forming the blocker of some family $\mathcal{A} := \{A_1, A_2, \dots, A_\alpha\}$ of subsets of the set $[m]$, that is of extracting all the minimal systems of representatives of the family $\mathcal{A}$. This is a well-known combinatorial problem; various algorithms for solving this problem are proposed. We present here one more algorithm; it takes into account its specific use in Algorithm CMB$(L, \{J_i \in \mathbf{J}:\ i \in T\})$.

An algorithm of forming the blocker of a set family

Let $\mathcal{A} := \{A_1, A_2, \dots, A_\alpha\}$ be a family of subsets of the set $[m]$. The blocker $\mathfrak{B}(\{A_1, A_2, \dots, A_k\})$ of a subfamily $\{A_1, A_2, \dots, A_k\} \subseteq \mathcal{A}$, $k \in [\alpha]$, will be denoted by $\{M_j^{(k)}:\ j \in [\beta_k]\}$. We will suppose by definition that the empty set is the unique system of representatives for the empty set family: $\mathfrak{B}(\emptyset) := \{M_1^{(0)}\} := \{\emptyset\}$. The following assertion is true:

Proposition 5.2. *Let $k \in [\alpha]$. For any index $s \in [\beta_k]$, there exists precisely one index $i_s \in [\beta_{k-1}]$ such that $M_{i_s}^{(k-1)} \subseteq M_s^{(k)}$.*

Proof. The existence of at least one index $i_s \in [\beta_{k-1}]$, such that $M_{i_s}^{(k-1)} \subseteq M_s^{(k)}$, is evident. Let us prove its uniqueness. Assume the converse: let $i_1, i_2 \in [\beta_{k-1}]$ be two different indices such that $M_{i_1}^{(k-1)} \subseteq M_s^{(k)}$ and $M_{i_2}^{(k-1)} \subseteq M_s^{(k)}$. Let us first show that

$$(M_{i_1}^{(k-1)} \cup M_{i_2}^{(k-1)}) \cap A_k = \emptyset\,. \tag{5.2}$$

Assume the converse: let, for example, $a \in M_{i_1}^{(k-1)} \cap A_k \neq \emptyset$. Then, because of the minimality of $M_s^{(k)}$, we have $M_{i_1}^{(k-1)} = M_s^{(k)}$ and, because of the inclusion $M_{i_2}^{(k-1)} \subseteq M_s^{(k)} = M_{i_1}^{(k-1)}$ and of the minimality of $M_{i_1}^{(k-1)}$, we obtain that $M_{i_1}^{(k-1)} = M_{i_2}^{(k-1)}$. By the definition of $M_s^{(k)}$, there exists $c \in M_s^{(k)} \cap A_k$, and $c \in M_s^{(k)} - M_{i_1}^{(k-1)}$, in view of (5.2). Since the sets $M_{i_1}^{(k-1)}$ and $M_{i_2}^{(k-1)}$ are inclusion-incomparable, there exists an element $b \in M_{i_2}^{(k-1)} - M_{i_1}^{(k-1)} \subset M_s^{(k)} - M_{i_1}^{(k-1)}$. Since $c \in A_k$ and $b \in M_{i_2}^{(k-1)}$, in view of (5.2), $c \neq b$. But then $M_s^{(k)} - \{b\} \supseteq M_{i_1}^{(k-1)} - \{c\}$, where $(M_{i_1}^{(k-1)} \cup \{c\}) \cap A_i \neq \emptyset$ for all $i \in [k]$, a contradiction with the minimality of $M_s^{(k)}$. This contradiction proves the proposition. □

Let us put in correspondence with the family $\mathcal{A}$ the directed graph $\mathbf{G}(\mathcal{A})$ with the vertex set $\{M_i^{(k)}:\ i \in [\beta_k],\ k \in \{0\}\dot\cup[\alpha]\}$, for which an arc from $M_{i_1}^{k_1}$ to $M_{i_2}^{k_2}$ exists if and only if $k_2 = k_1 + 1$ and $M_{i_1}^{k_1} \subseteq M_{i_2}^{k_2}$. Let us denote by $\Gamma(M_i^{(k)})$ the *out-neighborhood* of $M_i^{(k)}$, which is the set of the final vertices of all the arcs whose initial vertex is $M_i^{(k)}$; similarly, $\Gamma^{-1}(M_i^{(k)})$ will denote the *in-neighborhood* of $M_i^{(k)}$, which is the set of the initial vertices of all the arcs whose final vertex is $M_i^{(k)}$.

A graph is called a *rooted tree* with *root* y_0, if

- any vertex except y_0 has *in-degree one*, that is, such a vertex is the final vertex of precisely one arc;
- the vertex y_0 has in-degree zero;
- the vertex y_0 has nonzero *out-degree*, that is, y_0 is the initial vertex of at least one arc.

It follows immediately from Proposition 5.2 and from the definition of the graph $\mathbf{G}(\mathcal{A})$ that this is a rooted tree with the root $M_1^{(0)} = \emptyset$.

According to Proposition 5.2, for each vertex $M_i^{(k)}$ except the root, $\Gamma^{-1}(M_i^{(k)}) = 1$. Because of the minimality of $M_i^{(k)}$, we have $|M_i^{(k)} - \Gamma^{-1}(M_i^{(k)})| \leq 1$. We will call the number

$$\nu(M_i^{(k)}) := \begin{cases} 0, & \text{if } M_i^{(k)} - \Gamma^{-1}(M_i^{(k)}) = \emptyset\,, \\ a \in M_i^{(k)} - \Gamma^{-1}(M_i^{(k)}), & \text{if } M_i^{(k)} - \Gamma^{-1}(M_i^{(k)}) \neq \emptyset\,, \end{cases}$$

the *inner number* of the vertex $M_i^{(k)}$ of the rooted tree $\mathbf{G}(\mathcal{A})$. The rooted tree $\mathbf{G}(\mathcal{A})$ is uniquely determined if every vertex of the tree is marked by its inner number, because for each vertex $M_i^{(k)}$ in this rooted tree there exists a unique chain $(M_1^{(0)}, M_{i_1}^{(1)}, \ldots, M_{i_{k-1}}^{(k-1)}, M_i^{(k)})$ connecting $M_i^{(k)}$ with the root $M_1^{(0)}$ and, besides

$$M_i^{(k)} = \{\nu(M_{i_1}^{(1)}) \cup \ldots \cup \nu(M_i^{(k)})\} - \{0\}\,. \tag{5.3}$$

The idea of Algorithm ROOTEDTREE that extracts all the minimal systems of representatives of the family $\mathcal{A}$ with the use of the rooted tree $\mathbf{G}(\mathcal{A})$, consists in the construction of the rooted tree by means of sequential inspection of its vertices. We traverse the rooted tree moving each time along arcs of the rooted tree $\mathbf{G}(\mathcal{A})$ as far as possible, and coming one step back in the direction opposite to that of an arc when further movement in the direction of the arc does not lead us to an uninspected vertex of the rooted tree $\mathbf{G}(\mathcal{A})$. A step in the direction of an arc will be called *forward*, and a step in the opposite direction will be called *backward*. A traversal can be arranged in such a way that it will be necessary to store, at any current moment, a relatively small amount of information.

Let us suppose that we are currently at a vertex $M_i^{(k)}$. We will need the following data:

- LENGTH – equals k for the current vertex $M_i^{(k)}$.
- NUMBERS(i) – a one-dimensional array of the inner numbers of vertices, composing the chain that connects the current vertex with the root of the rooted tree, ordered in accordance with this chain; it uniquely determines the current vertex.
- FORESTEP – a variable taking the value *BACKWARD* if we have arrived at the current vertex by moving in the opposite direction of an arc of the rooted tree, and the value *FORWARD* if we have arrived by moving in the direction of the arc.
- FORENUM – a variable equal to the inner number of the vertex by moving from which we have arrived at the current vertex after performing a backward step.
- TRAVERSAL – a variable taking the value *COMPLETED* if the traversal of the rooted tree $\mathbf{G}(\mathcal{A})$ is completed, and the value *NOTCOMPLETED* otherwise.
- $m, \alpha, \mathcal{A}$ – defined earlier.
- M_{pr} – the current vertex, that is the set consisting of the nonzero elements of the array NUMBERS and of its first LENGTH items.

In the array NUMBERS a stack is organized; the LENGTH variable determines the length of this stack. A forward step corresponds to the pushing of the inner number of the next vertex into the NUMBERS stack; a backward step corresponds to the pulling of the tail element out of the stack.

The above information completely determines the state of the algorithm. In order to justify functionality of Algorithm ROOTEDTREE, it suffices to describe the process of passaging from the current vertex to the next vertex when traversing the rooted tree $\mathbf{G}(\mathcal{A})$:

Algorithm ROOTEDTREE($\mathcal{A}$) – *a passage from the current vertex of the rooted tree* $\mathbf{G}(\mathcal{A})$

0. The start of an elementary traversal of the rooted tree $\mathbf{G}(\mathcal{A})$.
1. If LENGTH $= \alpha$, then to retrieve M_{pr} as the immediate minimal system of representatives of the family $\mathcal{A}$.
2. If LENGTH $\neq \alpha$ or (FORESTEP $=$ *BACKWARD* and FORENUM $\in \{0, m\}$), go to instruction 6 – a backward move within the rooted tree $\mathbf{G}(\mathcal{A})$.
3. If FORESTEP = *FORWARD* and $M_{\mathrm{pr}} \cap A_{\mathrm{LENGTH}+1} \neq \emptyset$, then $i_0 = 0$; go to instruction 7 – a forward move within the rooted tree $\mathbf{G}(\{A_1, A_2, \ldots, A_{\mathrm{LENGTH}}\})$.
4. If FORESTEP = *BACKWARD*, then i := FORENUM; otherwise i := 0.
5. To inspect the integers starting with $i+1$, up to m, in the ascending order, up to the first number i_0 such that $M_{\mathrm{pr}} \cup \{i_0\}$ is a minimal system of representatives of the family $\{A_1, A_2, \ldots, A_{\mathrm{LENGTH}+1}\}$. If there does not exist such a number, go to instruction 6 – a backward move within the rooted tree $\mathbf{G}(\mathcal{A})$; else, go to instruction 7 – a forward move within the rooted tree $\mathbf{G}(\mathcal{A})$.
6. A backward move within the rooted tree $\mathbf{G}(\mathcal{A})$. If LENGTH = 0, then TRAVERSAL = *COMPLETED*; go to instruction 7. Else:
   FORESTEP := *BACKWARD*,
   FORENUM := NUMBERS(LENGTH),
   LENGTH := LENGTH – 1; go to instruction 8.

7. A forward move within the rooted tree $\mathbf{G}(\mathcal{A})$.
   FORESTEP := *FORWARD*,
   LENGTH := LENGTH + 1,
   NUMBERS(LENGTH) = i_0; go to instruction 8.
8. The end of an elementary traversal of the rooted tree $\mathbf{G}(\mathcal{A})$.

By repeating the above described actions until the variable TRAVERSAL takes the value *COMPLETED*, we will inspect all vertices of the rooted tree $\mathbf{G}(\mathcal{A})$ and find all the minimal systems of representatives $\mathcal{A}$; this is guaranteed by Proposition 5.2 and by the arrangement of the traversal.

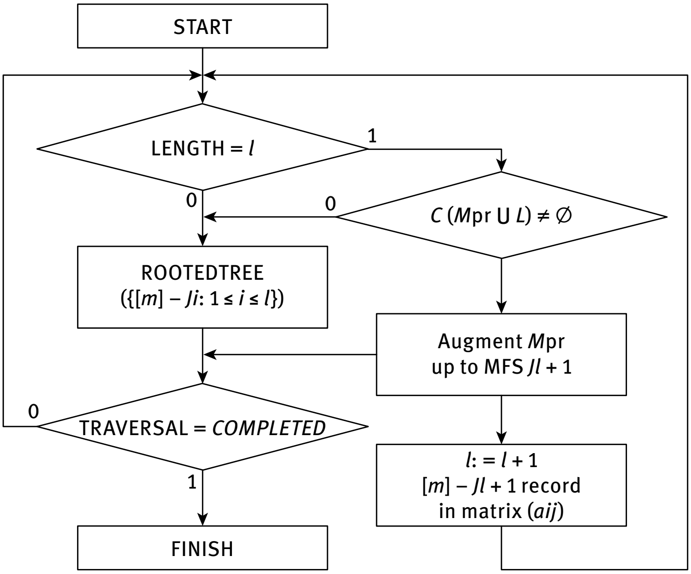
We now show how the above algorithm is used within Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon i \in T\}$), taking into account essential structural properties of the rooted tree of the minimal systems of representatives of the family $\{[m] - J_i\colon\ i \in T\}$. We will propose a realization of Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon\ i \in T\}$) more economical than its general scheme.

An economical realization of the combinatorial algorithm

Let us denote by $\{J_i\colon\ i \in T_{\mathrm{pr}}\}$ the family of the multi-indices of MFSs which are already known to the current moment. We emphasize the following observation: in the general scheme of Algorithm CMB, after the multi-index of a new MFS has been found, this multi-index is added to the family of the known multi-indices, and the process is repeated from the very beginning. As a matter of fact, it is possible to arrange Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon\ i \in T\}$) in such a way that we will find all multi-indices from the family $\{[m] - J_i\colon\ J_i \in \mathbf{J},\ J_i \supseteq L\}$ at one-pass traversal of the rooted tree of its minimal systems of representatives. Assume for simplicity that $\{J_i \in \mathbf{J}\colon\ i \in T\} = \{J_1, J_2, \dots, J_t\}$. The set T can be empty. Let the multi-indices of new MFSs get their indices starting with $t+1$, in order of their appearance during the work of the algorithm. Let information on the family $\{[m]-J_i\colon\ i \in T\}$ be given by the binary array $(a_{ij})_{m\times|T|}$; for every just found MFS with a multi-index J_j, the jth column of the matrix (a_{ij}) is automatically filled up; let l be the current number of columns of the matrix a_{ij}, that is the number of the already known MFSs. Let us use the algorithm of elementary traversal of the rooted tree $\mathbf{G}(\{[m] - J_i\colon\ i \in [l]\})$.

Let us show that Algorithm CMB(L, $\{J_i \in \mathbf{J}\colon\ i \in T\}$) depicted as follows finishes, all those multi-indices of MFSs will be found that contain the multi-index L as a subset, that is, $\{J_1, J_2, \dots, J_{l^*}\} = \{J_i \in \mathbf{J}\colon\ J_i \supseteq L\}$.

For this, it suffices to show, in accordance with Proposition 5.1, that $\mathbf{C}_>(M \cup L) = \emptyset$ for any minimal system of representatives $M \in \mathfrak{B}(\{[m] - J_1, [m] - J_2, \dots, [m] - J_{l^*}\})$. Suppose to the converse that there exists $M_i^{(l^*)} \in \mathfrak{B}(\{[m]-J_1, [m]-J_2, \dots, [m]-J_{l^*}\})$ such that the subsystem with the multi-index $M_i^{l^*} \cup L$ is feasible. Let us consider the chain that connects the vertex $M_i^{(l^*)}$ with the root $M_1^{(0)}$ of the rooted tree $\mathbf{G}(\{[m] - J_i\ :\ i \in [l]\})$, which is the chain $(M_1^{(0)}, M_{i_1}^{(0)}, \dots, M_{i_{l^*-1}}^{(l^*-1)}, M_{i_{l^*}}^{(l^*)})$. Let $k \in [l^*]$ be the max-



imal integer such that Algorithm ROOTEDTREE$(\{[m] - J_i : i \in [l]\})$ has led us to the vertex $M_{i_k}^{(k)}$.

Let us consider that moment of the execution of Algorithm ROOTEDTREE$(\{[m] - J_i : i \in [l]\})$ when we left the vertex $M_i^{(k)}$ last time. It is clear that such a passage was backward, that is, we left $M_i^{(k)}$ for the vertex $\Gamma^{-1}\left(M_{i_k}^{(k)}\right) = M_{i_{k-1}}^{(k-1)}$. The latter is possible in the two cases:

1. $k < l \le l^*$ – Algorithm ROOTEDTREE$(\{[m] - J_i : i \in [l]\})$ had already led us to all the vertices from $\Gamma\left(M_{i_k}^{(k)}\right)$; this contradicts the choice of k;
2. $k = l \le l^*$ and $\mathbf{C}_>(M^{(k)} \cup L) = \emptyset$; this contradicts the assumption $\emptyset \ne \mathbf{C}_>(M_i^{(l)} \cup L) \supseteq \mathbf{C}_>(M_i^{(k)} \cup L)$.

These contradictions prove that after Algorithm CMB$(L, \{J_i \in \mathbf{J} : i \in T\})$ finishes, the multi-indices of all desirable MFSs will be found, that is, $\{J_1, J_2, \ldots, J_{l^*}\} = \{J_i \in \mathbf{J} : J_i \supseteq L\}$.

The combinatorial algorithm of solving the $(L, \{J_i \in \mathbf{J} : J_i \supseteq L\})$-problem is difficult to use in practice because the computational burden, when extracting every new MFS, increases fast as the number of the already found MFSs increases. By the *combinatorial dimension* of the $(L, \{J_i \in \mathbf{J} : J_i \supseteq L\})$-problem, as well as of Algorithm CMB$(L, \{J_i \in \mathbf{J} : i \in T\})$, we will mean below the pair $(m, \#\{J_i \in \mathbf{J} : J_i \supseteq L\})$. This notion, being quite coarse, nevertheless reflects some properties of the $(L, \{J_i \in \mathbf{J} : i \in T\})$-problem that affect the rapid growth of the computational burden as the number of the already found MFSs increases.

Graph-combinatorial algorithms of extracting MFSs of an infeasible system of linear inequalities

We consider here a reduction of the $(L, \emptyset)$-problem to a sequence of $(L_i, \{J_s : s \in T_i\})$-problems of lower combinatorial dimension. For this purpose, we will also construct an approximate algorithm of solving the $(L, \emptyset)$-problem, such that the computational burden when finding a new MFS grows much more slower than in the case of Algorithm CMB$(L, \emptyset)$.

Let us first consider the $(\emptyset, \emptyset)$-problem, which is the problem of extracting all the MFSs of the system $\mathfrak{S}$.

We will describe Algorithm GRAPH-CMB of solving the $(\emptyset, \emptyset)$-problem based on the connectedness of the graph of MFSs of the system $\mathfrak{S}$; it involves the above described Algorithm CMB.

Algorithm GRAPH-CMB

1. We find the multi-index J_1 of the first MFS, by augmenting up to this MFS, the feasible subsystem with the multi-index $\{1\}$, that is the subsystem consisting of the first inequality. The multi-index J_1 is assigned the mark 0.
2. Among the multi-indices of the already found MFSs, we pick at random a multi-index with the mark 0 and go to instruction 3. If there are no such multi-indices, then we go to instruction 4.
3. For the chosen multi-index J_s, we distinguish among the found multi-indices the family $\{J_i : i \in T\}$ of those multi-indices that are adjacent with J_s in the graph of MFSs of the system $\mathfrak{S}$. We then solve the $([m]-J_s, \{J_i : i \in T\})$-problem by Algorithm CMB$([m]-J_s, \{J_i : i \in T\})$, that is, we find all the multi-indices of MFSs adjacent with J_s in the graph of MFSs; all the new found multi-indices of MFSs get the mark 0; the multi-index J_s gets the mark 1. We go to instruction 2.
4. The algorithm finishes.

After the algorithm finishes, the multi-indices of all MFSs have the mark 1, that is, for the multi-index of every found MFS we also obtain the multi-indices of all the MFSs that are adjacent with it in the graph of MFSs; because of the connectedness of the graph of MFSs (Proposition 2.20) of the system $\mathfrak{S}$, Algorithm GRAPH-CMB thus finds the multi-indices of all MFSs.

Algorithm GRAPH-CMB can be arranged in such a way that the property mentioned in Proposition 2.35 (iii) will be used: the diameter of the graph $\mathtt{MFSG}(\mathfrak{S})$ of the system $\mathfrak{S}$ does not exceed half the number of inequalities in this system. If, when executing instruction 3 of Algorithm GRAPH-CMB, the distance between the vertices J_1 and J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ turns out to be equal to $\lfloor \frac{m}{2} \rfloor$, then we can immediately assign the mark 1 to the vertex J_s, without solving the $([m]-J_s, \{J_i \in \mathbf{J} : i \in T\})$-problem, and we can go to instruction 2. In view of the mentioned Proposition 2.35 (iii), thus modified Algorithm GRAPH-CMB still finds the multi-indices of all MFSs of the system $\mathfrak{S}$.

Let us now turn to the constructing of a graph-combinatorial algorithm of solving the $(L, \emptyset)$-problem; let us denote it by GRAPH-CMB(L). This algorithm is based on the property: the subgraph $\mathtt{MFSG}^2\langle\{J \in \mathbf{J} : J \supseteq L\}\rangle$ is connected, for any multi-index L of a feasible subsystem of the system $\mathfrak{S}$.

Algorithm GRAPH-CMB(L)

1. By augmenting, up to a MFS, the subsystem with the multi-index L, we find the multi-index J_1 of this MFS; the multi-index J_1 is assigned the mark 0.
2. From the family of the multi-indices of already found MFSs, we pick at random a multi-index with the mark 0, say the multi-index J_s, and choose all those multi-indices of MFSs $\{J_i \in \mathbf{J}: i \in T_s\}$ found that are adjacent to J_s in the graph of MFSs. If there are no multi-indices of MFSs with the mark 0, then we go to instruction 3. If $J_s \supseteq L$, then we set $L' := [m] - J_s$; if $J_s \not\supseteq L$, then we set $L' := ([m] - J_s) \cup L$. By launching Algorithm CMB($L, \{J_i \in \mathbf{J}: i \in T_s\}$), we find all the multi-indices of MFSs that are adjacent with J_s in the graph of MFSs and contain L as a subset. The multi-indices of the new found MFSs are assigned the mark 0; the multi-index J_s is assigned the mark 1 when $J_s \not\supseteq L$, and it is assigned the mark 2 when $J_s \supseteq L$. We repeat the execution of instruction 2.
3. The algorithm finishes.

Proposition 5.3. *After Algorithm GRAPH-CMB(L) finishes, the multi-indices of the found MFSs, with the mark 2, compose the family of all the multi-indices of MFSs of the system $\mathfrak{S}$ that contain the multi-index L as a subset.*

Proof. Assume the converse. Then the family $\{J_i \in \mathbf{J}: J_i \supseteq L\}$ can be partitioned into two disjoint subfamilies: $\{J_i \in \mathbf{J}: J_i \supseteq L\} = \{J_i \in \mathbf{J}: i \in T\} \dot\cup \{J_i \in \mathbf{J}: i \in T'\}$, where $\{J_i \in \mathbf{J}: i \in T\}$ is the family of all the multi-indices of MFSs found by Algorithm GRAPH-CMB(L) and marked by 2. Since the subgraph $\mathtt{MFSG}^2\langle\{J \in \mathbf{J}: J \supseteq L\}\rangle$ is connected, there exists an edge $\{J_s, J_t\}$ in the square $\mathtt{MFSG}^2(\mathfrak{S})$ of the graph of MFSs of the system $\mathfrak{S}$ such that $J_s \in \{J_i \in \mathbf{J}: i \in T\}$ and $J_t \in \{J_i \in \mathbf{J}: i \in T'\}$. As a consequence, the following two mutually exclusive cases are only possible: (1) the pair $\{J_s, J_t\}$ is an edge of the graph $\mathtt{MFSG}(\mathfrak{S})$, and (2) there exists the multi-index J_r of a MFS such that the pairs $\{J_s, J_r\}$ and $\{J_r, J_t\}$ are edges of the graph $\mathtt{MFSG}(\mathfrak{S})$.

Let us consider the first case. The mark 2 is assigned in Algorithm GRAPH-CMB(L) to the multi-index J_s of some MFS if and only if all the multi-indices of MFSs that are adjacent with J_s in the graph $\mathtt{MFSG}(\mathfrak{S})$ have been found; as a consequence, the multi-index J_t has already been found by the algorithm. This contradicts the inclusion $J_t \in \{J_i \in \mathbf{J}: i \in T'\}$ because each found multi-index, after the algorithm finishes, has the mark 2 if it contains the multi-index L as a subset.

Let us consider the second case. As given earlier, the multi-index J_r was found by Algorithm GRAPH-CMB(L). Since at the finish of the algorithm the mark of the multi-index J_r is not 0, and the multi-index J_t is adjacent with J_r and it contains L as a subset, then the multi-index J_t was found by Algorithm GRAPH-CMB($[m] - J_r, \{J_i \in \mathbf{J}: i \in T_r\}$), and it was assigned the mark 2 at the finish of the algorithm, a contradiction with the inclusion $J_t \in \{J_i \in \mathbf{J}: i \in T'\}$. □

The difference of Algorithm GRAPH-CMB(L) from Algorithm GRAPH-CMB consists in the following: in the former algorithm, for every multi-index $J_s \not\supseteq L$ of a found MFS, the $(L \cup ([m] - J_s), \{J_i \in \mathbf{J}: i \in T_s\})$-problem is solved which, in the general case, has combinatorial dimension lower than that of $(([m] - J_s), \{J_i: i \in T_s\})$-problem solved by the latter algorithm. Thus, in the general case, Algorithm GRAPH-CMB(L) is more economical, when finding the MFSs that contain the subsystem with the multi-index L, than Algorithm GRAPH-CMB.

Approximate combinatorial and graph-combinatorial algorithms

In practice, it is often suffices to know just a subfamily of the family of the multi-indices of MFSs of the system $\mathfrak{S}$. Therefore, it is natural to discuss an approximate $(L, \{J_s \in \mathbf{J} : s \in T_i\})$-problem.

By $(L, \{J_s \in \mathbf{J} : s \in T_i\})^{(k)}$*-problem* for the system $\mathfrak{S}$, we will mean the problem of extracting arbitrary $\min\{k, \#\{J \in \mathbf{J} : J \supseteq L\}\}$ multi-indices of MFSs that contain L as a subset. The combinatorial algorithm CMB$(L, \{J_i \in \mathbf{J} : i \in T\})^{(k)}$ of solving the $(L, \{J_s \in \mathbf{J} : s \in T_i\})^{(k)}$-problem is obtained from Algorithm CMB$(L, \{J_i \in \mathbf{J} : i \in T\})$ when we require that the latter algorithm finishes in the case where k multi-indices of MFSs have already been found. By *combinatorial complexity* of the $(L, \{J_s \in \mathbf{J} : s \in T_i\})^{(k)}$-problem or of Algorithm CMB$(L, \{J_i \in \mathbf{J} : i \in T\})^{(k)}$, we will mean the pair $(m, \min\{k, \#\{J \in \mathbf{J} : J \supseteq L\}\})$. The combinatorial dimension of the $(\emptyset, \emptyset)^{(k)}$-problem, equal to $(m, \min\{k, \mathrm{q}\})$, will be still high; besides, Algorithm CMB$(\emptyset, \emptyset)^{(k)}$ naturally faces the same difficulties as earlier.

The graph-combinatorial algorithm GRAPH-CMB$^{(\mathrm{k})}$ differs from Algorithm GRAPH-CMB: in the appropriate place, the approximate algorithm CMB$(L, \{J_i \in \mathbf{J} : i \in T\})^{(k)}$ is used instead of the exact algorithm CMB$(L, \{J_i \in \mathbf{J} : i \in T\})$, that is, for every multi-index of a MFS found, one searches, in the general case, for not necessarily all the multi-indices, adjacent with this multi-index in the graph $\mathtt{MFSG}(\mathfrak{S})$, but for just a number of them. It is remarkable that even for quite small k's, say for those close to n, Algorithm GRAPH-CMB$^{(\mathrm{k})}$ finds, thanks to the connectedness of the graph $\mathtt{MFSG}(\mathfrak{S})$, a large number of the multi-indices of MFSs of the system $\mathfrak{S}$; besides, the computational burden when extracting a new MFS, grows slower for small k's.

In conclusion, let us make a few remarks concerning Algorithm GRAPH-CMB$^{(\mathrm{k})}$. Suppose that for some system $\mathfrak{S}$ it was found, with the help of Algorithm GRAPH-CMB$^{(\mathrm{k})}$, a quite large number of its MFSs. The search for new MFSs of the system $\mathfrak{S}$ can be arranged in the following way: to find the multi-indices of MFSs adjacent simultaneously with two (three, and so on) already found multi-indices of MFSs, thus applying Algorithm GRAPH-CMB$(([m] - J_s) \cup ([m] - J_t), \{J_i \in \mathbf{J} : i \in T_{st}\})^{(k)}$ to those pairs J_s, J_t of already found multi-indices, for which $\mathbf{C}_{>}\big(([m] - J_s) \cup ([m] - J_t)\big) \neq \emptyset$; here $\{J_i \in \mathbf{J} : i \in T_{st}\}$ are those multi-indices of MFSs already found that are adjacent with both J_s and J_t. Since the vertex degrees in the graphs of MFSs are quite high, for many multi-indices of MFSs, not found yet, there are two, three, or a larger number of already found multi-indices of MFSs, that are adjacent with them. For this reason, combinatorial complexity of the $(([m] - J_s) \cup ([m] - J_t), \{J_i \in \mathbf{J} : i \in T_{st}\})$-problem can turn out to be less than combinatorial complexity of the $(([m] - J_s), \{J_i \in \mathbf{J} : i \in T_s\})$- and $(([m] - J_t), \{J_i \in \mathbf{J} : i \in T_t\})$-problems.

Odd cycles in the graph of MFSs, and committees

The following fundamental property of the graph $\mathtt{MFSG}(\mathfrak{S})$ underlies well-known methods of constructing committees of the system $\mathfrak{S}$:

Theorem 5.4. *Let a sequence $(J_{i_1}, J_{i_2}, \dots, J_{i_{2k+1}}, J_{i_1})$ compose an odd cycle in the graph $\mathtt{MFSG}(\mathfrak{S})$ of the system $\mathfrak{S}$. Suppose that pairwise distinct vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}$ are solutions to the MFSs with the multi-indices $J_1, J_2, \dots, J_{2k+1}$ respectively. Then the collection of vectors $\mathcal{K} := \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}\}$ is a committee of the system $\mathfrak{S}$.*

Proof. Assume the converse. Since the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}$ are pairwise distinct, the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}\}$ contains $2k+1$ elements. Then there exists an integer $i_0 \in [m]$ such that the inequality with the index i_0 is satisfied by at most k vectors from the set $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{2k+1}$. As a consequence, in the sequence $(J_{i_1}, J_{i_2}, \dots, J_{i_{2k+1}})$ of the multi-indices of MFSs there exist $k+1$ multi-indices of MFSs which do not contain the element i_0 and thus are not adjacent in the graph of MFSs of the system $\mathfrak{S}$. This contradicts the assumption that the sequence $(J_{i_1}, J_{i_2}, \dots, J_{i_{2k+1}}, J_{i_1})$ composes a cycle in the graph of MFSs of the system $\mathfrak{S}$. □

5.2 The hypergraph of MFSs of an infeasible system of linear inequalities and committees

Let us consider a rank n infeasible system

$$\mathfrak{S}_{\boldsymbol{b}} := \{\langle \boldsymbol{a}_i, \mathbf{x}\rangle > b_i\colon\ \boldsymbol{b} := (b_1, \dots, b_m) \in \mathbb{R}^m,\ \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n;\ \|\boldsymbol{a}_i\| > 0,\ i \in [m]\} \quad (5.4)$$

of inhomogeneous strict linear inequalities over the real Euclidean space $\mathbb{R}^n$, with the property: each subsystem with two inequalities is feasible. Such a system has a *committee*, which is a finite set of vectors $\mathcal{K} \subset \mathbb{R}^n$ such that $|\{\boldsymbol{x} \in \mathcal{K}\colon\ \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle > b_i\}| > \frac{1}{2}|\mathcal{K}|$, for every vector $\boldsymbol{a}_i$, $i \in [m]$.

By a *multi-committee* of system (5.4) we mean a *sequence* (also considered, if necessary, as an unordered *multiset*) of vectors $\mathcal{K} \subset \mathbb{R}^n$ with the same property: $|\{\boldsymbol{x} \in \mathcal{K}\colon\ \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle > b_i\}| > \frac{1}{2}|\mathcal{K}|$, for every vector $\boldsymbol{a}_i$, $i \in [m]$.

A multi-committee $\mathcal{K}$ of a system $\mathfrak{S}_{\boldsymbol{b}}$ of linear inequalities (5.4) is called *minimal* if the system $\mathfrak{S}_{\boldsymbol{b}}$ has no multi-committee of cardinality less than $|\mathcal{K}|$.

Let us consider a linear operator $\Phi\colon\ \mathbb{R}^n \to \mathbb{R}^2$ such that the sequence of images $(\Phi(\boldsymbol{a}_1), \dots, \Phi(\boldsymbol{a}_m))$ of the vectors from the sequence $\boldsymbol{A}(\mathfrak{S}_{\boldsymbol{b}}) := \{\boldsymbol{a}_i\colon\ i \in [m]\}$ that determines the system $\mathfrak{S}_{\boldsymbol{b}}$ does not contain the origin and antipodal pairs. The system

$$\{\langle \Phi(\boldsymbol{a}_i), \mathbf{y}\rangle > 0\colon\ \mathbf{y} \in \mathbb{R}^2,\ i \in [m]\} \quad (5.5)$$

has a committee, for example, according to Theorem 5.4 and Proposition 2.33 and, thus, the system

$$\{\langle \Phi(\boldsymbol{a}_i), \mathbf{y}\rangle > b_i\colon\ \mathbf{y} \in \mathbb{R}^2,\ i \in [m]\} \quad (5.6)$$

also has a committee.

If a sequence $\mathcal{K}' := (\boldsymbol{y}_1, \dots, \boldsymbol{y}_q)$ is a multi-committee of system (5.6), then the sequence $(\Phi^*(\boldsymbol{y}_1), \dots, \Phi^*(\boldsymbol{y}_m))$ of the images of its elements under the map Φ^*, the adjoint of Φ, is a multi-committee of system (5.4).

Let $\{J_1^{\mathbf{0}}, \dots, J_q^{\mathbf{0}}\}$ and $\{J_1, \dots, J_r\}$ be the families of the multi-indices of MFSs of systems (5.5) and (5.6), respectively; see Section 2.4. The hypergraph of MFSs of system (5.5) is denoted by $\mathtt{MFSH}(\mathfrak{S}_{\mathbf{0}}, \Phi) := ((J_1^{\mathbf{0}}, \dots, J_q^{\mathbf{0}}), \mathcal{E}^{\mathbf{0}})$.

Let us consider the family

$$W := 2^{\{J_1^{\mathbf{0}}, \dots, J_q^{\mathbf{0}}\}} - (\emptyset, \{J_1^{\mathbf{0}}\}, \dots, \{J_q^{\mathbf{0}}\}) .$$

For each element $w := \{J_{i_1}^{\mathbf{0}}, \dots, J_{i_s}^{\mathbf{0}}\}$ there exists $k \in [s]$ such that $(i_{(k \pmod s)+1} - i_k) \pmod q > t + 1$, where $q := 2t + 1$. Let us define a map $\Lambda\colon\ W \to \mathbb{Z}$ by $\Lambda(w) := (i_k - i_{(k \pmod s)+1}) \pmod q$, the family

$$W' := \Big\{ w := \{J_{i_1}^{\mathbf{0}}, \dots, J_{i_s}^{\mathbf{0}}\} \colon\ \forall k \in [s]\ \exists J_{j_k} \colon\ J_{j_k} \supseteq J_{i_k}^{\mathbf{0}},\ \bigcup_{k \in [s]} J_{j_k} = [m] \Big\}$$

and the quantity

$$\delta(\mathfrak{S}_{\boldsymbol{b}}) := \begin{cases} \min\{\Lambda(w) \colon\ w \in W'\}, & \text{if } |W'| > 0 , \\ t, & \text{if } |W'| = 0 . \end{cases}$$

Proposition 5.5. *The number of elements in a minimal multi-committee of system* (5.4) *does not exceed* $2\delta(\mathfrak{S}_{\boldsymbol{b}}) + 1$.

In this proposition, a bound on the number of elements in a minimal multi-committee of systems $\mathfrak{S}_{\boldsymbol{b}}$ is justified, which depends on the set $\boldsymbol{A}(\mathfrak{S}_{\boldsymbol{b}})$ of determining vectors and on the vector $\boldsymbol{b}$. In the proof of Proposition 5.5, which we omit, one finds for system (5.6) a multi-committee, minimal among all the multi-committees formed from the solutions to MFSs whose multi-indices contain the multi-indices of MFSs of system (5.5), that compose a chain in the hypergraph $\mathtt{MFSH}(\mathfrak{S}_{\mathbf{0}}, \Phi)$ of maximal feasible subsystems of system (5.5).

5.3 Alternative covers

Let X be a nonempty set of any kind, and $\mathcal{M} \subseteq \mathbf{2}^X$ some family of subsets of the set X. Let $\mathcal{A}, \mathcal{B} \subset X$ be nonempty disjoint subsets of the set X.

An ordered pair $(\mathfrak{A}, \mathfrak{B})$ *of families* $\mathfrak{A}, \mathfrak{B} \subset \mathcal{M}$ *of subsets of the set X, picked from the admissible family* $\mathcal{M}$*, will be called an* alternative cover *of the pair* $(\mathcal{A}, \mathcal{B})$ *if* $\mathcal{A} = \bigcup_{A \in \mathfrak{A}} A$, $\mathcal{B} = \bigcup_{B \in \mathfrak{B}} B$*, and* $A \cap B = \emptyset$ *for any sets* $A \in \mathfrak{A}$ *and* $B \in \mathfrak{B}$.

An alternative cover will be called finite *if each of the families* $\mathfrak{A}$ *and* $\mathfrak{B}$ *is finite.*

By the cardinality *of a finite alternative cover* $(\mathfrak{A}, \mathfrak{B})$ *we mean the quantity* $|\mathfrak{A}| + |\mathfrak{B}|$.

This construction has a close relation with the pattern recognition subject. Suppose that in a space X two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ are fixed, and some admissible family $\mathcal{M}$ of subsets of the space X is predetermined. If there exists an alternative cover

$(\mathfrak{A}, \mathfrak{B})$ of the pair $(\mathcal{A}, \mathcal{B})$ then it *separates* the sets $\mathcal{A}$ and $\mathcal{B}$ in the space X. As a consequence, the problem of *efficient* separation of the subsets $\mathcal{A}$ and $\mathcal{B}$ of the space X, in the class of subsets from $\mathcal{M}$, can be stated as the problem of the search for a finite alternative cover, of *minimal cardinality*, of the pair $(\mathcal{A}, \mathcal{B})$.

Thus, we can symbolically write down the pattern recognition problem, in its geometric setting, as follows:

$$\mathrm{R}_1\colon\ \left(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} \subseteq \mathbf{2}^X\right) \to (\mathfrak{A}, \mathfrak{B})\,.$$

Substantially, alternative covers can differ; the next clarification of the pattern recognition problem consists in that a quality functional $f(\mathcal{A}, \mathcal{B})$ for an alternative cover is introduced which should be optimized, say minimized:

$$\mathrm{R}_2\colon\ \left(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} \subseteq \mathbf{2}^X,\ f\colon\ \mathbf{2}^{\mathcal{M}} \times \mathbf{2}^{\mathcal{M}} \to \mathbb{R}\right) \xrightarrow{\min f} (\mathfrak{A}, \mathfrak{B})\,.$$

One natural criterion of the quality of an alternative cover is its cardinality. Let us denote by f_{card} the map of the form $f\colon\ \mathbf{2}^{\mathcal{M}} \times \mathbf{2}^{\mathcal{M}} \to \mathbb{N}$, for which we have $f_{\mathrm{card}}(\mathcal{A}, \mathcal{B}) := |\mathcal{A}| + |\mathcal{B}|$. We state the problem

$$\mathrm{R}_3\colon\ \left(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} \subseteq \mathbf{2}^X,\ f_{\mathrm{card}}\right) \xrightarrow{\min f_{\mathrm{card}}} (\mathfrak{A}, \mathfrak{B})\,.$$

Interpretation of committees in terms of alternative covers

Suppose that in a space X finite disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ are fixed, as well as some class F of real-valued functions over X. Let us consider the inequality system

$$\begin{cases} f(\boldsymbol{x}) > 0, & \text{if } \boldsymbol{x} \in \mathcal{A}\,, \\ f(\boldsymbol{x}) < 0, & \text{if } \boldsymbol{x} \in \mathcal{B}\,. \end{cases} \tag{5.7}$$

Recall that by a *committee* of inequality system (5.7) we mean a finite collection of maps $\mathcal{K} := \{f_1, f_2, \ldots, f_q\} \subset F$, such that each inequality of system (5.7) is satisfied by more than half maps from $\mathcal{K}$.

For some subset of real-valued functions $F_0 \subseteq F$, the sets

$$\mathbf{C}_{>}(F_0) = \bigcap_{f \in F_0} \{\boldsymbol{x} \in X\colon\ f(\boldsymbol{x}) > 0\}$$

and

$$\mathbf{C}_{<}(F_0) = \bigcap_{f \in F_0} \{\boldsymbol{x} \in X\colon\ f(\boldsymbol{x}) < 0\}$$

will be called *F-polyhedra* of the space X. Let us denote by $\mathcal{M}(F, X)$ the class of all F-polyhedra of the space X. The pattern recognition problems R_1–R_3 can then be regarded in the situation when the class $\mathcal{M}$ is the class $\mathcal{M}(F, X)$, that is,

$$\mathrm{R}_1'\colon\ \left(X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} = \mathcal{M}(F, X)\right) \to (\mathfrak{A}, \mathfrak{B})\,,$$

and analogously for R_2 and R_3.

With each committee $\mathcal{K}$ of inequality system (5.7) can be put in correspondence an alternative cover $(\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K}))$ of the pair $(\mathcal{A}, \mathcal{B})$ as follows:

$$\begin{aligned}\mathfrak{A} &:= \left\{\mathbf{C}_{>}(\mathcal{K}')\colon\ |\mathcal{K}'| > \tfrac{1}{2}|\mathcal{K}|,\ \mathcal{K}' \subseteq \mathcal{K}\right\} ,\\ \mathfrak{B} &:= \left\{\mathbf{C}_{<}(\mathcal{K}')\colon\ |\mathcal{K}'| > \tfrac{1}{2}|\mathcal{K}|,\ \mathcal{K}' \subseteq \mathcal{K}\right\} .\end{aligned}$$

It follows from the definitions of a committee and family $(\mathfrak{A}, \mathfrak{B})$, that $(\mathfrak{A}, \mathfrak{B})$ is an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$. It is evident that $\mathbf{C}_{>}(\mathcal{K}') \cap \mathbf{C}_{<}(\mathcal{K}'') = \emptyset$ when $|\mathcal{K}'| > \frac{1}{2}|\mathcal{K}|$ and $|\mathcal{K}''| > \frac{1}{2}|\mathcal{K}|$. Let us denote by $\mathbf{max}_{\subseteq}\mathfrak{A}$ the set of maximal elements of the poset $(\mathfrak{A}, \subseteq)$, and suppose

$$\mathfrak{A} = \mathbf{max}_{\subseteq}\mathcal{A}\ , \quad \mathfrak{B} = \mathbf{max}_{\subseteq}\mathcal{B}$$

and

$$\begin{aligned}\mathfrak{A}(\mathcal{K}) &:= \{A' \in \mathbf{max}_{\subseteq}\mathfrak{A}\colon\ A' \cap \mathcal{A} \neq \emptyset\} ,\\ \mathfrak{B}(\mathcal{K}) &:= \{B' \in \mathbf{max}_{\subseteq}\mathfrak{B}\colon\ B' \cap \mathcal{B} \neq \emptyset\} .\end{aligned}$$

Thus, $(\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K}))$ is an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra.

With a committee $\mathcal{K}$ of system (5.7) we have put in one-to-one correspondence the alternative cover $(\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K}))$. Let us depict symbolically the scheme of constructing alternative covers of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra on the basis of the committee method:

$$\mathrm{R}'_{\mathrm{com}}\colon\ (X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} = \mathcal{M}(F, X)) \to \mathcal{K} \to (\mathfrak{A}(\mathcal{K}), \mathfrak{B}(\mathcal{K})) .$$

There exist systems of the form (5.7) such that for two distinct committees $\mathcal{K}_1$ and $\mathcal{K}_2$ of the same cardinality, the following relations hold:

$$\begin{gathered}|\mathcal{K}_1| = |\mathcal{K}_2| ,\\ |\mathfrak{A}(\mathcal{K}_1)| + |\mathfrak{B}(\mathcal{K}_1)| \neq |\mathfrak{A}(\mathcal{K}_2)| + |\mathfrak{B}(\mathcal{K}_2)| .\end{gathered}$$

As an example, let X be the space $\mathbb{R}^2$, and F the class of linear functionals; let us suppose

$$\begin{aligned}\mathcal{A} &:= \{\boldsymbol{a}_1 = (-1.5, 1.5),\ \boldsymbol{a}_2 = (1.5, 1.5),\ \boldsymbol{a}_3 = (0, -1)\} ,\\ \mathcal{B} &:= \{\boldsymbol{b}_1 = (-8, 3),\ \boldsymbol{a}_2 = (0, -5),\ \boldsymbol{a}_3 = (8, -3)\} ,\\ \mathcal{K}_1 &:= \{f_1 = \boldsymbol{x}_1 - \boldsymbol{x}_2 + 2,\ f_2 = -\boldsymbol{x}_2,\ f_3 = -\boldsymbol{x}_1 - \boldsymbol{x}_2 + 2\} ,\\ \mathcal{K}_2 &:= \{g_1 = \boldsymbol{x}_1 - \boldsymbol{x}_2 + 4,\ g_2 = -\boldsymbol{x}_2 - 2,\ g_3 = -\boldsymbol{x}_1 - \boldsymbol{x}_2 + 4\} ,\\ \mathfrak{A}(\mathcal{K}_1) &:= \{\mathbf{C}_{>}(\{f_1, f_2\}),\ \mathbf{C}_{>}(\{f_1, f_3\}),\ \mathbf{C}_{>}(\{f_2, f_3\})\} ,\\ \mathfrak{B}(\mathcal{K}_1) &:= \{\mathbf{C}_{<}(\{f_1, f_2\}),\ \mathbf{C}_{<}(\{f_1, f_3\}),\ \mathbf{C}_{<}(\{f_2, f_3\})\} ,\\ \mathfrak{A}(\mathcal{K}_2) &:= \{\mathbf{C}_{>}(\{f_1, f_2, f_3\})\} ,\\ \mathfrak{B}(\mathcal{K}_2) &:= \{\mathbf{C}_{<}(\{f_1, f_2\}),\ \mathbf{C}_{<}(\{f_1, f_3\}),\ \mathbf{C}_{<}(\{f_2, f_3\})\} .\end{aligned}$$

The committees $\mathcal{K}_1$ and $\mathcal{K}_2$ contain an equal and minimal possible number of members (namely, three), but the alternative covers corresponding to these committees have different cardinalities: $|\mathfrak{A}(\mathcal{K}_1)| + |\mathfrak{B}(\mathcal{K}_1)| = 6$, while $|\mathfrak{A}(\mathcal{K}_2)| + |\mathfrak{B}(\mathcal{K}_2)| = 4$.

Thus, we can conclude that a committee of system (5.7) represents a concise form of determining an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra, but not necessarily of minimal cardinality. This conclusion is rather important for the committee method, because it motivates us to take into account a very natural and relevant additional criterion of decision rule optimization. A simplest application of such an approach is as follows: if several committees of system (5.7) are found, then one should choose a committee such that the corresponding alternative cover has minimal cardinality.

In the case when $X := \mathbb{R}^n$, and F is the class of linear functionals, let us consider subsystems of system (5.7), defined as follows. Let $\mathcal{L} \subset \mathcal{A} \cup \mathcal{B}$; then an inequality $f(\boldsymbol{x}) > 0$ belongs to the subsystem under consideration when $\boldsymbol{x} \in \mathcal{A}$, and an inequality $f(\boldsymbol{x}) < 0$ belongs to this subsystem when $\boldsymbol{x} \in \mathcal{B}$. Let us denote this subsystem by $\mathrm{S}(\mathcal{A}', \mathcal{B}')$, where $\mathcal{A}' := \mathcal{L} \cap \mathcal{A}$ and $\mathcal{B}' := \mathcal{L} \cap \mathcal{B}$. Suppose that inequality system (5.7) is infeasible, and $\mathcal{K} := \{f_1, f_2, \dots, f_q\}$ is its committee. Let $(\mathfrak{A}', \mathfrak{B}')$ be the alternative cover that corresponds to the committee $\mathcal{K}$. The pair $(\mathfrak{A}', \mathfrak{B}')$ is an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of convex cones.

Alternative covers do not necessarily possess committees that generate them.

The following algorithmic problem can be stated. For a given inequality system of the form (5.7), to find a committee that generates an alternative cover of cardinality, minimal among all committees of system (5.7).

Let $\{\mathbf{J}(\mathcal{A}_i, \mathcal{B}_i) : \ i \in [\mathrm{q}]\}$ be the family of all inclusion-maximal feasible subsystems of system (5.7). Let us form the $|\mathcal{A} \,\dot\cup\, \mathcal{B}| \times \mathrm{q}$ incidence matrix $\mathbf{E} = (e_{ij})$: its first $|\mathcal{A}|$ rows are marked by the elements from $\mathcal{A}$, the last $|\mathcal{B}|$ rows are marked by the elements from $\mathcal{B}$, and the columns are marked by the maximal feasible subsystems of system (5.7).

In terms of the matrix $\mathbf{E}$, we can restate the problem as follows:

1. The problem of constructing a committee with the minimal number of members is reduced to choosing an inclusion-minimal collection of columns of the matrix $\mathbf{E}$ such that in each subrow, obtained as the intersection of the corresponding entire row with the above-mentioned columns, the number of units exceeds the number of zeros.
2. The problem of constructing a committee that generates an alternative cover of minimal cardinality is reduced to choosing a collection of columns of the matrix $\mathbf{E}$, such that condition 1 is fulfilled and, moreover, the total number of inclusion-minimal subrows of the upper and lower semimatrices is minimal among all such collections.

Interpretation of logical decision trees in terms of alternative covers

Suppose that in a space X finite disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ are fixed, as well as some class F of real-valued functions over X. Let us consider a binary tree $\mathcal{T} := (V, E)$ with root v_0, with each node of which a function from F is associated, that is, a map $\psi\colon V \to F$ is determined. Let v be a leaf of the tree $\mathcal{T}$ and $(v_0, v_1, \dots, v_{k-1}, v)$ the path from the root of the tree to the current node v. Then the leaf v is assigned the F-polyhedron

$$\mathbf{C}_v := \mathbf{C}_>\left(\left\{(-1)^{\sigma(v_0)}\psi(v_0), (-1)^{\sigma(v_1)}\psi(v_1), \dots, (-1)^{\sigma(v_{k-1})}\psi(v_{k-1})\right\}\right) ,$$

where $\sigma(v_i) := \begin{cases} 1, & \text{if } v_i \text{ is the left child of } v_{i-1}, \\ 0, & \text{if } v_i \text{ is the right child of } v_{i-1}. \end{cases}$

Let V' be the set of leaves of the tree $\mathcal{T}$. Let us suppose

$$\begin{aligned} \mathbf{C}_{V'}(\mathcal{T})\colon &= \{\mathbf{C}_v\colon\ v \in V'\} , \\ \mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T})\colon &= \{\mathbf{C}_v\colon\ v \in \mathbf{C}_{V'}(\mathcal{T}) \cap \mathcal{A}\} , \\ \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T})\colon &= \{\mathbf{C}_v\colon\ v \in \mathbf{C}_{V'}(\mathcal{T}) \cap \mathcal{B}\} . \end{aligned}$$

If $\mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T}) \cap \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T}) = \emptyset$ and $\mathcal{A} \cup \mathcal{B} \subset \bigcup_{v \in V'} \mathbf{C}_v$, then $\left(\mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T}), \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T})\right)$ is an alternative cover of $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra.

Let us depict symbolically the scheme of constructing an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in the class of F-polyhedra on the basis of logical decision trees:

$$\mathrm{R}'_{\mathrm{tree}}\colon\ (X,\ (\mathcal{A}, \mathcal{B}),\ \mathcal{M} = \mathcal{M}(F, X)) \to (\mathcal{T} := (V, E)) \to \left(\mathbf{C}^{\mathcal{A}}_{V'}(\mathcal{T}), \mathbf{C}^{\mathcal{B}}_{V'}(\mathcal{T})\right) .$$

Constructing an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$ in accordance with the scheme $\mathrm{R}'_{\mathrm{tree}}$ has, compared to the scheme $\mathrm{R}'_{\mathrm{com}}$, the following advantages:

(1) Suppose that for the pair $(\mathcal{A}, \mathcal{B})$ two trees $\mathcal{T}_1 := (V_1, E_1)$ and $\mathcal{T}_2 := (V_2, E_2)$ are given, such that $(\mathbf{C}^{\mathcal{A}}_{V'_1}(\mathcal{T}_1), \mathbf{C}^{\mathcal{B}}_{V'_1}(\mathcal{T}_1))$ and $(\mathbf{C}^{\mathcal{A}}_{V'_2}(\mathcal{T}_2), \mathbf{C}^{\mathcal{B}}_{V'_2}(\mathcal{T}_2))$ are alternative covers of the pair $(\mathcal{A}, \mathcal{B})$, and they are inclusion-minimal with respect to this property. If $|V_1| = |V_2|$, then the corresponding alternative covers also have the same cardinality.

(2) The obtained decision and the process itself have good substantial interpretation.

(3) In view of (2), organization of the constructing an alternative cover of the pair $(\mathcal{A}, \mathcal{B})$, in accordance with the scheme $\mathrm{R}'_{\mathrm{tree}}$, in interactive mode appears to be natural and efficient.

(4) When constructing an alternative cover in accordance with the scheme $\mathrm{R}'_{\mathrm{tree}}$, the problem of missing fragments of initial data, difficult in the case of the scheme $\mathrm{R}'_{\mathrm{com}}$, is solved naturally and quite easily.

Notes

The notion of committee of a linear inequality system was first formulated, in the explicit form, in notes [1, 2]. A systematical study of various committee constructions appeared in works [94, 95] and transformed over the past decades into an important independent branch of pure and applied mathematical investigations that exercise a significant influence on the arsenal of efficient methods in optimization and pattern recognition. We refer the reader to monograph [96] for a comprehensive review of fundamental results in the committee theory, as well as their applications. Key advances in the ever widening bounds of the theory of committee constructions are the research subject, for example, in surveys [76, 97–99].

On page 115, we briefly describe the two-class pattern recognition problem following [96].

The material in Section 5.1 is particularly based on the results of works [47, 49–51]. The combinatorial algorithm of extracting MFSs of an infeasible system of linear inequalities, as noted on page 117, repeatedly builds the blockers of set families; recall that constructing the blockers can also be interpreted as the finding of inclusion-minimal covers of the columns of (0, 1)-matrices that determine the families. See, for example, works [27, 61, 68, 109, 114, 157, 164] on the constructing representative systems.

The connectedness of the squares of subgraphs of the graphs of MFSs, mentioned on page 122, was proved by the second author of work [58].

In Section 5.2, we adopt fragments of surveys [76, 98]. We use the term *multi-committee* for committee constructions that represent, in the general case, multisets; the properties of the hypergraphs of MFSs are analyzed, in particular, in [74, 75].

The outline of alternative covers presented in Section 5.3 follows article [48].

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List of notation

- $:=$ – equals by definition
- $\delta(s,t)$ – Kronecker delta, equal to 1 when $s=t$, and 0 otherwise
- $\binom{n}{m}$ – binomial coefficient, equal to $\frac{n!}{m!(n-m)!}$
- $\mathbb{R}$ – real numbers

Sets and families
- $[m]$ – set of consecutive integers $\{1,2,\ldots,m\}$
- $\binom{U}{2}$ – family of unordered 2-subsets of a set U
- $A \dot\cup B$ – disjoint union of sets or families A and B
- $A \times B$ – Cartesian product of sets A and B
- $\mathrm{V}(\mathcal{A})$ – ground set $\bigcup_{i=1}^{\alpha} A_i$ of a family $\mathcal{A} := \{A_1,\ldots,A_\alpha\}$
- $|A|$ – cardinality (number of elements) of a set A
- $\#\mathcal{A}$ – number of sets in a family $\mathcal{A}$
- $\mathfrak{B}(\mathcal{A})$ – blocker of a family $\mathcal{A}$
- $\mathbf{2}^V$ – power set of a set V
- $A^\perp$ – complement, $\mathrm{V}(\mathcal{A}) - A$, of a set A from a family $\mathcal{A}$
- $\mathcal{A}^\perp$ – family of complements $\{A^\perp\colon\ A \in \mathcal{A}\}$

Topological spaces
- $\mathrm{Fr}(\cdot)$ – boundary of a subset of a topological space

Partially ordered sets (posets)
- $a \preceq b$ – elements a and b are comparable in a poset
- $\hat{0}$ and $\hat{1}$ – least and greatest elements of a lattice, respectively
- $\mathbb{B}(m)$ – Boolean lattice of rank m
- $\mathbb{B}(m)^{(1)}$ – atom set of the Boolean lattice $\mathbb{B}(m)$
- $\rho(\cdot)$ – rank function of a poset
- $\mathbf{min}\,\mathcal{V}$ and $\mathbf{max}\,\mathcal{V}$ – sets of minimal and maximal elements of a poset $\mathcal{V}$, respectively
- $\mathfrak{I}(\mathcal{V})$ and $\mathfrak{F}(\mathcal{V})$ – order ideal and order filter generated by a set $\mathcal{V}$, respectively

Maps
- $f\colon\ A \to B,\ a \mapsto b$ – map f from a set A to a set B; the image of an element $a \in A$ is an element $b := f(a) \in B$
- $f|_A\colon A \to C,\ a \mapsto f(a)$ – restriction of a map $f\colon B \to C$ to a subset $A \subseteq B$

Complexes and graphs
- Δ – abstract simplicial complex
- $\mathbf{max}\,\Delta$ – facet family of a complex Δ

- $\Delta(\mathcal{A})$ – complex with facet family $\mathcal{A}$
- (V, Δ) – complex on vertex set V
- $(V, \Delta) \simeq (V', \Delta')$ – complexes (V, Δ) and (V', Δ') are isomorphic
- $\Delta^{\vee}$ – Alexander dual of a complex Δ
- $\dim F$ – dimension of a face F of a complex
- $\dim \Delta$ – dimension of a complex Δ
- $f_j(\Delta)$ – number of j-dimensional faces of a complex Δ
- $(V, \mathcal{E})$ – simple graph with vertex set V and edge set $\mathcal{E}$, or the hypergraph with vertex set V and hyperedge family $\mathcal{E}$
- $\mathcal{N}(v)$ – neighborhood of a vertex v in a graph
- $\mathtt{ISG}(V, \Delta)$ and $\overrightarrow{\mathtt{ISG}}(V, \Delta)$ – undirected and oriented graphs of an independence system associated with a complex (V, Δ), respectively
- $\mathtt{MFSG}(\mathfrak{S})$ – graph of maximal feasible subsystems (the graph of MFSs) of an infeasible system of linear inequalities $\mathfrak{S}$

Vectors

- $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ – standard scalar product $\sum_{1 \le k \le n} a_k b_k$ of n-dimensional real vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
- $\|\boldsymbol{a}\| := \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$ – Euclidean norm of a real vector $\boldsymbol{a}$

Systems of constraints

- $\boldsymbol{A}(\mathfrak{S})$ – set of vectors $\{\boldsymbol{a}_i : \ i \in [m]\}$ that determine the system $\mathfrak{S} := \{\langle \boldsymbol{a}_i, \mathbf{x} \rangle > 0 : \boldsymbol{a}_i, \mathbf{x} \in \mathbb{R}^n; \ \|\boldsymbol{a}_i\| = 1\}$
- $\mathbf{J}$ and $\mathbf{I}$ – family of the multi-indices of maximal feasible subsystems (MFSs) and the family of the multi-indices of minimal (irreducible) infeasible subsystems (IISs), respectively
- ν_k and τ_k – number of feasible subsystems and the number of infeasible subsystems, of cardinality k, respectively

Boolean functions

- $\mathbf{B}$ – set $\{0, 1\}$
- $\mathbf{B}^m$ – unit discrete m-cube
- $\operatorname{supp}(\boldsymbol{x})$ – set $\{i \in [m] : \ x_i = 1\}$ corresponding to a tuple $\boldsymbol{x} \in \mathbf{B}^m$
- $|\boldsymbol{\alpha}|$ – number of units in a tuple $\boldsymbol{\alpha} \in \mathbf{B}^m$
- $\boldsymbol{\alpha} \oplus \boldsymbol{\beta}$ – coordinate-wise summation of tuples $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_2^m$ of length m over the field $\mathbb{F}_2$ with two elements that compose the set $\mathbf{B}$
- $\mathcal{M}_m$ – class of all monotone Boolean functions (MBFs) of m variables
- $\mathfrak{f}^{-1}(0)$ and $\mathfrak{f}^{-1}(1)$ – set of zeros and the set of units of a monotone Boolean function $\mathfrak{f}$, respectively
- $\mathfrak{Q}(\mathfrak{f}) := \mathbf{max}\, \mathfrak{f}^{-1}(0)$ and $\mathfrak{P}(\mathfrak{f}) := \mathbf{min}\, \mathfrak{f}^{-1}(1)$ – set of upper zeros and the set of lower units of a monotone Boolean function $\mathfrak{f}$, respectively

- $\mathcal{O}_{\mathfrak{f}}$ – operator that calculates, for a tuple from $\mathbf{B}^m$, the corresponding value of a monotone Boolean function $\mathfrak{f}$
- $\varphi(G, \mathfrak{f})$ – number of invocations, by an algorithm G, of the operator $\mathcal{O}_{\mathfrak{f}}$ when inferring a MBF $\mathfrak{f} \in \mathcal{M}_m$

Subspaces, hulls and convex sets
- $\operatorname{conv}(\boldsymbol{X})$ – convex hull of a set $\boldsymbol{X} \subset \mathbb{R}^n$
- $\operatorname{lin}(\boldsymbol{X})$ – linear hull of a set $\boldsymbol{X} \subset \mathbb{R}^n$
- $\mathbf{H}(T)$ – linear subspace $\bigcap_{t \in T} \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_t, \boldsymbol{x} \rangle = 0\}$
- $\mathbf{C}_>(T)$ – open cone $\bigcap_{t \in T} \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_t, \boldsymbol{x} \rangle > 0\}$
- $\mathbf{C}_<(T) := -\mathbf{C}_>(T)$
- $\overline{\mathbf{C}_>(T)}$ – closed cone $\bigcap_{t \in T} \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_t, \boldsymbol{x} \rangle \geq 0\}$
- $\mathcal{F}(L, T)$ – open, with respect to a subspace $\mathbf{H}(L)$, face $\mathbf{H}(L) \cap \mathbf{C}_>(T - L)$ of a closed cone $\overline{\mathbf{C}_>(T)}$
- $[\boldsymbol{x}, \boldsymbol{y}] := \operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\}$ – closed segment between points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
- $(\boldsymbol{x}, \boldsymbol{y})$ – open segment $[\boldsymbol{x}, \boldsymbol{y}] - \{\boldsymbol{x}, \boldsymbol{y}\}$

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